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ON SOME FOUR-DIMENSIONAL ALMOST KÄHLER EINSTEIN MANIFOLDS

Dedicated to the memory of Professor Shukichi Tanno

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Abstract

Concerning the intergarability of almost Kähler manifolds, it is known the conjecture by S. I. Goldberg that a compact almost Kähler Einstein manifold is Kähler. In this paper, we will give some positive partial answers to the conjecture.

1. Introduction

An almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if the Kähler form of M is closed or, equivalently, $\mathfrak{S}_{X,Y,Z}\, g((\nabla_X J)Y, Z) = 0$ for $X, Y, Z \in \mathfrak{X}(M)$, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum with respect to X, Y, Z. It follows immediately from the definition that a Kähler manifold ($\nabla J = 0$) is an almost Kähler manifold. A non-Kähler, almost Kähler manifold is called a strictly almost Kähler manifold. It is well-known that if the almost complex structure of an almost Kähler manifold is integrable, then it is a Kähler manifold. Concerning the integrability of almost Kähler manifold, the following conjecture by S. I. Goldberg is known ([5]).

Conjecture. *The almost complex structure of a compact almost Kähler Einstein manifold is integrable.*

The above conjecture is true in the case where the scalar curvature is nonnegative ([15]). However, it is still open in the remaining case and some progresses have been made by many authors under certain additional curvature conditions. The authors and A. Yamada proved the following Theorems A and B.

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THEOREM A ([14]). *A four-dimensional almost Kähler Einstein and $*$-Einstein manifold is integrable.*

THEOREM B ([14]). *A four-dimensional compact almost Kähler Einstein and weakly $*$-Einstein manifold is integrable.*

In the present paper, we shall prove the following Theorems 1 and 2 which improve Theorems A and B respectively.

THEOREM 1. *Let $M = (M, J, g)$ be a four-dimensional almost Kähler Einstein manifold of constant $*$-scalar curvature. Then M is a Kähler manifold.*

THEOREM 2. *Let $M = (M, J, g)$ be a four-dimensional compact almost Kähler Einstein manifold. If the norm of skew-symmetric part of the Ricci $*$-tensor is a constant, then M is a Kähler manifold.*

In [1], J. Armstrong proved that a four-dimensional compact almost Kähler Einstein manifold of constant $*$-scalar curvature is integrable. So, our Theorem 1 also improves his result. Further, he proved that if M is a four-dimensional compact almost Kähler Einstein manifold then the equality $\tau^* - \tau = 0$ holds at some point of M, where τ and τ^* are the scalar curvature and the $*$-scalar curvature of M respectively. We see also that if M is a compact four-dimensional almost Kähler Einstein manifold with constant negative scalar curvature, then there exist a constant δ (≤ 1) such that the inequality $\tau \leq \tau^* \leq \delta\tau$ holds. In this paper, we shall also prove the following.

THEOREM 3. *Let $M = (M, J, g)$ be a four-dimensional compact almost Kähler Einstein manifold with negative scalar curvature. Then, the $*$-scalar curvature satisfies $\tau \leq \tau^* \leq -\tau/6$ on M.*

The authors would like to express their thanks to the referee for his many valuable advice.

2. Preliminaries

In this section, we prepare several fundamental formulas which will be used in our argument. Some of them are established in [13] and [14].

Let $M = (M, J, g)$ be a four-dimensional almost Hermitian manifold with almost Hermitian structure (J, g). The Kähler form Ω of M is defined by $\Omega(X, Y) = g(X, JY)$ for $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M. We assume that M is oriented by the volume form $dM = \Omega^2/2$. We denote by ∇, R, ρ, and τ the Riemannian connection, the curvature tensor, the Ricci tensor and the scalar curvature of M respectively. The curvature tensor R is defined by $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$ for $X, Y, Z \in \mathfrak{X}(M)$. We denote by ρ^* the Ricci $*$-tensor of M defined by

$$\rho^*(x,y) = \frac{1}{2} \text{ trace of } (z \mapsto R(x,Jy)Jz) \tag{2.1}$$

for $x, y, z \in T_pM$, $p \in M$. Further, we denote by τ^* the $*$-scalar curvature of M which is the trace of the linear endomorphism Q^* defined by $g(Q^*x, y) = \rho^*(x, y)$ for $x, y \in T_pM$, $p \in M$. By the definition, we see immediately

$$\rho^*(x,y) = \rho^*(Jy, Jx), \tag{2.2}$$

and hence ρ^* is symmetric if and only if ρ^* is J-invariant. We may also note that if M is Kähler, then $\rho^* = \rho$ holds on M. An almost Hermitian manifold M is called a weakly $*$-Einstein manifold if $\rho^* = (\tau^*/4)g$ holds on M and a $*$-Einstein manifold if it is a weakly $*$-Einstein manifold of constant $*$-scalar curvature. It is known that the following identity holds for any four-dimensional almost Hermitian manifold ([7]):

$$\frac{1}{2}\{\rho(x,y) + \rho(Jx,Jy)\} - \frac{1}{2}\{\rho^*(x,y) + \rho^*(y,x)\} = \frac{\tau - \tau^*}{4} g(x,y) \tag{2.3}$$

for $x, y \in T_pM$, $p \in M$.

The curvature operator $\mathscr{R}$ is the symmetric endomorphism of the vector bundle $\bigwedge^2 M$ of real 2-forms over M defined by

$$g(\mathscr{R}(\iota(x) \wedge \iota(y)), \iota(z) \wedge \iota(w)) = -g(R(x,y)z, w) \tag{2.4}$$

for $x, y, z, w \in T_pM$, $p \in M$, where $\iota : TM \to T^*M$ denotes the duality defined by means of the metric g.

The following decomposition for the vector bundle $\bigwedge^2 M$ of real 2-forms on M is useful in our arguments:

$$\textstyle\bigwedge^2 M = \boldsymbol{R}\Omega \oplus \bigwedge_0^{1,1} M \oplus LM \tag{2.5}$$

where $\bigwedge_0^{1,1} M$ denotes the vector bundle of real primitive J-invariant 2-forms, LM the vector bundle of real primitive J-skew-invariant 2-forms over M respectively. The bundle LM is endowed with the natural complex structure (also denoted by J) which is defined by $J\Phi(X, Y) = -\Phi(JX, Y) (X, Y \in \mathfrak{X}(M))$ for any local section Φ of LM. The bundle $\bigwedge_0^{1,1} M$ is identified itself with the bundle $\bigwedge_-^2 M$ of anti-self-dual 2-forms, while the sum $\boldsymbol{R}\Omega \oplus LM$ is the bundle $\bigwedge_+^2 M$ of self-dual 2-forms. Further, it is well-known that M is Einstein if and only if both $\bigwedge_+^2 M$ and $\bigwedge_-^2 M$ are preserved by the curvature operator $\mathscr{R}$ ([8]). Let $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ be any local orthonormal bases of T_pM, $p \in M$. Then, the Kähler form is represented by $\Omega = -e^1 \wedge e^2 - e^3 \wedge e^4$ with respect to the dual bases $\{e^i\} = \{\iota(e_i)\}$. Further, we may observe that

$$\{\Phi, J\Phi\} = \left\{\frac{1}{\sqrt{2}}(e^1 \wedge e^3 - e^2 \wedge e^4), \frac{1}{\sqrt{2}}(e^1 \wedge e^4 + e^2 \wedge e^3)\right\}$$

and

$$\{\Psi_1, \Psi_2, \Psi_3\} = \left\{\frac{1}{\sqrt{2}}(e^1 \wedge e^2 - e^3 \wedge e^4), \frac{1}{\sqrt{2}}(e^1 \wedge e^3 + e^2 \wedge e^4), \frac{1}{\sqrt{2}}(e^1 \wedge e^4 - e^2 \wedge e^3)\right\}$$

are local orthonormal bases of LM and $\bigwedge_0^{1,1} M$ respectively.

We define 2-forms φ and ψ on M by

$$\varphi(x, y) = \text{trace of } (z \mapsto J(\nabla_x J)(\nabla_y J)z), \quad \psi(x, y) = \text{trace of } (z \mapsto R(x, y)Jz) \tag{2.6}$$

for $x, y, z \in T_pM$, $r \in M$. Then, the first Chern form γ of M is given by

$$8\pi\gamma = -\varphi + 2\psi. \tag{2.7}$$

It is well-known that the Chern class $c_1(M)$ of M is represented by γ in the de Rham cohomology group ([2]).

In this paper, for any orthonormal bases (resp. any local orthonormal frame field) $\{e_i\}$ at any point $p \in M$ (resp. on a neighborhood of p), we shall adopt the following notational convention:

$$\begin{aligned} R_{ijkl} &= g(R(e_i, e_j)e_k, e_l), \ldots, R_{\bar{i}\bar{j}\bar{k}\bar{l}} = g(R(Je_i, Je_j)Je_k, Je_l), \\ \rho_{ij} &= \rho(e_i, e_j), \ldots, \rho_{\bar{i}\bar{j}} = \rho(Je_i, Je_j), \\ \rho^*_{ij} &= \rho^*(e_i, e_j), \ldots, \rho^*_{\bar{i}\bar{j}} = \rho^*(Je_i, Je_j), \\ J_{ij} &= g(Je_i, e_j), \quad \nabla_i J_{jk} = g((\nabla_{e_i} J)e_j, e_k), \end{aligned} \tag{2.8}$$

and so on, where the Latin indices run over the range $1, \ldots, 4$. Then, we have

$$J_{ij} = -J_{ji}, \quad \nabla_i J_{jk} = -\nabla_i J_{kj}, \quad \nabla_i J_{\bar{j}\bar{k}} = -\nabla_i J_{jk}. \tag{2.9}$$

In the sequel, we assume that $M = (M, J, g)$ is a four-dimensional almost Kähler manifold. Then, the following curvature identity is known ([6]):

$$R_{ijkl} - R_{ij\bar{k}\bar{l}} - R_{\bar{i}\bar{j}kl} + R_{\bar{i}\bar{j}\bar{k}\bar{l}} + R_{\bar{i}j\bar{k}l} + R_{\bar{i}jk\bar{l}} + R_{i\bar{j}\bar{k}l} + R_{i\bar{j}k\bar{l}} = 2\sum_a (\nabla_a J_{ij})\nabla_a J_{kl}.$$

Thus, we immediately have

$$\rho^*_{ij} + \rho^*_{ji} - \rho_{ij} - \rho_{\bar{i}\bar{j}} = \sum_{a,k} (\nabla_a J_{ik})\nabla_a J_{jk}$$

and

$$\|\nabla J\|^2 = 2(\tau^* - \tau). \tag{2.10}$$

Since almost Kähler manifold is quasi-Kähler and hence semi-Kähler, the equalities

$$\nabla_i J_{jk} = -\nabla_{\bar{i}} J_{\bar{j}k}, \quad \sum_a \nabla_a J_{ai} = 0 \tag{2.11}$$

hold. Taking account of (2.9) and (2.11), we may observe

$$\nabla\Omega = \alpha \otimes \Phi - J\alpha \otimes J\Phi, \tag{2.12}$$

where α is a local 1-form, $\Phi, J\Phi \in LM$ and the local 1-form $J\alpha$ is defined by $J\alpha(X) = -\alpha(JX)$ $(X \in \mathfrak{X}(M))$. From this equality, we have

$$\|\alpha\|^2 = \frac{1}{2}\|\nabla\Omega\|^2 = \frac{1}{4}\|\nabla J\|^2 = \frac{\tau^* - \tau}{2}. \tag{2.13}$$

We put $g(\nabla_{e_i} e_j, e_k) = \Gamma_{ijk}$. Then, $\Gamma_{ijk} = -\Gamma_{ikj}$ hold. Further, taking account of (2.12), we have

$$\Gamma_{i24} - \Gamma_{i13} = -\frac{1}{\sqrt{2}}\alpha_{\bar{i}}, \quad \Gamma_{i23} + \Gamma_{i14} = -\frac{1}{\sqrt{2}}\alpha_i. \tag{2.14}$$

Now, we assume that $M = (M, J, g)$ is in addition an Einstein manifold. Then, we have

$$R_{1212} = R_{3434}, \quad R_{1313} = R_{2424}, \quad R_{1414} = R_{2323}. \tag{2.15}$$

Further, the equality (2.3) is reduced to

$$\rho^*_{ij} + \rho^*_{ji} = \frac{\tau^*}{2}\delta_{ij}. \tag{2.16}$$

Since $\mathscr{R}(\bigwedge^2_+ M) \subset \bigwedge^2_+ M$, we may put

$$\begin{aligned} \mathscr{R}(\Phi) &= u\Phi + wJ\Phi + B_0\Omega, \\ \mathscr{R}(J\Phi) &= w\Phi + vJ\Phi + B_1\Omega. \end{aligned} \tag{2.17}$$

Then, the equalities

$$\begin{aligned} u &= g(\mathscr{R}(\Phi), \Phi) = -(R_{1313} - R_{1324}), \\ v &= g(\mathscr{R}(J\Phi), J\Phi) = -(R_{1414} + R_{1423}), \\ w &= g(\mathscr{R}(\Phi), J\Phi) = -(R_{1314} + R_{1323}), \\ B_0 &= \frac{g(\mathscr{R}(\Phi), \Omega)}{g(\Omega, \Omega)} = \frac{1}{2\sqrt{2}}(\rho^*_{14} - \rho^*_{41}) = \frac{1}{\sqrt{2}}\rho^*_{14}, \\ B_1 &= \frac{g(\mathscr{R}(J\Phi), \Omega)}{g(\Omega, \Omega)} = -\frac{1}{2\sqrt{2}}(\rho^*_{13} - \rho^*_{31}) = -\frac{1}{\sqrt{2}}\rho^*_{13}. \end{aligned}$$

are derived from direct calculation. We have also

$$u + v = -(\rho^*_{11} - \rho_{11}) = -\frac{\tau^* - \tau}{4}. \tag{2.18}$$

Now, we define functions A, A', B, C, D, G and K on M respectively by

$$\begin{aligned}
A &= \sum (\nabla_a J_{ij}) R_{ijkl} \nabla_a J_{kl},\\
A' &= \sum (\nabla_a J_{ij})(\nabla_a R_{ijkl}) J_{kl},\\
B &= \sum (\nabla_a J_{ij})(\nabla_a J_{kl})(\nabla_b J_{ij}) \nabla_b J_{kl},\\
C &= \sum R_{ijkl} R_{\bar{i}\bar{j}k\bar{l}},\\
D &= \sum (R_{ijkl} - R_{ij\bar{k}\bar{l}})^2,\\
G &= \sum (\rho^*_{ij} - \rho^*_{ji})^2,\\
K &= (u - v)^2 + 4w^2.
\end{aligned}$$

Then, the equalities

$$\begin{aligned}
A &= \frac{1}{4} B = \frac{(\tau^* - \tau)^2}{2},\\
C &= -2K + \frac{(\tau^* - \tau)^2}{8},\\
G &= 4\|\rho^*\|^2 - (\tau^*)^2
\end{aligned} \tag{2.19}$$

are valid (see [13], (2.26), (3.9), (2.29)). Moreover, from (2.16), we have

$$G = 16\{(\rho^*_{13})^2 + (\rho^*_{14})^2\}. \tag{2.20}$$

We denote by $\mathscr{R}_{LM}$ the restriction of $\mathscr{R}$ to the subbundle LM and put $\mathscr{R}'_{LM} = P_{LM} \circ \mathscr{R}_{LM}$, where $P_{LM} : \bigwedge^2 M \to LM$ is the projection. Then, from (2.17) and (2.18), we obtain

$$K = (u + v)^2 + 4(w^2 - uv) = \frac{(\tau^* - \tau)^2}{16} - 4 \det \mathscr{R}'_{LM}.$$

We also note that

$$\|\mathscr{R}_{LM}\|^2 = \frac{1}{16} D, \quad \|\mathscr{R}'_{LM}\|^2 = \frac{1}{16}(D - G)$$

(see [13], (2.27), (2.28)).

Now, we denote by $\bar{\eta} = (\bar{\eta}_a)$ the smooth vector field on M defined by

$$\bar{\eta}_a = \sum_{i,j} (\nabla_a J_{ij}) \rho^*_{i\bar{j}}. \tag{2.21}$$

Then, by direct calculation, we have

$$
\begin{aligned}
(2.22)\qquad A + A' &= \sum (\nabla_a J_{ij}) R_{ijkl} \nabla_a J_{kl} + \sum (\nabla_a J_{ij})(\nabla_a R_{ijkl}) J_{kl} \\
&= \sum \nabla_a \{(\nabla_a J_{ij}) R_{ijkl} J_{kl}\} - \sum (\nabla^2_{aa} J_{ij}) R_{ijkl} J_{kl} \\
&= 2 \operatorname{div} \bar{\eta} + 2 \sum (\nabla^2_{ai} J_{ja} + \nabla^2_{aj} J_{ai}) \rho^*_{i\bar{j}} \\
&= 2 \operatorname{div} \bar{\eta} - 2 \sum \left\{ (R_{aijs} + R_{ajsi}) J_{sa} - \frac{\tau}{2} J_{ji} \right\} \rho^*_{i\bar{j}} \\
&= 2 \operatorname{div} \bar{\eta} - 2 \sum R_{a\bar{a}ij} \rho^*_{i\bar{j}} + \tau\tau^* \\
&= 2 \operatorname{div} \bar{\eta} - 4\|\rho^*\|^2 + \tau\tau^* \\
&= 2 \operatorname{div} \bar{\eta} - G - \tau^*(\tau^* - \tau),
\end{aligned}
$$

$$
\begin{aligned}
\sum (\nabla^2_{aa} J_{ij}) R_{ijk\bar{k}} &= -2 \sum (\nabla^2_{ai} J_{ja}) R_{ijk\bar{k}} \\
&= 2 \sum (R_{aijt} J_{ta} + R_{aiat} J_{jt}) R_{ijk\bar{k}} \\
&= \sum (R_{iatj} - R_{itaj}) J_{ta} R_{ijk\bar{k}} - \tau\tau^* \\
&= \sum R_{ija\bar{a}} R_{ijk\bar{k}} - \tau\tau^* \\
&= 4\|\rho^*\|^2 - \tau\tau^* \\
&= G + \tau^*(\tau^* - \tau),
\end{aligned}
$$

and

$$
\begin{aligned}
\sum J_{ij} (\nabla^2_{aa} R_{ijkl}) J_{kl} &= -2 \sum J_{ij} (\nabla^2_{ai} R_{jakl}) J_{kl} \\
&= 2 \sum J_{ij} (R_{aijt} R_{takl} + R_{aiat} R_{jtkl} + 2 R_{aikt} R_{jatl}) J_{kl} \\
&= -\sum J_{ij} R_{ijat} R_{takl} J_{kl} - \tau\tau^* + 4C \\
&= 4\|\rho^*\|^2 - \tau\tau^* + 4C \\
&= G - 8K + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{2}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
-2\Delta\tau^* &= \sum_a \nabla^2_{aa} \left(\sum_{i,k} R_{i\bar{i}k\bar{k}} \right) \\
&= 2 \sum (\nabla^2_{aa} J_{ij}) R_{ijk\bar{k}} + 2A + 4A' + \sum J_{ij} (\nabla^2_{aa} R_{ijkl}) J_{kl} \\
&= -G - 8K - \frac{(3\tau^* - \tau)(\tau^* - \tau)}{2} + 8 \operatorname{div} \bar{\eta},
\end{aligned}
$$

and hence

$$
(2.23)\qquad \Delta\tau^* = \frac{G}{2} + 4K + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{4} - 4 \operatorname{div} \bar{\eta}.
$$

Next, we denote by $\xi = (\xi_i)$ the smooth vector field on M defined by

$$\xi_i = \sum_a (\nabla_a \rho^*_{ai}).$$

Then, we obtain the following:

$$(2.24)\qquad \begin{aligned} \xi_1 &= \frac{3}{\sqrt{2}}(\alpha_1 \rho^*_{14} - \alpha_2 \rho^*_{13}), \\ \xi_2 &= \frac{3}{\sqrt{2}}(\alpha_1 \rho^*_{13} + \alpha_2 \rho^*_{14}), \\ \xi_3 &= \frac{3}{\sqrt{2}}(\alpha_3 \rho^*_{14} - \alpha_4 \rho^*_{13}) + \frac{1}{\sqrt{2}}\{-2\alpha_1 w + \alpha_2(u - v)\}, \\ \xi_4 &= \frac{3}{\sqrt{2}}(\alpha_4 \rho^*_{14} + \alpha_3 \rho^*_{13}) + \frac{1}{\sqrt{2}}\{\alpha_1(u - v) + 2\alpha_2 w\} \end{aligned}$$

(see [14], (2.43)).

Now, let $M = (M, J, g)$ be a four-dimensional strictly almost Kähler Einstein manifold. We put $M_0 = \{p \in M \mid (\tau^* - \tau)(p) > 0\}$, a non-empty open submanifold of M. For 1-forms α and $J\alpha$ in (2.12), we denote by $\mathscr{D}$ the 2-dimensional J-invariant smooth distribution on M_0 spanned by $\{\alpha^*, J\alpha^*\}$ $(\iota(\alpha^*) = \alpha)$ and by $\mathscr{D}^\perp$ the orthogonal complement of $\mathscr{D}$ in TM. We choose a local unitary frame field $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ on a neighborhood of any point of M_0 such that $e_1, e_2 \in \mathscr{D}$ and $e_3, e_4 \in \mathscr{D}^\perp$. Then, the equality

$$(2.25)\qquad \nabla\Omega = \alpha \otimes \frac{1}{\sqrt{2}}(e^1 \wedge e^3 - e^2 \wedge e^4) - J\alpha \otimes \frac{1}{\sqrt{2}}(e^1 \wedge e^4 + e^2 \wedge e^3)$$

holds. A pair $\{\{e_i\}_{i=1,\dots,4}, \{\alpha, J\alpha\}\}$ of unitary frame $\{e_i\}$ and 1-forms $\{\alpha, J\alpha\}$ is said to be adapted to $\nabla\Omega$ if it satisfies (2.25) and $\mathscr{D} = \mathrm{span}\{e_1, e_2\}$, $\mathscr{D}^\perp = \mathrm{span}\{e_3, e_4\}$. For an adapted pair, it is clear that

$$\alpha_3 = \alpha_4 = 0.$$

Further, we have the following equalities ([14], (3.1)$\sim$(3.4), (2.39), (2.40), (3.9), (3.11)):

$$(2.26)\qquad \begin{aligned} w &= \frac{1}{\sqrt{2}}\{-\Gamma_{131}\alpha_1 - \Gamma_{132}\alpha_2 - \nabla_3\alpha_1 + \alpha_2(\Gamma_{312} + \Gamma_{334})\}, \\ w &= \frac{1}{\sqrt{2}}\{\Gamma_{241}\alpha_1 + \Gamma_{242}\alpha_2 + \nabla_4\alpha_2 + \alpha_1(\Gamma_{412} + \Gamma_{434})\}, \\ w &= \frac{1}{\sqrt{2}}\{-\Gamma_{131}\alpha_1 - \Gamma_{132}\alpha_2 + \nabla_4\alpha_2 + \alpha_1(\Gamma_{412} + \Gamma_{434})\}, \\ w &= \frac{1}{\sqrt{2}}\{\Gamma_{241}\alpha_1 + \Gamma_{242}\alpha_2 - \nabla_3\alpha_1 + \alpha_2(\Gamma_{312} + \Gamma_{334})\}, \end{aligned}$$

$$
(2.27)\qquad \begin{aligned}
u &= \frac{1}{\sqrt{2}}\left\{\Gamma_{141}\alpha_1 + \Gamma_{142}\alpha_2 + \nabla_3\alpha_2 - \frac{\tau^* - \tau}{2\sqrt{2}} + \alpha_1(\Gamma_{312} + \Gamma_{334})\right\},\\
u &= \frac{1}{\sqrt{2}}\left\{\Gamma_{231}\alpha_1 + \Gamma_{232}\alpha_2 + \nabla_4\alpha_1 - \frac{\tau^* - \tau}{2\sqrt{2}} - \alpha_2(\Gamma_{412} + \Gamma_{434})\right\},
\end{aligned}
$$

$$
(2.28)\qquad \begin{aligned}
v &= \frac{1}{\sqrt{2}}\{-\Gamma_{141}\alpha_1 - \Gamma_{142}\alpha_2 - \nabla_4\alpha_1 + \alpha_2(\Gamma_{412} + \Gamma_{434})\},\\
v &= \frac{1}{\sqrt{2}}\{-\Gamma_{231}\alpha_1 - \Gamma_{232}\alpha_2 - \nabla_3\alpha_2 - \alpha_1(\Gamma_{312} + \Gamma_{334})\},
\end{aligned}
$$

$$
(2.29)\qquad \begin{aligned}
\rho^*_{13} &= \frac{1}{\sqrt{2}}\{\nabla_1\alpha_2 - \nabla_2\alpha_1 + \alpha_1(\Gamma_{112} + \Gamma_{134}) + \alpha_2(\Gamma_{212} + \Gamma_{234})\},\\
\rho^*_{13} &= \frac{1}{\sqrt{2}}\{(\Gamma_{431} - \Gamma_{341})\alpha_1 + (\Gamma_{432} - \Gamma_{342})\alpha_2\},\\
\rho^*_{14} &= -\frac{1}{\sqrt{2}}\{\nabla_1\alpha_1 + \nabla_2\alpha_2 + \alpha_1(\Gamma_{212} + \Gamma_{234}) - \alpha_2(\Gamma_{112} + \Gamma_{134})\},\\
\rho^*_{14} &= \frac{1}{\sqrt{2}}\{(\Gamma_{342} - \Gamma_{432})\alpha_1 - (\Gamma_{341} - \Gamma_{431})\alpha_2\}.
\end{aligned}
$$

$$
(2.30)\qquad \begin{aligned}
\Gamma_{142} - \Gamma_{131} &= \frac{\alpha_2}{\sqrt{2}}, \quad \Gamma_{242} - \Gamma_{231} = -\frac{\alpha_1}{\sqrt{2}},\\
\Gamma_{132} + \Gamma_{141} &= \frac{\alpha_1}{\sqrt{2}}, \quad \Gamma_{232} + \Gamma_{241} = \frac{\alpha_2}{\sqrt{2}},
\end{aligned}
$$

$$
(2.31)\qquad \Gamma_{131} - \Gamma_{232} = -\frac{\alpha_2}{\sqrt{2}}, \quad \Gamma_{132} + \Gamma_{231} = \frac{\alpha_1}{\sqrt{2}}.
$$

From (2.25), we have

$$
\begin{aligned}
-\frac{1}{2}\sum J_{ab}(\nabla_1 R_{abjk})J_{jk} &= \nabla_1\left(-\frac{1}{2}\sum J_{ab}R_{abjk}J_{jk}\right) + \sum(\nabla_1 J_{ab})R_{abjk}J_{jk}\\
&= e_1\tau^* + 2\sum(\nabla_1 J_{ab})\rho^*_{a\bar{b}}\\
&= e_1\tau^* - 4\sqrt{2}(\alpha_1\rho^*_{14} - \alpha_2\rho^*_{13}),
\end{aligned}
$$

and hence

$$
e_1\tau^* = 4\sqrt{2}(\alpha_1\rho^*_{14} - \alpha_2\rho^*_{13}) - \frac{1}{2}\sum J_{ab}(\nabla_1 R_{abjk})J_{jk}.
$$

From this equality, taking account of (2.16) and (2.25), we have

$$
\begin{aligned}
(2.32)\quad e_3(e_1\tau^*) &= 4\sqrt{2}\{(e_3\alpha_1)\rho^*_{14}+\alpha_1 e_3\rho^*_{14}-(e_3\alpha_2)\rho^*_{13}-\alpha_2 e_3\rho^*_{13}\}\\
&\quad -\sum(e_3J_{ab})(\nabla_1 R_{abjk})J_{jk}-\frac{1}{2}\sum J_{ab}(e_3\nabla_1 R_{abjk})J_{jk}\\
&= 4\sqrt{2}\{(\nabla_3\alpha_1+\Gamma_{312}\alpha_2)\rho^*_{14}+\alpha_1\nabla_3\rho^*_{14}-\alpha_1(\Gamma_{312}+\Gamma_{334})\rho^*_{13}\\
&\quad -(\nabla_3\alpha_2-\Gamma_{312}\alpha_1)\rho^*_{13}-\alpha_2\nabla_3\rho^*_{13}-\alpha_2(\Gamma_{312}+\Gamma_{334})\rho^*_{14}\}\\
&\quad -\frac{1}{2}\sum J_{ab}(\nabla^2_{31}R_{abjk})J_{jk}-\frac{1}{2}\sum J_{ab}\Gamma_{31t}(\nabla_t R_{abjk})J_{jk}\\
&= 4\sqrt{2}\{(\nabla_3\alpha_1-\Gamma_{334}\alpha_2)\rho^*_{14}-(\nabla_3\alpha_2+\Gamma_{334}\alpha_1)\rho^*_{13}+\alpha_1\nabla_3\rho^*_{14}-\alpha_2\nabla_3\rho^*_{13}\}\\
&\quad -\frac{1}{2}\sum J_{ab}(\nabla^2_{31}R_{abjk})J_{jk}-\frac{1}{2}\sum\Gamma_{31t}\nabla_t\Big(\sum J_{ab}R_{abjk}J_{jk}\Big)\\
&\quad +\sum\Gamma_{31t}(\nabla_t J_{ab})R_{abjk}J_{jk}\\
&= 4\sqrt{2}\{(\nabla_3\alpha_1-\Gamma_{334}\alpha_2)\rho^*_{14}-(\nabla_3\alpha_2+\Gamma_{334}\alpha_1)\rho^*_{13}+\alpha_1\nabla_3\rho^*_{14}-\alpha_2\nabla_3\rho^*_{13}\}\\
&\quad -\frac{1}{2}\sum J_{ab}(\nabla^2_{31}R_{abjk})J_{jk}+\sum\Gamma_{31t}e_t\tau^*+2\sum\Gamma_{312}(\nabla_2J_{ab})\rho^*_{a\bar{b}}.
\end{aligned}
$$

Here, we get

$$
\begin{aligned}
(2.33)\quad &\sum J_{ab}(\nabla^2_{31}R_{abjk})J_{jk}\\
&= \sum J_{ab}(\nabla^2_{13}R_{abjk})J_{jk}-4\sum J_{ab}R_{31at}R_{tbjk}J_{jk}\\
&= \sum J_{ab}(\nabla^2_{13}R_{abjk})J_{jk}+8\sum R_{31at}\rho^*_{ta}\\
&= \nabla_1\Big(\sum J_{ab}(\nabla_3 R_{abjk})J_{jk}\Big)-2\sum(\nabla_1J_{ab})(\nabla_3R_{abjk})J_{jk}+8\sum R_{31at}\rho^*_{ta}\\
&= -2\nabla_1(e_3\tau^*)-2\sum(\nabla_1J_{ab})\nabla_3(R_{abjk}J_{jk})+8\sum R_{31at}\rho^*_{ta}\\
&= -2e_1(e_3\tau^*)+2\sum\Gamma_{13t}e_t\tau^*-4\sum(\nabla_1J_{ab})\nabla_3\rho^*_{a\bar{b}}+8\sum R_{31at}\rho^*_{ta}\\
&= -2e_1(e_3\tau^*)+2\sum\Gamma_{13t}e_t\tau^*-8\sqrt{2}(\alpha_2\nabla_3\rho^*_{13}-\alpha_1\nabla_3\rho^*_{14})+8\sum R_{31at}\rho^*_{ta},
\end{aligned}
$$

and

$$
(2.34)\qquad 2\sum\Gamma_{312}(\nabla_2J_{ab})\rho^*_{a\bar{b}}=-4\sqrt{2}\Gamma_{312}(\alpha_1\rho^*_{13}+\alpha_2\rho^*_{14}).
$$

Thus, from (2.32)$\sim$(2.34), $(2.26)_1$ and $(2.28)_2$, we obtain

$$
\begin{aligned}
(2.35)\quad e_3(e_1\tau^*) &= 4\sqrt{2}\{(\nabla_3\alpha_1-\Gamma_{334}\alpha_2-\Gamma_{312}\alpha_2)\rho^*_{14}-(\nabla_3\alpha_2+\Gamma_{334}\alpha_1+\Gamma_{312}\alpha_1)\rho^*_{13}\}\\
&\quad +\sum\Gamma_{31t}e_t\tau^*+e_1(e_3\tau^*)-\sum\Gamma_{13t}e_t\tau^*-4\sum R_{13ta}\rho^*_{ta}
\end{aligned}
$$

$$= 4\sqrt{2}\{(-\sqrt{2}w - \Gamma_{131}\alpha_1 - \Gamma_{132}\alpha_2)\rho^*_{14} - (-\sqrt{2}v - \Gamma_{231}\alpha_1 - \Gamma_{232}\alpha_2)\rho^*_{13}\}$$
$$+ \sum \Gamma_{31t}e_t\tau^* - \sum \Gamma_{13t}e_t\tau^* + e_1(e_3\tau^*) - 4\sum R_{13ta}\rho^*_{ta}$$
$$= -8w\rho^*_{14} + 8v\rho^*_{13} - 4\sqrt{2}(\Gamma_{131}\alpha_1 + \Gamma_{132}\alpha_2)\rho^*_{14}$$
$$+ 4\sqrt{2}(\Gamma_{231}\alpha_1 + \Gamma_{232}\alpha_2)\rho^*_{13} - [e_1, e_3]\tau^* + e_1(e_3\tau^*) - 4\sum R_{13ta}\rho^*_{ta}.$$

Here, from (2.16), we have

$$(2.36) \quad -8w\rho^*_{14} + 8v\rho^*_{13} = 8(R_{1314} + R_{1323})\rho^*_{14} - 8(R_{1414} + R_{1423})\rho^*_{13}$$
$$= 8(R_{1314}\rho^*_{14} + R_{1323}\rho^*_{23}) - 8\left(-\frac{\tau}{4} - R_{1212} - R_{1313} - R_{1234} - R_{1342}\right)\rho^*_{13}$$
$$= 8(R_{1314}\rho^*_{14} + R_{1323}\rho^*_{23} + R_{1313}\rho^*_{13} + R_{1324}\rho^*_{24}) - 8\left(-\frac{\tau}{4} + \rho^*_{11}\right)\rho^*_{13}$$
$$= 4\sum R_{13ab}\rho^*_{ab} - 2(\tau^* - \tau)\rho^*_{13}.$$

Further, from (2.31), we have

$$(2.37) \quad \Gamma_{231}\alpha_1 + \Gamma_{232}\alpha_2 = \frac{\alpha_1^2}{\sqrt{2}} - \Gamma_{132}\alpha_1 + \frac{\alpha_2^2}{\sqrt{2}} + \Gamma_{131}\alpha_2$$
$$= \frac{\tau^* - \tau}{2\sqrt{2}} + \Gamma_{131}\alpha_2 - \Gamma_{132}\alpha_1.$$

Therefore, from (2.35) $\sim$ (2.37), we have finally

$$(2.38) \quad (\Gamma_{131}\alpha_1 + \Gamma_{132}\alpha_2)\rho^*_{14} - (-\Gamma_{132}\alpha_1 + \Gamma_{131}\alpha_2)\rho^*_{13} = 0.$$

Now, let $\{\{e_i\}_{i=1,\dots,4}, \{\alpha, J\alpha\}\}$ be an adapted pair. For arbitrary (local) functions θ and φ, we consider local 1-forms $\alpha(\theta,\varphi) = (\cos\theta)\alpha - (\sin\theta)J\alpha$, $J\alpha(\theta,\varphi) = (\sin\theta)\alpha + (\cos\theta)J\alpha$ and local unitary frame field $\{e_1(\theta,\varphi) = (\cos\varphi)e_1 - (\sin\varphi)e_2$, $e_2(\theta,\varphi) = (\sin\varphi)e_1 + (\cos\varphi)e_2, e_3(\theta,\varphi) = (\cos(\theta+\varphi))e_3 + (\sin(\theta+\varphi))e_4, e_4(\theta,\varphi) = -(\sin(\theta+\varphi))e_3 + (\cos(\theta+\varphi))e_4\}$. Then, it is easy to verify that $\{\{e_i(\theta,\varphi)\}_{i=1,\dots,4}$, $\{\alpha(\theta,\varphi), J\alpha(\theta,\varphi)\}\}$ is again an adapted pair and that the equalities

$$(2.39) \quad \|\alpha(\theta,\varphi)\| = \|\alpha\|$$

and

$$(2.40) \quad \begin{aligned} \alpha_1(\theta,\varphi) &= (\cos(\varphi-\theta))\alpha_1 - (\sin(\varphi-\theta))\alpha_2, \\ \alpha_2(\theta,\varphi) &= (\sin(\varphi-\theta))\alpha_1 + (\cos(\varphi-\theta))\alpha_2, \\ \alpha_3(\theta,\varphi) &= \alpha_4(\theta,\varphi) = 0 \end{aligned}$$

hold, where $\alpha_i(\theta,\varphi) = \alpha(\theta,\varphi)(e_i(\theta,\varphi))$. Moreover, from (2.16), we have

$$\begin{aligned} \rho^*_{13}(\theta,\varphi) &= \rho^*_{13}\cos\theta + \rho^*_{14}\sin\theta, \\ \rho^*_{14}(\theta,\varphi) &= -\rho^*_{13}\sin\theta + \rho^*_{14}\cos\theta, \end{aligned} \tag{2.41}$$

where $\rho^*_{ij}(\theta,\varphi) = \rho^*(e_i(\theta,\varphi), e_j(\theta,\varphi))$. We denote by $\Gamma_{ijk}(\theta,\varphi)$ the connection coefficients of g with respect to $\{e_i(\theta,\varphi)\}_{i=1,\dots,4}$. Then, from (2.30) and (2.31), we get

$$\begin{aligned} \Gamma_{131}(\theta,\varphi) &= \Gamma_{131}\cos(\theta+\varphi) - \Gamma_{132}\sin(\theta+\varphi) \\ &\quad + \frac{\alpha_1}{\sqrt{2}}\sin\theta\cos\varphi - \frac{\alpha_2}{\sqrt{2}}\sin\theta\sin\varphi, \\ \Gamma_{132}(\theta,\varphi) &= \Gamma_{131}\sin(\theta+\varphi) + \Gamma_{132}\cos(\theta+\varphi) \\ &\quad + \frac{\alpha_1}{\sqrt{2}}\sin\theta\sin\varphi + \frac{\alpha_2}{\sqrt{2}}\sin\theta\cos\varphi. \end{aligned} \tag{2.42}$$

Applying the same argument to obtain (2.38) with respect to $\{\{e_i(\theta,\varphi)\}_{i=1,\dots,4}, \{\alpha(\theta,\varphi), J\alpha(\theta,\varphi)\}\}$ instead of $\{\{e_i\}_{i=1,\dots,4}, \{\alpha, J\alpha\}\}$, we have

$$\begin{aligned} &(\Gamma_{131}(\theta,\varphi)\alpha_1(\theta,\varphi) + \Gamma_{132}(\theta,\varphi)\alpha_2(\theta,\varphi))\rho^*_{14}(\theta,\varphi) \\ &\quad - (-\Gamma_{132}(\theta,\varphi)\alpha_1(\theta,\varphi) + \Gamma_{131}(\theta,\varphi)\alpha_2(\theta,\varphi))\rho^*_{13}(\theta,\varphi) = 0. \end{aligned}$$

From (2.41) and (2.42), this equality becomes

$$\begin{aligned} &\{(\Gamma_{132}\alpha_1 - \Gamma_{131}\alpha_2)\rho^*_{13} + (\Gamma_{131}\alpha_1 + \Gamma_{132}\alpha_2)\rho^*_{14}\}\cos\theta \\ &\quad + \left\{(\Gamma_{131}\alpha_1 + \Gamma_{132}\alpha_2)\rho^*_{13} - \left(\Gamma_{132}\alpha_1 - \Gamma_{131}\alpha_2 - \frac{\tau^* - \tau}{2\sqrt{2}}\right)\rho^*_{14}\right\}\sin\theta = 0. \end{aligned}$$

Since this equality is valid for arbitrary θ, we obtain

$$(\Gamma_{131}\alpha_1 + \Gamma_{132}\alpha_2)\rho^*_{13} - \left(\Gamma_{132}\alpha_1 - \Gamma_{131}\alpha_2 - \frac{\tau^* - \tau}{2\sqrt{2}}\right)\rho^*_{14} = 0. \tag{2.43}$$

Now, taking account of (2.16), the 2-forms φ and ψ of (2.6) satisfy

$$\varphi(e_i, e_j) = 2(\alpha_{\bar{i}}\alpha_j - \alpha_i\alpha_{\bar{j}}), \quad \psi(e_i, e_j) = -2\rho^*_{i\bar{j}}.$$

Thus, the first Chern form γ is locally represented by

$$\begin{aligned} 8\pi\gamma &= \tau(e^1 \wedge e^2 + e^3 \wedge e^4) + (\tau^* - \tau)e^3 \wedge e^4 \\ &\quad - 4\rho^*_{14}(e^1 \wedge e^3 - e^2 \wedge e^4) + 4\rho^*_{13}(e^1 \wedge e^4 + e^2 \wedge e^3). \end{aligned} \tag{2.44}$$

Since γ is closed, we have in particular $d\gamma(e_1, e_2, e_3) = 0$. Thus, by using (2.14) and (2.16), we get

$$\begin{aligned}0 &= e_1(\gamma(e_2,e_3)) + e_2(\gamma(e_3,e_1)) + e_3(\gamma(e_1,e_2)) \\ &\quad - \{\gamma([e_1,e_2],e_3) + \gamma([e_2,e_3],e_1) + \gamma([e_3,e_1],e_2)\} \\ &= \frac{1}{8\pi}\{4e_1\rho^*_{13} + 4e_2\rho^*_{14} + 4(\Gamma_{212}+\Gamma_{234})\rho^*_{13} \\ &\quad - 4(\Gamma_{112}+\Gamma_{134})\rho^*_{14} - (\Gamma_{214}-\Gamma_{124})(\tau^*-\tau)\} \\ &= \frac{1}{8\pi}\{4\nabla_1\rho^*_{13} + 4\nabla_2\rho^*_{14} + (\tau^*-\tau)(\Gamma_{124}-\Gamma_{214})\},\end{aligned}$$

and hence

$$\nabla_1\rho^*_{13} + \nabla_2\rho^*_{14} = -\frac{(\tau^*-\tau)}{4}(\Gamma_{124}-\Gamma_{214}). \tag{2.45}$$

Similarly, $d\gamma(e_1,e_2,e_4)=0$, $d\gamma(e_1,e_3,e_4)=0$ and $d\gamma(e_2,e_3,e_4)=0$ yield

$$\begin{aligned}\nabla_1\rho^*_{14} - \nabla_2\rho^*_{13} &= \frac{\tau^*-\tau}{4}(\Gamma_{123}-\Gamma_{213}), \\ \nabla_3\rho^*_{13} + \nabla_4\rho^*_{14} &= \frac{1}{4}e_1\tau^* - \frac{\sqrt{2}}{2}(\alpha_1\rho^*_{14} - \alpha_2\rho^*_{13}) + \frac{\tau^*-\tau}{4}(\Gamma_{313}+\Gamma_{414}) \\ &= \frac{1}{4}e_1\tau^* - \sqrt{2}(\alpha_1\rho^*_{14} - \alpha_2\rho^*_{13}), \\ \nabla_3\rho^*_{14} - \nabla_4\rho^*_{13} &= \frac{1}{4}e_2\tau^* - \frac{\sqrt{2}}{2}(\alpha_1\rho^*_{13} + \alpha_2\rho^*_{14}) - \frac{\tau^*-\tau}{4}(\Gamma_{413}+\Gamma_{314}) \\ &= \frac{1}{4}e_2\tau^* - \sqrt{2}(\alpha_1\rho^*_{13} + \alpha_2\rho^*_{14}),\end{aligned} \tag{2.46}$$

where we use $(2.29)_{2,4}$ to obtain the last two equalities. From (2.14) and (2.45), we have

$$\begin{aligned}\xi_3 &= \nabla_1\rho^*_{13} + \nabla_2\rho^*_{23} + \nabla_3\rho^*_{33} + \nabla_4\rho^*_{43} \\ &= \nabla_1\rho^*_{13} + \nabla_2\rho^*_{14} + \frac{1}{4}e_3\tau^* + (\Gamma_{414}+\Gamma_{423})\rho^*_{13} - (\Gamma_{413}-\Gamma_{424})\rho^*_{14} \\ &= -\frac{\tau^*-\tau}{4}(\Gamma_{124}-\Gamma_{214}) + \frac{1}{4}e_3\tau^*.\end{aligned}$$

Comparing this equality with $(2.24)_3$, we obtain

$$e_3\tau^* = 2\sqrt{2}\{-2\alpha_1 w + \alpha_2(u-v)\} + (\tau^*-\tau)(\Gamma_{124}-\Gamma_{214}). \tag{2.47}$$

Applying the similar argument to ξ_4, from (2.14), $(2.46)_1$ and $(2.24)_4$, we also obtain

$$e_4\tau^* = 2\sqrt{2}\{\alpha_1(u-v) + 2\alpha_2 w\} - (\tau^*-\tau)(\Gamma_{123}-\Gamma_{213}). \tag{2.48}$$

Now for the vector field $\bar{\eta}$ defined by (2.21), we obtain

$$
\begin{aligned}
\text{(2.49)}\qquad \operatorname{div}\bar{\eta} &= \sum(\nabla^2_{ii}J_{ab})\rho^*_{a\bar{b}} + \sum(\nabla_i J_{ab})(\nabla_i J_{bc})\rho^*_{ac} + \sum(\nabla_i J_{ab})\nabla_i\rho^*_{a\bar{b}} \\
&= -\sum(\nabla^2_{ia}J_{bi})\rho^*_{a\bar{b}} - \sum(\nabla^2_{ib}J_{ia})\rho^*_{a\bar{b}} \\
&\quad -\frac{\tau^*}{2}\sum\{(\nabla_i J_{13})^2 + (\nabla_i J_{24})^2 + (\nabla_i J_{14})^2 + (\nabla_i J_{23})^2\} \\
&\quad + 2\sqrt{2}\sum(-\alpha_i\nabla_i\rho^*_{14} + \alpha_{\bar{i}}\nabla_i\rho^*_{13}) \\
&= \sum(R_{iabt}J_{ti} + R_{iait}J_{bt})\rho^*_{a\bar{b}} + \sum(R_{ibit}J_{ta} + R_{ibat}J_{it})\rho^*_{a\bar{b}} \\
&\quad -\frac{\tau^*}{2}\sum(\alpha_i^2 + \alpha_{\bar{i}}^2) - 2\sqrt{2}\{\alpha_1(\nabla_1\rho^*_{14} + \nabla_2\rho^*_{13}) - \alpha_2(\nabla_1\rho^*_{13} - \nabla_2\rho^*_{14})\} \\
&= \sum(R_{iabt} + R_{biat})J_{ti}\rho^*_{a\bar{b}} - \frac{(\tau^*)^2}{2} \\
&\quad - 2\sqrt{2}\{\alpha_1(\nabla_1\rho^*_{14} + \nabla_2\rho^*_{13}) - \alpha_2(\nabla_1\rho^*_{13} - \nabla_2\rho^*_{14})\} \\
&= \sum R_{abit}J_{it}\rho^*_{a\bar{b}} - \frac{(\tau^*)^2}{2} \\
&\quad - 2\sqrt{2}\{\alpha_1(\nabla_1\rho^*_{14} + \nabla_2\rho^*_{13}) - \alpha_2(\nabla_1\rho^*_{13} - \nabla_2\rho^*_{14})\} \\
&= 2\|\rho^*\|^2 - \frac{(\tau^*)^2}{2} - 2\sqrt{2}\{\alpha_1(\nabla_1\rho^*_{14} + \nabla_2\rho^*_{13}) - \alpha_2(\nabla_1\rho^*_{13} - \nabla_2\rho^*_{14})\} \\
&= \frac{G}{2} - 2\sqrt{2}\{\alpha_1(\nabla_1\rho^*_{14} + \nabla_2\rho^*_{13}) - \alpha_2(\nabla_1\rho^*_{13} - \nabla_2\rho^*_{14})\}.
\end{aligned}
$$

Since

$$
\nabla_k\rho^*_{ij} = -\frac{1}{2}\sum(\nabla_k J_{ju})R_{iut\bar{t}} - \frac{1}{2}\sum\nabla_k R_{i\bar{j}t\bar{t}} - \frac{1}{2}\sum R_{i\bar{j}ts}\nabla_k J_{ts},
$$

we have

$$
\begin{aligned}
\nabla_1\rho^*_{13} &= -\frac{1}{2}\sum(\nabla_1 J_{32})R_{12t\bar{t}} - \frac{1}{2}\sum\nabla_1 R_{14t\bar{t}} - \frac{1}{2}\sum R_{14ts}\nabla_1 J_{ts} \\
&= \frac{1}{\sqrt{2}}\alpha_2\rho^*_{11} - (\nabla_1 R_{1412} + \nabla_1 R_{1434}) \\
&\quad - \frac{1}{\sqrt{2}}\alpha_1(-R_{1314} + R_{1424}) + \frac{1}{\sqrt{2}}\alpha_2(R_{1414} + R_{1423}) \\
&= \frac{\tau^*}{4\sqrt{2}}\alpha_2 - (\nabla_1 R_{1412} + \nabla_1 R_{1434}) - \frac{1}{\sqrt{2}}(\alpha_1 w + \alpha_2 v),
\end{aligned}
$$

and hence,

$$\nabla_1 R_{1412} + \nabla_1 R_{1434} = -\nabla_1 \rho^*_{13} + \frac{\tau^*}{4\sqrt{2}}\alpha_2 - \frac{1}{\sqrt{2}}(\alpha_1 w + \alpha_2 v). \tag{2.50}$$

Similarly, we calculate $\nabla_1 \rho^*_{14}, \nabla_2 \rho^*_{13}$ and $\nabla_2 \rho^*_{14}$ to obtain

$$\begin{aligned} \nabla_1 R_{1312} + \nabla_1 R_{1334} &= \nabla_1 \rho^*_{14} + \frac{\tau^*}{4\sqrt{2}}\alpha_1 - \frac{1}{\sqrt{2}}(\alpha_1 u + \alpha_2 w), \\ \nabla_2 R_{1412} + \nabla_2 R_{1434} &= -\nabla_2 \rho^*_{13} - \frac{\tau^*}{4\sqrt{2}}\alpha_1 + \frac{1}{\sqrt{2}}(\alpha_1 v - \alpha_2 w), \\ \nabla_2 R_{1312} + \nabla_2 R_{1334} &= \nabla_2 \rho^*_{14} + \frac{\tau^*}{4\sqrt{2}}\alpha_2 + \frac{1}{\sqrt{2}}(\alpha_1 w - \alpha_2 u). \end{aligned} \tag{2.51}$$

From (2.18), (2.50) and $(2.51)_3$, we have

$$\begin{aligned} e_3(u-v) + 2e_4 w &= -e_3 R_{1313} + e_3 R_{1324} + e_3 R_{1414} + e_3 R_{1423} - 2e_4 R_{1314} - 2e_4 R_{1323} \\ &= -\nabla_3 R_{1313} + \nabla_3 R_{1324} + \nabla_3 R_{1414} + \nabla_3 R_{1423} - 2\nabla_4 R_{1314} - 2\nabla_4 R_{1323} \\ &\quad + 4w(\Gamma_{312} + \Gamma_{334}) - 2(u-v)(\Gamma_{412} + \Gamma_{434}) \\ &= \nabla_1 R_{1412} + \nabla_1 R_{1434} + \nabla_2 R_{1312} + \nabla_2 R_{1334} \\ &\quad + 4w(\Gamma_{312} + \Gamma_{334}) - 2(u-v)(\Gamma_{412} + \Gamma_{434}) \\ &= -\nabla_1 \rho^*_{13} + \nabla_2 \rho^*_{14} + \frac{\tau^*}{2\sqrt{2}}\alpha_2 - \frac{1}{\sqrt{2}}(u+v)\alpha_2 \\ &\quad + 4w(\Gamma_{312} + \Gamma_{334}) - 2(u-v)(\Gamma_{412} + \Gamma_{434}) \\ &= -\nabla_1 \rho^*_{13} + \nabla_2 \rho^*_{14} + \frac{3\tau^* - \tau}{4\sqrt{2}}\alpha_2 \\ &\quad + 4w(\Gamma_{312} + \Gamma_{334}) - 2(u-v)(\Gamma_{412} + \Gamma_{434}). \end{aligned} \tag{2.52}$$

Similarly, from (2.18) and $(2.51)_{1,2}$, we obtain

$$\begin{aligned} 2e_3 w - e_4(u-v) = -\nabla_1 \rho^*_{14} - \nabla_2 \rho^*_{13} - \frac{3\tau^* - \tau}{4\sqrt{2}}\alpha_1 \\ - 2(u-v)(\Gamma_{312} + \Gamma_{334}) - 4w(\Gamma_{412} + \Gamma_{434}). \end{aligned} \tag{2.53}$$

From $(2.26)_{1,3}$, we immediately have

$$2\sqrt{2}w = -2(\alpha_1 \Gamma_{131} + \alpha_2 \Gamma_{132}) - \nabla_3 \alpha_1 + \nabla_4 \alpha_2 + \alpha_1(\Gamma_{412} + \Gamma_{434}) + \alpha_2(\Gamma_{312} + \Gamma_{334}),$$

and hence

$$\begin{aligned} -\nabla_3 \alpha_1 + \nabla_4 \alpha_2 + \alpha_1(\Gamma_{412} + \Gamma_{434}) + \alpha_2(\Gamma_{312} + \Gamma_{334}) \\ = 2\sqrt{2}w + 2(\alpha_1 \Gamma_{131} + \alpha_2 \Gamma_{132}). \end{aligned} \tag{2.54}$$

Further, from (2.30), $(2.27)_1$ and $(2.28)_1$, we have

$$\begin{aligned}\sqrt{2}(u-v) &= 2(\alpha_1\Gamma_{141}+\alpha_2\Gamma_{142}) - \frac{\tau^*-\tau}{2\sqrt{2}} \\ &\quad + \nabla_3\alpha_2 + \nabla_4\alpha_1 + \alpha_1(\Gamma_{312}+\Gamma_{334}) - \alpha_2(\Gamma_{412}+\Gamma_{434}) \\ &= 2\left(\frac{1}{\sqrt{2}}\alpha_1^2 - \alpha_1\Gamma_{132} + \frac{1}{\sqrt{2}}\alpha_2^2 + \alpha_2\Gamma_{131}\right) - \frac{\tau^*-\tau}{2\sqrt{2}} \\ &\quad + \nabla_3\alpha_2 + \nabla_4\alpha_1 + \alpha_1(\Gamma_{312}+\Gamma_{334}) - \alpha_2(\Gamma_{412}+\Gamma_{434}) \\ &= \frac{\tau^*-\tau}{2\sqrt{2}} - 2(\alpha_1\Gamma_{132} - \alpha_2\Gamma_{131}) \\ &\quad + \nabla_3\alpha_2 + \nabla_4\alpha_1 + \alpha_1(\Gamma_{312}+\Gamma_{334}) - \alpha_2(\Gamma_{412}+\Gamma_{434}),\end{aligned}$$

and hence

$$\begin{aligned}&\nabla_3\alpha_2 + \nabla_4\alpha_1 + \alpha_1(\Gamma_{312}+\Gamma_{334}) - \alpha_2(\Gamma_{412}+\Gamma_{434}) \\ &\quad = \sqrt{2}(u-v) - \frac{\tau^*-\tau}{2\sqrt{2}} + 2(\alpha_1\Gamma_{132} - \alpha_2\Gamma_{131}).\end{aligned} \tag{2.55}$$

From (2.30), (2.31), (2.47) and (2.48), we have

$$\begin{aligned}&(e_3\tau^*)^2 + (e_4\tau^*)^2 \\ &= \{-4\sqrt{2}\alpha_1 w + 2\sqrt{2}\alpha_2(u-v) - (\tau^*-\tau)(\Gamma_{142}-\Gamma_{241})\}^2 \\ &\quad + \{2\sqrt{2}\alpha_1(u-v) + 4\sqrt{2}\alpha_2 w + (\tau^*-\tau)(\Gamma_{132}-\Gamma_{231})\}^2 \\ &= 8\{4w^2 + (u-v)^2\}(\alpha_1^2+\alpha_2^2) \\ &\quad + (\tau^*-\tau)^2\{(\Gamma_{132}-\Gamma_{231})^2 + (\Gamma_{142}-\Gamma_{241})^2\} \\ &\quad + 4\sqrt{2}(\tau^*-\tau)[(u-v)\{\alpha_1(\Gamma_{132}-\Gamma_{231}) - \alpha_2(\Gamma_{142}-\Gamma_{241})\} \\ &\quad + 2w\{\alpha_1(\Gamma_{142}-\Gamma_{241}) + \alpha_2(\Gamma_{132}-\Gamma_{231})\}] \\ &= 4(\tau^*-\tau)K + (\tau^*-\tau)^2\left\{\left(2\Gamma_{132} - \frac{1}{\sqrt{2}}\alpha_1\right)^2 + \left(2\Gamma_{131} + \frac{1}{\sqrt{2}}\alpha_2\right)^2\right\} \\ &\quad + 4\sqrt{2}(\tau^*-\tau)\left\{2(u-v)(\alpha_1\Gamma_{132} - \alpha_2\Gamma_{131}) - \frac{\tau^*-\tau}{2\sqrt{2}}(u-v)\right. \\ &\quad \left. + 4w(\alpha_1\Gamma_{131} + \alpha_2\Gamma_{132})\right\}\end{aligned} \tag{2.56}$$

$$
\begin{aligned}
&= 4(\tau^* - \tau)K + \frac{(\tau^* - \tau)^3}{4} - 2(\tau^* - \tau)^2(u - v) \\
&\quad + 2(\tau^* - \tau)^2\{2(\Gamma_{131}^2 + \Gamma_{132}^2) - \sqrt{2}(\alpha_1\Gamma_{132} - \alpha_2\Gamma_{131})\} \\
&\quad + 8\sqrt{2}(\tau^* - \tau)\{(u - v)(\alpha_1\Gamma_{132} - \alpha_2\Gamma_{131}) + 2w(\alpha_1\Gamma_{131} + \alpha_2\Gamma_{132})\}.
\end{aligned}
$$

Further, from (2.49), (2.52) $\sim$ (2.55), we get

$$
\begin{aligned}
(2.57)\qquad & e_3(e_3\tau^*) + e_4(e_4\tau^*) - \Gamma_{334}e_4\tau^* + \Gamma_{434}e_3\tau^* \\
&\quad - (\Gamma_{132} - \Gamma_{231})e_4\tau^* + (\Gamma_{142} - \Gamma_{241})e_3\tau^* \\
&= e_3\{-4\sqrt{2}\alpha_1 w + 2\sqrt{2}\alpha_2(u - v) - (\tau^* - \tau)(\Gamma_{142} - \Gamma_{241})\} \\
&\quad + e_4\{2\sqrt{2}\alpha_1(u - v) + 4\sqrt{2}\alpha_2 w + (\tau^* - \tau)(\Gamma_{132} - \Gamma_{231})\} \\
&\quad - \Gamma_{334}\{2\sqrt{2}\alpha_1(u - v) + 4\sqrt{2}\alpha_2 w + (\tau^* - \tau)(\Gamma_{132} - \Gamma_{231})\} \\
&\quad + \Gamma_{434}\{-4\sqrt{2}\alpha_1 w + 2\sqrt{2}\alpha_2(u - v) - (\tau^* - \tau)(\Gamma_{142} - \Gamma_{241})\} \\
&\quad - (\Gamma_{132} - \Gamma_{231})e_4\tau^* + (\Gamma_{142} - \Gamma_{241})e_3\tau^* \\
&= 2\sqrt{2}\{2w(-\nabla_3\alpha_1 + \nabla_4\alpha_2 - \alpha_1(\Gamma_{412} + \Gamma_{434}) - \alpha_2(\Gamma_{312} + \Gamma_{334})) \\
&\quad + (u - v)(\nabla_3\alpha_2 + \nabla_4\alpha_1 - \alpha_1(\Gamma_{312} + \Gamma_{334}) + \alpha_2(\Gamma_{412} + \Gamma_{434}))\} \\
&\quad + 2\sqrt{2}\{-\alpha_1(2e_3 w - e_4(u - v)) + \alpha_2(e_3(u - v) + 2e_4 w)\} + F \\
&= 2\sqrt{2}\Bigg\{2w(2\sqrt{2}w + 2(\alpha_1\Gamma_{131} + \alpha_2\Gamma_{132})) \\
&\quad + (u - v)\left(\sqrt{2}(u - v) - \frac{\tau^* - \tau}{2\sqrt{2}} + 2(\alpha_1\Gamma_{132} - \alpha_2\Gamma_{131})\right)\Bigg\} \\
&\quad + 2\sqrt{2}\{\alpha_1(\nabla_1\rho^*_{14} + \nabla_2\rho^*_{13}) - \alpha_2(\nabla_1\rho^*_{13} - \nabla_2\rho^*_{14})\} \\
&\quad + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{4} + F \\
&= 4K + 4\sqrt{2}\{2w(\alpha_1\Gamma_{131} + \alpha_2\Gamma_{132}) + (u - v)(\alpha_1\Gamma_{132} - \alpha_2\Gamma_{131})\} \\
&\quad - (\tau^* - \tau)(u - v) + \frac{G}{2} - \operatorname{div}\bar{\eta} + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{4} + F,
\end{aligned}
$$

where

$$
\begin{aligned}
(2.58)\qquad F = -(\tau^* - \tau)\{&e_3(\Gamma_{142} - \Gamma_{241}) - e_4(\Gamma_{132} - \Gamma_{231}) \\
&+ \Gamma_{334}(\Gamma_{132} - \Gamma_{231}) + \Gamma_{434}(\Gamma_{142} - \Gamma_{241})\}.
\end{aligned}
$$

Similarly, from (2.49), (2.52)$\sim$(2.55), taking account of (2.30), we obtain

$$\begin{aligned} e_3(e_3\tau^*) &+ e_4(e_4\tau^*) - \Gamma_{334}e_4\tau^* + \Gamma_{434}e_3\tau^* \\ &+ (\Gamma_{131} + \Gamma_{232})e_3\tau^* + (\Gamma_{141} + \Gamma_{242})e_4\tau^* \\ &= 4K + 4\sqrt{2}\{2w(\alpha_1\Gamma_{131} + \alpha_2\Gamma_{132}) + (u - v)(\alpha_1\Gamma_{132} - \alpha_2\Gamma_{131})\} \\ &\quad - (\tau^* - \tau)(u - v) + \frac{G}{2} - \operatorname{div}\bar{\eta} + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{4} + F, \end{aligned}$$

and hence

$$\begin{aligned} (2.59)\qquad \Delta\tau^* &= \sum e_i(e_i\tau^*) - \sum \Gamma_{iij}e_j\tau^* \\ &= \{e_1(e_1\tau^*) + e_2(e_2\tau^*) - \Gamma_{112}e_2\tau^* + \Gamma_{212}e_1\tau^* \\ &\quad - (\Gamma_{342} - \Gamma_{432})e_1\tau^* + (\Gamma_{341} - \Gamma_{431})e_2\tau^*\} \\ &\quad + 4K + 4\sqrt{2}\{2w(\alpha_1\Gamma_{131} + \alpha_2\Gamma_{132}) + (u - v)(\alpha_1\Gamma_{132} - \alpha_2\Gamma_{131})\} \\ &\quad - (\tau^* - \tau)(u - v) + \frac{G}{2} - \operatorname{div}\bar{\eta} + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{4} + F. \end{aligned}$$

3. Some Formulas

In this section, we assume that $M = (M, J, g)$ is a four-dimensional strictly almost Kähler Einstein manifold. We put $M_0 = \{p \in M \mid (\tau^* - \tau)(p) > 0\}$, a non-empty open submanifold of M, and $M_1 = \{p \in M_0 \mid G(p) > 0\}$.

In case $\tau < 0$, we observe that $M_0 - M_1$ has no interior point unless it is empty. In fact, if $p \in M_0 - M_1$ is an interior point, then $G = 0$ holds on some neighborhood U of p and hence M is locally a weakly $*$-Einstein manifold. Then, from Main Theorem in [14], $\tau = 0$ on U. This is a contradiction.

Now, we shall prove the following lemma, which states the existence of a 'good' unitary frame on M_1 in some sense.

LEMMA 4. *For each point $p \in M_1$, there exists a neighborhood U of p and an adapted pair $\{\{e_i\}_{i=1,\dots,4}, \{\alpha, J\alpha\}\}$ satisfying*

$$(3.1)\qquad \nabla\Omega = \|\alpha\|\left\{e^1 \otimes \frac{1}{\sqrt{2}}(e^1 \wedge e^3 - e^2 \wedge e^4) - e^2 \otimes \frac{1}{\sqrt{2}}(e^1 \wedge e^4 + e^2 \wedge e^3)\right\}$$

and

$$(3.2)\qquad \rho^*_{13} = \frac{\sqrt{G}}{4}, \quad \rho^*_{14} = 0.$$

We call such an adapted pair special.

Proof. From (2.20), there exists a local smooth function η such that

$$\rho^*_{13} = \frac{\sqrt{G}}{4}\cos\eta, \quad \rho^*_{14} = \frac{\sqrt{G}}{4}\sin\eta. \tag{3.3}$$

Thus, from (2.41) and (3.3), we have

$$\rho^*_{13}(\theta,\varphi) = \frac{\sqrt{G}}{4}\cos(\theta-\eta), \quad \rho^*_{14}(\theta,\varphi) = -\frac{\sqrt{G}}{4}\sin(\theta-\eta). \tag{3.4}$$

In particular, we consider an adapted pair $\{\{e_i(\eta)\}_{i=1,\dots,4}, \{\alpha(\eta), J\alpha(\eta)\}\}$ defined by $e_i(\eta) = e_i(\eta, 0)$, $\alpha(\eta) = \alpha(\eta, 0)$. Then, from (2.40), we have

$$\begin{aligned} \alpha_1(\eta) &= \alpha_1(\eta,0) = (\cos\eta)\alpha_1 + (\sin\eta)\alpha_2, \\ \alpha_2(\eta) &= \alpha_2(\eta,0) = -(\sin\eta)\alpha_1 + (\cos\eta)\alpha_2. \end{aligned} \tag{3.5}$$

On one hand, since

$$\mathrm{span}\{\alpha(\eta), J\alpha(\eta)\} = \mathrm{span}\{\alpha, J\alpha\} = \mathrm{span}\{e^1, e^2\} = \mathrm{span}\{e^1(\eta), e^2(\eta)\},$$

taking account of (2.39), we may put

$$\begin{aligned} \alpha(\eta) &= \|\alpha\|\{(\cos\zeta)e^1(\eta) - (\sin\zeta)e^2(\eta)\}, \\ J\alpha(\eta) &= \|\alpha\|\{(\sin\zeta)e^1(\eta) + (\cos\zeta)e^2(\eta)\} \end{aligned} \tag{3.6}$$

for some local smooth function ζ. The equalities (3.5) and (3.6) yield

$$\begin{aligned} \|\alpha\|\cos\zeta &= (\cos\eta)\alpha_1 + (\sin\eta)\alpha_2, \\ \|\alpha\|\sin\zeta &= (\sin\eta)\alpha_1 - (\cos\eta)\alpha_2. \end{aligned} \tag{3.7}$$

Now, we define a adapted pair $\{\{\tilde{e}_i\}_{i=1,\dots,4}, \{\tilde{\alpha}, J\tilde{\alpha}\}\}$ by $\tilde{e}_i = e_i(\eta,\zeta)$, $\tilde{\alpha} = \alpha(\eta,\zeta)$. Then, from (2.40) and (3.7), we have $\alpha_1(\eta,\zeta) = \|\alpha\|$, $\alpha_2(\eta,\zeta) = 0$, and hence

$$\tilde{\alpha} = \|\alpha\| e^1(\eta,\zeta) = \|\alpha\|\tilde{e}^1, \quad J\tilde{\alpha} = \|\alpha\|\tilde{e}^2.$$

Further, (3.4) yields

$$\rho^*_{13}(\eta,\zeta) = \frac{\sqrt{G}}{4}, \quad \rho^*_{14}(\eta,\zeta) = 0.$$

The lemma follows. □

Now, we deduce several formulas in terms of special adapted pair which play an important role in the proofs of Theorems 1 and 2. Let $\{\{e_i\}_{i=1,\dots,4}, \{\alpha, J\alpha\}\}$ be a special adapted pair on a neighborhood of any point in M_1. From (3.1), we see $\alpha = \|\alpha\| e^1$ and hence

$$\alpha_1^2 = \|\alpha\|^2 = \frac{\tau^* - \tau}{2}, \quad \alpha_2 = \alpha_3 = \alpha_4 = 0. \tag{3.8}$$

From (2.38), (2.43), (3.2) and (3.8), we obtain

$$\Gamma_{113} = \Gamma_{123} = 0. \tag{3.9}$$

From (2.14), (2.30), (2.31), (3.8) and (3.9), we have further

$$\begin{aligned} &\Gamma_{124} = \Gamma_{223} = \Gamma_{214} = \Gamma_{224} = 0, \\ &\Gamma_{114} = \Gamma_{213} = -\frac{\alpha_1}{\sqrt{2}} = -\frac{\|\alpha\|}{\sqrt{2}}, \\ &\Gamma_{324} - \Gamma_{313} = 0, \quad \Gamma_{323} + \Gamma_{314} = 0, \\ &\Gamma_{424} - \Gamma_{413} = 0, \quad \Gamma_{423} + \Gamma_{414} = 0. \end{aligned} \tag{3.10}$$

From (3.8) and (3.10), the equality $(2.29)_2$ is reduced to

$$\rho^*_{13} = \frac{1}{\sqrt{2}}(\Gamma_{314} - \Gamma_{413})\alpha_1 = -\frac{1}{\sqrt{2}}(\Gamma_{323} + \Gamma_{424})\alpha_1 \tag{3.11}$$

and $(2.29)_4$ yields

$$\Gamma_{324} - \Gamma_{423} = 0, \quad \Gamma_{313} + \Gamma_{414} = 0. \tag{3.12}$$

Further, (3.8)$\sim$(3.10) reduce $(2.26)_{1,2}$, (2.27), (2.28) and $(2.29)_{1,3}$ respectively to

$$\begin{aligned} &w = -\frac{1}{\sqrt{2}}e_3\alpha_1, \quad w = \frac{\alpha_1}{\sqrt{2}}(2\Gamma_{412} + \Gamma_{434}), \\ &u = \frac{\alpha_1}{\sqrt{2}}(2\Gamma_{312} + \Gamma_{334}), \quad u = \frac{1}{\sqrt{2}}e_4\alpha_1, \\ &v = -\frac{1}{\sqrt{2}}\left(e_4\alpha_1 + \frac{\tau^* - \tau}{2\sqrt{2}}\right), \quad v = -\frac{1}{\sqrt{2}}\left\{\frac{\tau^* - \tau}{2\sqrt{2}} + \alpha_1(2\Gamma_{312} + \Gamma_{334})\right\}, \\ &\rho^*_{13} = \frac{1}{\sqrt{2}}\{-e_2\alpha_1 + \alpha_1(2\Gamma_{112} + \Gamma_{134})\}, \\ &0 = -\frac{1}{\sqrt{2}}\{e_1\alpha_1 + \alpha_1(2\Gamma_{212} + \Gamma_{234})\}, \end{aligned} \tag{3.13}$$

and hence, we obtain

$$\begin{aligned} &e_1\alpha_1 = -\alpha_1(2\Gamma_{212} + \Gamma_{234}), \\ &e_2\alpha_1 = \alpha_1(2\Gamma_{112} + \Gamma_{134}) - \sqrt{2}\rho^*_{13}, \\ &e_3\alpha_1 = -\alpha_1(2\Gamma_{412} + \Gamma_{434}), \\ &e_4\alpha_1 = \alpha_1(2\Gamma_{312} + \Gamma_{334}). \end{aligned} \tag{3.14}$$

From (3.8), we see $e_i\tau^* = e_i(\tau^* - \tau) = 4\alpha_1 e_i\alpha_1$ $(i = 1, \dots, 4)$. Substituting (3.14) into this equality, we have

$$
(3.15)\qquad
\begin{aligned}
e_1\tau^* &= e_1(\tau^*-\tau) = -2(\tau^*-\tau)(2\Gamma_{212}+\Gamma_{234}),\\
e_2\tau^* &= e_2(\tau^*-\tau) = 2(\tau^*-\tau)(2\Gamma_{112}+\Gamma_{134}) - 4\sqrt{2}\alpha_1\rho^*_{13},\\
e_3\tau^* &= e_3(\tau^*-\tau) = -2(\tau^*-\tau)(2\Gamma_{412}+\Gamma_{434}),\\
e_4\tau^* &= e_4(\tau^*-\tau) = 2(\tau^*-\tau)(2\Gamma_{312}+\Gamma_{334}).
\end{aligned}
$$

Taking account of (2.16), (3.2), (3.8)$\sim$(3.10) and (3.15), we find that (2.45) and (2.46) are reduced to

$$
(3.16)\qquad
\begin{aligned}
e_1\rho^*_{13} &= -(\Gamma_{212}+\Gamma_{234})\rho^*_{13},\\
e_2\rho^*_{13} &= (\Gamma_{112}+\Gamma_{134})\rho^*_{13} - \frac{\tau^*-\tau}{4\sqrt{2}}\alpha_1,\\
e_3\rho^*_{13} &= -(\Gamma_{412}+\Gamma_{434})\rho^*_{13} + \frac{1}{4}e_1\tau^*\\
&= -(\Gamma_{412}+\Gamma_{434})\rho^*_{13} - \frac{\tau^*-\tau}{2}(2\Gamma_{212}+\Gamma_{234}),\\
e_4\rho^*_{13} &= (\Gamma_{312}+\Gamma_{334})\rho^*_{13} - \frac{1}{4}e_2\tau^* + \sqrt{2}\alpha_1\rho^*_{13}\\
&= (\Gamma_{312}+\Gamma_{334})\rho^*_{13} - \frac{\tau^*-\tau}{2}(2\Gamma_{112}+\Gamma_{134}) + 2\sqrt{2}\alpha_1\rho^*_{13}.
\end{aligned}
$$

Thus, since $G = 16(\rho^*_{13})^2$, we obtain

$$
(3.17)\qquad
\begin{aligned}
e_1 G &= -2(\Gamma_{212}+\Gamma_{234})G,\\
e_2 G &= 2(\Gamma_{112}+\Gamma_{134})G - 4\sqrt{2}(\tau^*-\tau)\alpha_1\rho^*_{13},\\
e_3 G &= -2(\Gamma_{412}+\Gamma_{434})G - 16(\tau^*-\tau)(2\Gamma_{212}+\Gamma_{234})\rho^*_{13},\\
e_4 G &= 2(\Gamma_{312}+\Gamma_{334})G - 16(\tau^*-\tau)(2\Gamma_{112}+\Gamma_{134})\rho^*_{13} + 4\sqrt{2}\alpha_1 G,
\end{aligned}
$$

and hence

$$
(3.18)\qquad
\begin{aligned}
\frac{1}{2}e_1\log G &= -(\Gamma_{212}+\Gamma_{234}),\\
\frac{1}{2}e_2\log G &= (\Gamma_{112}+\Gamma_{134}) - \frac{\sqrt{2}}{8\rho^*_{13}}(\tau^*-\tau)\alpha_1,\\
\frac{1}{2}e_3\log G &= -(\Gamma_{412}+\Gamma_{434}) - \frac{\tau^*-\tau}{2\rho^*_{13}}(2\Gamma_{212}+\Gamma_{234})\\
&= -(\Gamma_{412}+\Gamma_{434}) + \frac{1}{4\rho^*_{13}}e_1\tau^*,\\
\frac{1}{2}e_4\log G &= (\Gamma_{312}+\Gamma_{334}) - \frac{\tau^*-\tau}{2\rho^*_{13}}(2\Gamma_{112}+\Gamma_{134}) + 2\sqrt{2}\alpha_1\\
&= (\Gamma_{312}+\Gamma_{334}) - \frac{1}{4\rho^*_{13}}e_2\tau^* + \sqrt{2}\alpha_1.
\end{aligned}
$$

From (2.49) and (2.54) ~ (2.59), taking account of (3.2), (3.8) ~ (3.10), (3.12), (3.14) and (3.16), we easily obtain the following equalities:

$$(e_3\tau^*)^2 + (e_4\tau^*)^2 = 4(\tau^* - \tau)K + \frac{(\tau^* - \tau)^3}{4} - 2(\tau^* - \tau)^2(u - v), \tag{3.19}$$

$$\begin{aligned} & e_3(e_3\tau^*) + e_4(e_4\tau^*) - \Gamma_{334}e_4\tau^* + \Gamma_{434}e_3\tau^* + \Gamma_{231}e_4\tau^* \\ &\quad = 4K - (\tau^* - \tau)(u - v) + \frac{G}{2} - \operatorname{div}\bar{\eta} + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{4} + F, \end{aligned} \tag{3.20}$$

$$\begin{aligned} \Delta\tau^* &= \{e_1(e_1\tau^*) + e_2(e_2\tau^*) - \Gamma_{112}e_2\tau^* + \Gamma_{212}e_1\tau^* + (\Gamma_{341} - \Gamma_{431})e_2\tau^*\} \\ &\quad + 4K - (\tau^* - \tau)(u - v) + \frac{G}{2} - \operatorname{div}\bar{\eta} + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{4} + F, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} F &= -\sqrt{2}(\tau^* - \tau)\alpha_1\Gamma_{312}, \\ u - v &= \frac{\tau^* - \tau}{4} + \sqrt{2}\alpha_1(2\Gamma_{312} + \Gamma_{334}), \\ 2w &= \sqrt{2}\alpha_1(2\Gamma_{412} + \Gamma_{434}), \\ \operatorname{div}\bar{\eta} &= \frac{G}{2} + \frac{(\tau^* - \tau)^2}{4} - 4\sqrt{2}(\Gamma_{112} + \Gamma_{134})\alpha_1\rho^*_{13}. \end{aligned} \tag{3.22}$$

Next, we put

$$\begin{aligned} \Delta_1(\tau^* - \tau) &= e_1(e_1(\tau^* - \tau)) + e_2(e_2(\tau^* - \tau)) \\ &\quad - \sum \Gamma_{11j}e_j(\tau^* - \tau) - \sum \Gamma_{22j}e_j(\tau^* - \tau) \\ &= e_1(e_1(\tau^* - \tau)) + e_2(e_2(\tau^* - \tau)) + \frac{\alpha_1}{\sqrt{2}}e_4(\tau^* - \tau) \\ &\quad - \Gamma_{112}e_2(\tau^* - \tau) + \Gamma_{212}e_1(\tau^* - \tau), \end{aligned} \tag{3.23}$$

$$\begin{aligned} \Delta_2(\tau^* - \tau) &= e_3(e_3(\tau^* - \tau)) + e_4(e_4(\tau^* - \tau)) \\ &\quad - \sum \Gamma_{33j}e_j(\tau^* - \tau) - \sum \Gamma_{44j}e_j(\tau^* - \tau) \\ &= e_3(e_3(\tau^* - \tau)) + e_4(e_4(\tau^* - \tau)) - \frac{\sqrt{2}}{\alpha_1}\rho^*_{13}e_2(\tau^* - \tau) \\ &\quad - \Gamma_{334}e_4(\tau^* - \tau) - \Gamma_{443}e_3(\tau^* - \tau) \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} \|\operatorname{grad}_1(\tau^* - \tau)\|^2 &= (e_1(\tau^* - \tau))^2 + (e_2(\tau^* - \tau))^2, \\ \|\operatorname{grad}_2(\tau^* - \tau)\|^2 &= (e_3(\tau^* - \tau))^2 + (e_4(\tau^* - \tau))^2. \end{aligned}$$

Then, $\Delta(\tau^* - \tau) = \Delta_1(\tau^* - \tau) + \Delta_2(\tau^* - \tau)$ and $\|\mathrm{grad}(\tau^* - \tau)\|^2 = \|\mathrm{grad}_1(\tau^* - \tau)\|^2 + \|\mathrm{grad}_2(\tau^* - \tau)\|^2$. Further, we put

$$f = (\tau^* - \tau)\Delta(\tau^* - \tau),$$
$$f_1 = (\tau^* - \tau)\Delta_1(\tau^* - \tau),$$
$$f_2 = (\tau^* - \tau)\Delta_2(\tau^* - \tau),$$

and

$$H_1 = (\tau^* - \tau)\alpha_1\rho^*_{13}(2\Gamma_{112} + \Gamma_{134}),$$
$$H_2 = (\tau^* - \tau)\alpha_1\rho^*_{13}\Gamma_{112}.$$

Taking account of (3.9), (3.10), (3.14)$\sim$(3.16) and (3.23), by direct calculation, we have

$$\begin{aligned}
(3.25)\quad R_{1212} - \frac{\tau^*}{4}
&= 2R_{1212} + R_{1234} \\
&= 2\Big\{e_1\Gamma_{212} - e_2\Gamma_{112} + \sum \Gamma_{21t}\Gamma_{1t2} - \sum \Gamma_{11t}\Gamma_{2t2} - \sum(\Gamma_{12t} - \Gamma_{21t})\Gamma_{t12}\Big\} \\
&\quad + e_1\Gamma_{234} - e_2\Gamma_{134} + \sum \Gamma_{23t}\Gamma_{1t4} - \sum \Gamma_{13t}\Gamma_{2t4} - \sum(\Gamma_{12t} - \Gamma_{21t})\Gamma_{t34} \\
&= e_1(2\Gamma_{212} + \Gamma_{234}) - e_2(2\Gamma_{112} + \Gamma_{134}) + \Gamma_{112}(2\Gamma_{112} + \Gamma_{134}) \\
&\quad + \Gamma_{212}(2\Gamma_{212} + \Gamma_{234}) - \frac{\alpha_1}{\sqrt{2}}(2\Gamma_{312} + \Gamma_{334}) - \frac{\tau^* - \tau}{4} \\
&= e_1\left(-\frac{e_1(\tau^* - \tau)}{2(\tau^* - \tau)}\right) - e_2\left(\frac{e_2(\tau^* - \tau)}{2(\tau^* - \tau)} + \frac{2\sqrt{2}\alpha_1\rho^*_{13}}{\tau^* - \tau}\right) \\
&\quad + \Gamma_{112}\left(\frac{e_2(\tau^* - \tau)}{2(\tau^* - \tau)} + \frac{2\sqrt{2}\alpha_1\rho^*_{13}}{\tau^* - \tau}\right) + \Gamma_{212}\left(-\frac{e_1(\tau^* - \tau)}{2(\tau^* - \tau)}\right) \\
&\quad - \frac{\alpha_1}{\sqrt{2}}\left(\frac{e_4(\tau^* - \tau)}{2(\tau^* - \tau)}\right) - \frac{\tau^* - \tau}{4} \\
&= -\frac{1}{2(\tau^* - \tau)}\{e_1(e_1(\tau^* - \tau)) + e_2(e_2(\tau^* - \tau))\} \\
&\quad + \frac{1}{2(\tau^* - \tau)^2}\{(e_1(\tau^* - \tau))^2 + (e_2(\tau^* - \tau))^2\} \\
&\quad - \frac{\sqrt{2}}{\alpha_1^2}(\alpha_1 e_2\rho^*_{13} - \rho^*_{13}e_2\alpha_1) + \frac{1}{2(\tau^* - \tau)}\Gamma_{112}e_2(\tau^* - \tau) + \frac{\sqrt{2}}{\alpha_1}\Gamma_{112}\rho^*_{13} \\
&\quad - \frac{1}{2(\tau^* - \tau)}\Gamma_{212}e_1(\tau^* - \tau) - \frac{\alpha_1}{2\sqrt{2}(\tau^* - \tau)}e_4(\tau^* - \tau) - \frac{\tau^* - \tau}{4}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2(\tau^*-\tau)}\Delta_1(\tau^*-\tau) + \frac{1}{2(\tau^*-\tau)^2}\|\mathrm{grad}_1(\tau^*-\tau)\|^2 \\
&\quad - \frac{\sqrt{2}}{\alpha_1}\left\{(\Gamma_{112}+\Gamma_{134})\rho^*_{13} - \frac{\tau^*-\tau}{4\sqrt{2}}\alpha_1\right\} \\
&\quad + \frac{\sqrt{2}}{\alpha_1^2}\rho^*_{13}\{\alpha_1(2\Gamma_{112}+\Gamma_{134}) - \sqrt{2}\rho^*_{13}\} + \frac{\sqrt{2}}{\alpha_1}\Gamma_{112}\rho^*_{13} - \frac{\tau^*-\tau}{4} \\
&= -\frac{1}{2(\tau^*-\tau)}\Delta_1(\tau^*-\tau) + \frac{1}{2(\tau^*-\tau)^2}\|\mathrm{grad}_1(\tau^*-\tau)\|^2 \\
&\quad - \frac{G}{4(\tau^*-\tau)} + \frac{2\sqrt{2}}{\alpha_1}\rho^*_{13}\Gamma_{112}.
\end{aligned}
$$

Further, from (3.10)$\sim$(3.12) and (3.14), we have

$$
\begin{aligned}
(3.26)\qquad & -\frac{\tau}{4} - R_{1212} \\
&= R_{1313} + R_{1414} \\
&= e_1\Gamma_{313} - e_3\Gamma_{113} + \sum \Gamma_{31t}\Gamma_{1t3} \\
&\quad - \sum \Gamma_{11t}\Gamma_{3t3} - \sum(\Gamma_{13t} - \Gamma_{31t})\Gamma_{t13} \\
&\quad + e_1\Gamma_{414} - e_4\Gamma_{114} + \sum \Gamma_{41t}\Gamma_{1t4} \\
&\quad - \sum \Gamma_{11t}\Gamma_{4t4} - \sum(\Gamma_{14t} - \Gamma_{41t})\Gamma_{t14} \\
&= e_4\left(\frac{\alpha_1}{\sqrt{2}}\right) - \frac{\alpha_1}{\sqrt{2}}(\Gamma_{334}+\Gamma_{312}) + \Gamma_{112}(\Gamma_{314}-\Gamma_{413}) \\
&\quad + \Gamma_{313}^2 + \Gamma_{414}^2 + 2\Gamma_{314}\Gamma_{413} + \frac{\alpha_1^2}{2} \\
&= \frac{\alpha_1}{\sqrt{2}}\Gamma_{312} + \frac{\sqrt{2}}{\alpha_1}\rho^*_{13}\Gamma_{112} + \frac{\tau^*-\tau}{4} + H - \frac{G}{4(\tau^*-\tau)},
\end{aligned}
$$

where we put $H = \Gamma_{314}^2 + \Gamma_{413}^2 + \Gamma_{324}^2 + \Gamma_{423}^2$. From (3.25), (3.26) and (3.22), we have

$$
\begin{aligned}
2(\tau^*-\tau)^2\left(R_{1212} - \frac{\tau^*}{4}\right) &= -f_1 + \|\mathrm{grad}_1(\tau^*-\tau)\|^2 - \frac{\tau^*-\tau}{2}G + 8\sqrt{2}H_2, \\
-2(\tau^*-\tau)^2\left(\frac{\tau}{4} + R_{1212}\right) &= -(\tau^*-\tau)F + 4\sqrt{2}H_2 + \frac{(\tau^*-\tau)^3}{2} \\
&\quad - \frac{\tau^*-\tau}{2}G + 2(\tau^*-\tau)^2 H.
\end{aligned}
$$

Adding each side of the above equalities, we obtain

$$(3.27)\qquad f_1 = \tau^*(\tau^*-\tau)^2 - (\tau^*-\tau)G - (\tau^*-\tau)F + 2(\tau^*-\tau)^2H + 12\sqrt{2}H_2 + \|\mathrm{grad}_1(\tau^*-\tau)\|^2.$$

On one hand, from (2.23), (3.21) and (3.23), we have

$$\Delta_1(\tau^*-\tau) = -\Gamma_{114}e_4(\tau^*-\tau) - (\Gamma_{341}-\Gamma_{431})e_2(\tau^*-\tau) + (\tau^*-\tau)(u-v) - 3\,\mathrm{div}\,\bar{\eta} - F.$$

Thus, taking account of (3.11), (3.15) and (3.22), we obtain

$$\begin{aligned}(3.28)\qquad f_1 &= \sqrt{2}\alpha_1(\tau^*-\tau)^2(2\Gamma_{312}+\Gamma_{334}) \\ &\quad + \frac{\sqrt{2}}{\alpha_1}(\tau^*-\tau)\rho^*_{13}\{2(\tau^*-\tau)(2\Gamma_{112}+\Gamma_{134}) - 4\sqrt{2}\alpha_1\rho^*_{13}\} \\ &\quad + (\tau^*-\tau)^2(u-v) - (\tau^*-\tau)F - \frac{3}{2}(\tau^*-\tau)G \\ &\quad - \frac{3}{4}(\tau^*-\tau)^3 + 12\sqrt{2}(\tau^*-\tau)(\Gamma_{112}+\Gamma_{134})\alpha_1\rho^*_{13} \\ &= (\tau^*-\tau)^2\left\{(u-v) - \frac{\tau^*-\tau}{4}\right\} + 4\sqrt{2}H_1 - \frac{\tau^*-\tau}{2}G \\ &\quad + (\tau^*-\tau)^2(u-v) - (\tau^*-\tau)F - \frac{3}{2}(\tau^*-\tau)G \\ &\quad - \frac{3}{4}(\tau^*-\tau)^3 + 12\sqrt{2}(H_1-H_2) \\ &= 2(\tau^*-\tau)^2(u-v) - (\tau^*-\tau)^3 - 2(\tau^*-\tau)G - (\tau^*-\tau)F \\ &\quad + 16\sqrt{2}H_1 - 12\sqrt{2}H_2.\end{aligned}$$

Further, from (3.10), (3.20) and (3.24), we have

$$\Delta_2(\tau^*-\tau) = -\frac{\sqrt{2}}{\alpha_1}\rho^*_{13}e_2(\tau^*-\tau) - \frac{\alpha_1}{\sqrt{2}}e_4(\tau^*-\tau) + 4K - (\tau^*-\tau)(u-v) + \frac{G}{2} - \mathrm{div}\,\bar{\eta} + \frac{(3\tau^*-\tau)(\tau^*-\tau)}{4} + F.$$

Thus, taking account of (3.15) and (3.22), we obtain

$$
\begin{aligned}
(3.29) \qquad f_2 = & -\frac{\sqrt{2}}{\alpha_1}(\tau^* - \tau)\rho^*_{13}\{2(\tau^* - \tau)(2\Gamma_{112} + \Gamma_{134}) - 4\sqrt{2}\alpha_1\rho^*_{13}\} \\
& - \sqrt{2}\alpha_1(\tau^* - \tau)^2(2\Gamma_{312} + \Gamma_{334}) + 4(\tau^* - \tau)K \\
& - (\tau^* - \tau)^2(u - v) - \frac{(\tau^* - \tau)^3}{4} + 4\sqrt{2}\alpha_1\rho^*_{13}(\tau^* - \tau)(\Gamma_{112} + \Gamma_{134}) \\
& + \frac{(3\tau^* - \tau)(\tau^* - \tau)^2}{4} + (\tau^* - \tau)F \\
= & -2(\tau^* - \tau)^2(u - v) + \frac{\tau^* - \tau}{2}G + (\tau^* - \tau)F \\
& + 4(\tau^* - \tau)K + \frac{(3\tau^* - \tau)(\tau^* - \tau)^2}{4} - 4\sqrt{2}H_2.
\end{aligned}
$$

From (2.21), (3.1) and (3.2), we have

$$
\begin{aligned}
\bar{\eta}_1 &= \sum(\nabla_1 J_{ab})\rho^*_{a\bar{b}} \\
&= -\frac{\alpha_1}{\sqrt{2}}\sum(\delta_{1a}\delta_{3b} - \delta_{1b}\delta_{3a} - \delta_{2a}\delta_{4b} + \delta_{2b}\delta_{4a})\rho^*_{a\bar{b}} = 0, \\
\bar{\eta}_2 &= \sum(\nabla_2 J_{ab})\rho^*_{a\bar{b}} \\
&= \frac{\alpha_1}{\sqrt{2}}\sum(\delta_{1a}\delta_{4b} - \delta_{1b}\delta_{4a} + \delta_{2a}\delta_{3b} - \delta_{2b}\delta_{3a})\rho^*_{a\bar{b}} = -2\sqrt{2}\alpha_1\rho^*_{13}, \\
\bar{\eta}_3 &= \bar{\eta}_4 = 0.
\end{aligned}
$$

From these equalities and (3.15), we obtain

$$
\begin{aligned}
(3.30) \qquad \sum(\nabla_i\tau^*)\bar{\eta}_i &= (\nabla_2\tau^*)\bar{\eta}_2 = -2\sqrt{2}\alpha_1\rho^*_{13}e_2\tau^* \\
&= -2\sqrt{2}\alpha_1\rho^*_{13}(2(\tau^* - \tau)(2\Gamma_{112} + \Gamma_{134}) - 4\sqrt{2}\alpha_1\rho^*_{13}) \\
&= -4\sqrt{2}H_1 + \frac{\tau^* - \tau}{2}G.
\end{aligned}
$$

4. Proof of Theorem 1

We assume that M is not a Kähler manifold. Taking account of Theorem A, it suffices to consider the case where M is not weakly $*$-Einstein, namely $G \not\equiv 0$ on M.

Let $\{\{e_i\}_{i=1,\dots,4}, \{\alpha, J\alpha\}\}$ be a special adapted pair on some neighborhood of any point in M_1 (see Lemma 4). With respect to the unitary frame $\{e_i\}$, from (3.9) and (3.10), we have

$$(4.1)\quad \begin{aligned} R_{2312} &= e_2\Gamma_{312} - e_3\Gamma_{212} + \sum_t \Gamma_{31t}\Gamma_{2t2} - \sum_t \Gamma_{21t}\Gamma_{3t2} - \sum_t (\Gamma_{23t} - \Gamma_{32t})\Gamma_{t12} \\ &= e_2\Gamma_{312} - e_3\Gamma_{212} + \frac{\alpha_1}{\sqrt{2}}\Gamma_{314} - \frac{\alpha_1}{\sqrt{2}}\Gamma_{112} \\ &\quad - \Gamma_{234}\Gamma_{412} - \Gamma_{312}\Gamma_{112} - \Gamma_{314}\Gamma_{312} + \Gamma_{313}\Gamma_{412}, \end{aligned}$$

$$(4.2)\quad \begin{aligned} R_{2334} &= e_2\Gamma_{334} - e_3\Gamma_{234} - \frac{\alpha_1}{\sqrt{2}}\Gamma_{314} - \frac{\alpha_1}{\sqrt{2}}\Gamma_{134} \\ &\quad - \Gamma_{234}\Gamma_{434} - \Gamma_{312}\Gamma_{134} - \Gamma_{314}\Gamma_{334} + \Gamma_{313}\Gamma_{434}, \end{aligned}$$

$$(4.3)\quad R_{2312} = R_{1223} = \frac{\alpha_1}{\sqrt{2}}\Gamma_{314} - \frac{\alpha_1}{\sqrt{2}}\Gamma_{112}.$$

Since τ^* is constant, from (3.8), the equalities (3.15) are reduced to

$$(4.4)\quad \begin{aligned} &2\Gamma_{112} + \Gamma_{134} = \frac{\sqrt{2}}{\alpha_1}\rho^*_{13}, \quad 2\Gamma_{212} + \Gamma_{234} = 0, \\ &2\Gamma_{312} + \Gamma_{334} = 0, \quad 2\Gamma_{412} + \Gamma_{434} = 0. \end{aligned}$$

Thus, from (4.1)$\sim$(4.4), we have

$$\begin{aligned} -\rho^*_{13} &= \rho^*_{24} = R_{2312} + R_{2334} \\ &= e_2(\Gamma_{312} + \Gamma_{334}) - e_3(\Gamma_{212} + \Gamma_{234}) - \frac{\alpha_1}{\sqrt{2}}(\Gamma_{112} + \Gamma_{134}) - \Gamma_{234}(\Gamma_{412} + \Gamma_{434}) \\ &\quad - \Gamma_{312}(\Gamma_{112} + \Gamma_{134}) - \Gamma_{314}(\Gamma_{312} + \Gamma_{334}) + \Gamma_{313}(\Gamma_{412} + \Gamma_{434}) \\ &= -e_2\Gamma_{312} + e_3\Gamma_{212} - \frac{\alpha_1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\alpha_1}\rho^*_{13} - \Gamma_{112}\right) + \Gamma_{234}\Gamma_{412} \\ &\quad - \Gamma_{312}\left(\frac{\sqrt{2}}{\alpha_1}\rho^*_{13} - \Gamma_{112}\right) + \Gamma_{314}\Gamma_{312} - \Gamma_{313}\Gamma_{412} \\ &= -(e_2\Gamma_{312} - e_3\Gamma_{212} - \Gamma_{234}\Gamma_{412} - \Gamma_{312}\Gamma_{112} - \Gamma_{314}\Gamma_{312} + \Gamma_{313}\Gamma_{412}) \\ &\quad - \rho^*_{13} + \frac{\alpha_1}{\sqrt{2}}\Gamma_{112} - \frac{\sqrt{2}}{\alpha_1}\Gamma_{312}\rho^*_{13} \\ &= -\rho^*_{13} + \frac{\alpha_1}{\sqrt{2}}\Gamma_{112} - \frac{\sqrt{2}}{\alpha_1}\Gamma_{312}\rho^*_{13}, \end{aligned}$$

and hence

$$(4.5)\quad (\tau^* - \tau)\Gamma_{112} = 4\Gamma_{312}\rho^*_{13}.$$

If we put

$$h_1 = \frac{\sqrt{2}}{\alpha_1}\Gamma_{112}\rho^*_{13} \quad \text{and} \quad h_2 = \frac{\alpha_1}{\sqrt{2}}\Gamma_{312},$$

then (4.5) becomes

$$(\tau^* - \tau)^2 h_1 = G h_2. \tag{4.6}$$

Since τ^* is constant, from (3.19), $(3.22)_2$ and (4.4), we have

$$K = \frac{(\tau^* - \tau)^2}{16}. \tag{4.7}$$

From (3.20), $(3.22)_4$, (4.4) and (4.7), we have

$$\begin{aligned} 0 &= -\frac{(\tau^* - \tau)^2}{4} + 4\sqrt{2}\left(\frac{\sqrt{2}}{\alpha_1}\rho^*_{13} - \Gamma_{112}\right)a_1\rho^*_{13} \\ &\quad + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{4} - \sqrt{2}(\tau^* - \tau)\alpha_1\Gamma_{312} \\ &= \frac{\tau^*(\tau^* - \tau)}{2} + \frac{G}{2} - 2(\tau^* - \tau)h_1 - 2(\tau^* - \tau)h_2, \end{aligned}$$

and hence

$$h_1 + h_2 = \frac{G}{4(\tau^* - \tau)} + \frac{\tau^*}{4}. \tag{4.8}$$

Thus, from (4.6) and (4.8), we obtain

$$\begin{aligned} h_1 &= \frac{G\{G + \tau^*(\tau^* - \tau)\}}{4(\tau^* - \tau)\{G + (\tau^* - \tau)^2\}}, \\ h_2 &= \frac{(\tau^* - \tau)\{G + \tau^*(\tau^* - \tau)\}}{4\{G + (\tau^* - \tau)^2\}}. \end{aligned} \tag{4.9}$$

From (2.23), (3.22), (4.4) and (4.7), we have

$$\begin{aligned} 0 &= \frac{G}{2} + \frac{(\tau^* - \tau)^2}{4} + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{4} \\ &\quad - 2G - (\tau^* - \tau)^2 + 16\sqrt{2}\left(\frac{\sqrt{2}}{\alpha_1}\rho^*_{13} - \Gamma_{112}\right)\alpha_1\rho^*_{13} \\ &= \frac{G}{2} + \frac{\tau(\tau^* - \tau)}{2} - 8(\tau^* - \tau)h_1. \end{aligned} \tag{4.10}$$

Substituting $(4.9)_1$ into (4.10), we get

$$3G^2 + 3\tau^*(\tau^* - \tau)G - \tau(\tau^* - \tau)^3 = 0, \tag{4.11}$$

which implies that G is (and hence ρ^*_{13} is) constant.

Now, since ρ^*_{13} and α_1 are constant, from $(3.14)_2$ and $(3.16)_2$, we have

$$2\Gamma_{112} + \Gamma_{134} = \frac{\sqrt{2}}{\alpha_1}\rho^*_{13},$$
$$(\Gamma_{112} + \Gamma_{134})\rho^*_{13} = \frac{\tau^* - \tau}{4\sqrt{2}}\alpha_1,$$

and hence

$$\Gamma_{112}\rho^*_{13} = \frac{\sqrt{2}}{16\alpha_1}G - \frac{\tau^* - \tau}{4\sqrt{2}}\alpha_1.$$

Thus, we obtain

$$h_1 = \frac{\sqrt{2}}{\alpha_1}\Gamma_{112}\rho^*_{13} = \frac{G - (\tau^* - \tau)^2}{4(\tau^* - \tau)}. \tag{4.12}$$

Comparing $(4.9)_1$ with (4.12), we obtain

$$G\tau^* = -(\tau^* - \tau)^3. \tag{4.13}$$

Since $G > 0$ and $\tau^* - \tau > 0$, this equality implies $0 > \tau^* > \tau$. On one hand, from (4.11) and (4.13), we obtain

$$7\tau(\tau^*)^2 - 9\tau^2\tau^* + 3\tau^3 = 0.$$

But, since $\tau \neq 0$, no real number τ^* satisfies this equality. This is a contradiction and it completes the proof.

5. Proof of Theorem 2

We assume that M is not a Kähler manifold. By the assumption, G is constant. Taking account of Theorem B, it suffices to consider the case $G > 0$.

Let $\{\{e_i\}_{i=1,\dots,4}, \{\alpha, J\alpha\}\}$ be a special adapted pair on some neighborhood of any point in M_1 (see Lemma 4). With respect to the unitary frame $\{e_i\}$, since G is (and hence ρ^*_{13} is) a non-zero constant, the equalities in (3.18) are reduced to

$$\begin{aligned}
&\Gamma_{112} + \Gamma_{134} = \frac{\sqrt{2}\alpha_1}{8\rho^*_{13}}(\tau^* - \tau),\\
&\Gamma_{212} + \Gamma_{234} = 0,\\
&\Gamma_{312} + \Gamma_{334} = \frac{1}{4\rho^*_{13}}e_2\tau^* - \sqrt{2}\alpha_1,\\
&\Gamma_{412} + \Gamma_{434} = \frac{1}{4\rho^*_{13}}e_1\tau^*.
\end{aligned} \tag{5.1}$$

Substituting $(5.1)_1$ into $(3.22)_4$, we obtain

$$\operatorname{div} \bar{\eta} = \frac{G}{2} - \frac{(\tau^* - \tau)^2}{4}. \tag{5.2}$$

From (2.22), we may observe that this equality is also valid on the interior of the complement of M_1. Thus, we have

$$\int_M \left\{ \frac{G}{2} - \frac{(\tau^* - \tau)^2}{4} \right\} dM = 0. \tag{5.3}$$

Now, let $p_0 \in M$ be a point at which the function $\tau^* - \tau$ attains its maximum. Since G is a positive constant, (5.3) implies

$$\frac{G}{2} - \frac{(\tau^* - \tau)^2}{4} \leq 0 \tag{5.4}$$

at p_0. On one hand, from (2.23) and (3.21), we have

$$L = (\tau^* - \tau)(u - v) - F - 3 \operatorname{div} \bar{\eta} \tag{5.5}$$

at p_0, where $L = e_1(e_1\tau^*) + e_2(e_2\tau^*)$. From $(3.14)_4$ and $(5.1)_3$, we have

$$2\Gamma_{312} + \Gamma_{334} = 0, \quad \Gamma_{312} + \Gamma_{334} = -\sqrt{2}\alpha_1 \tag{5.6}$$

at p_0, and hence, from $(3.22)_{1,2}$ and (5.6), we have

$$F = -\sqrt{2}(\tau^* - \tau)\alpha_1\Gamma_{312} = -(\tau^* - \tau)^2,$$

$$u - v = \frac{\tau^* - \tau}{4}$$

at p_0. Substituting these equalities and (5.2) into (5.5), we obtain

$$L = -\frac{3}{2}G + 2(\tau^* - \tau)^2 \tag{5.7}$$

at p_0. Since $L \leq 0$ at p_0,

$$\frac{3}{2}G \geq 2(\tau^* - \tau)^2 \tag{5.8}$$

at p_0. Thus, from (5.4) and (5.8), we conclude

$$0 < \frac{5}{12}(\tau^* - \tau)^2 = \frac{2}{3}(\tau^* - \tau)^2 - \frac{(\tau^* - \tau)^2}{4} \leq \frac{G}{2} - \frac{(\tau^* - \tau)^2}{4} \leq 0 \tag{5.9}$$

at p_0. This is a contradiction and it completes the proof.

6. Proof of Theorem 3

We may assume that M is a strictly almost Kähler manifold. Since $\tau < 0$, as we have seen in section 3, M_1 is open and dense in M_0. Thus, taking account of (2.57)$\sim$(2.59) and (3.19)$\sim$(3.30), we see that the functions H_1 and H_2 (defined on

M_1) can be extended to the ones on M_0, respectively (and hence, the equalities (3.19)$\sim$(3.30) (on M_1) can be regarded as the ones on M_0, respectively).

Let $p_0 \in M_0$ be a point such that the function $\tau^* - \tau$ attains its maximum at p_0. With respect to a special adapted frame field, from (3.11) and (3.12), we have

$$(\Gamma_{314} - \Gamma_{413})^2 + (\Gamma_{324} - \Gamma_{423})^2 = \frac{2}{\alpha_1^2}(\rho_{13}^*)^2 = \frac{G}{4(\tau^* - \tau)}.$$

From this equality, if we put $H = \Gamma_{314}^2 + \Gamma_{413}^2 + \Gamma_{324}^2 + \Gamma_{423}^2$, then we have

$$2(\Gamma_{314}\Gamma_{413} + \Gamma_{324}\Gamma_{423}) = H - \frac{G}{4(\tau^* - \tau)}. \tag{6.1}$$

From the definition of H, we may put

$$\Gamma_{314} = a\cos\theta, \quad \Gamma_{413} = a\sin\theta, \quad \Gamma_{324} = b\cos\zeta, \quad \Gamma_{423} = b\sin\zeta,$$

where $a, b \geq 0$ and $a^2 + b^2 = H$. Then, (6.1) becomes

$$a^2 \sin 2\theta + b^2 \sin 2\zeta = H - \frac{G}{4(\tau^* - \tau)}.$$

This implies

$$-H \leq H - \frac{G}{4(\tau^* - \tau)} \leq H,$$

and hence

$$(0 \leq)\ \frac{G}{8(\tau^* - \tau)} \leq H. \tag{6.2}$$

From (3.30), (3.15) and (3.22), we have

$$4\sqrt{2}H_1 = \frac{\tau^* - \tau}{2}G, \tag{6.3}$$

$$u - v = \frac{\tau^* - \tau}{4}, \quad w = 0, \tag{6.4}$$

at p_0, and thus, from (2.18) and the definition of K,

$$K = \frac{(\tau^* - \tau)^2}{16}, \tag{6.5}$$

$$u = 0, \quad v = -\frac{\tau^* - \tau}{4}, \quad w = 0, \tag{6.6}$$

at p_0.

From (3.27), (3.29), (6.5) and (6.6), we have

$$f_1(p_0) = \tau^*(\tau^* - \tau)^2 - (\tau^* - \tau)G - (\tau^* - \tau)F + 2(\tau^* - \tau)^2 H + 12\sqrt{2}H_2, \tag{6.7}$$

$$f_2(p_0) = \frac{1}{2}\tau^*(\tau^* - \tau)^2 + \frac{1}{2}(\tau^* - \tau)G + (\tau^* - \tau)F - 4\sqrt{2}H_2, \tag{6.8}$$

and hence

$$\begin{aligned} f(p_0) &= f_1(p_0) + f_2(p_0) \\ &= \frac{3}{2}\tau^*(\tau^* - \tau)^2 - \frac{1}{2}(\tau^* - \tau)G + 2(\tau^* - \tau)^2 H + 8\sqrt{2}H_2. \end{aligned} \tag{6.9}$$

On one hand, from (2.23), (3.22) and (6.3), we have

$$\begin{aligned} f(p_0) &= -\frac{3}{2}(\tau^* - \tau)G + \frac{1}{2}\tau(\tau^* - \tau)^2 + 16\sqrt{2}(H_1 - H_2) \\ &= \frac{1}{2}(\tau^* - \tau)G + \frac{1}{2}\tau(\tau^* - \tau)^2 - 16\sqrt{2}H_2. \end{aligned} \tag{6.10}$$

Comparing (6.9) with (6.10), we obtain

$$24\sqrt{2}H_2 = (\tau^* - \tau)G - \frac{1}{2}(3\tau^* - \tau)(\tau^* - \tau)^2 - 2(\tau^* - \tau)^2 H \tag{6.11}$$

at p_0. Substituting this equality into (6.7) and (6.8) and taking account of (6.2), we have

$$\begin{aligned} f_1(p_0) &= \frac{1}{4}(\tau^* + \tau)(\tau^* - \tau)^2 - \frac{1}{2}(\tau^* - \tau)G + (\tau^* - \tau)^2 H - (\tau^* - \tau)F \\ &\geq \frac{1}{4}(\tau^* + \tau)(\tau^* - \tau)^2 - \frac{3}{8}(\tau^* - \tau)G - (\tau^* - \tau)F, \end{aligned} \tag{6.12}$$

$$\begin{aligned} f_2(p_0) &= \frac{1}{12}(9\tau^* - \tau)(\tau^* - \tau)^2 + \frac{1}{3}(\tau^* - \tau)G + (\tau^* - \tau)F + \frac{1}{3}(\tau^* - \tau)^2 H \\ &\geq \frac{1}{12}(9\tau^* - \tau)(\tau^* - \tau)^2 + \frac{3}{8}(\tau^* - \tau)G + (\tau^* - \tau)F. \end{aligned} \tag{6.13}$$

Since $f_1(p_0) \leq 0$ and $f_2(p_0) \leq 0$, (6.12) and (6.13) imply

$$\frac{1}{4}(\tau^* + \tau)(\tau^* - \tau)^2 + \frac{1}{12}(9\tau^* - \tau)(\tau^* - \tau)^2 \leq 0$$

at p_0. From this inequality, we obtain $\tau^* \leq -\tau/6$ at p_0. Since $\tau^*(p_0)$ is maximum, this inequality holds on M and it completes the proof.

References

[1] J. Armstrong, On four-dimensional almost Kähler manifolds, Quart. J. Math., **48** (1997), 405–415.

[2] A. Gray, M. Barros, A. M. Naveira and L. Vanhecke, The Chern numbers of holomorphic vector bundles and formally holomorphic connections of complex vector bundles over almost complex manifolds, J. Reine Angew. Math., **314** (1980), 84–98.

[3] T. Draghici, On the almost Kähler manifolds with Hermitian Ricci tensor, Houston J. Math., **20** (1994), 293–298.

[4] ———, On some 4-dimensional almost Kähler manifolds, Kodai Math. J., **18** (1995), 156–168.

[5] S. I. Goldberg, Integrability of almost Kähler manifolds, Proc. Amer. Math. Soc., **21** (1969), 96–100.

[6] A. Gray, Curvature identities for Hermitian and almost Hermitian manifolds, Tôhoku Math. J., **28** (1969), 601–612.

[7] A. Gray and L. M. Hervella, The sixteen classes of almost Hermitian manifolds, Ann. Mat. Pura Appl., **123** (1980), 35–58.

[8] N. Hitchin, Compact four-dimensional Einstein manifolds, J. Differential Geom., **9** (1974), 435–441.

[9] N. Murakoshi, T. Oguro and K. Sekigawa, Four-dimensional almost Kähler locally symmetric spaces, Differential Geom. Appl., **6** (1996), 237–244.

[10] A. Newlander and L. Nirenberg, Complex analytic coordinates in almost complex manifolds, Ann. of Math., **65** (1957), 391–404.

[11] T. Oguro and K. Sekigawa, Non-existence of almost Kähler structure on hyperbolic spaces of dimension $2n (\geq 4)$, Math. Ann., **350** (1994), 317–329.

[12] ———, Almost Kähler structures on the Riemannian product of a 3-dimensional hyperbolic space and a real line, Tsukuba J. Math., **20** (1996), 151–161.

[13] ———, Four-dimensional almost Kähler Einstein and $*$-Einstein manifolds, Geom. Dedicata, **69** (1998), 91–112.

[14] T. Oguro, K. Sekigawa and A. Yamada, Four-dimensional almost Kähler Einstein and weakly $*$-Einstein manifolds, Yokohama Math. J., **47** (1999), 75–92.

[15] K. Sekigawa, On some compact Einstein almost Kähler manifolds, J. Math. Soc. Japan, **36** (1987), 677–684.

[16] K. Sekigawa and L. Vanhecke, Four-dimensional almost Kähler Einstein manifolds, Ann. Math. Pura Appl., **157** (1990), 149–160.

[17] W. P. Thurston, Some simple examples of symplectic manifolds, Proc. Amer. Math. Soc., **55** (1976), 467–468.

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