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INTEGRAL MEANS FOR THE FIRST DERIVATIVE OF BLASCHKE PRODUCTS

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Abstract

Suppose that $B(z, \{a_k\})$ is a Blaschke product associated with the sequence $\{a_k\}$. If $\sum_{k=1}^{\infty}(1-|a_k|)^{\alpha} < \infty$ for some α in $(0, 1/2)$, it is known that $B'(z, \{a_k\}) \in H^p$ for all p in $(0, 1-\alpha]$. We extend this result by considering the situation when $p > 1-\alpha$. In these cases we obtain that $B'(z, \{a_k\}) \notin H^p$. We then give some counterexamples to indicate to what extent our results may be regarded as being best possible.

1. Introduction and preliminaries

A Blaschke product $B(z, \{a_k\})$ associated with the sequence $\{a_k\}$ is a function defined by the formula

$$B(z, \{a_k\}) = \prod_{k=1}^{\infty} \frac{\bar{a}_k}{|a_k|}\left(\frac{a_k - z}{1 - \bar{a}_k z}\right), \tag{1}$$

where $\sum_{k=1}^{\infty}(1-|a_k|) < \infty$, and $0 < |a_k| < 1$ for all k in $\boldsymbol{N}$. This function is regular and bounded whenever z is in $U = \{z : |z| < 1\}$.

For $0 < p < \infty$, the Hardy space H^p is defined to be the class of all functions f regular in U for which $\sup_{0<r<1}\{1/(2\pi)\int_0^{2\pi}|f(re^{i\theta})|^p\, d\theta\}^{1/p}$ is finite.

The purpuse of this paper is to extend a result due to Protas [5], which says that, if

$$\sum_{k=1}^{\infty}(1-|a_k|)^{\alpha} < \infty, \tag{2}$$

for some α in $(0, 1/2)$, then $B'(z, \{a_k\}) \in H^p$ for all p in $(0, 1-\alpha]$. In §2, we show that, for all $p > 1-\alpha$, the integral means $I(r) = \int_0^{2\pi}|B'(re^{i\theta}, \{a_k\})|^p\, d\theta = o\{(1-r)^{1-\alpha-p}\}$ as $r \to 1$. We then present, in §3, a specific example which leads to methods that can be employed and gives guide to a procedure that can be adopted in the general counterexamples that we encounter later in §4 and §5 while illustrating that our estimates are essentially best possible of their type. In these counterexamples we find it convenient to construct Blaschke products which have

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groups of factors contributing in a dominant way to the estimation of the integral $I(r)$ for a set of values r increasing to 1.

Finally, we see in §6 that the sequence of values r for which the general counterexamples hold may be widely dispersed. If this is the case, then these counterexamples can hold everywhere on an interval $[r_0, 1)$.

Throughout this paper, the symbol C denotes a positive constant, and $C(\cdots)$ a positive constant depending on the parameters indicated in $(\cdots)$. For clarification we will, from time to time, distinguish various appearances of the constants by using suffixes such as $C_1, C_2, \ldots, C_1(\cdots), C_2(\cdots), \ldots$. Also the symbol $\varepsilon(r)$ denotes a positive function that tends to zero as r increases to one. However it should be noted that for $j = 1, 2, \ldots,$ the terms $C, C_j, C_j(\cdots)$ and $\varepsilon(r)$ may vary from one appearance to the next, but always there will be some implicit means of finding them at each stage.

For later convenience, we list the following standard lemmas.

Lemma 1 (See [6], p. 333). *Let $B(z, \{a_k\})$ be a Blaschke product. Then*

$$|B(z, \{a_k\})| < 1 \quad \textit{for all } z \textit{ in } U.$$

Lemma 2 (See [7], p. 226). *If $x \in (0, 1) \cup (1, \infty)$, and y is a natural number, then there are two positive constants $C_1(x)$ and $C_2(x)$ not dependent on y such that*

$$C_2(x) < (1 - r)^{h(x)} \int_0^{2\pi} |1 - re^{iy\theta}|^{-x}\, d\theta < C_1(x) \quad (0 < r < 1),$$

where $h(x) = \max(0, x - 1)$.

In view of Lemma 1, we note that for

$$B_m(z, \{a_k\}) = \prod_{\substack{k=1\\k\neq m}}^{\infty} \frac{\bar{a}_k}{|a_k|}\left(\frac{a_k - z}{1 - \bar{a}_k z}\right), \tag{3}$$

we also get

$$|B_m(z, \{a_k\})| < 1 \quad (|z| < 1). \tag{4}$$

2. Some extensions of Protas' theorem

Here we show that, Protas' theorem can be extended for all $p > 1 - \alpha$ while $0 < \alpha < 1/2$. First we present some interesting useful lemmas.

Lemma 3. *Let $\{a_k\}$ be a Blaschke sequence such that* (2) *holds for some α in $(0, 1/2)$. Then for any number $p > 1 - \alpha$, we have*

$$\sum_{k=1}^{\infty} \frac{(1 - r_k)^p}{(1 - r_k r)^{2p-1}} < \frac{\varepsilon(r)}{(1 - r)^{p+\alpha-1}} \quad (0 < r < 1),$$

where $\lim_{r\to 1} \varepsilon(r) = 0$.

Proof. The inequality $p - \alpha > 0$ implies

$$\frac{(1-r_k)^p}{(1-r_k r)^{2p-1}} = \frac{(1-r_k)^\alpha}{(1-r_k r)^{p+\alpha-1}} \left(\frac{1-r_k}{1-r_k r}\right)^{p-\alpha} < \frac{(1-r_k)^\alpha}{(1-r)^{p+\alpha-1}}. \tag{5}$$

Also we have

$$\frac{(1-r_k)^p}{(1-r_k r)^{2p-1}} < (1-r_k)^{1-p}. \tag{6}$$

Now, for any number $\lambda > 0$ we can find a sufficiently large number N, depending on λ, such that $\sum_{k=N+1}^{\infty}(1-r_k)^\alpha < \lambda$. Together with the inequalities (5) and (6) this shows that

$$\sum_{k=1}^{\infty} \frac{(1-r_k)^p}{(1-r_k r)^{2p-1}} < \left\{\sum_{k=1}^{N}(1-r_k)^{1-p}\right\} + \frac{\lambda}{(1-r)^{p+\alpha-1}} < \frac{2\lambda}{(1-r)^{p+\alpha-1}},$$

since, for a suitable choice of R, depending on N and $\{r_k\}$, the value $(1-r)^{p+\alpha-1}$ can be made so small that

$$\sum_{k=1}^{N}(1-r_k)^{1-p} < \frac{\lambda}{(1-r)^{p+\alpha-1}} \quad (0 < R < r < 1).$$

The last inequality is equivalent to

$$1 - \left\{\frac{\lambda}{\sum_{k=1}^{N}(1-r_k)^{1-p}}\right\}^{1/(p+\alpha-1)} < r < 1.$$

Hence we conclude that

$$\sum_{k=1}^{\infty} \frac{(1-r_k)^p}{(1-r_k r)^{2p-1}} < \frac{\varepsilon(r)}{(1-r)^{p+\alpha-1}} \quad (0 < r < 1),$$

where $\lim_{r\to 1} \varepsilon(r) = 0$. □

Lemma 4. *Let $B(z, \{a_k\})$ be a Blaschke product. Then, for any $z \in U$,*

$$|B'(z, \{a_k\})| < \frac{C}{(1-r)} \quad (|z| = r).$$

Proof. Since $|B(ve^{i\phi}, \{a_k\})| < 1$ for all v in $(0, 1)$ and $0 \le \phi < 2\pi$. Then by the Cauchy's integral formula for derivatives [4, p. 123] we get

$$B'(z, \{a_k\}) = \frac{1}{2\pi i}\int_{|w|=v} \frac{B(w, \{a_k\})}{(w-z)^2}\, dw \quad (0 < r < v < 1),$$

where $z = re^{i\theta}$ and $w = ve^{i\phi}$

Thus, for $\mu = \phi - \theta$, we have

$$|B'(re^{i\theta}, \{a_k\})| = \left|\frac{1}{2\pi i}\int_0^{2\pi} \frac{B(ve^{i\phi}, \{a_k\})}{(ve^{i\phi} - re^{i\theta})^2} ive^{i\phi}\, d\phi\right| < \frac{1}{2\pi v}\int_0^{2\pi} \frac{d\mu}{|1 - (r/v)e^{i\mu}|^2}.$$

Put $v = (1/2)(1 + r)$. Then, $r/v = 2r/(1 + r) < 1$, and so Lemma 2 yields

$$|B'(re^{i\theta}, \{a_k\})| < \frac{1}{2\pi v}\frac{C_1}{(1 - r/v)} \leq \frac{C}{(1 - r)}.$$

This completes the proof. □

Using Lemma 3, we prove the following result which extends Protas' theorem for $1 - \alpha < p \leq 1$.

THEOREM 1. *Let $B(z, \{a_k\})$ be a Blaschke product such that the condition* (2) *holds for some α in* $(0, 1/2)$. *Then, for $1 - \alpha < p \leq 1$, there is a positive function $\varepsilon(r)$ such that*

$$\int_0^{2\pi} |B'(re^{i\theta}, \{a_k\})|^p\, d\theta < \frac{\varepsilon(r)}{(1 - r)^{p+\alpha-1}} \quad (0 < r < 1), \tag{7}$$

where

$$\lim_{r\to 1} \varepsilon(r) = 0. \tag{8}$$

Proof. Since $1/2 < 1 - \alpha < p \leq 1$, then the logarithmic derivative of $B(z, \{a_k\})$ together with (4) (replacing m with k) yields

$$|B'(z, \{a_k\})|^p = \left|-\sum_{k=1}^{\infty} B_k(z, \{a_k\})\frac{\bar{a}_k}{|a_k|}\frac{1 - |a_k|^2}{(1 - \bar{a}_k z)^2}\right|^p < \sum_{k=1}^{\infty}\left|\frac{1 - |a_k|^2}{(1 - \bar{a}_k z)^2}\right|^p.$$

Therefore, for $z = re^{i\theta}$ and $a_k = r_k e^{i\gamma_k}$, Lemma 2 implies

$$\int_0^{2\pi} |B'(re^{i\theta}, r_k e^{i\gamma_k})|^p\, d\theta < C(p)\sum_{k=1}^{\infty}\frac{(1 - r_k)^p}{(1 - r_k r)^{2p-1}}.$$

Applying Lemma 3 to the last inequality, and absorbing the constant $C(p)$ in the function $\varepsilon(r)$, we complete the proof of the Theorem. □

Now utilizing our previous results, we obtain the following more general result, extending the Protas' theorem for $p > 1$.

THEOREM 2. *Let $B(z, \{a_k\})$ be a Blaschke product, such that the condition* (2) *holds for some α in* $(0, 1/2)$. *Then, for all $p > 1$, there is a positive function $\varepsilon(r)$ such that*

$$\int_0^{2\pi} |B'(re^{i\theta}, \{a_k\})|^p\, d\theta < \frac{\varepsilon(r)}{(1 - r)^{p+\alpha-1}} \quad (0 < r < 1),$$

where $\lim_{r\to 1}\varepsilon(r) = 0$.

Proof. Choose a suitable number γ with $1-\alpha < \gamma \le 1 < p$. Then by applying Theorem 1 and Lemma 4, we get

$$\int_0^{2\pi} |B'(re^{i\theta},\{a_k\})|^p\, d\theta = \int_0^{2\pi} |B'(re^{i\theta},\{a_k\})|^{p-\gamma}|B'(re^{i\theta},\{a_k\})|^{\gamma}\, d\theta$$
$$< \frac{C(\gamma,p)}{(1-r)^{p-\gamma}}\int_0^{2\pi} |B'(re^{i\theta},\{a_k\})|^{\gamma}\, d\theta \le \frac{\varepsilon(r)}{(1-r)^{p+\alpha-1}},$$

if $C(\gamma,p)\le 1$. If not, then replacing the given function $\varepsilon(r)$ by $\varepsilon(r)/C(\gamma,p)$ ends the proof. □

3. Some counterexamples

The question that will now be considered here is whether (7) with (8) is the best possible result that can be obtained, or can we find stronger conditions than (8) satisfied by $\varepsilon(r)$? For example, is $\varepsilon(r) < (1-r)^{\tau}$ possible for some positive number τ, and all sequences $\{a_k\}$ satisfying (2)?

We will see that this can be disproved by example, and further examples will be considered to test whether $\varepsilon(r)$ must tend to 0 more slowly as $r\to 1$. e.g. Is it always true that

$$\varepsilon(r) < \{\log\log(1-r)^{-1}\}^{-\tau} \quad (0<r<1), \tag{9}$$

for some $\tau > 0$?

If a result such as (9) is true for all sequences $\{a_k\}$, Protas' theorem might be modified to show this. If (9) is not generally true for all sequences $\{a_k\}$, then some suitable examples should be sought out.

The following particular example shows that (9) is, in fact, not generally true.

Example 1. Let $B(z,\{a_k\})$ be a Blaschke product with zeros $\{a_k\}$ such that

$$B(z,\{a_k\}) = \prod_{k=1}^{\infty}\frac{r_k^{q_k} - z^{q_k}}{1 - r_k^{q_k} z^{q_k}} = \prod_{k=1}^{\infty} b_k(z,\{a_k\}), \tag{10}$$

where $q_k = 2^{p_k}$ and $p_k = 2^{2^k}$ for all $k\ge 1$ $(k\in \boldsymbol{N})$.

Further, for some α in $(0,1/2)$ and $1-\alpha < p \le 1$, define

$$r_k = 1 - 2^{-p_k/\alpha}2^{-k\beta} \quad (\beta > 0),$$

then

$$I(r) = \int_0^{2\pi} |B'(re^{i\theta},\{a_k\})|^p\, d\theta > \frac{\varepsilon(r)}{(1-r)^{p+\alpha-1}},$$

where $r = r_t$, for sufficiently large values of t, and

$$\varepsilon(r) > C(p)\{\log\log(1-r)^{-1}\}^{-\tau} \quad (\tau > 0).$$

For convenience log will denote $\log_2$.

Similar examples have been considered previously by Linden [3], and it is not difficult to follow similar applications.

This result could be extended to include values of p in $(1, \infty)$. We shall, however, generalize our considerations of the problem, and cover such cases later, by making some refinements of this simpler method.

4. General counterexamples for $1 - \alpha < p \le 1$

In the following theorem we shall investigate the sense in which our result in Theorem 1 is the best possible of its type.

Theorem 3. *Let* $\alpha \in (0, 1/2)$, *and let* $\varepsilon(r)$ *be a given positive function which is continuous on* $(0, 1)$ *with* $\lim_{r\to 1} \varepsilon(r) = 0$. *Then, there exist an infinite Blaschke product* $B(z, \{a_k\})$ *whose zeros satisfy the convergence condition* (2), *and for any number* p *in* $(1-\alpha, 1]$, *we have*

$$I(r) = \int_0^{2\pi} |B'(re^{i\theta}, \{a_k\})|^p \, d\theta > \frac{\varepsilon(r)}{(1-r)^{p+\alpha-1}}, \tag{11}$$

for a sequence of values r *increasing to* 1.

In fact estimating $I(r)$ would be difficult in general. So, it will be an advantage to look at a Blaschke product $B(z, \{a_k\})$ which can be factorized so that for certain values of r the contribution to $I(r)$ arising from some of the terms is dominant.

One way of doing this is to consider, as in Example 1, clusters of zeros of equal modulus which will give rise to major contributions in the formation of the Blaschke product $B(z, \{a_k\})$ and the corresponding integral $I(r)$ for certain values of z and r.

So, for a sequence of numbers $\{r_k\}$ increasing to 1 on $(0, 1)$ and for a sequence of integers $\{q_k\}$ we consider $B(z, \{a_k\})$ to have q_k zeros of the form $r_k e^{i(2\pi j/q_k)}$ $(0 \le j \le q_k - 1)$ equally distributed on $\{z : |z| = r_k\}$ according to definition (10).

Note, however, that the general convergence condition (2) will be here equivalent to

$$\sum_{k=1}^{\infty} q_k (1-r_k)^{\alpha} < \infty, \quad \alpha \in \left(0, \frac{1}{2}\right). \tag{12}$$

Although q_k will be chosen as a large integer, we must have, at least, $q_k = o\{(1-r_k)^{-\alpha}\}$ in order that (12) is satisfied. We will also choose the ratio

$(1-r_{k+1})/(1-r_k)$ to be so small as to give the dominance of one term in $B(z,\{a_k\})$ for appropriate values of z.

In verifying this theorem we may assume, without loss of generality, that the given function $\varepsilon(r)$ is decreasing. If not, then we can replace it by $\varepsilon_1(r) = \sup\{\varepsilon(x) : x \geq r\}$ which satisfies the same conditions as $\varepsilon(r)$ with the addition that $\varepsilon_1(r)$ is decreasing and $\varepsilon_1(r) \geq \varepsilon(r)$.

First we state the following lemma which will be used to show that, without loss of generality, we can replace the decreasing function ε by a smoother function to make our deductions easier to pick out dominant terms in $B(z,\{a_k\})$. We omit the proof for brevity.

LEMMA 5. *For a given positive decreasing function $\varepsilon(r)$ which is continuous on $(0,1)$ with $\lim_{r\to 1}\varepsilon(r) = 0$, there is a positive decreasing function $\delta(r)$ satisfying the following conditions*:

(i) $\delta(r) \geq \varepsilon(r)$ $(r \in (0,1))$,
(ii) $\lim_{r\to 1}\delta(r) = 0$,
(iii) *for each given number σ in* $(0,1)$

$$\lim_{r\to 1}\frac{\delta(r)}{\delta(1-\sigma(1-r))} = 1.$$

It would be interesting to note that, under the given conditions, the function $\delta(r)$ cannot be smaller than any positive power of $(1-r)$.

Proof of Theorem 3. Let $B(z,\{a_k\})$ be defined as in (10) while condition (12) holds. We will estimate the size of $I(r)$ in (11) at a suitable set of values $r = r_t$ $(t \geq t_0; t, t_0 \in \boldsymbol{N})$ by examining the expansion

$$B'(z,\{a_k\}) = -\sum_{k=1}^{\infty} B_k(z,\{a_k\})\frac{q_k z^{q_k-1}(1-r_k^{2q_k})}{(1-r_k^{q_k}z^{q_k})^2}, \tag{13}$$

where

$$B_k(z,\{a_k\}) = \prod_{\substack{j=1\\ j\neq k}}^{\infty} b_j(z,\{a_k\}), \tag{14}$$

and $b_j(z,\{a_k\})$ is defined as in (10) with j in place of k.

Now, for $0 < \sigma < 1$, we will first choose the sequence of numbers $\{r_k\}$ in $(0,1)$ so that

$$1 - r_{k+1} = \sigma(1-r_k) \quad (k = 1,2,3,\ldots). \tag{15}$$

Later, however, this value of σ will be further qualified.

By applying Lemma 5 to $\varepsilon(r)$, we can find $\delta(r)$ which satisfies the conditions (i), (ii), (iii). It will be convenient to prove our result in terms of δ rather than ε, which will be acceptable in view of condition (i). Consequently we define the sequence of integers $\{q_k\}$ so that

$$q_k = [(1 - r_k)^{-\alpha}\delta(r_k)] \quad (k = 1, 2, 3, \ldots).$$

This means that, for large value of k, we can write

$$\frac{\delta(r_k)}{2} < q_k(1 - r_k)^{\alpha} \le \delta(r_k) \quad (k \ge k_0; k, k_0 \in \boldsymbol{N}). \tag{16}$$

For large natural number t, let

$$A_1(z, \{a_k\}) = \prod_{k=1}^{t-1} b_k(z, \{a_k\}) \quad \text{and} \quad A_2(z, \{a_k\}) = \prod_{k=t+1}^{\infty} b_k(z, \{a_k\}),$$

where $b_k(z, \{a_k\})$ is defined as in (10). Further, for $k = t$, $r = r_t$ and $1 - \alpha < p \le 1$, define

$$J(r_t) = \int_0^{2\pi} \left| B_t(r_t e^{i\theta}, \{a_k\}) \frac{q_t(r_t e^{i\theta})^{q_t - 1}(1 - r_t^{2q_t})}{(1 - r_t^{2q_t} e^{iq_t\theta})^2} \right|^p d\theta, \tag{17}$$

where $B_t(z, \{a_k\})$ is defined by (14) with t in place of k.

We show that (as $t \to \infty$) $J(r_t)$ gives a dominant contribution to $I(r)$ where $r = r_t$. We begin by noting the following property of Blaschke product (see Frostman [1]):

$$|B_t(r_t e^{i\theta}, \{a_k\})|^2 \ge 1 - \sum_{\substack{j=1 \\ j \ne t}}^{\infty} \frac{(1 - r_t^{2q_j})(1 - r_j^{2q_j})}{|1 - r_j^{q_j} r_t^{q_j} e^{iq_j\theta}|^2}, \tag{18}$$

and also since

$$r_j^{q_j} \to 1 \quad \text{as } j \to \infty,$$

then we may assume (without loss of generality) that $r_j^{q_j} \ge 1/2$ for $j \ge 1$. Observe that

$$T_1 = \sum_{j=1}^{t-1} \frac{(1 - r_t^{2q_j})(1 - r_j^{2q_j})}{|1 - r_j^{q_j} r_t^{q_j} e^{iq_j\theta}|^2} < 8 \sum_{j=1}^{t-1} \frac{1 - r_t}{1 - r_j} < \frac{8\sigma}{1 - \sigma}. \tag{19}$$

Similarly, for $r_t^{q_j} \ge 1/2$, we get

$$\begin{aligned} T_2 &= \sum_{j=t+1}^{\infty} \frac{(1 - r_t^{2q_j})(1 - r_j^{2q_j})}{|1 - r_j^{q_j} r_t^{q_j} e^{iq_j\theta}|^2} \\ &< \sum_{j=t+1}^{\infty} \frac{(1 - r_t^{2q_j})(1 - r_j^{2q_j})}{(1 - r_t^{q_j})^2} < 4 \sum_{j=t+1}^{\infty} \frac{1 - r_j^{q_j}}{1 - r_t^{q_j}} \\ &< 4 \sum_{j=t+1}^{\infty} \frac{1 - r_j^{q_j}}{1 - r_t^{q_t}} < \frac{8}{q_t(1 - r_t)} \sum_{j=t+1}^{\infty} q_j(1 - r_j)^{\alpha}(1 - r_j)^{1-\alpha} \end{aligned} \tag{20}$$

$$< \frac{16}{\delta(r_t)(1-r_t)^{1-\alpha}} \sum_{j=t+1}^{\infty} \delta(r_j)(1-r_j)^{1-\alpha}$$

$$< \frac{16\delta(r_{t+1})}{\delta(r_t)(1-r_t)^{1-\alpha}} \sum_{j=1}^{\infty} (1-r_t)^{1-\alpha} \sigma^{j(1-\alpha)}$$

$$= 16 \frac{\delta(r_{t+1})}{\delta(r_t)} \sum_{j=1}^{\infty} \sigma^{j(1-\alpha)} < \frac{16\sigma^{(1-\alpha)}}{1-\sigma^{(1-\alpha)}},$$

since $1 - r_j = \sigma^{(j-t)}(1-r_t)$ for all $j > t$. This together with (18) and (19) gives

$$|B_t(r_t e^{i\theta}, \{a_k\})| > 1 - (T_1 + T_2) > 1 - 8\left[\frac{\sigma}{1-\sigma} + \frac{2\sigma^{(1-\alpha)}}{1-\sigma^{(1-\alpha)}}\right].$$

Thus for a suitable choice of σ in $(0, 1)$ depending on α, the value of $[\sigma/(1-\sigma) + 2\sigma^{(1-\alpha)}/(1-\sigma^{(1-\alpha)})]$ can be made so small that

$$|B_t(r_t e^{i\theta}, \{a_k\})| > 2^{-1} \quad (0 \le \theta < 2\pi). \tag{21}$$

Consequently, substituting (21) in (17) and applying Lemma 2, we obtain

$$J(r_t) > C_1(p) q_t (1-r_t)^{1-p}. \tag{22}$$

Now, for $p \le 1$; (See [2], pp. 146–147), we have

$$J_1(r_t) = \int_0^{2\pi} |A_1'(r_t e^{i\theta}, \{a_k\})|^p \, d\theta < C_2(p) \sum_{k=1}^{t-1} q_k (1-r_k)^{1-p}. \tag{23}$$

Similarly, we obtain

$$J_2(r_t) = \int_0^{2\pi} |A_2'(r_t e^{i\theta}, \{a_k\})|^p \, d\theta < \frac{C_3(p)}{(1-r_t)^{2p-1}} \sum_{k=t+1}^{\infty} q_k (1-r_k)^p \quad (p \le 1). \tag{24}$$

The triangle inequality yields that

$$\begin{aligned} I(r_t) &\ge J(r_t) - J_1(r_t) - J_2(r_t) \\ &> q_t (1-r_t)^{1-p} \left\{ C_1(p) - C_2(p) \sum_{k=1}^{t-1} \frac{q_k (1-r_k)^{1-p}}{q_t (1-r_t)^{1-p}} \right. \\ &\qquad \left. - C_3(p) \sum_{k=t+1}^{\infty} \frac{q_k (1-r_k)^p}{q_t (1-r_t)^p} \right\}. \end{aligned} \tag{25}$$

To estimate the finite sum in (25), we note that

$$\lim_{k \to \infty} \frac{\delta(r_k)}{\delta(1 - \sigma(1-r_k))} = 1,$$

that is, for any $\Omega > 0$ there is a corresponding positive number $K = K(\Omega)$ such that

$$\left|\frac{\delta(r_k)}{\delta(1-\sigma(1-r_k))} - 1\right| < \Omega \quad (k > K).$$

Therefore, for a suitable choice of σ such that $(1+\Omega)\sigma^{p+\alpha-1} < 1$, we obtain

$$\sum_{k=1}^{t-1} \frac{q_k(1-r_k)^{1-p}}{q_t(1-r_t)^{1-p}} < \frac{4(1+\Omega)\sigma^{p+\alpha-1}}{1-(1+\Omega)\sigma^{p+\alpha-1}}. \tag{26}$$

Next, we can easily estimate the infinite sum, and get

$$\sum_{k=t+1}^{\infty} \frac{q_k(1-r_k)^{p}}{q_t(1-r_t)^{p}} < \frac{2\sigma^{p-\alpha}}{1-\sigma^{p-\alpha}}. \tag{27}$$

If we now further restrict the number σ so that

$$C_2(p)\frac{4(1+\Omega)\sigma^{p+\alpha-1}}{1-(1+\Omega)\sigma^{p+\alpha-1}} + C_3(p)\frac{2\sigma^{p-\alpha}}{1-\sigma^{p-\alpha}} \le \frac{C_1(p)}{2},$$

we obtain from (25), (26) and (27)

$$I(r_t) > \frac{C_1(p)}{2} q_t(1-r_t)^{1-p} > \frac{\varepsilon(r_t)}{(1-r_t)^{p+\alpha-1}}. \tag{28}$$

Now the required result follows as long as $\sum_{k=1}^{\infty} \delta(r_k)$ converges. However, our specification of δ does not ensure that this is so. Thus, we note that, due to the condition $\lim_{r\to 1} \delta(r) = 0$, we can choose a subsequence of natural numbers $\{k_s\}$ such that $\sum_{s=1}^{\infty} \delta(r_{k_s}) < \infty$.

Further, we note that, the general inequalities obtained relating to $B(z, \{a_k\})$ are also applicable correspondingly to the following Blaschke subproduct:

$$\beta(z, \{a_k\}) = \prod_{s=1}^{\infty} \frac{r_{k_s}^{q_{k_s}} - z^{q_{k_s}}}{1 - r_{k_s}^{q_{k_s}} z^{q_{k_s}}}.$$

This completes the proof. □

5. General counterexamples for $p > 1$

The following theorem indicates that our result in Theorem 2 is essentially the best possible result that can be obtained.

THEOREM 4. *Let* $\alpha \in (0, 1/2)$, *and let* $\varepsilon(r)$ *be a given positive function which is continuous on* $(0,1)$ *with* $\lim_{r\to 1} \varepsilon(r) = 0$. *Then we can find an infinite Blaschke product* $B(z, \{a_k\})$ *whose zeros satisfy the associated convergence condition* (2), *and for any number* $p > 1$, *we have*

$$I(r) = \int_0^{2\pi} |B'(re^{i\theta}, \{a_k\})|^p \, d\theta > \frac{\varepsilon(r)}{(1-r)^{p+\alpha-1}},$$

for a sequence of values r *increasing to* 1.

Because of similarity of the proof with that of Theorem 3, we shall forego the corresponding details.

6. Integral mean inequalities valid everywhere on $[r_0, 1)$

We note that, if $\varepsilon(r)$ is fairly small, then Theorems 3 and 4 can hold everywhere on an interval $[r_0, 1)$. As a sample of what is possible, we obtain the following theorem.

Theorem 5. *Let* $\alpha \in (0, 1/2)$, *and let* $\varepsilon(r)$ *be a given continuous function on* $(0, 1)$ *and which for a constant number* η (> 1), *satisfies*

$$0 < \varepsilon(r) \le \{\log(1-r)^{-1}\}^{-\eta} \quad (0 < r < 1), \tag{29}$$

then, for some r_0 *in* $(0, 1)$, *and for any number* $p > 1 - \alpha$, *we can find an infinite Blaschke product* $B(z, \{a_k\})$ *whose zeros satisfy the associated convergence condition* (2) *such that*

$$I(r) = \int_0^{2\pi} |B'(re^{i\theta}, \{a_k\})|^p \, d\theta > \frac{\varepsilon(r)}{(1-r)^{p+\alpha-1}}, \quad (r_0 \le r < 1). \tag{30}$$

Proof. Since $\varepsilon(r)$ is assumed to satisfy the condition (29), then we can follow through the proofs of Theorems 3 and 4 by putting

$$\delta(r) = \{\log(1-r)^{-1}\}^{-\eta}. \tag{31}$$

So, for the suitably chosen constant σ in $(0, 1)$ we have

$$\sum_{k=1}^{\infty} \delta(r_k) = \sum_{k=1}^{\infty} \left\{ (k-1) \log\left(\frac{1}{\sigma}\right) + \log\left(\frac{1}{1-r_1}\right) \right\}^{-\eta} < \infty, \tag{32}$$

as long as we assume that $\eta > 1$ and

$$1 - r_{k+1} = \sigma(1 - r_k) \quad (k = 1, 2, \ldots). \tag{33}$$

Consequently, there is no need to take a subsequence $\{r_{k_s}\}$ of $\{r_k\}$ to ensure the convergence of the series $\sum_{s=1}^{\infty} \delta(r_{k_s})$.

We point out that $B(z, \{a_k\})$ was considered to have q_k zeros, for which we obtain

$$\sum_{k=1}^{\infty} (1 - |a_k|)^{\alpha} = \sum_{k=1}^{\infty} q_k (1 - r_k)^{\alpha} < \infty \quad \left(0 < \alpha < \frac{1}{2}\right), \tag{34}$$

where $q_k = [(1 - r_k)^{-\alpha} \delta(r_k)]$ for $k = 1, 2, \ldots$.

Thus, we have a Blaschke product $B(z, \{a_k\})$ with zeros satisfying the convergence condition (34), while the inequality (30) holds for the sequence of values $r = r_t$ increasing to 1, when t is sufficiently large. However, if for any natural number t, we have $r_t \le r < r_{t+1}$, then we can use the increasing property of $I(r)$ (as a function of r) to get

$$I(r) = \int_0^{2\pi} |B'(re^{i\theta}, \{a_k\})|^p \, d\theta \ge \int_0^{2\pi} |B'(r_t e^{i\theta}, \{a_k\})|^p \, d\theta$$
$$> \frac{\delta(r_t)}{(1 - r_t)^{p+\alpha-1}} > \frac{\sigma^{p+\alpha-1}\delta(r)}{(1 - r)^{p+\alpha-1}}.$$

By redesignating the function δ or initially writing $\delta(r) = \sigma^{1-p-\alpha} \cdot \{\log(1 - r)^{-1}\}^{-\eta}$ in place of (31), we end the proof of the Theorem. □

Thus, we can see that this type of results may be extended by the introducing of a logarithmic scale which ensures the convergence property (32). For example, the condition (29) might be replaced by

$$0 < \varepsilon(r) \le \{\log(1 - r)^{-1}\}^{-1}\{\log\log(1 - r)^{-1}\}^{-\eta}$$

for $\eta > 1$ and $0 < r < 1$.

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