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ANOTHER INVOLUTION ON MODULI OF SEXTICS

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1. Introduction

Let $\mathcal{M}$ be the moduli space of sextics with 6 cusps and 3 nodes. A sextic C is called *of a* $(2,3)$-*torus type* (or briefly *of a torus type*) if its defining polynomial f has the expression $f(x,y) = f_2(x,y)^3 + f_3(x,y)^2$ for some polynomials f_2, f_3 of degree 2, 3 respectively. We denote by $\mathcal{M}_{torus}$ the component of $\mathcal{M}$ which consists of curves of a torus type and by $\mathcal{M}_{gen}$ the curves of a general type (= not of a torus type). We denote the dual curve of C by C^*. Recall that C^* is the image of the Gauss map $\mathrm{dual}_C : C \to \boldsymbol{P}^{2*}$, $(X,Y,Z) \mapsto (F_X(X,Y,Z), F_Y(X,Y,Z), F_Z(X,Y,Z))$. In our previous paper [O3], we have shown that *the dual curve operation* $C \mapsto C^*$ *gives an involution on* $\mathcal{M}$ *and it preserves the type of the curve in* $\mathcal{M}$, i.e., $C^* \in \mathcal{M}_{torus}$ *if and only if* $C \in \mathcal{M}_{torus}$. Let $G := \mathrm{PGL}(3, \boldsymbol{C})$. The quotient moduli spaces are by definition the quotient spaces of the moduli spaces by the action of G.

The purpose of this note is to show that *there exists an involution* $\bar{\imath}$ *on* $\mathcal{M}/G$ *such that* $\bar{\imath}$ *is different from the dual curve operation and* $\bar{\imath}$ *preserves the types of the sextics* (Theorem 2.4).

For the construction of $\bar{\imath}$, we consider the moduli space $\tilde{\mathcal{M}}$ of plane curves of degree 12 with 24 cusps and 24 nodes. This moduli space is also self-dual in the sense that $C^* \in \tilde{\mathcal{M}}$ if $C \in \tilde{\mathcal{M}}$. The construction of $\bar{\imath}$ is done as follows. First observe that C has three bi-tangent lines for any $C \in \mathcal{M}$. We take $g \in G$ so that the three coordinate lines $X = 0$, $Y = 0$, $Z = 0$ are the bi-tangent lines of C^g and let $F(X,Y,Z) = 0$ be the defining homogeneous polynomial of degree 6. Then consider the curve $\tilde{C}$ defined by $F(X^2, Y^2, Z^2) = 0$. It turns out that $\tilde{C}$ is contained in $\tilde{\mathcal{M}}$. This operation defines a rational mapping $\psi : \mathcal{M}/G \to \tilde{\mathcal{M}}/G$. We define $\bar{\imath}(C) = \psi^{-1}(\psi(C^g)^*)$.

2. Involution on the quotient moduli $\mathcal{M}/G$

Let $\mathcal{M}$ and $\tilde{\mathcal{M}}$ be the moduli space of sextics with 6 cusps and three nodes and the moduli space of irreducible plane curves of degree 12 with 24 cusps and 24 nodes respectively. Note that the genus of a generic curve in $\mathcal{M}$ (respectively in $\tilde{\mathcal{M}}$) is 1 (resp. 7). By the class formula ([N] or [O3]), it is easy to see that

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for a generic $C \in \tilde{\mathcal{M}}$, the dual curve C^* is also in $\tilde{\mathcal{M}}$. We consider the mapping $\pi : \boldsymbol{P}^2 \to \boldsymbol{P}^2$, defined by $\pi(X, Y, Z) = (X^2, Y^2, Z^2)$, which is a 4-fold covering branched along the coordinate axes $\{X = 0\} \cup \{Y = 0\} \cup \{Z = 0\}$. Take a generic curve $C \in \mathcal{M}$ and let $F(X, Y, Z)$ be the defining homogeneous polynomial of degree 6. As C^* has three nodes, C has three bi-tangent lines. We denote by $\mathcal{M}^{nml}$ the subset of $\mathcal{M}$ which consists of curves $C \in \mathcal{M}$ whose three bi-tangent lines are $X = 0$, $Y = 0$ and $Z = 0$. We define a mapping $\psi : \mathcal{M}^{nml} \to \tilde{\mathcal{M}}$ as follows. Let $C \in \mathcal{M}^{nml}$ and let $F(X, Y, Z)$ be the defining homogeneous polynomial. We define $\psi(C) := \pi^{-1}(C)$. Note that $\psi(C)$ is defined by $\tilde{F}(X, Y, Z) := F(X^2, Y^2, Z^2)$. Each cusp of C produces 4 cusps on $\psi(C)$. Thus $\psi(C)$ has 24 cusps. Each node of C also gives 4 nodes on $\psi(C)$, thus we get 12 nodes on $\psi(C)$ which are mapped onto the nodes of C. As the restriction of π to the affine chart $\{Z \neq 0\}$ is the composition of double coverings $(x, y) \mapsto (x, y^2)$ and $(x, y) \mapsto (x^2, y)$, each simple tangent on the coordinate axis $X = 0$, $Y = 0$ gives 2 nodes on $\psi(C)$ ([O1]). This is the same for the simple tangents for $Z = 0$. Thus there are 12 nodes on $\psi(C)$ which are on the three coordinate axes and they are mapped to simple tangents on coordinate axis by π. Thus $\psi(C)$ has 24 nodes. Thus $\psi(C) \in \tilde{\mathcal{M}}$.

Now for $C \in \mathcal{M}$, we define $\bar{\psi}(C)$ as $\psi(C^g)$ by choosing $g \in G$ such that $C^g \in \mathcal{M}^{nml}$. The ambiguity for the choice of $g \in G$ is in the stabilizer $G_{\mathcal{M}^{nml}}$ of $\mathcal{M}^{nml}$ which is a direct product of $\mathfrak{S}_3$ (the permutations of coordinates) and $\boldsymbol{C}^* \times \boldsymbol{C}^* \times \boldsymbol{C}^*$ (scalar multiplications). Thus the polynomials $F(X, Y, Z)$ and $\tilde{F}(X, Y, Z)$ are unique up to a $G_{\mathcal{M}^{nml}}$ action. Thus $\mathcal{M}^{nml}/G_{\mathcal{M}^{nml}} \cong \mathcal{M}/G$ and $\bar{\psi} : \mathcal{M}/G \to \tilde{\mathcal{M}}/G$ is well-defined.

Recall that a polynomial $F(X, Y, Z)$ is called *even* in X (respectively *symmetric* in X, Y) if $F(-X, Y, Z) = F(X, Y, Z)$ (resp. $F(Y, X, Z) = F(X, Y, Z)$). Thus the polynomial $F(X^2, Y^2, Z^2)$ is even in X, Y, Z. Note that evenness (or symmetricity) is preserved by the dual curve operation.

Lemma 2.1. *Assume that $C = \{F(X, Y, Z) = 0\}$ is defined by an even polynomial $F(X, Y, Z)$ in X (respectively symmetric polynomial in X, Y). Then the dual curve C^* is defined by an even polynomial $F^*(X^*, Y^*, Z^*)$ in X^* (resp. in X^*, Y^*).*

Proof. Assume for example that $F(X, Y, Z)$ is even in X. Then for any point $P = (X, Y, Z) \in C$, let $P' := (-X, Y, Z)$ is also in C. Then it is easy to see that

$$\mathrm{dual}_C(P') = (F_X(P'), F_Y(P'), F_Z(P')) = (-F_X(P), F_Y(P), F_Z(P)) = \mathrm{dual}_C(P)'$$

This implies that $F^*(X^*, Y^*, Z^*)$ is even in X. The symmetric case is proved similarly. □

Assume that $C \in \mathcal{M}$ is defined by $F(X, Y, Z) = 0$. If F is an even polynomial in the variable X (respectively a symmetric polynomial in X, Y), then 6 cusps are stable by the involution $(X, Y, Z) \mapsto (-X, Y, Z)$ (resp. by $(X, Y, Z) \mapsto$

(Y, X, Z)). Then there exists a homogeneous polynomial $F_2(X, Y, Z)$ of degree 2 which is even in X (resp. symmetric in X, Y) such that the conic $F_2(X, Y, Z) = 0$ passes through the 6 cusps of C. By the criterion of Degtyarev [D], the sextic $F(X, Y, Z) = 0$ is of a torus type.

Now we take a generic $C \in \mathcal{M}^{nml}$ and consider the dual curve $\psi(C)^*$ and let $\tilde{G}(X^*, Y^*, Z^*)$ be a defining homogeneous polynomial of degree 12, where (X^*, Y^*, Z^*) is the dual coordinates of (X, Y, Z). As $\tilde{F}(X, Y, Z)$ is even in X, Y, Z, so is $\tilde{G}(X^*, Y^*, Z^*)$ in X, Y, Z by Lemma 2.1.

Proposition 2.2. *$\psi(C)^*$ has 4 nodes on each coordinate axis $X^* = 0$, $Y^* = 0$ or $Z^* = 0$.*

Proof. Let $C = \{F(X, Y, Z) = 0\}$ and let us consider the discriminant polynomial $\Delta_Y F$ with respect to Y-variable. This is a homogeneous polynomial of degree 30 in X, Z ([O2]). We assume that the singularities of the sextic $F(X, Y, Z) = 0$ are not on the coordinate axis. Assume that $P := (\alpha, \beta, \gamma) \in C$ is a singular point of C with Milnor number μ and multiplicity m. Then $\Delta_Y F(X, Z)$ has a linear term $(\gamma X - \alpha Z)^\rho$ with $\rho \geq \mu + m - 1$ and the equality holds if the line $\gamma Y - \beta Z = 0$ is generic with respect to C (see [O3]). Thus to each cusp (respectively to each node), there is an associated linear term with multiplicity 3 (resp. with multiplicity 2). The factor $X = 0$ and $Z = 0$ has also multiplicity 2 in $\Delta_Y F(X, Z) = 0$, as they are bi-tangent lines. Assume C is generic in $\mathcal{M}$. Then the sum of degrees is $18 + 6 + 4 = 28$ by the above consideration. Thus there exists two simple tangent lines of the form $X - \eta_1 Z = 0$ and $X - \eta_2 Z = 0$ for some $\eta_1, \eta_2 \neq 0$. Then four lines $X = \pm\sqrt{\eta_i} Z$, $i = 1, 2$ are bi-tangent lines for the curve $\psi(C)$. This implies that $(1, 0, \pm\sqrt{\eta_i})$, $i = 1, 2$ are nodes of the dual curve $\psi(C)^*$. Thus the coordinate axis $Y^* = 0$ contains 4 nodes of $\psi(C)^*$. By the same argument, $X^* = 0$ and $Z^* = 0$ contains also 4 nodes respectively. The non-emptiness of "generic" curves in $\mathcal{M}$ in the above sense is not obvious but it follows from the example below. □

Definition 2.3. For $C \in \mathcal{M}^{nml}$, we define a polynomial of degree 6 by $G(X^*, Y^*, Z^*) := \tilde{G}(\sqrt{X^*}, \sqrt{Y^*}, \sqrt{Z^*})$ and we define $\iota(C)$ by the sextics defined by $G(X^*, Y^*, Z^*) = 0$. For $C \in \mathcal{M}$, take $g \in G$ so that $C^g \in \mathcal{M}^{nml}$ and we define an involution $\bar{\iota} : \mathcal{M}/G \to \mathcal{M}/G$ by $\bar{\iota}(C) = \iota(C^g)$.

Claim 1. *$\bar{\iota}(C) \in \mathcal{M}$ for a generic $C \in \mathcal{M}$ and $\bar{\iota}$ is an involution which preserves the type of sextics, that is, we have the commutative diagram:*

$$\begin{array}{ccc} \mathcal{M}/G & \xrightarrow{\bar{\iota}} & \mathcal{M}/G \\ \downarrow{\scriptstyle\bar{\psi}} & & \downarrow{\scriptstyle\bar{\psi}} \\ \tilde{\mathcal{M}}/G & \xrightarrow{dual} & \tilde{\mathcal{M}}/G \end{array} \qquad \begin{array}{ccc} \mathcal{M}_{torus}/G & \xrightarrow{\bar{\iota}} & \mathcal{M}_{torus}/G \\ \downarrow{\scriptstyle\bar{\psi}} & & \downarrow{\scriptstyle\bar{\psi}} \\ \tilde{\mathcal{M}}_{torus}/G & \xrightarrow{dual} & \tilde{\mathcal{M}}_{torus}/G \end{array}$$

Proof. We may assume that $C \in \mathcal{M}^{nml}$. By the above consideration, we have seen that the dual curve $\psi(C)^*$ of $\psi(C)$ is defined by a polynomial $G(X^*, Y^*, Z^*)$ of degree 12 which is even in each of the three variables and it has 24 cusps and 12 nodes outside of coordinate axis and 4 nodes on each coordinate axis. Thus $\iota(C)$ has 6 cusps and 3 nodes. Note that nodes of $\psi(C)^*$ on the coordinate axes are mapped on simple tangents on the corresponding coordinate axes of $\iota(C)$. Thus the curve $\iota(C)$, defined by $g(\sqrt{x^*}, \sqrt{y^*}) = 0$, belongs to $\mathcal{M}^{nml}$. Finally we will show that ι keeps the type of the curve. As the curves $\{\bar{\iota}(C); C \in \mathcal{M}_{torus}/G\}$ are topologically equivalent, the image is contained in a connected component. Thus it is enough to show that there exists a $C \in \mathcal{M}_{torus}/G$ such that $\bar{\iota}(C) \in \mathcal{M}_{torus}/G$. To see this, it is enough to take $C \in \mathcal{M}_{torus}^{nml}$ whose defining polynomial $F(X, Y, Z)$ is symmetric in each of X, Y. Then $\tilde{F}(X, Y, Z)$ is also symmetric in X, Y. This implies also that $\tilde{G}(X^*, Y^*, Z^*)$ and $G(X^*, Y^*, Z^*)$ symmetric in X^*, Y^*. By Degtyarev's criterion, this implies that $\iota(C)$ is a sextic of a torus type. □

Thus we have proved the following:

THEOREM 2.4. *There exists an involution $\bar{\iota}$ on the quotient moduli space $\mathcal{M}/G$ which is defined on generic points such that $\bar{\iota}$ is different from the dual curve operation and $\bar{\iota}$ preserves the types of the sextics, that is $\bar{\iota}(C) \in \mathcal{M}_{torus}/G \Leftrightarrow C \in \mathcal{M}_{torus}/G$.*

The following example shows that $\bar{\iota}(C) \neq C^*$ in general.

Example 2.5. Let $C \in \mathcal{M}_{torus}^{nml}$ be the sextic defined by the symmetric polynomial:

$$f := -684(x^3y + xy^3) - 1055(x^3 + y^3) + 2235(x^2 + y^2) - 2178(x + y) + (819/16)(x^5y + y^5x) + (1767/16)(x^4y^2 + x^2y^4) + (881/8)y^3x^3 + (405/16)(x^6 + y^6) - (873/8)(x^5 + y^5) + (2001/4)(x^4 + y^4) - (971/8)(x^4y + xy^4) - (6947/2)y^2x^2 + 2268 + 1038(x^2y + xy^2) - 4883yx - (375/2)(x^2y^3 + x^3y^2).$$

Then $\psi(C)$ is defined by $f(x^2, y^2)$ and $\psi(C)^*$ is defined by $g(x^{*2}, y^{*2}) = 0$ and $\iota(C)$ is the sextic defined by the symmetric polynomial

$$g(x^*, y^*) := 908294x^{*2}y^{*2} - 354000(x^*y^{*2} + x^{*2}y^*) + 302745(y^{*4} + x^{*4}) + 529284(x^{*4}y^{*2} + y^{*4}x^{*2}) - 396458(x^*y^{*4} + y^*x^{*4}) - 722148(x^{*3}y^{*2} + y^{*3}x^{*2}) + 11340(y^{*6} + x^{*6}) - 109170(x^{*5} + y^{*5}) + 86296x^*y^* + 482724(x^{*3}y^* + y^{*3}x^*) - 158508(y^*x^{*5} + y^{*5}x^*) + 103096y^{*3}x^{*3} - 22230(x^* + y^*) - 203920(y^{*3} + x^{*3}) + 90570(y^{*2} + x^{*2}) + 2025$$

The dual curve C^* of C is defined by the following symmetric polynomial and we can easily check that $\bar{\iota}(C) \neq C^*$ in $\mathcal{M}/G$.

$$h(x^*, y^*) := 3(x^{*4} + y^{*4}) + 14(x^{*3} + y^{*3}) + 3(x^{*2} + y^{*2}) + 4(y^*x^{*4} + x^*y^{*4}) + 36(y^*x^{*3} + x^*y^{*3}) + 6(y^*x^{*2} + x^*y^{*2}) - 2y^*x^* + 12(y^{*2}x^{*4} + x^{*2}y^{*4}) + 84(y^{*2}x^{*3} + x^{*2}y^{*3}) + 14y^{*2}x^{*2} + 88y^{*3}x^{*3}$$

Proof. Put $C' := \iota(C)$. We can see that C^* and C' are not in the same orbit of $\mathrm{PSL}(3;\mathbf{C})$. In fact, assume that there exists a $A \in \mathrm{PSL}(3;\mathbf{C})$ such that $(C^*)^A = C'$. Then A maps nodes to nodes. This implies that A permutes the three points $(0,0,1)$, $(0,1,0)$, and $(1,0,0)$. Thus A is a scalar multiplication of the coordinates $(X^*, Y^*, Z^*) \mapsto (\alpha X^*, \beta Y^*, \gamma Z^*)$, followed by a persutaion $\sigma \in \mathfrak{S}_3$. These actions does not change the number of monomials in x^* and y^*. Thus $h^A = g$ is impossible as g has 28 monomials while h has only 19 monomials. □

Remark. We know that $\mathscr{M}_{torus}/G$ is irreducible of dimension 4, in which one dimensional subvariety comes as the image of Gauss map (= dual curves) of 3 $(3,4)$-cuspidal sextics of torus type ([O4]). We do not know either the irreduciblity of $\mathscr{M}_{gen}/G$ or the dimension. Only thing we know is that it contains one dimensional subvariety coming from 3 $(3,4)$-cuspidal non-torus sextics as the image of Gauss map. However any such curve C is special in the sense C^* is not contained in $\mathscr{M}_{gen}/G$. We do not have any explicit example of a generic element $C \in \mathscr{M}_{gen}/G$ which has three bitangent lines.

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