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KODAI MATH. J.
27 (2004), 214–226

ISOSPECTRAL HYPERSURFACES IN EUCLIDEAN SPHERES

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Abstract

The aim of this work is to present a classification of some compact hypersurfaces M^n of a unit sphere S^{n+1} provided the spectra of the Laplacian of p-forms of M^n, which we denote by $\mathrm{Spec}^p(M)$, is equal to the spectra $\mathrm{Spec}^p(M_0)$, of a given hypersurface M_0^n.

1 Introduction

Let M be a compact Riemannian manifold without boundary of dimension n. We will denote the spectrum of the Laplacian of p-forms in M by

$$\mathrm{Spec}^p(M) := \{0 \le \lambda_0^p \le \lambda_1^p \le \cdots \uparrow +\infty\}, \quad p = 0, 1, \ldots, n.$$

One hard problem in Riemannian Geometry is to decide whether two isospectral Riemannian manifolds are isometric. The existence of flat tori which are isospectral but are not isometric (see [3]) is a counterexample to the validity in general of a positive answer to this question. The principal ingredient used to deal with this problem is the asymptotic expansion formula of the heat kernel due to Minakshisundaram-Pleijel (see [3] or [8]) which asserts

$$\sum_{i=1}^{\infty} e^{-(\lambda_i^p)t} \sim (4\pi t)^{-n/2}(a_{0,n}^p + a_{1,n}^p t + a_{2,n}^p t^2 + \cdots), \quad t \to 0^+,$$

where $a_{i,n}^p$ are geometric constants depending on M.

However, if we consider an isometric immersion of M into the Euclidean sphere S^{n+1} with some geometric properties, this problem comes less dificult. For instance, Q. Ding [7] proved that if M is a closed, orientable minimal hypersurface of S^4 and $\mathrm{Spec}^p(M) = \mathrm{Spec}^p(M_0)$, for a given $p \in \{0, 1, 2, 3\}$, where M_0 is the totally geodesic sphere, or the Clifford torus $S^1(\sqrt{1/3}) \times S^2(\sqrt{2/3})$, or the Cartan minimal hypersurface, then M is isometric to M_0. On the other hand, J. Wang [10] had shown that if M is a closed, orientable hypersurface in S^4 with constant mean curvature H, M_H is an isoparametric hypersurface in S^4 with

Mathematics Subject Classifications (1991): 53 A 10, 53 C 42.
Received August 8, 2003.

the same mean curvature H and $\mathrm{Spec}^p(M) = \mathrm{Spec}^p(M_H)$, $\forall p \in \{0,1\}$, then M is isometric to M_H.

We will denote by k_i, $i = 1, \dots, n$, the principal curvatures of an immersed hypersurface $M \hookrightarrow S^{n+1}$. In that way, the symmetric functions of k_i are defined by

$$\sigma_m = \sum_{\substack{i_1,\dots,i_m=1 \\ i_1<\cdots<i_m}}^{n} k_{i_1} \cdots k_{i_m},$$

with $m = 1, \dots, n$. The square of the length of the second fundamental form is given by

$$S = \sum_{i=1}^{n} k_i^2.$$

Finally, $\mathrm{d}v$ stands for the element of volume of M.

Now, we are able to state the main theorem of this work:

Theorem 1. *Let $M, M_0 \hookrightarrow S^{n+1}$, $n \geq 3$, be closed hypersurfaces of S^{n+1} with mean curvatures H and H_0, and scalar curvatures ρ and ρ_0, respectively. We require that one of the curvatures H and H_0 is nonnull and ρ_0 is constant. Suppose in addition that*

(i) $\mathrm{Spec}^p(M) = \mathrm{Spec}^p(M_0)$, $\forall p \in \{0,1\}$, *if* $n = 3$;

(ii) $\mathrm{Spec}^p(M) = \mathrm{Spec}^p(M_0)$, $\forall p \in \{0,1,2\}$, *if* $n \geq 4$.

Then $\rho = \rho_0$, i.e., M has also the same constant scalar curvature as M_0. Moreover the following integral equalities hold:

$$\int_M H\sigma_3 \,\mathrm{d}v = \int_{M_0} H_0 \sigma_3^0 \,\mathrm{d}v_0, \quad \textit{if } n \geq 3,$$

$$\int_M \sigma_4 \,\mathrm{d}v = \int_{M_0} \sigma_4^0 \,\mathrm{d}v_0, \quad \textit{if } n \geq 4, \tag{1}$$

where σ_m^0 and $\mathrm{d}v_0$ denote the values of σ_m and $\mathrm{d}v$ correspondent to M_0, respectively. In particular, we have

$$n^2 H^2 - S = n^2 H_0^2 - S_0, \tag{2}$$

where S_0 is the square of the length of the second fundamental form of M_0.

A consequence of our calculations is the next result about the case $H = H_0 = 0$, whose proof follows closely techniques presented before by Q. Ding in his paper [7].

Theorem 2. *Let $M, M_0 \hookrightarrow S^{n+1}$, $n \geq 3$, be closed minimal hypersurfaces of S^{n+1} whose scalar curvatures are ρ and ρ_0, respectively, with ρ_0 constant. Suppose that*

(i) $\mathrm{Spec}^p(M) = \mathrm{Spec}^p(M_0)$, *for some* $p \in \{0,1,2,3\}$, *if* $n = 3$;
(ii) $\mathrm{Spec}^p(M) = \mathrm{Spec}^p(M_0)$, $\forall p \in \{0,1\}$, *if* $n \geq 4$.

Then $\rho = \rho_0$. *Moreover, for* $n \geq 4$, *we have*

$$\int_M \sigma_4 \, dv = \int_{M_0} \sigma_4^0 \, dv_0.$$

Given $r \in (0,1)$ and $m \in \{1,\dots,n-1\}$ we will denote by $M^r_{n-m,m}(H)$, the hypersurface of S^{n+1} with constant mean curvature H, obtained by considering the standard immersions $S^{n-m}(r) \subset \boldsymbol{R}^{n-m+1}$, $S^m(\sqrt{1-r^2}) \subset \boldsymbol{R}^{m+1}$ of spheres with radius r and $\sqrt{1-r^2}$ and dimensions $n-m$ and m, respectively, and taking the product immersion

$$S^{n-m}(r) \times S^m(\sqrt{1-r^2}) \hookrightarrow \boldsymbol{R}^{n-m+1} \times \boldsymbol{R}^{m+1}.$$

Thus we have that $M^r_{n-m,m}(H)$ is contained in S^{n+1} and has principal curvatures k_i, $i = 1,\dots,n$, and mean curvature, respectively, given by

$$k_1 = \cdots = k_{n-m} = \frac{\sqrt{1-r^2}}{r}, \quad k_{n-m+1} = \cdots = k_n = -\frac{r}{\sqrt{1-r^2}},$$

and

$$H = \frac{n-m-nr^2}{nr\sqrt{1-r^2}},$$

or the negative of these values when we choose the opposite orientation. The hypersurface $M^r_{n-m,m}(H)$ is usually known as $H(r)$-torus or generalized Clifford Totus.

Let $\mathscr{F}_H$ be the set consisting of isoparametric hypersurfaces in S^4 with constant mean curvature H. E. Cartan proved in [5] that if $M \in \mathscr{F}_H$ then M is totally umbilical, or a $H(r)$-torus $M^r_{3-k,k}(H)$, or a Cartan hypersurface (that is, the isoparametric hypersurface obtained from the Cartan minimal hypersurface). Using Theorem 1 we will show that the assumption $H = H_0$ is not necessary in the theorem proved by J. Wang, above mentioned. More precisely, we will prove the following result:

THEOREM 3. *Let* $M \hookrightarrow S^4$ *be a closed and orientable hypersurface with constant mean curvature in* S^4 *and* $M_0 \in \mathscr{F}_{H_0}$. *If* $\mathrm{Spec}^p(M) = \mathrm{Spec}^p(M_0)$, *for* $p \in \{0,1\}$, *then* $H = H_0$ *and* M *is isometric to* M_0.

For dimension $n \geq 4$, we will derive also from Theorem 1 the following result:

THEOREM 4. *Let* $M \hookrightarrow S^{n+1}$, $n \geq 4$, *be a closed and orientable hypersurface in* S^{n+1} *with the same constant mean curvature* H_0 *of an isoparametric hypersurface* M_0 *in* S^{n+1}. *If* $\mathrm{Spec}^p(M) = \mathrm{Spec}^p(M_0)$, $\forall p \in \{0,1,2\}$, *then* M *is also isoparametric. Moreover,*

(i) *if* M_0 *is either totally umbilical or the* $H_0(r)$*-torus* $M^r_{n-1,1}(H_0)$, *with* $r^2 \le (n-1)/n$, *then* $M = M_0$.

(ii) *When* $n = 4$ *the principal curvatures of* M *and* M_0 *coincide.*

Finally, we will prove the following theorem:

Theorem 5. *Let* $M \hookrightarrow S^{n+1}$ *a closed hypersurface of* S^{n+1} *with nonnegative sectional curvature and* $M_0 \hookrightarrow S^{n+1}$ *a totally umbilical hypersurface or a* $H_0(r_0)$*-torus* $M^{r_0}_{n-1,1}(H_0)$, *with* $r_0 \le (n-2)/n$. *Suppose that*

(i) $\mathrm{Spec}^p(M) = \mathrm{Spec}^p(M_0)$, $\forall p \in \{0,1\}$, *if* $n = 3$;

(ii) $\mathrm{Spec}^p(M) = \mathrm{Spec}^p(M_0)$, $\forall p \in \{0,1,2\}$, *if* $n \ge 4$.

Then M *is isometric to* M_0.

2 Preliminaries

Let $M \subset S^{n+1}$ be a closed hypersurface with mean curvature H. Choose a local orthonormal frame field $\{e_1, \dots, e_n\}$ and let $\{\omega_1, \dots, \omega_n\}$ be the corresponding dual frame. We consider the second fundamental form

$$h = \sum_{i,j=1}^{n} h_{ij}\omega_i\omega_j.$$

Let R and $\tilde{R}$ be respectively the curvature and Ricci curvature tensors of M and denote by R_{ijkl} and $\tilde{R}_{ij}$, $i,j,k,l = 1,\dots,n$, their respective components with respect to the above frame. If we choose $\{e_1,\dots,e_n\}$ such that $h_{ij} = k_i\delta_{ij}$, then

$$H = \frac{1}{n}\sum_{i=1}^{n} k_i,$$

$$R_{ijkl} = (1 + k_ik_j)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{3}$$

$$\tilde{R}_{ij} = [(n-1) + nHk_i - k_ik_j]\delta_{ij}. \tag{4}$$

Let ρ and S be, respectively, the scalar curvature of M and the square of the length of the second fundamental form h. The Gauss formula yields

$$\rho = n(n-1) + n^2H^2 - S \tag{5}$$

and taking into account (3) and (4) we have

$$|R|^2 = 2S^2 - 2f_4 + 4n^2H^2 - 4S + 2n(n-1), \tag{6}$$

$$|\tilde{R}|^2 = n^2H^2S + f_4 + n(n-1)^2 - 2nHf_3 + 2n^2(n-1)H^2 - 2(n-1)S, \tag{7}$$

where

$$f_m = \sum_{i=1}^{n} k_i^m.$$

The expressions of f_m can be calculated using the formulas (see e.g. [9], p. 101)

$$f_m - f_{m-1}\sigma_1 + f_{m-2}\sigma_2 - \cdots + (-1)^{m-1} f_1\sigma_{m-1} + (-1)^m m\sigma_m = 0, \quad \text{for } m \le n,$$

$$f_m - f_{m-1}\sigma_1 + \cdots + (-1)^n f_{m-n}\sigma_n = 0, \quad \text{for } m > n.$$

When $n = 3$ we get

$$f_3 = \frac{9}{2}HS - \frac{27}{2}H^3 + 3\sigma_3,$$

$$f_4 = \frac{1}{2}S^2 + 9H^2S - \frac{81}{2}H^4 + 12H\sigma_3, \tag{8}$$

and for $n \ge 4$,

$$f_3 = \frac{3n}{2}HS - \frac{n^3}{2}H^3 + 3\sigma_3,$$

$$f_4 = \frac{1}{2}S^2 + n^2H^2S - \frac{n^4}{2}H^4 + 4nH\sigma_3 - 4\sigma_4. \tag{9}$$

If H is constant, the Simons formula for M is given by

$$\frac{1}{2}\Delta S = |\nabla h|^2 + S(n - S) - n^2H^2 + nHf_3. \tag{10}$$

Since M is compact, using Minakshisundaram-Pleijel's asymptotic expansion formula of the heat kernel stated in the introduction we can write

$$\sum_{i=1}^{\infty} e^{-(\lambda_i^p)t} \sim (4\pi t)^{-n/2}(a_{0,n}^p + a_{1,n}^p t + a_{2,n}^p t^2 + \cdots), \quad t \to 0^+, \tag{11}$$

where

$$a_{0,n}^p = \binom{n}{p}\operatorname{vol}(M), \quad a_{1,n}^p = \left[\frac{1}{6}\binom{n}{p} - \binom{n-2}{p-1}\right]\int_M \rho \, \mathrm{d}v,$$

$$a_{2,n}^p = \int_M (E_n^p\rho^2 + F_n^p|\tilde{R}|^2 + G_n^p|R|^2)\, \mathrm{d}v,$$

and

$$E_n^p = \frac{1}{72}\binom{n}{p} - \frac{1}{6}\binom{n-2}{p-1} + \frac{1}{2}\binom{n-4}{p-2}$$

$$F_n^p = -\frac{1}{180}\binom{n}{p} + \frac{1}{2}\binom{n-2}{p-1} - 2\binom{n-4}{p-2}$$

$$G_n^p = \frac{1}{180}\binom{n}{p} - \frac{1}{12}\binom{n-2}{p-1} + \frac{1}{2}\binom{n-4}{p-2},$$

where dv and $\mathrm{vol}(M)$ represent respectively the volume form and volume of M, with respect to the induced Riemannian metric of S^{n+1}. We point out that these coefficients were calculated in [8]. Moreover we will decree here that $\binom{l}{q} = 0$ if $l < 0$ or $q < 0$ or $l < q$.

3 Proof of Theorems

We use the same notation for the geometric data of M as in the previous section. We indicate with a subscript "0" the corresponding data for M_0.

Proof of Theorem 1: By hypothesis, the asymptotic expansion formula of M and M_0 coincide. Thus

$$\mathrm{vol}(M) = \mathrm{vol}(M_0), \tag{12}$$

$$\int_M \rho \, dv = \int_{M_0} \rho_0 \, dv_0 \tag{13}$$

and

$$\int_M (E_n^p \rho^2 + F_n^p |\tilde{R}|^2 + G_n^p |R|^2) \, dv = \int_{M_0} (E_n^p \rho_0^2 + F_n^p |\tilde{R}_0|^2 + G_n^p |R_0|^2) \, dv_0. \tag{14}$$

Therefore taking in account (5) and (13) we obtain

$$\int_M [n(n-1) + n^2 H^2 - S] \, dv = \int_{M_0} [n(n-1) + n^2 H_0^2 - S_0] \, dv_0$$

Since

$$\int_M dv = \mathrm{vol}(M) = \mathrm{vol}(M_0) = \int_{M_0} dv_0$$

we conclude that

$$\int_M (n^2 H^2 - S) \, dv = \int_{M_0} (n^2 H_0^2 - S_0) \, dv_0. \tag{15}$$

We first consider the case $n \geq 4$. Replacing the expressions of f_3 and f_4 in (6) and (7) we obtain

$$|R|^2 = (n^2 H^2 - S)^2 + 4(n^2 H^2 - S) - 8nH\sigma_3 + 2n(n-1) + 8\sigma_4, \tag{16}$$

$$|\tilde{R}|^2 = \frac{1}{2}(n^2 H^2 - S)^2 + 2(n-1)(n^2 H^2 - S) - 2nH\sigma_3 + n(n-1)^2 - 4\sigma_4. \tag{17}$$

Therefore, for $p \in \{0, 1, 2\}$, we can write

$$\int_M (E_n^p \rho^2 + F_n^p |\tilde{R}|^2 + G_n^p |R|^2)\,\mathrm{d}v$$
$$= E_n^p \int_M [(n^2H^2 - S)^2 + 2n(n-1)(n^2H^2 - S) + n^2(n-1)^2]\,\mathrm{d}v$$
$$+ F_n^p \int_M \left[\frac{1}{2}(n^2H^2 - S)^2 + 2(n-1)(n^2H^2 - S) - 2nH\sigma_3 + n(n-1)^2 - 4\sigma_4\right]\mathrm{d}v$$
$$+ G_n^p \int_M [(n^2H^2 - S)^2 + 4(n^2H^2 - S) - 8nH\sigma_3 + 2n(n-1) + 8\sigma_4]\,\mathrm{d}v. \quad (18)$$

Analogously, we have a similar identity for M_0. Therefore considering this equations in equality (14) and using (15) we derive the system of equations

$$\alpha_n^p \mathbf{X} + \beta_n^p \mathbf{Y} + \gamma_n^p \mathbf{Z} = 0, \quad p = 0, 1, 2,$$

where

$$\alpha_n^p = \left(E_n^p + \frac{1}{2}F_n^p + G_n^p\right), \quad \beta_n^p = -2n(F_n^p + 4G_n^p), \quad \gamma_n^p = -4(F_n^p - 2G_n^p)$$

and

$$\mathbf{X} := \int_M (n^2H^2 - S)^2\,\mathrm{d}v - \int_{M_0} (n^2H_0^2 - S_0)^2\,\mathrm{d}v_0,$$
$$\mathbf{Y} := \int_M H\sigma_3\,\mathrm{d}v - \int_{M_0} H_0\sigma_3^0\,\mathrm{d}v_0,$$
$$\mathbf{Z} := \int_M \sigma_4\,\mathrm{d}v - \int_{M_0} \sigma_4^0\,\mathrm{d}v_0.$$

Now a straightforward calculation, using the expressions for E_n^p, F_n^p and G_n^p, yields

$$\det\begin{pmatrix} \alpha_n^0 & \beta_n^0 & \gamma_n^0 \\ \alpha_n^1 & \beta_n^1 & \gamma_n^1 \\ \alpha_n^2 & \beta_n^2 & \gamma_n^2 \end{pmatrix} \neq 0.$$

We conclude that $\mathbf{X} = \mathbf{Y} = \mathbf{Z} = 0$. Therefore,

$$\int_M (n^2H^2 - S)^2\,\mathrm{d}v = \int_{M_0} (n^2H_0^2 - S_0)^2\,\mathrm{d}v_0, \quad (19)$$

$$\int_M H\sigma_3\,\mathrm{d}v = \int_{M_0} H_0\sigma_3^0\,\mathrm{d}v_0 \quad \text{and} \quad \int_M \sigma_4\,\mathrm{d}v = \int_{M_0} \sigma_4^0\,\mathrm{d}v_0.$$

Since ρ_0 is constant, we have $n^2H_0^2 - S_0$ constant. Then combining (15), (19) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|n^2H_0^2 - S_0| \operatorname{vol}(M_0) &= \left|\int_M (n^2H^2 - S)\,\mathrm{d}v\right| \\ &\le \left[\int_M (n^2H^2 - S)^2\,\mathrm{d}v\right]^{1/2} \left[\int_M \mathrm{d}v\right]^{1/2} \\ &= |n^2H_0^2 - S_0| \operatorname{vol}(M_0).\end{aligned}$$

Thus, $n^2H^2 - S = n^2H_0^2 - S_0$, which is equivalent to $\rho = \rho_0$. This concludes the proof of the theorem for $n \ge 4$.

In the case $n = 3$, the calculations are entirely analogous. The similar formula to (18) is given by

$$\begin{aligned}&\int_M (E_3^p\rho^2 + F_3^p|\tilde{R}|^2 + G_3^p|R|^2)\,\mathrm{d}v \\ &\quad = E_3^p \int_M [(9H^2 - S)^2 - 12(9H^2 - S) + 36]\,\mathrm{d}v \\ &\qquad + F_3^p \int_M \left[\frac{1}{2}(9H^2 - S)^2 + 4(9H^2 - S) - 6H\sigma_3 + 12\right]\mathrm{d}v \\ &\qquad + G_3^p \int_M [(9H^2 - S)^2 + 4(9H^2 - S) - 24H\sigma_3 + 12]\,\mathrm{d}v, \qquad (20)\end{aligned}$$

whereas the corresponding system of equations is

$$\alpha_3^p\tilde{\mathbf{X}} + \beta_3^p\tilde{\mathbf{Y}} = 0, \quad p = 0, 1,$$

where

$$\begin{aligned}\tilde{\mathbf{X}} &:= \int_M (9H^2 - S)^2\,\mathrm{d}v - \int_{M_0} (9H_0^2 - S_0)^2\,\mathrm{d}v_0, \\ \tilde{\mathbf{Y}} &:= \int_M H\sigma_3\,\mathrm{d}v - \int_{M_0} H_0\sigma_3^0\,\mathrm{d}v_0.\end{aligned}$$

It is easily checked that

$$\det\begin{pmatrix} \alpha_3^0 & \beta_3^0 \\ \alpha_3^1 & \beta_3^0 \end{pmatrix} \neq 0.$$

So, proceeding as in the case $n \ge 4$, we complete the proof. □

Proof of Theorem 3: It follows from Theorem 1 that $9H^2 - S = 9H_0^2 - S_0$. Since H is constant, S is also constant and we can make use the theorems of S. Almeida, F. Brito [2] and S. Chang [5] to conclude that M is isoparametric and belongs to $\mathscr{F}_H$. In particular σ_3 is also constant. Considering (1) and (12) we conclude

$$H\sigma_3 = H_0\sigma_3^0. \tag{21}$$

We now analyze separatedly three cases.

Case 1) M_0 is a Cartan hypersurface.

It is known that $S_0 = 6 + 9H_0^2$ and $\sigma_3^0 = -3H_0$ (see e.g. [4]). Thus, using (2) we have $S = 6 + 9H^2$ and, by E. Cartan [4], M is a Cartan hypersurface. By using (21) we obtain $-3H^2 = H\sigma_3 = H_0\sigma_3^0 = -3H_0^2$, that is, $H = \pm H_0$. We now use the same theorem of Cartan [4] to conclude that $M = M_0$.

Case 2) M_0 is totally umbilical.

Since $S_0 = 3H_0^2$ and $\sigma_3^0 = H_0^3$, the expressions (2) and (21) yield

$$S = 9H^2 - 6H_0^2, \tag{22}$$

$$H\sigma_3 = H_0^4. \tag{23}$$

From case 1, it follows that M can not be a Cartan hypersurface, otherwise M_0 is also a Cartan hypersurface. Since $M \in \mathscr{F}_H$, M is either a $H(r)$-torus $M_{2,1}^r(H)$ or totally umbilical. Suppose $M = M_{2,1}^r(H)$, for some r. Then the principal curvatures of M are

$$k_1 = k_2 = \frac{\sqrt{1-r^2}}{r}, \quad k_3 = -\frac{r}{\sqrt{1-r^2}},$$

or the symmetric of these values for the opposite orientation. We can see now that, independently of the orientation, H and S satisfy

$$H^2 = \frac{9r^4 - 12r^2 + 4}{9r^2(1-r^2)}, \quad S = \frac{3r^4 - 4r^2 + 2}{r^2(1-r^2)}, \tag{24}$$

$$H\sigma_3 = \frac{3r^2 - 2}{3r^2}. \tag{25}$$

By using (22) and (24) we conclude that $r^2 = 1/3(H_0^2 + 1) < 2/3$. Hence (25) guarantees $H\sigma_3 < 0$. So, we have a contradiction with (23). Thus, M is totally umbilical and $S = 3H^2 = 9H^2 - 6H_0^2$. Therefore $H = \pm H_0$ and similarly $M = M_0$.

Case 3) M_0 is a $H_0(r_0)$-torus.

Let us suppose $M_0 = M_{2,1}^{r_0}(H_0)$. From cases (1) and (2) M is neither totally umbilical nor Cartan hypersurface. Thus M is an $H(r)$-torus $M^r_{2,1}(H)$. It follows from (25) that

$$H\sigma_3 = \frac{3r^2 - 2}{3r^2} \quad \text{and} \quad H_0\sigma_3^0 = \frac{3r_0^2 - 2}{3r_0^2}.$$

Since H is constant, (1) and (12) yield $H\sigma_3 = H_0\sigma_3^0$, from where we conclude that $r = r_0$. This finishes the proof of the theorem. □

Proof of Theorem 4: First we will consider $H_0 \neq 0$. Using Theorem 1 for $n \geq 4$ and $H = H_0$ we obtain that $S = S_0$,

$$\int_M \sigma_3 \,\mathrm{d}v = \int_{M_0} \sigma_3^0 \,\mathrm{d}v_0, \tag{26}$$

$$\int_M \sigma_4 \,\mathrm{d}v = \int_{M_0} \sigma_4^0 \,\mathrm{d}v_0. \tag{27}$$

We use now formula (9) to obtain

$$f_3 = \frac{3n}{2} H_0 S_0 - \frac{n^3}{2} H_0^3 + 3\sigma_3,$$

$$f_3^0 = \frac{3n}{2} H_0 S_0 - \frac{n^3}{2} H_0^3 + 3\sigma_3^0.$$

From (26) and the fact that $\mathrm{vol}(M) = \mathrm{vol}(M_0)$ we conclude

$$\int_M f_3 \,\mathrm{d}v = \int_{M_0} f_3^0 \,\mathrm{d}v_0. \tag{28}$$

Since $H = H_0$, $S = S_0$ and $\nabla h_0 = 0$ (h_{ij}^0 are constants, $i, j = 1, \ldots, n$) the respective Simons formulae (10) for M and M_0 read as follows

$$0 = \frac{1}{2}\Delta S_0 = |\nabla h|^2 + S_0(n - S_0) - n^2 H_0^2 + nH_0 f_3,$$

$$0 = \frac{1}{2}\Delta S_0 = S_0(n - S_0) - n^2 H_0^2 + nH_0 f_3^0,$$

from where we conclude that

$$\int_M |\nabla h|^2 = nH_0 \left(\int_M f_3 \,\mathrm{d}v - \int_{M_0} f_3^0 \,\mathrm{d}v_0 \right) = 0.$$

When $H_0 = 0$ the Theorem 2 carries that $S = S_0$ and the Simons formulae for M and M_0 still imply that $\int_M |\nabla h|^2 = 0$. Hence, whatever it is the value of H_0, we have $\nabla h = 0$, that is, $h_{ijk} = 0$, for $i, j, k = 1, \ldots, n$. Since M is a hypersurface, it follows from formula (2.10) of [6] that

$$\sum_{k=1}^{n} h_{ijk}\omega_k = \mathrm{d}h_{ij} - \sum_{l=1}^{n} h_{il}\omega_{jl} - \sum_{l=1}^{n} h_{lj}\omega_{il}.$$

Since $h_{ij} = k_i\delta_{ij}$ and $h_{ijk} = 0$, $i, j, k = 1, \ldots, n$, we have

$$0 = \mathrm{d}h_{ij} + (k_i - k_j)\omega_{ij}$$

and setting $i = j$, we conclude $\mathrm{d}k_i = \mathrm{d}h_{ii} = 0$. Thus, k_i is constant, $i = 1, \ldots, n$, and M is isoparametric.

On the other hand, the Theorem 1.5 of [1] due to H. Alencar and M. do Carmo gives us that the totally umbilical hypersurfaces of S^{n+1} as well as the $H(r)$-torus $M^r_{n-1,1}(H)$ with $r^2 \leq (n-1)/n$, are characterized by the constant mean curvature and the square of the length of the second fundamental form. Thus, since $H = H_0$ and $S = S_0$, we can apply the Alencar-do Carmo Theorem to conclude (i).

Let us suppose now that $n = 4$ to prove (ii). Since M is isoparametric, σ_3 and σ_4 are both constants. Joining the expressions (26), (27) and the fact that $\mathrm{vol}(M) = \mathrm{vol}(M_0)$ we have $\sigma_3 = \sigma_3^0$ and $\sigma_4 = \sigma_4^0$. On the other hand,

$$\sigma_1 = 4H = 4H_0 = \sigma_1^0 \quad \text{and} \quad \sigma_2 = \frac{16H^2 - S}{2} = \frac{16H_0^2 - S_0}{2} = \sigma_2^0.$$

Therefore the four symmetric functions for M and M_0 agree and we conclude that $k_i = k_i^0$, for $i = 1, \ldots, 4$, which conclude the proof of (ii) of Theorem 4. □

Proof of Theorem 5: It follows from Theorem 1 that $\rho = \rho_0$. If M_0 is totally umbilical, then $\rho_0 = n(n-1)(H_0^2 + 1)$, whereas for $M_0 = M^{r_0}_{n-1,1}(H_0)$ we have that

$$\rho_0 = \frac{(n-1)(n-2)}{r_0^2}.$$

It follows in both cases that $\rho_0 \geq n(n-1)$, i.e., the normalized scalar curvature of M_0, and hence of M, is constant and greater than or equal to 1. This fact and the assumption that M has nonnegative sectional curvature imply, from Theorem 2 of [11], that M is either totally umbilical or a product of two totally umbilical constantly curved submanifolds. In the last case, M is a $H(r)$-torus. Hence, H, S and σ_3 are constant, as well as $\nabla h = 0$. Therefore, Simons formula (10) for M yields

$$0 = S(n - S) - n^2H^2 + nHf_3.$$

The relations (8) and (9) for M, allow us to rewrite this formula as

$$0 = S(n - S) - n^2H^2 + \frac{3}{2}n^2H^2S - \frac{1}{2}n^4H^4 + 3nH\sigma_3. \tag{29}$$

Since $\rho = \rho_0$, the Gauss formula implies $S - n^2H^2 = S_0 - n^2H_0^2 = c_0$. Then, we have $S = c_0 + n^2H^2$ and the equality (29) becomes

$$0 = (n - c_0)c_0 + n^2\left(n - 1 - \frac{1}{2}c_0\right)H^2 + 3nH\sigma_3. \tag{30}$$

Analogously, the Simons formula for M_0 give us

$$0 = (n - c_0)c_0 + n^2\left(n - 1 - \frac{1}{2}c_0\right)H_0^2 + 3nH_0\sigma_3^0. \tag{31}$$

On the other hand, it follows from Theorem 1 that $\int_M H\sigma_3\, dv = \int_{M_0} H_0\sigma_3^0\, dv_0$ and with the same argument contained in its proof we conclude $\mathrm{vol}(M) = \mathrm{vol}(M_0)$. Since H and σ_3 are constant, we have that $H\sigma_3 = H_0\sigma_3^0$. Therefore, putting together the equalities (30) and (31) we obtain

$$\left(n - 1 - \frac{1}{2}c_0\right)(H^2 - H_0^2) = 0.$$

We will show that $n - 1 - (1/2)c_0 \neq 0$. Indeed, otherwise $\rho_0 = (n-1)(n-2)$, since

$$\rho_0 = n(n-1) + n^2H^2 - S = n(n-1) - c_0.$$

But if M_0 is totally umbilical, then $\rho_0 = n(n-1)(H_0^2 + 1) \neq (n-1)(n-2)$ while for $M_0 = M_{n-1,1}^{r_0}(H_0)$, we have $\rho_0 = (n-1)(n-2)/r_0^2 \neq (n-1)(n-2)$ for $0 < r_0 < 1$. Hence, $n - 1 - (1/2)c_0 \neq 0$ and we can conclude that $H = \pm H_0$. Therefore, $S = S_0$. Now, we can make use of Alencar-do Carmo's Theorem mentioned above to finish the proof of theorem. □

Acknowledgments. The author would like to thank Abdênago Barros, Aldir Brasil and Jorge Lira for careful readings and helpful sugestions during the preparation of this work.

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