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SUBMANIFOLDS WITH CONSTANT SCALAR CURVATURE

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Abstract

Let M^n be a compact submanifold of $S^{n+p}(c)$ with constant scalar curvature. In this paper, we prove that if the squared norm S of the second fundamental form satisfies a certain inequality, then M^n is a totally umbilic or eqality holds and we described all M^n that satisfy this equality.

1. Introduction

Let $S^{n+p}(c)$ be a sphere of constant sectional curvature c and let M^n be a compact submanifold of $S^{n+p}(c)$ with constant scalar curvature. Let $S = \sum_{i,j,\alpha}(h_{ij}^{\alpha})^2$ and ξ be the square norm of the second fundamental form and the mean curvature vector of M respectively. We set $H = \|\xi\|$, $S_\alpha = \sum_{i,j}(h_{ij}^{\alpha})^2$ and $H^\alpha = (h_{ij}^{\alpha})_{n\times n}$. In [1], Li Haizhong studied some properties for $p = 1$. The purpose of this paper is to extend the result of [1] to higher codimensions. In other words, we shall prove the following:

Theorem 1. *Let M^n $(2 \le n \le 4,$ when $n = 2,$ $p \ne 1, 2)$ be a compact submanifold of $S^{n+p}(c)$ with constant scalar curvature ρ. Assume the normalized mean curvature vector is parallel with respect to the normal connection. If*

$$S - n\bar{R} \le \frac{B_p}{n-1-B_p}\left\{n^2c + n^2\bar{R} - \frac{n-2}{\sqrt{n-1}}\sqrt{[S+n(n-1)\bar{R}][nS_{n-1} - S - n(n-1)\bar{R}]}\right\}, \tag{1.1}$$

where

$$B_p = \begin{cases} 1, & p = 1, 2 \\ \dfrac{2}{3}, & p \ge 3 \end{cases}$$

and $\bar{R} = \rho/n(n-1) - c$, then either

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(1) $S = nH^2$ *and* M *is totally umbilic.*

or

(2) *The equality holds in* (1.1), H *is a constant and one of the following cases occurs*:

(a) $H = 0$, $p = 1$ *and* M *is a minimal Clifford hypersurface,*

$$M^n = S^m\left(\left(\frac{m}{nc}\right)^{1/2}\right) \times S^{n-m}\left(\left(\frac{n-m}{nc}\right)^{1/2}\right) \hookrightarrow S^{n+p}(c).$$

(b) $H \neq 0$, $p = 1$ *and* $M^n = S^{n-1}(r_1) \times S^1(r_2)$, *where* $r_1^2 + r_2^2 = c^{-1}$, $r_1^2 < (n-1)/nc$.

(c) $H \neq 0$, $p = 2$ *and* M *is a minimal Clifford hypersurface in a hypersphere,*

$$M^n = S^m\left(\left(\frac{m}{n(c+H^2)}\right)^{1/2}\right) \times S^{n-m}\left(\left(\frac{n-m}{n(c+H^2)}\right)^{1/2}\right) \subset S^{n+1}(c+H^2) \hookrightarrow S^{n+2}(c).$$

(d) $H \neq 0$, $p = 2$ *and for all* $0 < H_2 \leq H$, M *is an* H_1*-torus,*

$$M^n = S^{n-1}(r_1) \times S^1(r_2) \subset S^{n+1}(c + H_2^2) \hookrightarrow S^{n+2}(c),$$

where $H_1^2 + H_2^2 = H^2$, $r-1^2 + r_2^2 = (c+H_2^2)^{-1}$, $r_1^2 < (n-1)/n(c+H_1^2)$, *and an* H_1*-torus is defined by Walcy Santos* (*see* [3]).

(e) $H \neq 0$, $n = 2$, $p = 3$ *and* M^2 *is a Veronese surface in a hypersphere,*

$$M^2 \subset S^4(c+H^2) \hookrightarrow S^5(c).$$

for $n \geq 5$, *we have*

THEOREM 2. *Let* M^n $(n \geq 5)$ *be a compact submanifold of* $S^{n+p}(c)$ *with constant scalar curvature* ρ. *Assume the normalized mean curvature vector is parallel with respect to the normal connection and* $\rho \geq n(n-1)c$. *If* S *and* H *satisfied* (1.1), *then either*

(1) $S = nH^2$ *and* M *is totally umbilic.*

or

(2) *The equality holds in* (1.1), H *is a constant and one of the following cases occurs*:

(a) $H \neq 0$, $p = 1$ *and* $M^n = S^{n-1}(r_1) \times S^1(r_2)$, *where* $r_1^2 + r_2^2 = c^{-1}$, $r_1^2 < (n-1)/nc$.

(b) $H \neq 0$, $p = 2$ *and* M *is a minimal Clifford hypersurface in a hypersphere,*

$$M^n = S^m\left(\left(\frac{m}{n(c+H^2)}\right)^{1/2}\right) \times S^{n-m}\left(\left(\frac{n-m}{n(c+H^2)}\right)^{1/2}\right) \subset S^{n+1}(c+H^2) \hookrightarrow S^{n+2}(c).$$

(c) $H \neq 0$, $p = 2$ *and for all* $0 < H_2 \leq H$, M *is an* H_1*-torus,*

$$M^n = S^{n-1}(r_1) \times S^1(r_2) \subset S^{n+1}(c + H_2^2) \hookrightarrow S^{n+2}(c),$$

where $H_1^2 + H_2^2 = H^2$, $r-1^2 + r_2^2 = (c+H_2^2)^{-1}$, $r_1^2 < (n-1)/n(c+H_1^2)$.

Remark 1. When $p = 1$, $n = 3, 4$, our assumption condition (1.1), i.e.

$$S - n\bar{R} \leq \frac{1}{n-2}\{n^2 c + n^2 \bar{R} - (n-2)\sqrt{[S + n(n-1)\bar{R}][S - n\bar{R}]}\}$$

is equivalent to

$$(n-2)^2[S + n(n-1)\bar{R}][S - n\bar{R}] \leq \{n^2 c + 2n(n-1)\bar{R} - (n-2)S\}^2. \quad (1.1')$$

On the other hand, (1.1′) is equivalent to

$$S \leq \frac{n}{(n-2)(n\bar{R}+2)}[n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n],$$

so Theorem 1 improve Theorem 2 of [1].

Remark 2. In Theorem 2, when the equality holds in (1.1), we have that $H \neq 0$ by $\rho \geq n(n-1)c$ and the *Guass* equation.

2. Priliminaries

We choose local field of orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ in such way that e_{n+1} is the normalized mean curvature vector and $h_{ij}^{n+1} = \lambda_i \delta_{ij}$. We need the following Lemma:

Lemma 1. *If* $2 \leq n \leq 4$, *then*

$$\int_M \sum_{i,j,k,\alpha} \{(h_{ijk}^\alpha)^2 + h_{ij}^\alpha h_{kkij}^\alpha\} \geq 0.$$

Proof.

$$\begin{aligned}
\sum_{i,j,k,\alpha} h_{ij}^\alpha h_{kkij}^\alpha &= \sum_j \nabla_{e_j}\left(\sum_{i,k,\alpha} h_{ij}^\alpha h_{kki}^\alpha\right) - \sum_{i,j,k,\alpha} h_{ijj}^\alpha h_{kki}^\alpha \\
&= \sum_j \nabla_{e_j}\left(\sum_{i,k,\alpha} h_{ij}^\alpha h_{kki}^\alpha\right) - \sum_{i,\alpha}\left(\sum_j h_{jji}^\alpha\right)^2 \\
&= \sum_j \nabla_{e_j}\left(\sum_{i,k,\alpha} h_{ij}^\alpha h_{kki}^\alpha\right) - \sum_{i,\alpha}(h_{iii}^\alpha)^2 - \sum_{i,\alpha}\left(\sum_{j\neq i} h_{jji}^\alpha\right)^2, \qquad (2.1)
\end{aligned}$$

$$\sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2 = \sum_{i,\alpha}(h_{iii}^\alpha)^2 + 3\sum_{i\neq j,\alpha}(h_{jji}^\alpha)^2 + \sum_{i,j,k\neq\alpha}(h_{ijk}^\alpha)^2. \qquad (2.2)$$

For $n \leq 4$ and $\forall i$, we have

$$3\sum_{j\neq i}(h^{\alpha}_{jji})^2 \geq (n-1)\sum_{j\neq i}(h^{\alpha}_{jji})^2 \geq \left(\sum_{j\neq i}h^{\alpha}_{jji}\right)^2,$$

so we have

$$3\sum_{j\neq i,\alpha}(h^{\alpha}_{jji})^2 \geq \sum_{i,\alpha}\left(\sum_{j\neq i}h^{\alpha}_{jji}\right)^2. \tag{2.3}$$

By (2.1), (2.2), (2.3) and the compactness of M, the Lemma 1 is true.

Lemma 2 [1]. *Assume the scalar curvature $\rho = constant$ and $\rho \geq n(n-1)c$, then*

$$\sum_{i,j,k,\alpha}\left\{(h^{\alpha}_{ijk})^2 - \sum_i (nH_i)^2\right\} \geq 0.$$

Lemma 3 [2]. *M^n is a compact submanifold of $S^{n+p}(c)$ with parallel normalized mean curvature vector if and only if $H^{n+1}H^{\alpha} = H^{\alpha}H^{n+1}$.*

Lemma 4 [3]. *Let $A, B : R^n \to R^n$ be symmetric linear maps such that $[A, B] = 0$ and $\operatorname{tr} A = \operatorname{tr} B = 0$, then*

$$-\frac{n-2}{\sqrt{n(n-1)}}(\operatorname{tr} A^2)(\operatorname{tr} B^2)^{1/2} \leq \operatorname{tr} A^2 B \leq \frac{n-2}{\sqrt{n(n-1)}}(\operatorname{tr} A^2)(\operatorname{tr} B^2)^{1/2},$$

and the equality holds on the right hand (resp. the left hand) side if and only if $n-1$ of the eigenvalues x_i of A and the corresponding eigenvalues y_i of B satisfy

$$|x_i| = \frac{(\operatorname{tr} A^2)^{1/2}}{\sqrt{n(n-1)}}, \quad x_i x_j \geq o,$$

$$y_i = \frac{(\operatorname{tr} B^2)^{1/2}}{\sqrt{n(n-1)}}. \quad \left(resp.\ \ y_i = -\frac{(\operatorname{tr} B^2)^{1/2}}{\sqrt{n(n-1)}}\right)$$

Lemma 5 [4]. *Let $A_{n+1}, \dots, A_{n+p}$ be symmetric $(n \times n)$-matrices. Denote $S_{\alpha\beta} = \operatorname{tr}(A_\alpha A_\beta)$, $S_\alpha = S_{\alpha\alpha} = N(A_\alpha)$, $S = \sum_\alpha S_\alpha$, then*

$$\sum_{\alpha,\beta} N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha,\beta} S^2_{\alpha\beta} \leq \left(1 + \frac{1}{2}\operatorname{sgn}(p-1)\right)S^2.$$

3. The of Theorem

We define a self-adjoint operator $\square$ by [1]

$$\square f = \sum_{i,j}(nH\delta_{ij} - h^{n+1}_{ij})f_{ij}.$$

we have [2]

$$\frac{1}{2}\triangle S=\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2+\sum_{i,j,k,\alpha}h^{\alpha}_{ij}h^{\alpha}_{kkij}+2\sum_{\alpha,\beta}\mathrm{tr}(H^{\alpha}H^{\beta})^2-2\sum_{\alpha,\beta}\mathrm{tr}(H^{\alpha})^2(H^{\beta})^2$$
$$-\sum_{\alpha,\beta}[\mathrm{tr}(H^{\alpha}H^{\beta})]^2+\sum_{\alpha,\beta}\mathrm{tr}\ H^{\beta}[\mathrm{tr}(H^{\alpha})^2H^{\beta}]+cnS-cn^2H^2. \tag{3.1}$$

By the *Guass* equation and the scalar curvature $\rho=\text{constant}$, we get

$$\begin{aligned}\square(nH)&=\frac{1}{2}\triangle(nH)^2-\sum_i(nH_i)^2-\sum_i(nH)_{ii}\lambda_i\\&=\frac{1}{2}\triangle S-\sum_i(nH_i)^2-\sum_i(nH)_{ii}\lambda_i\\&=\sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2-\sum_i(nH_i)^2+2\sum_{\alpha,\beta}\mathrm{tr}(H^{\alpha}H^{\beta})^2-2\sum_{\alpha,\beta}\mathrm{tr}(H^{\alpha})^2(H^{\beta})^2\\&\quad-\sum_{\alpha,\beta}[\mathrm{tr}(H^{\alpha}H^{\beta})]^2+\sum_{\alpha,\beta}\mathrm{tr}\ H^{\beta}[\mathrm{tr}(H^{\alpha})^2H^{\beta}]+cnS-cn^2H^2.\end{aligned} \tag{3.2}$$

From Lemma 3 and Lemma 5, we have, $p\geq 2$,

$$\begin{aligned}&2\sum_{\alpha,\beta}\mathrm{tr}(H^{\alpha}H^{\beta})^2-2\sum_{\alpha,\beta}\mathrm{tr}(H^{\alpha})^2(H^{\beta})^2-\sum_{\alpha,\beta}[\mathrm{tr}(H^{\alpha}H^{\beta})]^2\\&\quad=2\sum_{\alpha\neq\beta\neq n+1}\mathrm{tr}(H^{\alpha}H^{\beta})^2-2\sum_{\alpha\neq\beta\neq n+1}\mathrm{tr}(H^{\alpha})^2(H^{\beta})^2\\&\qquad-\sum_{\alpha\neq n+1,\beta\neq n+1}[\mathrm{tr}(H^{\alpha}H^{\beta})]^2-S_{n+1}^2-2\sum_{\alpha\neq n+1}[\mathrm{tr}(H^{\alpha}H^{n+1})]^2\\&\quad=-N_{\alpha\neq\beta\neq n+1}(H^{\alpha}H^{\beta}-H^{\beta}H^{\alpha})-\sum_{\alpha\neq n+1,\beta\neq n+1}S_{\alpha\beta}^2\\&\qquad-S_{n+1}^2-2\sum_{\alpha\neq n+1}[\mathrm{tr}(H^{\alpha}H^{\beta})]^2\\&\quad\geq-\left(1+\frac{1}{2}\ \mathrm{sgn}(p-2)\right)\left(\sum_{\alpha\neq n+1}S_{\alpha}\right)^2-S_{n+1}^2-2\sum_{\alpha\neq n+1}\left[\sum_{i,j}(h^{n+1}_{ij}-H\delta_{ij})h^{\alpha}_{ij}\right]^2\\&\quad\geq-\frac{1}{B_p}(S-S_{n+1})^2-S_{n+1}^2-2\sum_{i,j}(h^{n+1}_{ij}-H\delta_{ij})^2\sum_{i,j,\alpha\neq n+1}(h^{\alpha}_{ij})^2\\&\quad\geq-\frac{1}{B_p}(S-S_{n+1})^2-S_{n+1}^2-2(S_{n+1}-nH^2)(S-S_{n+1}).\end{aligned} \tag{3.3}$$

Obviously, the inequality (3.3) holds for $p=1$.

By Lemma 4, we take

$$\begin{aligned}
\sum_{\alpha,\beta} \operatorname{tr} H^{\beta} \operatorname{tr}[(H^{\alpha})^2 H^{\beta}]
&= nH \sum_i \lambda_i^3 + nH \sum_{\alpha \neq n+1} \operatorname{tr}[(H^{\alpha})^2 H^{n+1}] \\
&= 3nH^2 S_{n+1} - 2n^2 H^4 - nH \sum_i (H - \lambda_i)^3 \\
&\quad + nH \sum_{\alpha \neq n+1} \operatorname{tr}[(H^{\alpha})^2 (H^{n+1} - HI_{n\times n})] + nH \sum_{\alpha \neq n+1} \operatorname{tr}[(H^{\alpha})^2 HI_{n\times n}] \\
&\geq 3nH^2 S_{n+1} - 2n^2 H^4 - \frac{n(n-2)}{\sqrt{n(n-1)}} H (S_{n+1} nH^2)^{3/2} \\
&\quad - \frac{n(n-2)}{\sqrt{n(n-1)}} H (S - S_{n+1})(S_{n+1} - nH^2)^{1/2} + nH^2 (S - S_{n+1}) \\
&= 2nH^2 S_{n+1} - 2n^2 H^4 + nH^2 S - \frac{n(n-2)}{\sqrt{n(n-1)}} H (S - nH^2)(S_{n+1} - nH^2)^{1/2},
\end{aligned} \tag{3.4}$$

and the equation holds in (3.4) if and only if

$$H - \lambda_i = -\frac{S^{1/2}}{\sqrt{n(n-1)}}, \quad (i = 1, \ldots, n-1). \tag{3.5}$$

It follows from (3.1), (3.2), (3.3) and (3.4) that

$$\begin{aligned}
\frac{1}{2} \triangle S &\geq \sum_{i,j,k,\alpha} (h_{ijk}^{\alpha})^2 + \sum_{i,j,k,\alpha} h_{ij}^{\alpha} h_{kkij}^{\alpha} - \frac{1}{B_p} S^2 - \left(\frac{1}{B_p} - 1\right) S_{n+1}^2 \\
&\quad + 2\left(\frac{1}{B_p} - 1\right) S S_{n+1} + 3nH^2 S - 2n^2 H^4 \\
&\quad - \frac{n(n-2)}{\sqrt{n(n-1)}} H (S - nH^2)(S_{n+1} - nH^2)^{1/2} + nc(S - nH^2) \\
&= \sum_{i,j,k,\alpha} (h_{ijk}^{\alpha})^2 + \sum_{i,j,k,\alpha} h_{ij}^{\alpha} h_{kkij}^{\alpha} \\
&\quad + \left(\frac{1}{B_p} - 1\right)(S_{n+1} - nH^2)(2S - S_{n+1} - nH^2) \\
&\quad - \frac{1}{B_p}(S - nH^2)^2 + (nH^2 + nc)(S - nH^2) \\
&\quad - \frac{n(n-2)}{\sqrt{n(n-1)}} H (S - nH^2)(S_{n+1} - nH^2)^{1/2}
\end{aligned}$$

$$\geq \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2+\sum_{i,j,k,\alpha}h^\alpha_{ij}h^\alpha_{kkij}$$
$$+(S-nH^2)\left\{-\frac{1}{B_p}(S-nH^2)+nH^2+nc\right.$$
$$\left.-\frac{n(n-2)}{\sqrt{n(n-1)}}H(S_{n+1}-nH^2)^{1/2}\right\}, \qquad (3.6)$$

and

$$\square(nH)\geq \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2-\sum_i(nH_i)^2$$
$$+(S-nH^2)\left\{-\frac{1}{B_p}(S-nH^2)+nH^2+nc\right.$$
$$\left.-\frac{n(n-2)}{\sqrt{n(n-1)}}H(S_{n+1}-nH^2)^{1/2}\right\}. \qquad (3.7)$$

When $2\leq n\leq 4$, it follows from (3.6) and Lemma 1 that

$$\int_M\frac{1}{2}\triangle S\,dv\geq\int_M(S-nH^2)\left\{-\frac{1}{B_p}(S-nH^2)+nH^2+nc\right.$$
$$\left.-\frac{n(n-2)}{\sqrt{n(n-1)}}H(S_{n+1}-nH^2)^{1/2}\right\}dv. \qquad (3.8)$$

When $n\geq 5$, it follows from (3.7) and Lemma 2 that

$$\square(nH)\geq(S-nH^2)\left\{-\frac{1}{B_p}(S-nH^2)+nH^2+nc\right.$$
$$\left.-\frac{n(n-2)}{\sqrt{n(n-1)}}H(S_{n+1}-nH^2)^{1/2}\right\}. \qquad (3.9)$$

By $\bar{R}=(\rho-n(n-1)c)/n(n-1)=(n^2H^2-S)/n(n-1)$, it can be seen that our assumption condition (1.1), i.e.

$$S-n\bar{R}\leq\frac{B_p}{n-1-B_p}\left\{n^2c+n^2\bar{R}\right.$$
$$\left.-\frac{n-2}{\sqrt{n-1}}\sqrt{[S+n(n-1)\bar{R}][nS_{n-1}-S-n(n-1)\bar{R}]}\right\}$$

is equivalent to

$$\frac{nS-n^2H^2}{n-1}\leq\frac{B_p}{n-1-B_p}\left\{n^2c+n^2H^2-\frac{n}{n-1}(S-nH^2)\right.$$
$$\left.-\frac{n^2(n-2)}{\sqrt{n(n-1)}}H(S_{n+1}-nH^2)^{1/2}\right\},$$

so it is also equivalent to

$$-\frac{1}{B_p}(S-nH^2)+nH^2+nc-\frac{n(n-2)}{\sqrt{n(n-1)}}H(S_{n+1}-nH^2)^{1/2}\geq 0,$$

therefore the right hand side of (3.8), (3.9) is non-negative. Since M is compact, we have that either $S=nH^2$, M is a totally umbilic or

$$S-n\bar{R}=\frac{B_p}{n-1-B_p}\left\{n^2c+n^2\bar{R}\right.$$
$$\left.-\frac{n-2}{\sqrt{n-1}}\sqrt{[S+n(n-1)\bar{R}][nS_{n-1}-S-n(n-1)\bar{R}]}\right\}. \tag{3.10}$$

In the latter case, by (3.5), we have

$$S_{n+1}=(n-1)\left(H+\frac{S^{1/2}}{\sqrt{n(n-1)}}\right)^2+\left(H-\sqrt{\frac{(n-1)S}{n}}\right)^2. \tag{3.11}$$

By the *Guass* equation, we get

$$S=n^2H^2+n(n-1)c-\rho. \tag{3.12}$$

By (3.10), (3.11), (3.12) and $\bar{R}=\rho/n(n-1)-c=\text{constant}$, we take that H is a constant, thus we complete the proof of Theorem 1, 2 by Theorem 1.6 of [3].

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