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# SHARP ENDPOINT INEQUALITY FOR MULTILINEAR LITTLEWOOD-PALEY OPERATOR

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**Abstract**

We establish a sharp inequality for multilinear Littlewood-Paley operator. As application, we obtain the weighted norm inequalities and $L \log L$ type endpoint estimate for the multilinear operator.

## 1. Introduction and result

Let $\psi$ be a function on $R^n$ which satisfies the following properties:
(1) $\int \psi(x)\,dx = 0$,
(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,
(3) $|\psi(x+y) - \psi(x)| \le C|y|(1+|x|)^{-(n+2)}$ when $2|y| < |x|$;

Let $m$ be a positive integer and $A$ be a function on $R^n$. We denote that $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The multilinear Littlewood-Paley operator is defined by

$$S_\psi^A(f)(x) = \left[\iint_{\Gamma(x)} |F_t^A(f)(y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^A(f)(x,y) = \int_{R^n} \frac{f(z)\psi_t(y-z)}{|x-z|^m} R_{m+1}(A;x,z)\,dz,$$

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha| \le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha,$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. We write $F_t(f)(x) = f * \psi_t(x)$. We also define that

$$S_\psi(f)(x) = \left(\iint_{\Gamma(x)} |F_t(f)(x)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

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which is the Littlewood-Paley operator (see [12]).

Let $H$ be the Hilbert space $H = \left\{ h : \|h\| = \left( \int\int_{R_+^{n+1}} |h(t)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} < \infty \right\}$. Then for each fixed $x \in R^n$, $F_t^A(f)(x)$ and $F_t(f)(x)$ may be viewed as a mapping from $(0, +\infty)$ to $H$, and it is clear that

$$S_\psi^A(f)(x) = \|\chi_{\Gamma(x)} F_t^A(f)(x, y)\|.$$

Note that when $m = 0$, $S_\psi^A$ is just the commutator of Littlewood-Paley operator (see [9]), while when $m > 0$, it is non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [1–5]). In [8], authors establish a variant sharp estimate for the multilinear singular integral operators. The main purpose of this paper is to establish a sharp estimate for the multilinear Littlewood-Paley operator, then the weighted norm inequalities and the $L \log L$ type endpoint estimate for the multilinear operator are obtained by using the sharp estimate. We point out that some of our ideas come from [8] and [10]. First, let us introduce some notation (see [6] [7] [10]).

For any locally integrable function $f$, the sharp function of $f$ is defined by

$$f^{\#}(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy,$$

where, and in what follows, $Q$ will denote a cube with sides parallel to the axes, and $f_Q = |Q|^{-1} \int_Q f(x) \, dx$. It is well-known that

$$f^{\#}(x) = \sup_{x \in Q} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| \, dy.$$

We say that $f$ belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$. For $0 < r < \infty$, we denote $f_r^{\#}$ by

$$f_r^{\#}(x) = [(|f|^r)^{\#}(x)]^{1/r}.$$

Let $M$ be the Hardy-Littlewood maximal operator, that is

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

we write that $M_p f = (M(f^p))^{1/p}$, for $k \in N$, we denote by $M^k$ the operator $M$ iterated $k$ times, i.e., $M^1 f(x) = Mf(x)$ and

$$M^k f(x) = M(M^{k-1} f)(x) \quad \text{when } k \geq 2.$$

Let $B$ be a Young function and $\tilde{B}$ be the complementary associated to $B$, we denote that, for a function $f$

$$\|f\|_{B,Q} = \inf\left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q B\left( \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}$$

and the maximal function by

$$M_B f(x) = \sup_{x \in Q} \|f\|_{B,Q};$$

The main Young function to be using in this paper is $B(t) = t(1 + \log^+ t)$ and its complementary $\tilde{B} = \exp t$, the corresponding maximal denoted by $M_{L \log L}$ and $M_{\exp L}$. We have the generalized Holder's inequality (see [10])

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|\,dy \le \|f\|_{B,Q}\|g\|_{\tilde{B},Q}$$

and the following inequality (in fact they are equivalent), for any $x \in R^n$

$$M_{L \log L} f(x) \le CM^2 f(x)$$

and the following inequalities, for all cube $Q$ any $b \in BMO(R^n)$

$$\|b - b_Q\|_{\exp L, Q} \le C\|b\|_{BMO}$$

and

$$|b_{2^{k+1}Q} - b_{2Q}| \le 2k\|b\|_{BMO}.$$

We denote the Muckenhoupt weights by $A_p$ for $1 \le p < \infty$ (see [6]).

Now we state the results in this paper as following.

Theorem 1. *Let $D^\alpha A \in BMO(R^n)$ for all $\alpha$ with $|\alpha| = m$. Then for any $0 < r < 1$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $x \in R^n$,*

$$(S_\psi^A(f))_r^\#(x) \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} M^2 f(x).$$

Theorem 2. *Let $1 < p < \infty$ and $D^\alpha A \in BMO(R^n)$ for all $\alpha$ with $|\alpha| = m$, $w \in A_p$. Then $S_\psi^A$ is bounded on $L^p(w)$, that is*

$$\|S_\psi^A(f)\|_{L^p(w)} \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^p(w)}.$$

Theorem 3. *Let $D^\alpha A \in BMO(R^n)$ for all $\alpha$ with $|\alpha| = m$, $w \in A_1$. Then there exists a constant $C > 0$ such that for each $\lambda > 0$,*

$$w(\{x \in R^n : S_\psi^A(f)(x) > \lambda\}) \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \int_{R^n} \frac{|f(x)|}{\lambda}\left(1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right) w(x)\,dx.$$

As in [10], Theorem 2 and 3 follow from Theorem 1 and the boundedness of $S_\psi$ with $M$. So we only need to prove Theorem 1.

## 2. Some lemmas

We begin with some preliminary lemmas.

LEMMA 1 (Kolmogorov, [7, p. 485]). *Let $0 < p < q < \infty$ and for any function $f \geq 0$. We define that*

$$\|f\|_{WL^q} = \sup_{\lambda>0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q},$$

$$N_{p,q}(f) = \sup_E \|f\chi_E\|_{L^p} / \|\chi_E\|_{L^r}, \quad (1/r = 1/p - 1/q)$$

*where the sup is taken for all measurable sets $E$ with $0 < |E| < \infty$. Then*

$$\|f\|_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p} \|f\|_{WL^q}.$$

LEMMA 2 ([10, p. 165]). *Let $w \in A_1$. Then there exists a constant $C > 0$ such that for any function $f$ and for all $\lambda > 0$,*

$$w(\{y \in R^n : M^2 f(y) > \lambda\}) \leq C\lambda^{-1} \int_{R^n} |f(y)|(1 + \log^+(\lambda^{-1}|f(y)|)) w(y)\, dy.$$

LEMMA 3 ([3, p. 448]). *Let $A$ be a function on $R^n$ and $D^\alpha A \in L^q(R^n)$ for all $\alpha$ with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \leq C|x-y|^m \sum_{|\alpha|=m} \left( \frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q \, dz \right)^{1/q},$$

*where $\tilde{Q}(x,y)$ is the cube centered at $x$ and having side length $5\sqrt{n}|x-y|$.*

LEMMA 4. *Let $1 < p < \infty$, $1 < r \leq \infty$, $1/q = 1/p + 1/r$ and $D^\alpha A \in BMO(R^n)$ for all $\alpha$ with $|\alpha| = m$. Then $S_\psi^A$ is bound from $L^p(R^n)$ to $L^q(R^n)$, that is*

$$\|S_\psi^A(f)\|_{L^q} \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^p}.$$

*Proof.* By Minkowski inequality and the condition of $\psi$, we have

$$\begin{aligned}
S_\psi^A(f)(x) &\leq \int_{R^n} \frac{|f(z)|\,|R_{m+1}(A; x, z)|}{|x-z|^m} \left( \int_{\Gamma(x)} |\psi_t(y-z)|^2 \frac{dydt}{t^{1+n}} \right)^{1/2} dz \\
&\leq C \int_{R^n} \frac{|f(z)|\,|R_{m+1}(A; x, z)|}{|x-z|^m} \left( \int_0^\infty \int_{|x-y|\leq t} \frac{t^{-2n}}{(1+|y-z|/t)^{2n+4}} \frac{dydt}{t^{1+n}} \right)^{1/2} dz \\
&\leq C \int_{R^n} \frac{|f(z)|\,|R_{m+1}(A; x, z)|}{|x-z|^m} \left( \int_0^\infty \int_{|x-y|\leq t} \frac{t^{-2n}}{(1+|y-z|/t)^{2n+4}} \frac{dydt}{t^{n+1}} \right)^{1/2} dz, \\
&\leq C \int_{R^n} \frac{|f(z)|\,|R_{m+1}(A; x, z)|}{|x-z|^m} \left( \int_0^\infty \int_{|x-y|\leq t} \frac{2^{2n+4} \cdot t^{1-n}}{(2t+|y-z|)^{2n+2}} \, dydt \right)^{1/2} dz,
\end{aligned}$$

noting that $2t+|y-z| \geq 2t+|x-z|-|x-y| \geq t+|x-z|$ when $|x-y| \leq t$ and

$$\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2}} = C|x-z|^{-2n},$$

we obtain

$$\begin{aligned} S_\psi^A(f)(x) &\leq C\int_{R^n} \frac{|f(z)|}{|x-z|^m}|R_{m+1}(A;x,z)|\left(\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2}}\right)^{1/2} dz \\ &= C\int_{R^n} \frac{|f(z)|}{|x-z|^{m+n}}|R_{m+1}(A;x,z)|\,dz, \end{aligned}$$

thus, the lemma follows from [4] [5].

## 3. Proof of Theorems

We first prove Theorem 1.

*Proof of Theorem* 1. For $\tilde{x} \in R^n$, let $Q = Q(x_0, l)$ be a cube centered at $x_0$ and having side length $l$ such that $\tilde{x} \in Q$. It is suffice to prove for $f \in C_0^\infty(R^n)$ and some constant $C_0$, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q |S_\psi^A(f)(x) - C_0|^r\,dx\right)^{1/r} \leq CM^2 f(\tilde{x}).$$

Set $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}}x^\alpha$, then $R_m(A;x,y) = R_m(\tilde{A};x,y)$ and $D^\alpha\tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n\setminus\tilde{Q}}$,

$$\begin{aligned} F_t^A(f)(x,y) &= \int \frac{R_{m+1}(A;x,z)}{|x-z|^m}\psi_t(y-z)f(z)\,dz \\ &= \int \frac{R_{m+1}(\tilde{A};x,z)}{|x-z|^m}\psi_t(y-z)f_2(z)\,dz + \int \frac{R_m(\tilde{A};x,z)}{|x-z|^m}\psi_t(y-z)f_1(z)\,dz \\ &\quad - \sum_{|\alpha|=m}\frac{1}{\alpha!}\int \frac{(x-z)^\alpha D^\alpha\tilde{A}(z)}{|x-z|^m}\psi_t(y-z)f_1(z)\,dz \end{aligned}$$

then

$$\begin{aligned} &|S_\psi^A(f)(x) - S_\psi^{\tilde{A}}(f_2)(x_0)| \\ &\quad = |\,\|\chi_{\Gamma(x)}F_t^A(f)(x,y)\| - \|\chi_{\Gamma(x_0)}F_t^{\tilde{A}}(f)(x_0,y)\|\,| \\ &\quad \leq \|\chi_{\Gamma(x)}F_t^A(f)(x,y) - \chi_{\Gamma(x_0)}F_t^{\tilde{A}}(f)(x_0,y)\| \end{aligned}$$

$$\leq \left\| \chi_{\Gamma(x)} F_t \left( \frac{R_m(\tilde{A}; x, \cdot)}{|x - \cdot|^m} f_1 \right)(y) \right\| + \sum_{|\alpha|=m} \frac{1}{\alpha!} \left\| \chi_{\Gamma(x)} F_t \left( \frac{(x - \cdot)^\alpha}{|x - \cdot|^m} D^\alpha \tilde{A} f_1 \right)(y) \right\|$$
$$+ \|\chi_{\Gamma(x)} F_t^{\tilde{A}}(f_2)(x, y) - \chi_{\Gamma(x_0)} F_t^{\tilde{A}}(f_2)(x_0, y)\|$$
$$\equiv I(x) + II(x) + III(x),$$

thus,

$$\left( \frac{1}{|Q|} \int_Q |S_\psi^A(f)(x) - S_\psi^{\tilde{A}}(f_2)(x_0)|^r \, dx \right)^{1/r}$$
$$\leq \left( \frac{C}{|Q|} \int_Q I(x)^r \, dx \right)^{1/r} + \left( \frac{C}{|Q|} \int_Q II(x)^r \, dx \right)^{1/r} + \left( \frac{C}{|Q|} \int_Q III(x)^r \, dx \right)^{1/r}$$
$$\equiv I + II + III.$$

Now, let us estimate $I, II$ and $III$, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, using Lemma 2, we get

$$R_m(\tilde{A}; x, y) \leq C|x - y|^m \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO},$$

thus, by Lemma 1 and the weak type $(1, 1)$ of $S_\psi$ (see [9] [12]), we obtain

$$I \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} |Q|^{-1} \frac{\|S_\psi(f_1)\chi_Q\|_{L^r}}{\|\chi_Q\|_{L^{r/(1-r)}}}$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} |Q|^{-1} \|S_\psi(f_1)(f_1)\|_{WL^1}$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} |\tilde{Q}|^{-1} \int_{\tilde{Q}} |f(y)| \, dy$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} M(f)(\tilde{x});$$

For $II$, similar to the proof of $I$, we get

$$II \leq C \sum_{|\alpha|=m} |Q|^{-1} \frac{\|S_\psi(D^\alpha \tilde{A} f_1)\chi_Q\|_{L^r}}{\|\chi_Q\|_{L^{r/(1-r)}}} \leq C \sum_{|\alpha|=m} |Q|^{-1} \|S_\psi(D^\alpha \tilde{A} f_1)\|_{WL^1}$$
$$\leq C \sum_{|\alpha|=m} |Q|^{-1} \int_{\tilde{Q}} |D^\alpha \tilde{A}(y)| \, |f(y)| \, dy \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\exp L, \tilde{Q}} \|f\|_{L \log L, \tilde{Q}}$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} M_{L \log L} f(\tilde{x}) \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} M^2 f(\tilde{x});$$

Now let us estimate $III$. We write

$$\begin{aligned}
&\chi_{\Gamma(x)}F_t^{\tilde{A}}(f_2)(x,y) - \chi_{\Gamma(x_0)}F_t^{\tilde{A}}(f_2)(x_0,y) \\
&\quad = \int_{R^n}\left[\frac{1}{|x-z|^m} - \frac{1}{|x_0-z|^m}\right]\chi_{\Gamma(x)}\psi_t(y-z)R_m(\tilde{A};x,z)f_2(z)\,dz \\
&\qquad + \int \frac{\chi_{\Gamma(x)}\psi_t(y-z)f_2(z)}{|x_0-z|^m}[R_m(\tilde{A};x,z) - R_m(\tilde{A};x_0,z)]\,dz \\
&\qquad + \int (\chi_{\Gamma(x)} - \chi_{\Gamma(x_0)})\frac{\psi_t(y-z)R_m(\tilde{A};x_0,z)f_2(z)}{|x_0-z|^m}\,dz \\
&\qquad - \sum_{|\alpha|=m}\frac{1}{\alpha!}\int\left[\frac{\chi_{\Gamma(x)}(x-z)^\alpha}{|x-z|^m} - \frac{\chi_{\Gamma(x_0)}(x_0-z)^\alpha}{|x_0-z|^m}\right]\psi_t(y-z)D^\alpha\tilde{A}(z)f_2(z)\,dz \\
&\quad = III_1 + III_2 + III_3 + III_4.
\end{aligned}$$

Note that $|x-z| \sim |x_0-z|$ for $x \in Q$ and $z \in R^n \setminus \tilde{Q}$. By Lemma 3 and the following inequality (see [7])

$$|b_{Q_1} - b_{Q_2}| \le C\log(|Q_2|/|Q_1|)\|b\|_{BMO}, \quad \text{for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $z \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$\begin{aligned}
|R_m(\tilde{A};x,z)| &\le C|x-z|^m \sum_{|\alpha|=m}(\|D^\alpha A\|_{BMO} + |(D^\alpha A)_{\tilde{Q}(x,z)} - (D^\alpha A)_{\tilde{Q}}|) \\
&\le Ck|x-z|^m \sum_{|\alpha|=m}\|D^\alpha A\|_{BMO};
\end{aligned}$$

For $III_1$, by the condition on $\psi$ and similar to the proof of Lemma 4, we get

$$\begin{aligned}
\|III_1\| &\le \int_{R^n\setminus\tilde{Q}} \frac{|x-x_0|\,|f(z)|}{|x-z|^{n+m+1}}|R_m(\tilde{A};x,z)|\,dz \\
&\le \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|x-x_0|\,|f(z)|}{|x-z|^{n+m+1}}|R_m(\tilde{A};x,z)|\,dz \\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=1}^\infty k2^{-k}M(f)(\tilde{x}) \\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}M(f)(\tilde{x});
\end{aligned}$$

For $III_2$, by the formula (see [3]):

$$R_m(\tilde{A};x,z) - R_m(\tilde{A};x_0,z) = \sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x,x_0)(x-z)^\beta$$

and Lemma 3, we have

$$|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)| \le C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-z|^{|\beta|}\|D^\alpha A\|_{BMO}$$
$$\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}|x-x_0|\,|x-z|^{m-1},$$

thus, similar to the proof of Lemma 4

$$\|III_2\| \le \int_{R^n\setminus\tilde{Q}}\frac{|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|}{|x_0-z|^{m+n}}|f(z)|\,dz$$
$$\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|Q|^{1/n}}{|x_0-z|^{n+1}}|f(z)|\,dz$$
$$\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=1}^{\infty}2^{-k}\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(z)|\,dz$$
$$\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}M(f)(\tilde{x});$$

For $III_3$, similar to the proof of Lemma 4, we obtain

$$\|III_3\| \le C\int_{R^n}\left(\iint_{R^{n+1}_+}\left[\frac{|\psi_t(y-z)|\,|f_2(z)|\,|R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}\right.\right.$$
$$\left.\left.\times|\chi_{\Gamma(x)}(y,t)-\chi_{\Gamma(x_0)}(y,t)|\right]^2\frac{dydt}{t^{n+1}}\right)^{1/2}dz$$
$$\le C\int_{R^n}\frac{|f_2(z)|\,|R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}$$
$$\times\left|\iint_{\Gamma(x)}\frac{t^{1-n}\,dydt}{(t+|y-z|)^{2n+2}}-\iint_{\Gamma(x_0)}\frac{t^{1-n}\,dydt}{(t+|y-z|)^{2n+2}}\right|^{1/2}dz$$
$$\le C\int_{R^n}\frac{|f_2(z)|\,|R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}$$
$$\times\left(\iint_{|y|\le t}\left|\frac{1}{(t+|x+y-z|)^{2n+2}}-\frac{1}{(t+|x_0+y-z|)^{2n+2}}\right|\frac{dydt}{t^{n-1}}\right)^{1/2}dz$$
$$\le C\int_{R^n}\frac{|f_2(z)|\,|R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}\left(\iint_{|y|\le t}\frac{|x-x_0|t^{1-n}\,dydt}{(t+|x+y-z|)^{2n+3}}\right)^{1/2}dz$$

$$\le C\int_{R^n}\frac{|f_2(z)|\,|x-x_0|^{1/2}|R_m(\tilde{A};x_0,z)|}{|x_0-z|^{m+n+1/2}}\,dz$$

$$\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=1}^{\infty}2^{-k/2}\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(z)|\,dz$$

$$\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}M(f)(\tilde{x});$$

For $III_4$, similar to the proof of $III_1$ and $III_3$, we get

$$\|III_4\|\le C\sum_{|\alpha|=m}\int_{R^n}\left(\frac{|x-x_0|}{|x_0-z|^{n+1}}+\frac{|x-x_0|^{1/2}}{|x_0-z|^{n+1/2}}\right)|D^\alpha\tilde{A}(y)|\,|f_2(z)|\,dz$$

$$\le C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}(2^{-k}+2^{-k/2})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(z)|\,|D^\alpha A(z)-(D^\alpha A)_{\tilde{Q}}|\,dz$$

$$\le C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}k(2^{-k}+2^{-k/2})(\|D^\alpha A\|_{\exp L,2^k\tilde{Q}}\|f\|_{L\log L,2^k\tilde{Q}}$$

$$+\|D^\alpha A\|_{BMO}M(f)(\tilde{x}))$$

$$\le C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}k(2^{-k}+2^{-k/2})\|D^\alpha A\|_{BMO}M_{L\log L}(f)(\tilde{x})$$

$$\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}M^2(f)(\tilde{x}).$$

Thus,

$$III\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}M^2(f)(\tilde{x}).$$

This completes the proof of Theorem 1.

From Theorem 1 and the weighted boundedness of $S_\psi$ and $M$, we may obtain the conclusion of Theorem 2.

From Theorem 1 and Lemma 2, we may obtain the conclusion of Theorem 3.

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