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INEQUALITIES GENERATED BY CHAINS OF JENSEN INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract

In this paper, we study the monotonicity of some weighted differences between any two terms in chains of Jensen's inequalities for convex functions. The problem is reduced to the solvability of some weight equations or weight inequalities. Our proofs are based on the classical Jensen's inequality and some elementary identity involved combinatorial numbers.

1 Introduction

Let f be a given convex function defined on a non-empty interval $\boldsymbol{I} \subset \mathbf{R}$. For any given $n \in \boldsymbol{N}$, $x_i \in \boldsymbol{I}$ and $t_i > 0$ $(i = 1, 2, \ldots, n)$, it is well-known that the following Jensen's inequality holds

$$\sum_{i=1}^{n} t_i f(x_i) \geq \left(\sum_{i=1}^{n} t_i\right) f\left(\sum_{i=1}^{n} t_i x_i \Big/ \sum_{i=1}^{n} t_i\right). \tag{1.1}$$

Jensen's inequality is one of the classical inequalities and has a lot of applications. Also, there exist extensive works which were devoted to generalize or improve Jensen's inequality. In this respect, we refer the reader to [1], [2], [3] and [6] and the reference cited therein for updated results.

For any $j \in \boldsymbol{N} \cup \{0\}$, we recall the definition of combinatorial number C_n^j:

$$C_n^j = \begin{cases} 1 & \text{if } j = 0, \\ \dfrac{n!}{j!(n-j)!} & \text{if } 0 < j \leq n, \\ 0 & \text{if } j > n. \end{cases} \tag{1.2}$$

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Now, for any positive integers s, k and r with $1 \leq k,\ r \leq n$, put

$$M(n,s;f) \equiv M(n,s;f,t_1,\ldots,t_n,x_1,\ldots,x_n)$$
$$\triangleq \sum_{1\leq j_1<\cdots<j_s\leq n}\left(\sum_{p=1}^{s} t_{j_p}\right) f\left(\sum_{p=1}^{s} t_{j_p}x_{j_p}\Big/\sum_{p=1}^{s} t_{j_p}\right), \tag{1.3}$$

$$J_k^n(f) \equiv J_k^n(f,t_1,\ldots,t_n,x_1,\ldots,x_n) \triangleq \frac{1}{C_{n-1}^{k-1}} M(n,k;f), \tag{1.4}$$

and

$$F(n,k,r;f) \equiv F(n,k,r;f,t_1,\ldots,t_n,x_1,\ldots,x_n)$$
$$\triangleq C_{n-1}^{r-1}M(n,k;f) - C_{n-1}^{k-1}M(n,r;f). \tag{1.5}$$

It is easy to see that

$$F(n,k,r;f) = C_{n-1}^{r-1}C_{n-1}^{k-1}(J_k^n(f) - J_r^n(f)).$$

The first named author of this paper showed in [5] that the following chain of Jensen inequalities hold

$$J_1^n(f) \geq \cdots \geq J_k^n(f) \geq \cdots \geq J_r^n(f) \geq \cdots \geq J_n^n(f), \quad 1 \leq k < r \leq n. \tag{1.6}$$

Note that

$$J_n^n(f) = \left(\sum_{i=1}^{n} t_i\right) f\left(\sum_{i=1}^{n} t_i x_i\Big/\sum_{i=1}^{n} t_i\right), \quad J_1^n(f) = \sum_{i=1}^{n} t_i f(x_i),$$

Therefore, (1.6) is obtained by inserting $n-2$ terms $J_2^n(f),\ldots,J_{n-1}^n(f)$ in (1.1).

On the other hand, it is easy to see that

$$F(n,1,n;f,t_1,\ldots,t_n,x_1,\ldots,x_n) = \sum_{i=1}^{n} t_i f(x_i) - \left(\sum_{i=1}^{n} t_i\right) f\left(\sum_{i=1}^{n} t_i x_i\Big/\sum_{i=1}^{n} t_i\right)$$
$$= J_1^n(f) - J_n^n(f) \tag{1.7}$$

can be obtained by taking the difference between the left hand side and the right one of (1.1) (or between the first term and the n-th one in (1.6)). Vasić, Mijalković and Pečarić found an interesting property on the monotonicity of $F(n,1,n;f,1,\ldots,1,x_1,\ldots,x_n)$ with respect to n. In [7] and [8], they showed the following inequality

$$F(n,1,n;f,1,\ldots,1,x_1,\ldots,x_n) = \sum_{i=1}^{n} f(x_i) - nf\left(\sum_{i=1}^{n} x_i\Big/n\right)$$
$$\geq \sum_{i=1}^{n-1} f(x_i) - (n-1)f\left(\sum_{i=1}^{n-1} x_i\Big/(n-1)\right)$$
$$= F(n-1,1,n-1;f,1,\ldots,1,x_1,\ldots,x_{n-1}). \tag{1.8}$$

Wang ([4]) gave a generalization of inequality (1.8). He obtained the monotonicity of $F(n, 1, n; f)$ with respect to n, i.e.

$$\begin{aligned}
&F(n, 1, n; f, t_1, \ldots, t_n, x_1, \ldots, x_n) \\
&\quad = \sum_{i=1}^{n} t_i f(x_i) - \left(\sum_{i=1}^{n} t_i\right) f\left(\sum_{i=1}^{n} t_i x_i \Big/ \sum_{i=1}^{n} t_i\right) \\
&\quad \geq \sum_{i=1}^{n-1} t_i f(x_i) - \left(\sum_{i=1}^{n-1} t_i\right) f\left(\sum_{i=1}^{n-1} t_i x_i \Big/ \sum_{i=1}^{n-1} t_i\right) \\
&\quad = F(n-1, 1, n-1; f, t_1, \ldots, t_{n-1}, x_1, \ldots, x_{n-1}).
\end{aligned} \tag{1.9}$$

In view of (1.7), one may re-write (1.9) as

$$J_1^n(f) - J_n^n(f) \geq J_1^{n-1}(f) - J_{n-1}^{n-1}(f). \tag{1.10}$$

Therefore, it seems to be natural to expect that

$$J_k^n(f) - J_r^n(f), \quad 1 \leq k < r \leq n$$

i.e., the difference between the k-th term and the r-th one in the chain of inequalities (1.6), has a similar monotonicity property with respect to n. However, this is not the case. Indeed, we have the following simple counterexample.

Example 1.1. Let us take $\boldsymbol{I} = [0, 4]$. Then it is clear that the function

$$f(x) \triangleq \begin{cases} 0, & 0 \leq x \leq 3, \\ x - 3, & 3 < x \leq 4 \end{cases}$$

is convex in $\boldsymbol{I}$. Take $x_1 = 4$, $t_1 = 1$, $x_2 = 3$, $t_2 = 1$, $x_3 = 2$, $t_3 = 1$, $x_4 = 1$ and $t_4 = 1$. Then, noting that $f(x) = 0$ whenever $x \in [0, 3]$, we get

$$\begin{aligned}
&J_2^4(f) - J_3^4(f) \\
&\quad = \frac{1}{C_{4-1}^{2-1}} \sum_{1 \leq i < j \leq 4} (t_i + t_j) f\left(\frac{t_i x_i + t_j x_j}{t_i + t_j}\right) \\
&\qquad - \frac{1}{C_{4-1}^{3-1}} \sum_{1 \leq i < j < k \leq 4} (t_i + t_j + t_k) f\left(\frac{t_i x_i + t_j x_j + t_k x_k}{t_i + t_j + t_k}\right) \\
&\quad = \frac{2}{C_{4-1}^{2-1}} f\left(\frac{x_1 + x_2}{2}\right) = \frac{1}{3}
\end{aligned}$$

and

$$\begin{aligned}
&J_2^3(f) - J_3^3(f) \\
&\quad = \frac{1}{C_{3-1}^{2-1}} \sum_{1 \leq i < j \leq 3} (t_i + t_j) f\left(\frac{t_i x_i + t_j x_j}{t_i + t_j}\right)
\end{aligned}$$

$$- \frac{1}{C_{3-1}^{3-1}} \sum_{1 \le i<j<k \le 3} (t_i + t_j + t_k) f\left(\frac{t_i x_i + t_j x_j + t_k x_k}{t_i + t_j + t_k}\right)$$
$$= \frac{2}{C_{3-1}^{2-1}} f\left(\frac{x_1 + x_2}{2}\right) = \frac{1}{2}.$$

Therefore

$$J_2^4(f) - J_3^4(f) < J_2^3(f) - J_3^3(f).$$

On the other hand, take $x_1 = 1$, $t_1 = 1$, $x_2 = 2$, $t_2 = 1$, $x_3 = 3$, $t_3 = 1$, $x_4 = 4$ and $t_4 = 1$. Then similar to the above computation, we obtain the following converse inequality:

$$J_2^4(f) - J_3^4(f) = \frac{1}{3} > J_2^3(f) - J_3^3(f) = 0.$$

Example 1.1 shows that $J_k^n(f) - J_r^n(f)$ does not have monotonicity with respect to n without further assumptions. Stimulated by this example, we will consider in this paper some weighted differences between $J_k^n(f)$ and $J_r^n(f)$, or equivalently between $M(n,k;f)$ and $M(n,r;f)$, and analyze their monotonicity under suitable conditions.

The rest of this paper is organized as follows. In Section 2, we will state our main results, Theorems 2.1–2.5. Section 3 is devoted to the proof of Theorems 2.1–2.2. In Section 4, we will prove Theorems 2.3–2.5.

2 Main results

2.1 Monotonicity of weighted differences

Fix two functions $u, v : \boldsymbol{N} \times \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{R}$. For any $k, r \in \{1, 2, \dots, n\}$, we set

$$\begin{aligned}
\mathscr{F}(n,k,r;f) &\triangleq u(n,k,r)M(n,k;f) - v(n,k,r)M(n,r;f), \\
A(n,k,r) &\triangleq (u(n,k,r) - u(n-1,k,r))C_{n-1}^k C_{r-1}^{k-1} \\
&\quad - (v(n,k,r) - v(n-1,k,r))C_{n-1}^r C_r^k - v(n,k,r)C_{n-1}^{r-1} C_{r-1}^k, \\
B(n,k,r) &\triangleq u(n,k,r)C_{n-1}^{k-1} - v(n,k,r)C_{n-1}^{r-1}.
\end{aligned} \tag{2.1}$$

Obviously, $\mathscr{F}(n,k,r;f)$ is a weighted difference between $M(n,k;f)$ and $M(n,r;f)$ (with weight functions u and v). We have the following monotonicity result on $\mathscr{F}(n,k,r;f)$ with respect to n:

Theorem 2.1. *Let f be a convex function defined on $\boldsymbol{I}$, $n \ge 3$, $1 \le k < r < n$. Assume v is non-negative and $v(n,k,r) \ge v(n-1,k,r)$. Then*

$$\mathscr{F}(n,k,r;f) \ge \mathscr{F}(n-1,k,r;f) \tag{2.2}$$

provided one of the following three class of conditions holds

$$(1) \qquad A(n,k,r) = B(n,k,r) = 0; \tag{2.3}$$

$$(2) \qquad A(n,k,r) \ge 0, \quad B(n,k,r) \ge 0 \tag{2.4}$$

and f is non-negative; *or*

$$(3) \qquad A(n,k,r) \le 0, \quad B(n,k,r) \le 0 \tag{2.5}$$

and f is non-positive.

Note, however, that in Theorem 2.1, we assume $r < n$, which excludes the important case of $r = n$. Therefore, Theorem 2.1 does not cover the monotonicity result in inequality (1.9).

In order to include the case of $r = n$, we will go a little bit further. Fix two functions $\hat{u}, \hat{v} : \boldsymbol{N} \times \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{R}$. For any $k, r \in \{1, 2, \dots, n-1\}$, we put

$$\begin{aligned}
\mathscr{G}(n,k,r;f) &\triangleq \hat{u}(n,k,r)M(n,k;f) - \hat{v}(n,k,r)M(n,n-r+1;f),\\
\hat{A}(n,k,r) &\triangleq C_{n-r}^{k-1}C_{n-1}^{k}(\hat{u}(n,k,r) - \hat{u}(n-1,k,r))\\
&\qquad - C_{n-1}^{n-r+1}C_{n-r+1}^{k}\hat{v}(n,k,r),\\
\hat{B}(n,k,r) &\triangleq C_{n-1}^{k-1}\hat{u}(n,k,r) - C_{n-1}^{r-1}\hat{v}(n,k,r),\\
\hat{C}(n,k,r) &\triangleq C_{n-r}^{k-1}\hat{v}(n-1,k,r) - C_{n-r-1}^{k-1}\hat{v}(n,k,r),\\
\tilde{A}(n,k,r) &\triangleq \hat{u}(n,k,r) - \hat{u}(n-1,k,r),\\
\tilde{B}(n,k,r) &\triangleq C_{n-1}^{k-1}\hat{u}(n,k,r) - C_{n-1}^{r-1}\hat{v}(n,k,r),\\
\tilde{C}(n,k,r) &\triangleq (n-r)\hat{v}(n-1,k,r) - (n-k)\hat{v}(n,k,r).
\end{aligned} \tag{2.6}$$

Clearly, $\mathscr{G}(n,k,r;f)$ is a weighted difference between $M(n,k;f)$ and $M(n, n-r+1;f)$ (with weight functions $\hat{u}$ and $\hat{v}$). We have the following monotonicity result on $\mathscr{G}(n,k,r;f)$ with respect to n:

Theorem 2.2. *Let f be a convex function defined on $\boldsymbol{I}$, $\hat{v}$ be non-negative, $n \ge 3$ and $1 < k + r < n + 1$. Then*

$$\mathscr{G}(n,k,r;f) \ge \mathscr{G}(n-1,k,r;f) \tag{2.7}$$

provided one of the following six class of conditions holds

$$(1) \qquad \hat{A}(n,k,r) = \hat{B}(n,k,r) = \hat{C}(n,k,r) = 0; \tag{2.8}$$

$$(2) \qquad \tilde{A}(n,k,r) = \tilde{B}(n,k,r) = \tilde{C}(n,k,r) = 0; \tag{2.9}$$

$$(3) \qquad \hat{A}(n,k,r) \ge 0, \quad \hat{B}(n,k,r) \ge 0, \quad \hat{C}(n,k,r) \ge 0 \tag{2.10}$$

and f is non-negative;

$$(4) \qquad \tilde{A}(n,k,r) \ge 0, \quad \tilde{B}(n,k,r) \ge 0, \quad \tilde{C}(n,k,r) \ge 0 \tag{2.11}$$

and f is non-negative;

(5) $$\hat{A}(n,k,r) \le 0, \quad \hat{B}(n,k,r) \le 0, \quad \hat{C}(n,k,r) \le 0 \tag{2.12}$$

and f is non-positive; *or*

(6) $$\tilde{A}(n,k,r) \le 0, \quad \tilde{B}(n,k,r) \le 0, \quad \tilde{C}(n,k,r) \le 0 \tag{2.13}$$

and f is non-positive.

Theorems 2.1–2.2 reduce the monotonicity of weighted differences between $M(n,k;f)$ and $M(n,r;f)$, or equivalently between $J_k^n(f)$ and $J_r^n(f)$, to the solvability of suitable weight equations or weight inequalities. The solutions of weight equations will be studied in the next subsection.

The proof of Theorems 2.1–2.2 will be given in Section 3.

Remark 2.1. Theorems 2.1–2.2 remain true for more general case when interval $\boldsymbol{I} \subset \mathbf{R}$ is replaced by any non-empty convex subset $\boldsymbol{E}$ in a linear space $\mathbf{X}$ over $\boldsymbol{R}$.

Remark 2.2. To the best of our knowledge, the following weight inequalities (with unknowns u and v)

$$\begin{cases} A(n,k,r) \ge 0, \\ B(n,k,r) \ge 0, \end{cases}$$

and the following two weight inequalities (with unknowns $\hat{u}$ and $\hat{v}$)

$$\begin{cases} \hat{A}(n,k,r) \ge 0, \\ \hat{B}(n,k,r) \ge 0, \\ \hat{C}(n,k,r) \ge 0, \end{cases} \quad \begin{cases} \tilde{A}(n,k,r) \ge 0, \\ \tilde{B}(n,k,r) \ge 0, \\ \tilde{C}(n,k,r) \ge 0, \end{cases}$$

are new, and very little is known about their solutions. (The other weight inequalities appeared in Theorems 2.1–2.2 can be easily reduced to the above ones). It would be interesting to analyze the structure of their solutions. But this is by now an open problem.

2.2 Solutions of weight equations

We have the following three results, which characterize the structure of solutions of weight equations (2.3), (2.8) and (2.9).

THEOREM 2.3. *Let $n \ge 3$, $1 \le k < r < n$. Then $u(n,k,r)$ and $v(n,k,r)$ satisfy*

$$A(n,k,r) = B(n,k,r) = 0 \tag{2.14}$$

if and only if

$$u(n,k,r) = \frac{C_{n-1}^{r-1}}{C_{n-1}^{k-1}} v(n,k,r). \tag{2.15}$$

THEOREM 2.4. *Let* $n \geq 3$, $1 < k + r < n + 1$. *Then* $\hat{u}(n,k,r)$ *and* $\hat{v}(n,k,r)$ *satisfy*

$$\hat{A}(n,k,r) = \hat{B}(n,k,r) = \hat{C}(n,k,r) = 0 \tag{2.16}$$

if and only if

$$\hat{v}(n,k,r) = \frac{C_{n-r}^{k-1}}{C_{n-r-1}^{k-1}} \hat{v}(n-1,k,r), \quad \hat{u}(n,k,r) = \frac{C_{n-1}^{r-1}}{C_{n-1}^{k-1}} \hat{v}(n,k,r). \tag{2.17}$$

THEOREM 2.5. *Let* $n \geq 3$, $1 < k + r < n + 1$. *Then* $\hat{u}(n,k,r)$ *and* $\hat{v}(n,k,r)$ *satisfy*

$$\tilde{A}(n,k,r) = \tilde{B}(n,k,r) = \tilde{C}(n,k,r) = 0 \tag{2.18}$$

if and only if

$$\hat{u}(n,k,r) = \hat{u}(n-1,k,r), \quad \hat{v}(n,k,r) = \frac{C_{n-1}^{k-1}}{C_{n-1}^{r-1}} \hat{u}(n,k,r). \tag{2.19}$$

The proof of Theorems 2.3–2.5 will be given in Section 4.

Remark 2.3. By Theorem 2.5, in order that $\tilde{A}(n,k,r) = \tilde{B}(n,k,r) = \tilde{C}(n,k,r) = 0$ (and therefore inequality (2.7) holds), without loss of generality, one may choose the weight functions $\hat{u}(n,k,r)$ and $\hat{v}(n,k,r)$ in $\mathscr{G}(n,k,r;f)$ as follows:

$$\hat{u}(n,k,r) = 1, \quad \hat{v}(n,k,r) = \frac{C_{n-1}^{k-1}}{C_{n-1}^{r-1}}.$$

This is the unique "linearly" independent solution of weight equation (2.18).

2.3 Several corollaries

First, combining Theorem 2.1 and Theorem 2.3, we get

COROLLARY 2.1. *Let* f *be a convex function defined on* $\boldsymbol{I}$, $n \geq 3$, $1 \leq k < r < n$. *Let* $u(n,k,r) = C_{n-1}^{r-1} v(n,k,r) / C_{n-1}^{k-1}$, *where* $v : \boldsymbol{N} \times \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{R}$ *is any given non-negative function such that* $v(n,k,r) \geq v(n-1,k,r)$. *Then*

$$\mathscr{F}(n,k,r;f) \geq \mathscr{F}(n-1,k,r;f).$$

Let us choose $v(n,k,r) = C_{n-1}^{k-1}$ in Corollary 2.1. Then $u(n,k,r) = C_{n-1}^{r-1}$. As a direct consequence of Corollary 2.1, we obtain the monotonicity of

$$F(n,k,r;f) \quad (1 \leq k < r < n)$$

with respect to n, i.e., we have

COROLLARY 2.2. *Let* f *be a convex function defined on* $\boldsymbol{I}$, $n \geq 3$ *and* $1 \leq k < r < n$. *Then*

$$F(n,k,r;f) \geq F(n-1,k,r;f). \tag{2.20}$$

Next, choosing $r = 1$, $\hat{u}(n,k,r) = 1$ and $\hat{v}(n,k,r) = C_{n-1}^{k-1}$ in Theorem 2.2 and Theorem 2.4, and noting (1.5) and (2.6), we get

COROLLARY 2.3. *Let f be a convex function defined on $\boldsymbol{I}$, $n \geq 3$ and $1 \leq k < n$. Then*

$$F(n,k,n;f) \geq F(n-1,k,n-1;f). \tag{2.21}$$

Remark 2.4. It is easy to see that inequality (1.9) is a special case of Corollary 2.3.

Similarly, choosing $r = k$, $\hat{u}(n,k,r) = \hat{v}(n,k,r) = 1$ in Theorem 2.2 and Theorem 2.5, we get

COROLLARY 2.4. *Let f be a convex function defined on $\boldsymbol{I}$, $n \geq 4$ and $1 \leq k \leq [n/2]$, the integer part of $n/2$. Then*

$$M(n,k;f) - M(n,n-k+1;f) \geq M(n-1,k;f) - M(n-1,n-k;f). \tag{2.22}$$

3 Proof of Theorems 2.1–2.2

This section is devoted to prove Theorems 2.1–2.2. For this purpose, we need some simple preliminaries.

LEMMA 3.1. *Fix $m \in \boldsymbol{N}$ with $1 \leq m \leq n$. Let $y_1, y_2, \ldots, y_n \in \boldsymbol{R}$. Then*

$$\frac{1}{C_{n-1}^{m-1}} \sum_{1 \leq s_1 < \cdots < s_m \leq n} \left(\sum_{j=1}^{m} y_{s_j} \right) = \sum_{i=1}^{n} y_i. \tag{3.1}$$

Proof. In view of the symmetry, it is easy to see that

$$\sum_{1 \leq s_1 < \cdots < s_m \leq n} \left(\sum_{j=1}^{m} y_{s_j} \right) \tag{3.2}$$

is equal to some integer times of $\sum_{i=1}^{n} y_i$. Obviously, this integer is equal to the times that y_1 appears in (3.2), which in turn is C_{n-1}^{m-1}. This completes the proof of Lemma 3.1. □

LEMMA 3.2. *Let f be a convex function defined on $\boldsymbol{I}$. Then for any $m, \ell \in \boldsymbol{N}$ with $1 \leq m \leq \ell \leq n$, it holds*

$$\left(\sum_{i=1}^{\ell} t_i \right) f\left(\sum_{i=1}^{\ell} t_i x_i \Big/ \sum_{i=1}^{\ell} t_i \right) \leq \frac{1}{C_{\ell-1}^{m-1}} \sum_{1 \leq s_1 < \cdots < s_m \leq \ell} \left(\sum_{j=1}^{m} t_{s_j} \right) f\left(\sum_{j=1}^{m} t_{s_j} x_{s_j} \Big/ \sum_{j=1}^{m} t_{s_j} \right). \tag{3.3}$$

Proof. By (3.1), we have

$$\sum_{i=1}^{\ell} t_i x_i = \frac{1}{C_{\ell-1}^{m-1}} \sum_{1\le s_1<\cdots<s_m\le\ell} \left(\sum_{j=1}^{m} t_{s_j} x_{s_j}\right) \tag{3.4}$$

and

$$\sum_{i=1}^{\ell} t_i = \frac{1}{C_{\ell-1}^{m-1}} \sum_{1\le s_1<\cdots<s_m\le\ell} \left(\sum_{j=1}^{m} t_{s_j}\right). \tag{3.5}$$

Now, by (3.4), (3.5) and (1.1), we conclude that

$$\begin{aligned} f\left(\sum_{i=1}^{\ell} t_i x_i \Big/ \sum_{i=1}^{\ell} t_i\right) &= f\left(\frac{1}{C_{\ell-1}^{m-1}} \sum_{1\le s_1<\cdots<s_m\le\ell} \left(\sum_{j=1}^{m} t_{s_j} x_{s_j}\right) \Big/ \sum_{i=1}^{\ell} t_i\right) \\ &= f\left(\sum_{1\le s_1<\cdots<s_m\le\ell} \left(\frac{1}{C_{\ell-1}^{m-1}} \sum_{j=1}^{m} t_{s_j}\right)\left(\sum_{j=1}^{m} t_{s_j} x_{s_j} \Big/ \sum_{j=1}^{m} t_{s_j}\right) \Big/ \sum_{i=1}^{\ell} t_i\right) \\ &\le \frac{1}{C_{\ell-1}^{m-1}} \sum_{1\le s_1<\cdots<s_m\le\ell} \left(\sum_{j=1}^{m} t_{s_j} \Big/ \sum_{i=1}^{\ell} t_i\right) f\left(\sum_{j=1}^{m} t_{s_j} x_{s_j} \Big/ \sum_{j=1}^{m} t_{s_j}\right), \end{aligned}$$

which yields (3.3). This completes the proof of Lemma 3.2. □

LEMMA 3.3. *Let $k, r, m \in \boldsymbol{N}$ satisfy $1 \le k \le r \le m$. Then for any function $g : \boldsymbol{N}^k \to \boldsymbol{R}$, it holds*

$$\sum_{1\le i_1<\cdots<i_r\le m} \sum_{\{i_1',\dots,i_k'\}\subset\{i_1,\dots,i_r\}} g(i_1',\dots,i_k') = \frac{C_m^r C_r^k}{C_m^k} \sum_{1\le j_1<\cdots<j_k\le m} g(j_1,\dots,j_k). \tag{3.6}$$

Proof. Similar to the proof of Lemma 3.1, by symmetry, it is easy to see that

$$\sum_{1\le i_1<\cdots<i_r\le m} \sum_{\{i_1',\dots,i_k'\}\subset\{i_1,\dots,i_r\}} g(i_1',\dots,i_k') \tag{3.7}$$

is equal to some integer times of the following summation

$$\sum_{1\le j_1<\cdots<j_k\le m} g(j_1,\dots,j_k) \tag{3.8}$$

To compute this integer, we note that there are $C_m^r C_r^k$ terms (including repeated terms) in (3.7); while in (3.8) there are C_m^k terms. Therefore, the desired integer is equal to $C_m^r C_r^k / C_m^k$. This completes the proof of Lemma 3.3. □

LEMMA 3.4. *Let f be a convex function defined on $\boldsymbol{I}$. Then for any $s, h, m \in \boldsymbol{N}$ with $1 \le s \le h \le m \le n$, it holds*

$$\sum_{1\le i_1<\cdots<i_h\le m}\left(\sum_{q=1}^{h} t_{i_q}\right) f\left(\sum_{q=1}^{h} t_{i_q}x_{i_q}\Big/\sum_{q=1}^{h} t_{i_q}\right) \le \frac{C_m^h C_h^s}{C_{h-1}^{s-1}C_m^s}\sum_{1\le j_1<\cdots<j_s\le m}\left(\sum_{p=1}^{s} t_{j_p}\right) f\left(\sum_{p=1}^{s} t_{j_p}x_{j_p}\Big/\sum_{p=1}^{s} t_{j_p}\right). \tag{3.9}$$

Proof. Thanks to Lemma 3.2, and by $s \le h$, we see that

$$\left(\sum_{q=1}^{h} t_{i_q}\right) f\left(\sum_{q=1}^{h} t_{i_q}x_{i_q}\Big/\sum_{q=1}^{h} t_{i_q}\right) \le \frac{1}{C_{h-1}^{s-1}}\sum_{\{i'_1,\dots,i'_s\}\subset\{i_1,\dots,i_h\}}\left(\sum_{q=1}^{s} t_{i'_q}\right) f\left(\sum_{q=1}^{s} t_{i'_q}x_{i'_q}\Big/\sum_{q=1}^{s} t_{i'_q}\right).$$

Therefore, we have

$$\sum_{1\le i_1<\cdots<i_h\le m}\left(\sum_{q=1}^{h} t_{i_q}\right) f\left(\sum_{q=1}^{h} t_{i_q}x_{i_q}\Big/\sum_{q=1}^{h} t_{i_q}\right) \le \frac{1}{C_{h-1}^{s-1}}\sum_{1\le i_1<\cdots<i_h\le m}\ \sum_{\{i'_1,\dots,i'_s\}\subset\{i_1,\dots,i_h\}}\left(\sum_{q=1}^{s} t_{i'_q}\right) f\left(\sum_{q=1}^{s} t_{i'_q}x_{i'_q}\Big/\sum_{q=1}^{s} t_{i'_q}\right). \tag{3.10}$$

However, by Lemma 3.3, we get

$$\sum_{1\le i_1<\cdots<i_h\le m}\ \sum_{\{i'_1,\dots,i'_s\}\subset\{i_1,\dots,i_h\}}\left(\sum_{q=1}^{s} t_{i'_q}\right) f\left(\sum_{q=1}^{s} t_{i'_q}x_{i'_q}\Big/\sum_{q=1}^{s} t_{i'_q}\right) = \frac{C_m^h C_h^s}{C_m^s}\sum_{1\le j_1<\cdots<j_s\le m}\left(\sum_{p=1}^{s} t_{j_p}\right) f\left(\sum_{p=1}^{s} t_{j_p}x_{j_p}\Big/\sum_{p=1}^{s} t_{j_p}\right). \tag{3.11}$$

Now, combining (3.10) and (3.11), we arrive at the desired inequality (3.9). □

Now, we can prove Theorems 2.1–2.2.

Proof of Theorem 2.1. First, we assume $k > 1$. The proof is divided into several steps.

STEP 1. It is easy to see that

$$
\begin{aligned}
&\mathscr{F}(n,k,r;f)\\
&= u(n,k,r)\left[\sum_{1\le j_1<\cdots<j_k\le n-1}\left(\sum_{p=1}^{k} t_{j_p}\right) f\left(\sum_{p=1}^{k} t_{j_p}x_{j_p}\Big/\sum_{p=1}^{k} t_{j_p}\right)\right.\\
&\quad\left. + \sum_{1\le j_1<\cdots<j_k=n}\left(t_n+\sum_{p=1}^{k-1} t_{j_p}\right) f\left(\left(t_nx_n+\sum_{p=1}^{k-1} t_{j_p}x_{j_p}\right)\Big/\left(t_n+\sum_{p=1}^{k-1} t_{j_p}\right)\right)\right]\\
&\quad - v(n,k,r)\left[\sum_{1\le i_1<\cdots<i_r\le n-1}\left(\sum_{q=1}^{r} t_{i_q}\right) f\left(\sum_{q=1}^{r} t_{i_q}x_{i_q}\Big/\sum_{q=1}^{r} t_{i_q}\right)\right.\\
&\quad\left. + \sum_{1\le i_1<\cdots<i_r=n}\left(t_n+\sum_{q=1}^{r-1} t_{i_q}\right) f\left(\left(t_nx_n+\sum_{q=1}^{r-1} t_{i_q}x_{i_q}\right)\Big/\left(t_n+\sum_{q=1}^{r-1} t_{i_q}\right)\right)\right] \qquad (3.12)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathscr{F}(n-1,k,r;f)\\
&\qquad = u(n-1,k,r)\sum_{1\le j_1<\cdots<j_k\le n-1}\left(\sum_{p=1}^{k} t_{j_p}\right) f\left(\sum_{p=1}^{k} t_{j_p}x_{j_p}\Big/\sum_{p=1}^{k} t_{j_p}\right)\\
&\qquad\quad - v(n-1,k,r)\sum_{1\le i_1<\cdots<i_r\le n-1}\left(\sum_{q=1}^{r} t_{i_q}\right) f\left(\sum_{q=1}^{r} t_{i_q}x_{i_q}\Big/\sum_{q=1}^{r} t_{i_q}\right). \qquad (3.13)
\end{aligned}
$$

From (3.12) and (3.13), we see that

$$
\begin{aligned}
&\mathscr{F}(n,k,r;f)-\mathscr{F}(n-1,k,r;f)\\
&= (u(n,k,r)-u(n-1,k,r))\sum_{1\le j_1<\cdots<j_k\le n-1}\left(\sum_{p=1}^{k} t_{j_p}\right) f\left(\sum_{p=1}^{k} t_{j_p}x_{j_p}\Big/\sum_{p=1}^{k} t_{j_p}\right)\\
&\quad - (v(n,k,r)-v(n-1,k,r))\sum_{1\le i_1<\cdots<i_r\le n-1}\left(\sum_{q=1}^{r} t_{i_q}\right) f\left(\sum_{q=1}^{r} t_{i_q}x_{i_q}\Big/\sum_{q=1}^{r} t_{i_q}\right)\\
&\quad + u(n,k,r)\sum_{1\le j_1<\cdots<j_k=n}\left(t_n+\sum_{p=1}^{k-1} t_{j_p}\right) f\left(\left(t_nx_n+\sum_{p=1}^{k-1} t_{j_p}x_{j_p}\right)\Big/\left(t_n+\sum_{p=1}^{k-1} t_{j_p}\right)\right)\\
&\quad - v(n,k,r)\sum_{1\le i_1<\cdots<i_r=n}\left(t_n+\sum_{q=1}^{r-1} t_{i_q}\right) f\left(\left(t_nx_n+\sum_{q=1}^{r-1} t_{i_q}x_{i_q}\right)\Big/\left(t_n+\sum_{q=1}^{r-1} t_{i_q}\right)\right).
\end{aligned}
\qquad (3.14)
$$

Step 2. Let us analyze the second term in the right side of (3.14). Applying Lemma 3.4 with $s=k$, $h=r$ and $m=n-1$, and by $v(n,k,r)\ge v(n-1,k,r)$, we get

$$-(v(n,k,r)-v(n-1,k,r))\sum_{1\le i_1<\cdots<i_r\le n-1}\left(\sum_{q=1}^{r}t_{i_q}\right)f\left(\sum_{q=1}^{r}t_{i_q}x_{i_q}\Big/\sum_{q=1}^{r}t_{i_q}\right)$$
$$\ge -\frac{(v(n,k,r)-v(n-1,k,r))C_{n-1}^{r}C_{r}^{k}}{C_{r-1}^{k-1}C_{n-1}^{k}}\sum_{1\le j_1<\cdots<j_k\le n-1}\left(\sum_{p=1}^{k}t_{j_p}\right)$$
$$\times f\left(\sum_{p=1}^{k}t_{j_p}x_{j_p}\Big/\sum_{p=1}^{k}t_{j_p}\right). \tag{3.15}$$

STEP 3. Let us analyze the last term in the right side of (3.14). Thanks to Lemma 3.2, and recalling that $k<r$, we see that

$$\left(t_n+\sum_{q=1}^{r-1}t_{i_q}\right)f\left(\left(t_nx_n+\sum_{q=1}^{r-1}t_{i_q}x_{i_q}\right)\Big/\left(t_n+\sum_{q=1}^{r-1}t_{i_q}\right)\right)$$
$$\le \frac{1}{C_{r-1}^{k-1}}\sum_{\{j'_1,\ldots,j'_k\}\subset\{i_1,\ldots,i_{r-1},n\}}\left(\sum_{q=1}^{k}t_{j'_q}\right)f\left(\sum_{q=1}^{k}t_{j'_q}x_{j'_q}\Big/\sum_{q=1}^{k}t_{j'_q}\right).$$

Therefore,

$$-v(n,k,r)\sum_{1\le i_1<\cdots<i_r=n}\left(t_n+\sum_{q=1}^{r-1}t_{i_q}\right)f\left(\left(t_nx_n+\sum_{q=1}^{r-1}t_{i_q}x_{i_q}\right)\Big/\left(t_n+\sum_{q=1}^{r-1}t_{i_q}\right)\right)$$
$$\ge -\frac{v(n,k,r)}{C_{r-1}^{k-1}}\sum_{1\le i_1<\cdots<i_{r-1}\le n-1}\;\sum_{\{j'_1,\ldots,j'_k\}\subset\{i_1,\ldots,i_{r-1},n\}}\left(\sum_{q=1}^{k}t_{j'_q}\right)f\left(\sum_{q=1}^{k}t_{j'_q}x_{j'_q}\Big/\sum_{q=1}^{k}t_{j'_q}\right)$$
$$=-\frac{v(n,k,r)}{C_{r-1}^{k-1}}\sum_{1\le i_1<\cdots<i_{r-1}\le n-1}\left[\sum_{\{j'_1,\ldots,j'_k\}\subset\{i_1,\ldots,i_{r-1}\}}\left(\sum_{q=1}^{k}t_{j'_q}\right)f\left(\sum_{q=1}^{k}t_{j'_q}x_{j'_q}\Big/\sum_{q=1}^{k}t_{j'_q}\right)\right.$$
$$\left.+\sum_{\{j'_1,\ldots,j'_{k-1}\}\subset\{i_1,\ldots,i_{r-1}\}}\left(t_n+\sum_{q=1}^{k-1}t_{j'_q}\right)f\left(\left(t_nx_n+\sum_{q=1}^{k-1}t_{j'_q}x_{j'_q}\right)\Big/\left(t_n+\sum_{q=1}^{k-1}t_{j'_q}\right)\right)\right]. \tag{3.16}$$

In view of Lemma 3.3, it is easy to see that

$$\sum_{1\le i_1<\cdots<i_{r-1}\le n-1}\;\sum_{\{j'_1,\ldots,j'_k\}\subset\{i_1,\ldots,i_{r-1}\}}\left(\sum_{q=1}^{k}t_{j'_q}\right)f\left(\sum_{q=1}^{k}t_{j'_q}x_{j'_q}\Big/\sum_{q=1}^{k}t_{j'_q}\right)$$
$$=\frac{C_{n-1}^{r-1}C_{r-1}^{k}}{C_{n-1}^{k}}\sum_{1\le j_1<\cdots<j_k\le n-1}\left(\sum_{q=1}^{k}t_{j_q}\right)f\left(\sum_{q=1}^{k}t_{j_q}x_{j_q}\Big/\sum_{q=1}^{k}t_{j_q}\right) \tag{3.17}$$

and

$$\sum_{1\le i_1<\cdots<i_{r-1}\le n-1}\ \sum_{\{j'_1,\ldots,j'_{k-1}\}\subset\{i_1,\ldots,i_{r-1}\}}\left(t_n+\sum_{q=1}^{k-1}t_{j'_q}\right)$$
$$\times f\left(\left(t_nx_n+\sum_{q=1}^{k-1}t_{j'_q}x_{j'_q}\right)\Big/\left(t_n+\sum_{q=1}^{k-1}t_{j'_q}\right)\right)$$
$$=\frac{C_{n-1}^{r-1}C_{r-1}^{k-1}}{C_{n-1}^{k-1}}\sum_{1\le j_1<\cdots<j_{k-1}\le n-1}\left(t_n+\sum_{q=1}^{k-1}t_{j_q}\right)f\left(\left(t_nx_n+\sum_{q=1}^{k-1}t_{j_q}x_{j_q}\right)\Big/\left(t_n+\sum_{q=1}^{k-1}t_{j_q}\right)\right). \tag{3.18}$$

Now, combining (3.16), (3.17) and (3.18), we conclude that

$$-v(n,k,r)\sum_{1\le i_1<\cdots<i_r=n}\left(t_n+\sum_{q=1}^{r-1}t_{i_q}\right)f\left(\left(t_nx_n+\sum_{q=1}^{r-1}t_{i_q}x_{i_q}\right)\Big/\left(t_n+\sum_{q=1}^{r-1}t_{i_q}\right)\right)$$
$$\ge-\frac{v(n,k,r)}{C_{r-1}^{k-1}}\left[\frac{C_{n-1}^{r-1}C_{r-1}^{k}}{C_{n-1}^{k}}\sum_{1\le j_1<\cdots<j_k\le n-1}\left(\sum_{q=1}^{k}t_{j_q}\right)f\left(\sum_{q=1}^{k}t_{j_q}x_{j_q}\Big/\sum_{q=1}^{k}t_{j_q}\right)\right.$$
$$+\frac{C_{n-1}^{r-1}C_{r-1}^{k-1}}{C_{n-1}^{k-1}}\sum_{1\le j_1<\cdots<j_{k-1}<j_k=n}\left(t_n+\sum_{q=1}^{k-1}t_{j_q}\right)$$
$$\left.\times f\left(\left(t_nx_n+\sum_{q=1}^{k-1}t_{j_q}x_{j_q}\right)\Big/\left(t_n+\sum_{q=1}^{k-1}t_{j_q}\right)\right)\right]. \tag{3.19}$$

STEP 4. Let us complete the proof. By (3.14), (3.15) and (3.19), and recalling the definitions of $A(n,k,r)$ and $B(n,k,r)$ in (2.1), we end up with

$$\mathscr{F}(n,k,r;f)-\mathscr{F}(n-1,k,r;f)$$
$$\ge\frac{A(n,k,r)}{C_{r-1}^{k-1}C_{n-1}^{k}}\sum_{1\le j_1<\cdots<j_k\le n-1}\left(\sum_{p=1}^{k}t_{j_p}\right)f\left(\sum_{p=1}^{k}t_{j_p}x_{j_p}\Big/\sum_{p=1}^{k}t_{j_p}\right)$$
$$+\frac{B(n,k,r)}{C_{n-1}^{k-1}}\sum_{1\le j_1<\cdots<j_k=n}\left(t_n+\sum_{q=1}^{k-1}t_{j_q}\right)$$
$$\times f\left(\left(t_nx_n+\sum_{q=1}^{k-1}t_{j_q}x_{j_q}\right)\Big/\left(t_n+\sum_{q=1}^{k-1}t_{j_q}\right)\right). \tag{3.20}$$

However, from the proof of (3.20), it is easy to see that the same inequalities hold for $k=1$ if we replace

$$\sum_{1\le j_1<\cdots<j_k=n}\left(t_n+\sum_{q=1}^{k-1}t_{j_q}\right)f\left(\left(t_nx_n+\sum_{q=1}^{k-1}t_{j_q}x_{j_q}\right)\Big/\left(t_n+\sum_{q=1}^{k-1}t_{j_q}\right)\right)$$

in (3.20) by $t_nf(x_n)$.

Now, noting our assumptions on $A(n,k,r)$, $B(n,k,r)$ and f, the desired result (2.2) follows from (3.20) immediately. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. We proceed as in the proof of Theorem 2.1. First, we assume $k > 1$. By (2.6) and (1.3), we see that

$$\begin{aligned}
&\mathscr{G}(n,k,r;f)\\
&\triangleq \hat{u}(n,k,r)M(n,k;f) - \hat{v}(n,k,r)M(n,n-r+1;f)\\
&= \hat{u}(n,k,r)\Bigg[\sum_{1\le j_1<\cdots<j_k\le n-1}\left(\sum_{p=1}^{k} t_{j_p}\right) f\left(\sum_{p=1}^{k} t_{j_p}x_{j_p}\Big/\sum_{p=1}^{k} t_{j_p}\right)\\
&+ \sum_{1\le j_1<\cdots<j_{k-1}\le n-1}\left(t_n+\sum_{p=1}^{k-1} t_{j_p}\right) f\left(\left(t_nx_n+\sum_{p=1}^{k-1} t_{j_p}x_{j_p}\right)\Big/\left(t_n+\sum_{p=1}^{k-1} t_{j_p}\right)\right)\Bigg]\\
&- \hat{v}(n,k,r)\Bigg[\sum_{1\le i_1<\cdots<i_{n-r+1}\le n-1}\left(\sum_{q=1}^{n-r+1} t_{i_q}\right) f\left(\sum_{q=1}^{n-r+1} t_{i_q}x_{i_q}\Big/\sum_{q=1}^{n-r+1} t_{i_q}\right)\\
&+ \sum_{1\le i_1<\cdots<i_{n-r}\le n-1}\left(t_n+\sum_{q=1}^{n-r} t_{i_q}\right) f\left(\left(t_nx_n+\sum_{q=1}^{n-r} t_{i_q}x_{i_q}\right)\Big/\left(t_n+\sum_{q=1}^{n-r} t_{i_q}\right)\right)\Bigg]. \qquad (3.21)
\end{aligned}$$

Here, we agree that when $r = 1$,

$$\sum_{1\le i_1<\cdots<i_{n-r+1}\le n-1}\left(\sum_{q=1}^{n-r+1} t_{i_q}\right) f\left(\sum_{q=1}^{n-r+1} t_{i_q}x_{i_q}\Big/\sum_{q=1}^{n-r+1} t_{i_q}\right) = 0 \qquad (3.22)$$

(since in this case there are no integers $i_1,\ldots,i_{n-r+1}$ such that $1 \le i_1 < \cdots < i_{n-r+1} \le n-1$).

Now, when $r > 1$, applying Lemma 3.4 with $s = k$, $h = n-r+1$ and $m = n-1$, we conclude that

$$\begin{aligned}
&\sum_{1\le i_1<\cdots<i_{n-r+1}\le n-1}\left(\sum_{q=1}^{n-r+1} t_{i_q}\right) f\left(\sum_{q=1}^{n-r+1} t_{i_q}x_{i_q}\Big/\sum_{q=1}^{n-r+1} t_{i_q}\right)\\
&\qquad \le \frac{C_{n-1}^{n-r+1}C_{n-r+1}^{k}}{C_{n-r}^{k-1}C_{n-1}^{k}}\sum_{1\le j_1<\cdots<j_k\le n-1}\left(\sum_{p=1}^{k} t_{j_p}\right) f\left(\sum_{p=1}^{k} t_{j_p}x_{j_p}\Big/\sum_{p=1}^{k} t_{j_p}\right). \qquad (3.23)
\end{aligned}$$

Note that by our convention in (3.22), inequality (3.23) is also valid for $r = 1$ (recall that, by definition (1.2), $C_{n-1}^{n} = 0$).

Replacing k by $n-r$ in (3.23), we get

$$\sum_{1\le i_1<\cdots<i_{n-r+1}\le n-1}\left(\sum_{q=1}^{n-r+1} t_{i_q}\right) f\left(\sum_{q=1}^{n-r+1} t_{i_q}x_{i_q}\Big/\sum_{q=1}^{n-r+1} t_{i_q}\right)$$

$$\le \frac{(n-r+1)C_{n-1}^{n-r+1}}{(n-r)C_{n-1}^{n-r}}\sum_{1\le j_1<\cdots<j_{n-r}\le n-1}\left(\sum_{p=1}^{n-r} t_{j_p}\right) f\left(\sum_{p=1}^{n-r} t_{j_p}x_{j_p}\Big/\sum_{p=1}^{n-r} t_{j_p}\right). \quad (3.24)$$

On the other hand, by Lemma 3.1, we see that (recall that $k+r<1+n$)

$$\sum_{\{i'_1,\ldots,i'_{k-1}\}\subset\{i_1,\ldots,i_{n-r}\}}\left(t_n+\sum_{q=1}^{k-1} t_{i'_q}\right)+C_{n-r-1}^{k-1}\sum_{q=1}^{n-r} t_{i_q}$$

$$= C_{n-r}^{k-1}t_n+\sum_{1\le s_1<\cdots<s_{k-1}\le n-r}\left(\sum_{j=1}^{k-1} t_{i_{s_j}}\right)+C_{n-r-1}^{k-1}\sum_{q=1}^{n-r} t_{i_q}$$

$$= C_{n-r}^{k-1}t_n+(C_{n-r-1}^{k-2}+C_{n-r-1}^{k-1})\sum_{q=1}^{n-r} t_{i_q}$$

$$= C_{n-r}^{k-1}\left(t_n+\sum_{q=1}^{n-r} t_{i_q}\right).$$

Similarly,

$$\sum_{\{i'_1,\ldots,i'_{k-1}\}\subset\{i_1,\ldots,i_{n-r}\}}\left(t_nx_n+\sum_{q=1}^{k-1} t_{i'_q}x_{i'_q}\right)+C_{n-r-1}^{k-1}\sum_{q=1}^{n-r} t_{i_q}x_{i_q} = C_{n-r}^{k-1}\left(t_nx_n+\sum_{q=1}^{n-r} t_{i_q}x_{i_q}\right).$$

Noting the above two identities, and using (1.1), we obtain

$$\left(t_n+\sum_{q=1}^{n-r} t_{i_q}\right) f\left(\left(t_nx_n+\sum_{q=1}^{n-r} t_{i_q}x_{i_q}\right)\Big/\left(t_n+\sum_{q=1}^{n-r} t_{i_q}\right)\right)$$

$$= \left(t_n+\sum_{q=1}^{n-r} t_{i_q}\right) f\left(\left[\sum_{\{i'_1,\ldots,i'_{k-1}\}\subset\{i_1,\ldots,i_{n-r}\}}\left(t_nx_n+\sum_{q=1}^{k-1} t_{i'_q}x_{i'_q}\right)\right.\right.$$

$$\left.\left.+\,C_{n-r-1}^{k-1}\sum_{q=1}^{n-r} t_{i_q}x_{i_q}\right]\Big/\left[C_{n-r}^{k-1}\left(t_n+\sum_{q=1}^{n-r} t_{i_q}\right)\right]\right)$$

$$= \left(t_n + \sum_{q=1}^{n-r} t_{i_q}\right) f\left(\left\{\sum_{\{i'_1,\dots,i'_{k-1}\}\subset\{i_1,\dots,i_{n-r}\}} \left[\left(t_n + \sum_{q=1}^{k-1} t_{i'_q}\right)\cdot \frac{t_n x_n + \sum_{q=1}^{k-1} t_{i'_q} x_{i'_q}}{t_n + \sum_{q=1}^{k-1} t_{i'_q}}\right]\right.\right.$$

$$\left.\left. + \left(C_{n-r-1}^{k-1} \sum_{q=1}^{n-r} t_{i_q}\right)\cdot \frac{\sum_{q=1}^{n-r} t_{i_q} x_{i_q}}{\sum_{q=1}^{n-r} t_{i_q}}\right\} \Big/ \left[C_{n-r}^{k-1}\left(t_n + \sum_{q=1}^{n-r} t_{i_q}\right)\right]\right)$$

$$\le \frac{1}{C_{n-r}^{k-1}} \sum_{\{i'_1,\dots,i'_{k-1}\}\subset\{i_1,\dots,i_{n-r}\}} \left(t_n + \sum_{q=1}^{k-1} t_{i'_q}\right) f\left(\left(t_n x_n + \sum_{q=1}^{k-1} t_{i'_q} x_{i'_q}\right)\Big/\left(t_n + \sum_{q=1}^{k-1} t_{i'_q}\right)\right)$$

$$+ \frac{C_{n-r-1}^{k-1}}{C_{n-r}^{k-1}} \left(\sum_{q=1}^{n-r} t_{i_q}\right) f\left(\sum_{q=1}^{n-r} t_{i_q} x_{i_q} \Big/ \sum_{q=1}^{n-r} t_{i_q}\right). \tag{3.25}$$

However, by Lemma 3.3, we have

$$\sum_{1\le i_1<\cdots<i_{n-r}\le n-1} \sum_{\{i'_1,\dots,i'_{k-1}\}\subset\{i_1,\dots,i_{n-r}\}} \left(t_n + \sum_{q=1}^{k-1} t_{i'_q}\right)$$

$$\times f\left(\left(t_n x_n + \sum_{q=1}^{k-1} t_{i'_q} x_{i'_q}\right)\Big/\left(t_n + \sum_{q=1}^{k-1} t_{i'_q}\right)\right)$$

$$= \frac{C_{n-1}^{n-r} C_{n-r}^{k-1}}{C_{n-1}^{k-1}} \sum_{1\le i_1<\cdots<i_{k-1}\le n-1} \left(t_n + \sum_{q=1}^{k-1} t_{i_q}\right)$$

$$\times f\left(\left(t_n x_n + \sum_{q=1}^{k-1} t_{i_q} x_{i_q}\right)\Big/\left(t_n + \sum_{q=1}^{k-1} t_{i_q}\right)\right). \tag{3.26}$$

Now, by (3.25) and (3.26), we get

$$- \sum_{1\le i_1<\cdots<i_{n-r}\le n-1} \left(t_n + \sum_{q=1}^{n-r} t_{i_q}\right) f\left(\left(t_n x_n + \sum_{q=1}^{n-r} t_{i_q} x_{i_q}\right)\Big/\left(t_n + \sum_{q=1}^{n-r} t_{i_q}\right)\right)$$

$$\ge -\frac{C_{n-1}^{n-r}}{C_{n-1}^{k-1}} \sum_{1\le i_1<\cdots<i_{k-1}\le n-1} \left(t_n + \sum_{q=1}^{k-1} t_{i_q}\right) f\left(\left(t_n x_n + \sum_{q=1}^{k-1} t_{i_q} x_{i_q}\right)\Big/\left(t_n + \sum_{q=1}^{k-1} t_{i_q}\right)\right)$$

$$-\frac{C_{n-r-1}^{k-1}}{C_{n-r}^{k-1}} \sum_{1\le i_1<\cdots<i_{n-r}\le n-1} \left(\sum_{q=1}^{n-r} t_{i_q}\right) f\left(\sum_{q=1}^{n-r} t_{i_q} x_{i_q} \Big/ \sum_{q=1}^{n-r} t_{i_q}\right). \tag{3.27}$$

On the other hand, it follows from (2.6) and (1.3) that

$$\begin{aligned}
&\mathscr{G}(n-1,k,r;f)\\
&\quad\triangleq \hat{u}(n-1,k,r)M(n-1,k;f)-\hat{v}(n-1,k,r)M(n-1,n-r;f)\\
&\quad= \hat{u}(n-1,k,r)\sum_{1\le j_1<\cdots<j_k\le n-1}\left(\sum_{p=1}^{k} t_{j_p}\right) f\left(\sum_{p=1}^{k} t_{j_p}x_{j_p}\Big/\sum_{p=1}^{k} t_{j_p}\right)\\
&\qquad - \hat{v}(n-1,k,r)\sum_{1\le i_1<\cdots<i_{n-r}\le n-1}\left(\sum_{q=1}^{n-r} t_{i_q}\right) f\left(\sum_{q=1}^{n-r} t_{i_q}x_{i_q}\Big/\sum_{q=1}^{n-r} t_{i_q}\right). \qquad (3.28)
\end{aligned}$$

Therefore, combining (3.21), (3.23), (3.27) and (3.28), and recalling the definitions of $\hat{A}(n,k,r)$, $\hat{B}(n,k,r)$ and $\hat{C}(n,k,r)$ in (2.6), we conclude that for any $1<k<n+1-r$, it holds

$$\begin{aligned}
&\mathscr{G}(n,k,r;f)-\mathscr{G}(n-1,k,r;f)\\
&\quad\ge \frac{\hat{A}(n,k,r)}{C_{n-r}^{k-1}C_{n-1}^{k}}\sum_{1\le j_1<\cdots<j_k\le n-1}\left(\sum_{p=1}^{k} t_{j_p}\right) f\left(\sum_{p=1}^{k} t_{j_p}x_{j_p}\Big/\sum_{p=1}^{k} t_{j_p}\right)\\
&\qquad+\frac{\hat{B}(n,k,r)}{C_{n-1}^{k-1}}\sum_{1\le i_1<\cdots<i_{k-1}\le n-1}\left(t_n+\sum_{q=1}^{k-1} t_{i_q}\right)\\
&\qquad\times f\left(\left(t_nx_n+\sum_{q=1}^{k-1} t_{i_q}x_{i_q}\right)\Big/\left(t_n+\sum_{q=1}^{k-1} t_{i_q}\right)\right)\\
&\qquad+\frac{\hat{C}(n,k,r)}{C_{n-r}^{k-1}}\sum_{1\le i_1<\cdots<i_{n-r}\le n-1}\left(\sum_{p=1}^{n-r} t_{i_p}\right) f\left(\sum_{p=1}^{n-r} t_{i_p}x_{i_p}\Big/\sum_{p=1}^{n-r} t_{i_p}\right). \qquad (3.29)
\end{aligned}$$

Similarly, combining (3.21), (3.24), (3.27) and (3.28), and recalling the definitions of $\tilde{A}(n,k,r)$, $\tilde{B}(n,k,r)$ and $\tilde{C}(n,k,r)$ in (2.6), we conclude that for any $1<k<n+1-r$, it holds

$$\begin{aligned}
&\mathscr{G}(n,k,r;f)-\mathscr{G}(n-1,k,r;f)\\
&\quad\ge \tilde{A}(n,k,r)\sum_{1\le j_1<\cdots<j_k\le n-1}\left(\sum_{p=1}^{k} t_{j_p}\right) f\left(\sum_{p=1}^{k} t_{j_p}x_{j_p}\Big/\sum_{p=1}^{k} t_{j_p}\right)\\
&\qquad+\frac{\tilde{B}(n,k,r)}{C_{n-1}^{k-1}}\sum_{1\le i_1<\cdots<i_{k-1}\le n-1}\left(t_n+\sum_{q=1}^{k-1} t_{i_q}\right)\\
&\qquad\times f\left(\left(t_nx_n+\sum_{q=1}^{k-1} t_{i_q}x_{i_q}\right)\Big/\left(t_n+\sum_{q=1}^{k-1} t_{i_q}\right)\right)\\
&\qquad+\frac{\tilde{C}(n,k,r)}{n-r}\sum_{1\le i_1<\cdots<i_{n-r}\le n-1}\left(\sum_{p=1}^{n-r} t_{i_p}\right) f\left(\sum_{p=1}^{n-r} t_{i_p}x_{i_p}\Big/\sum_{p=1}^{n-r} t_{i_p}\right). \qquad (3.30)
\end{aligned}$$

However, from the proof of (3.29) and (3.30), it is easy to see that the same inequalities hold for $k = 1$ if replacing

$$\sum_{1 \le i_1 < \cdots < i_{k-1} \le n-1} \left(t_n + \sum_{q=1}^{k-1} t_{i_q} \right) f\left(\left(t_n x_n + \sum_{q=1}^{k-1} t_{i_q} x_{i_q} \right) \Big/ \left(t_n + \sum_{q=1}^{k-1} t_{i_q} \right) \right)$$

in (3.29) and (3.30) by $t_n f(x_n)$. These facts, combined with our assumptions on $\hat{A}(n,k,r)$, $\hat{B}(n,k,r)$, $\hat{C}(n,k,r)$, $\tilde{A}(n,k,r)$, $\tilde{B}(n,k,r)$, $\tilde{C}(n,k,r)$ and f, yield the desired inequality (2.7). This completes the proof of Theorem 2.2. □

4 Proof of Theorems 2.3–2.5

This section is devoted to the proof of Theorems 2.3–2.5.

Proof of Theorem 2.3. The "only if" part is a direct consequence of $B(n,k,r) = 0$.

The "if" part. It suffices to show $A(n,k,r) = 0$. By (2.15) and the definition of $A(n,k,r)$ in (2.1), we have

$$\begin{aligned} A(n,k,r) &= \left(\frac{C_{n-1}^{r-1}}{C_{n-1}^{k-1}} v(n,k,r) - \frac{C_{n-2}^{r-1}}{C_{n-2}^{k-1}} v(n-1,k,r) \right) C_{n-1}^{k} C_{r-1}^{k-1} \\ &\quad - (v(n,k,r) - v(n-1,k,r)) C_{n-1}^{r} C_{r}^{k} - v(n,k,r) C_{n-1}^{r-1} C_{r-1}^{k} \\ &= \left(\frac{C_{n-1}^{r-1} C_{n-1}^{k} C_{r-1}^{k-1}}{C_{n-1}^{k-1}} - C_{n-1}^{r} C_{r}^{k} - C_{n-1}^{r-1} C_{r-1}^{k} \right) v(n,k,r) \\ &\quad + \left(C_{n-1}^{r} C_{r}^{k} - \frac{C_{n-2}^{r-1} C_{n-1}^{k} C_{r-1}^{k-1}}{C_{n-2}^{k-1}} \right) v(n-1,k,r). \end{aligned} \tag{4.1}$$

However,

$$\begin{aligned} &\frac{C_{n-1}^{r-1} C_{n-1}^{k} C_{r-1}^{k-1}}{C_{n-1}^{k-1}} - C_{n-1}^{r} C_{r}^{k} - C_{n-1}^{r-1} C_{r-1}^{k} \\ &\quad = \frac{(n-k)(n-1)!}{k!(n-r)!(r-k)!} - \frac{(n-1)!}{k!(n-r-1)!(r-k)!} - \frac{(n-1)!}{k!(n-r)!(r-k-1)!} \\ &\quad = 0. \end{aligned} \tag{4.2}$$

Similarly,

$$C_{n-1}^{r} C_{r}^{k} - \frac{C_{n-2}^{r-1} C_{n-1}^{k} C_{r-1}^{k-1}}{C_{n-2}^{k-1}} = 0. \tag{4.3}$$

Hence, combining (4.1)–(4.3), we see that $A(n,k,r) = 0$. This completes the proof of Theorem 2.3. □

Proof of Theorem 2.4. It suffices to analyze the structure of solutions of $\hat{A}(n,k,r) = \hat{B}(n,k,r) = \hat{C}(n,k,r) = 0$.

Obviously, by (2.6), $\hat{B}(n,k,r)=0$ is equivalent to

$$\hat{u}(n,k,r)=\frac{C_{n-1}^{r-1}}{C_{n-1}^{k-1}}\hat{v}(n,k,r). \tag{4.4}$$

Similarly, $\hat{C}(n,k,r)=0$ is equivalent to

$$\hat{v}(n-1,k,r)=\frac{C_{n-r-1}^{k-1}}{C_{n-r}^{k-1}}\hat{v}(n,k,r). \tag{4.5}$$

Therefore, by (4.4) and (4.5), we see that

$$\hat{u}(n-1,k,r)=\frac{C_{n-2}^{r-1}}{C_{n-2}^{k-1}}\hat{v}(n-1,k,r)=\frac{C_{n-2}^{r-1}C_{n-r-1}^{k-1}}{C_{n-2}^{k-1}C_{n-r}^{k-1}}\hat{v}(n,k,r). \tag{4.6}$$

Now, by (4.4) and (4.6), and using the definition of $\hat{A}(n,k,r)$ in (2.6), we get

$$\hat{A}(n,k,r)=\left(\frac{C_{n-r}^{k-1}C_{n-1}^{k}C_{n-1}^{r-1}}{C_{n-1}^{k-1}}-\frac{C_{n-1}^{k}C_{n-2}^{r-1}C_{n-r-1}^{k-1}}{C_{n-2}^{k-1}}-C_{n-1}^{n-r+1}C_{n-r+1}^{k}\right)\hat{v}(n,k,r) \tag{4.7}$$

A direct computation shows that for any $r>1$, it holds

$$\begin{aligned}&\frac{C_{n-r}^{k-1}C_{n-1}^{k}C_{n-1}^{r-1}}{C_{n-1}^{k-1}}-\frac{C_{n-1}^{k}C_{n-2}^{r-1}C_{n-r-1}^{k-1}}{C_{n-2}^{k-1}}-C_{n-1}^{n-r+1}C_{n-r+1}^{k}\\&\quad=\frac{(n-1)!(n-k)}{(n-k-r+1)!k!(r-1)!}-\frac{(n-1)!}{(n-k-r)!k!(r-1)!}\\&\qquad-\frac{(n-1)!}{(n-k-r+1)!k!(r-2)!}\\&\quad=0.\end{aligned} \tag{4.8}$$

However, when $r=1$, by (1.2), we see that $C_{n-1}^{n-r+1}=C_{n-1}^{n}=0$. Thus

$$\begin{aligned}&\frac{C_{n-r}^{k-1}C_{n-1}^{k}C_{n-1}^{r-1}}{C_{n-1}^{k-1}}-\frac{C_{n-1}^{k}C_{n-2}^{r-1}C_{n-r-1}^{k-1}}{C_{n-2}^{k-1}}-C_{n-1}^{n-r+1}C_{n-r+1}^{k}\\&\quad=\frac{(n-1)!(n-k)}{(n-k)!k!}-\frac{(n-1)!}{(n-k-1)!k!}=0.\end{aligned}$$

Therefore, (4.8) holds for $r\geq 1$ such that $1<k+r<n+1$.

Now, from (4.7) and (4.8), it is easy to see that the desired result holds. This completes the proof of Theorem 2.4. □

Proof of Theorem 2.5. Similar to the proof of Theorem 2.4, let us analyze the structure of solutions of $\tilde{A}(n,k,r)=\tilde{B}(n,k,r)=\tilde{C}(n,k,r)=0$.

Obviously, by (2.6), $\tilde{A}(n,k,r)=0$ is equivalent to

$$\hat{u}(n,k,r)=\hat{u}(n-1,k,r). \tag{4.9}$$

Similarly, $\tilde{B}(n,k,r)=0$ is equivalent to

$$\hat{v}(n,k,r) = \frac{C_{n-1}^{k-1}}{C_{n-1}^{r-1}} \hat{u}(n,k,r). \tag{4.10}$$

Therefore, by (4.9) and (4.10), we see that

$$\hat{v}(n-1,k,r) = \frac{C_{n-2}^{k-1}}{C_{n-2}^{r-1}} \hat{u}(n-1,k,r) = \frac{C_{n-2}^{k-1}}{C_{n-2}^{r-1}} \hat{u}(n,k,r). \tag{4.11}$$

Now, by (4.10) and (4.11), and using the definition of $\tilde{C}(n,k,r)$ in (2.6), we get

$$\tilde{C}(n,k,r) = \left((n-r) \frac{C_{n-2}^{k-1}}{C_{n-2}^{r-1}} - (n-k) \frac{C_{n-1}^{k-1}}{C_{n-1}^{r-1}} \right) \hat{u}(n,k,r) = 0,$$

which implies the desired result. This completes the proof of Theorem 2.5. □

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