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MEROMORPHIC FUNCTION OF INFINITE ORDER WITH MAXIMUM DEFICIENCY SUM*

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Abstract

In this paper we prove the following theorem: Let $f(z)$ be a meromorphic function of infinite order. If $\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2$, then for each positive integer k, we have $K(f^{(k)}) = 2k(1 - \delta(\infty, f))/(1 + k - k\delta(\infty, f))$, where $K(f^{(k)}) = \lim_{r\to\infty}(N(r, 1/f^{(k)}) + N(r, f^{(k)}))/T(r, f^{(k)})$ exists. This result improves the results by S. K. Singh and V. N. Kulkarni [1] and Mingliang Fang [2].

1 Introduction and results

In this paper, we assume that $f(z)$ is a nonconstant meromorphic function in the complex plane C. We shall use the standard notations of the Nevanlinna theory of meromorphic functions (see [3]).

$$T(r, f), m(r, f), N(r, f), \overline{N}(r, f), \delta(a, f), S(r, f) \text{ and so on.}$$

We shall also use the following notations (see [4]):

$$T_0(r, f) = \int_1^r \frac{T(t, f)}{t}\,dt, \quad N_0(r, f) = \int_1^r \frac{N(t, f)}{t}\,dt,$$

$$m_0(r, f) = \int_1^r \frac{m(t, f)}{t}\,dt, \quad S_0(r, f) = \int_1^r \frac{S(t, f)}{t}\,dt.$$

Similarly, we use the notations $m_0(r, a, f)$ and $N_0(r, a, f)$. Set

$$\delta_0(a, f) = 1 - \limsup_{r\to\infty} \frac{N_0(r, a, f)}{T_0(r, a, f)},$$

$$K(f^{(k)}) = \limsup_{r\to\infty} \frac{N(r, 1/f^{(k)}) + N(r, f^{(k)})}{T(r, f^{(k)})}, \quad (k = 0, 1, 2, \ldots).$$

In 1973, S. K. Singh and V. N. Kulkarni [1] proved the following result:

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Theorem 1.1. *Suppose that f is a transcendental meromorphic function of finite order satisfying*

$$\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2.$$

Then

$$\frac{1 - \delta(\infty, f)}{2 - \delta(\infty, f)} \leq K(f') \leq \frac{2(1 - \delta(\infty, f))}{2 - \delta(\infty, f)}.$$

In 2000, Mingliang Fang [2] proved the following result:

Theorem 1.2. *Suppose that f is a transcendental meromorphic function of finite order satisfying*

$$\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2.$$

Then

$$K(f^{(k)}) = \frac{2k(1 - \delta(\infty, f))}{1 + k - k\delta(\infty, f)}.$$

In this paper, we shall prove the following theorem:

Theorem 1.3. *Suppose that f is a meromorphic function of infinite order. If*

$$\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2,$$

then for each positive integer k,

$$K(f^{(k)}) = \frac{2k(1 - \delta(\infty, f))}{1 + k - k\delta(\infty, f)}.$$

2 Lemmas

Lemma 1 ([4]). *Suppose that f is a meromorphic function of infinite order. Then for each complex number a,*

$$0 \leq \delta(a, f) \leq \delta_0(a, f) \leq 1, \quad \sum_{a \neq \infty} \delta_0(a, f) + \delta_0(\infty, f) \leq 2.$$

Lemma 2 ([5]). *Suppose that f is a meromorphic function of infinite order satisfying $\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2$. Then for each $k \in N$,*

$$T_0(r, f^{(k)}) = ((k+1) - k\delta_0(\infty, f) + o(1))T_0(r, f),$$

as $r \to \infty$ *through all values.*

Using Lemma 2, we can prove

Lemma 3 ([5]). *Suppose that* f *is a meromorphic function of infinite order satisfying* $\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2$. *Then for each* $k \in N$,

$$T(r, f^{(k)}) = ((k+1) - k\delta(\infty, f) + o(1))T(r, f),$$

as $r \to \infty$ *through all values.*

Proof. Because the paper [5] is written in Chinese, we give here a sketch of the proof of Lemma 3. Set $b = (k+1) - k\delta(\infty, f)$, $F_1(x) = T_0(e^x, f^{(k)})$ and $G_1 = b \cdot T_0(e^x, f)$. Then by Lemma 2 and by a similar method to the part (II) of the proof of Theorem 1.3 below, we see that $F_1(x)$ and $G_1(x)$ satisfy the conditions of Lemma 9. Thus we have

$$\lim_{x \to \infty} \frac{F_1'(x)}{G_1'(x)} = 1.$$

Hence we obtain the conclusion of Lemma 3.

Lemma 4 ([6]). *Suppose that* f *is a transcendental meromorphic function, and* a_i $(i = 1, 2, \dots, p)$ *be* p *distinct complex numbers. Then for each* $k \in N$,

$$\sum_{i=1}^{p} m(r, a_i, f) \leq m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f).$$

Lemma 5 ([6]). *Suppose that* f *is a transcendental meromorphic function of infinite order,* $S(r, f)$ *be any quantity satisfying*

$$S(r, f) \leq A \log^+ T(R, f) + B \log^+ \frac{1}{R - r} + C \log^+ R + D,$$

where $0 < r < R$, *and* A, B, C, D *are positive constants. Then* $\lim_{r \to \infty} S_0(r, f)/T_0(r, f) = 0$.

Lemma 6 ([7]). *Suppose that* f *is a transcendental meromorphic function. Then for each positive number* ε_0 *and for each* $k \in N$,

$$\overline{N}(r, f) < \frac{1}{k} N\left(r, \frac{1}{f^{(k)}}\right) + \frac{1}{k} N(r, f) + \varepsilon_0 T(r, f) + S(r, f).$$

Lemma 7. *Suppose that* f *is a meromorphic function of infinite order. If* $\sum_{a \neq \infty} \delta_0(a, f) + \delta_0(\infty, f) = 2$, *then for each positive integer* k,

$$\lim_{r\to\infty}\frac{m_0(r,1/f^{(k)})}{T_0(r,f)}=2-\delta_0(\infty,f)\quad\text{and}\quad\lim_{r\to\infty}\frac{N_0(r,f)}{T_0(r,f)}=\lim_{r\to\infty}\frac{\overline{N}_0(r,f)}{T_0(r,f)}.$$

Proof. Since $\sum_{a\neq\infty}\delta_0(a,f)+\delta_0(\infty,f)=2$, then for any positive number ε_0, there exist distinct complex numbers a_i $(i=1,2,\dots,p)$ such that

$$\sum_{i=1}^{p}\delta_0(a_i,f)+\delta_0(\infty,f)\geq 2-\varepsilon_0. \tag{1}$$

By Lemma 4, we have

$$\sum_{i=1}^{p}m(r,a_i,f)\leq m\left(r,\frac{1}{f^{(k)}}\right)+S(r,f). \tag{2}$$

From (1), (2) and Lemma 5, we have

$$\liminf_{r\to\infty}\frac{m_0(r,1/f^{(k)})}{T_0(r,f)}\geq\sum_{i=1}^{p}\delta_0(a_i,f)\geq 2-\delta_0(\infty,f)-\varepsilon_0.$$

Taking $\varepsilon_0\to 0$, we deduce

$$\liminf_{r\to\infty}\frac{m_0(r,1/f^{(k)})}{T_0(r,f)}\geq 2-\delta_0(\infty,f). \tag{3}$$

On the other hand, by Lemma 6, we have

$$\begin{aligned}m\left(r,\frac{1}{f^{(k)}}\right)&\leq T(r,f^{(k)})-N\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\&\leq T(r,f)+k\overline{N}(r,f)-N\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\&\leq T(r,f)+N(r,f)+k\varepsilon_0T(r,f)+S(r,f).\end{aligned} \tag{4}$$

Thus we have

$$m_0\left(r,\frac{1}{f^{(k)}}\right)\leq T_0(r,f)+N_0(r,f)+k\varepsilon_0T_0(r,f)+S_0(r,f).$$

Hence we obtain

$$\limsup_{r\to\infty}\frac{m_0(r,1/f^{(k)})}{T_0(r,f)}\leq 2-\delta_0(\infty,f)+k\varepsilon_0.$$

Taking $\varepsilon_0\to 0$, we deduce

$$\limsup_{r\to\infty}\frac{m_0(r,1/f^{(k)})}{T_0(r,f)}\leq 2-\delta_0(\infty,f). \tag{5}$$

From (3) and (5), we have

$$\lim_{r\to\infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f)} = 2 - \delta_0(\infty, f).$$

Choosing $k = 1$, from (2) and (4) we have

$$\sum_{i=1}^{p} \delta_0(a_i, f) \le 1 + \liminf_{r\to\infty} \frac{\overline{N}_0(r, f)}{T_0(r, f)} \le 1 + \limsup_{r\to\infty} \frac{N_0(r, f)}{T_0(r, f)} = 2 - \delta_0(\infty, f).$$

From (1), we have

$$2 - \delta_0(\infty, f) - \varepsilon_0 \le \sum_{i=1}^{p} \delta_0(a_i, f).$$

Taking $\varepsilon_0 \to 0$, we obtain

$$\liminf_{r\to\infty} \frac{\overline{N}_0(r, f)}{T_0(r, f)} = \limsup_{r\to\infty} \frac{N_0(r, f)}{T_0(r, f)},$$

hence

$$\lim_{r\to\infty} \frac{\overline{N}_0(r, f)}{T_0(r, f)} = \lim_{r\to\infty} \frac{N_0(r, f)}{T_0(r, f)}.$$

This completes the proof of Lemma 7.

Lemma 8 ([8]). *Suppose that $f(r)$ is a nonnegative and increasing function with $\lim_{r\to\infty} f(r)/r^{\alpha} = \infty$ for each positive number α. We set $F(x) = \int_0^x f(r)\, dr$. Then*

$$\lim_{x\to\infty} \frac{F(x)}{f^2(x)/f'(x)} = 1.$$

Lemma 9 ([9]). *Let $f(x)$ and $g(x)$ satisfy the following four conditions for $x \ge 0$:*

(i) *$f'(x)$ and $g'(x)$ are two continuous functions*
(ii) *$f(x)$ is an increasing convex function*
(iii) *$1/g(x)$ is a convex function*
(iv) $\lim_{x\to\infty} f'(x)/g'(x) = 1$.

Then $\lim_{x\to\infty} f'(x)/g'(x) = 1$.

3 Proof of Theorem 1.3

Since $\sum_{a\ne\infty} \delta(a, f) + \delta(\infty, f) = 2$, by Lemma 1 and Lemma 3, we have $\delta(\infty, f) = \delta_0(\infty, f)$ and

$$T(r, f^{(k)}) = ((k+1) - k\delta(\infty, f) + o(1))T(r, f), \tag{6}$$

as $r \to \infty$ through all values.

From (6), Lemma 3 and Lemma 7, we have

$$\lim_{r\to\infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f)} = 2 - \delta_0(\infty, f), \tag{7}$$

$$\lim_{r\to\infty} \frac{N_0(r, 1/f^{(k)})}{T_0(r, f^{(k)})} = 1 - \lim_{r\to\infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f)} \lim_{r\to\infty} \frac{T_0(r, f)}{T_0(r, f^{(k)})} = A_0, \tag{8}$$

$$\begin{aligned}\lim_{r\to\infty} \frac{N_0(r, f^{(k)})}{T_0(r, f^{(k)})} &= \lim_{r\to\infty} \frac{N_0(r, f) + k\overline{N}_0(r, f)}{T_0(r, f^{(k)})} \\ &= \lim_{r\to\infty} \frac{(k+1)N_0(r, f)}{T_0(r, f)} \lim_{r\to\infty} \frac{T_0(r, f)}{T_0(r, f^{(k)})} = B_0,\end{aligned} \tag{9}$$

where $A_0 = (k-1)(1-\delta(\infty, f))/(1+k-k\delta(\infty, f))$ and $B_0 = (k+1)\cdot(1-\delta(\infty, f))/(1+k-k\delta(\infty, f))$.

(I) We shall first prove that either for any positive number β

$$\lim_{r\to\infty} \frac{N(r, 1/f^{(k)})}{r^\beta} = \infty \quad \text{and} \quad \lim_{r\to\infty} \frac{N(r, f^{(k)})}{r^\beta} = \infty,$$

or the conclusion of Theorem 1.3 holds. Since $\sum_{a\neq\infty} \delta(a, f) + \delta(\infty, f) = 2$, $f(z)$ is of regular growth [6]. Note that $f(z)$ is a meromorphic function of infinite order and regular growth. Then for any positive number β, we have $\lim_{r\to\infty} T(r, f^{(k)})/r^\beta = \infty$. Thus we have

$$\lim_{r\to\infty} \frac{T(r, 1/f^{(k)})}{r^\beta} = \infty. \tag{10}$$

(i) If

$$\liminf_{r\to\infty} \frac{N(r, 1/f^{(k)})}{T(r, f^{(k)})} = B \neq 0,$$

then

$$\liminf_{r\to\infty} \frac{N(r, 1/f^{(k)})}{r^\beta} \geq \liminf_{r\to\infty} \frac{N(r, 1/f^{(k)})}{T(r, f^{(k)})} \liminf_{r\to\infty} \frac{T(r, f^{(k)})}{r^\beta} = \infty.$$

Thus we have $\lim_{r\to\infty} N(r, 1/f^{(k)})/r^\beta = \infty$.

(ii) If

$$\limsup_{r\to\infty} \frac{N(r, 1/f^{(k)})}{T(r, f^{(k)})} = 0,$$

then

$$\lim_{r\to\infty} \frac{N(r, 1/f^{(k)})}{T(r, f^{(k)})} = 0 \quad \text{and} \quad \lim_{r\to\infty} \frac{m(r, 1/f^{(k)})}{T(r, f^{(k)})} = 1.$$

Thus we have

$$\lim_{r\to\infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f^{(k)})} = 1.$$

From (6) and (7), we obtain $\delta(\infty, f) = 1$. Therefore the conclusion of Theorem 1.3 holds.

(iii) Suppose that

$$\limsup_{r\to\infty} \frac{N(r, 1/f^{(k)})}{T(r, f^{(k)})} = B \neq 0 \quad \text{and} \quad \liminf_{r\to\infty} \frac{N(r, 1/f^{(k)})}{T(r, f^{(k)})} = 0.$$

Then we have

$$\limsup_{r\to\infty} \frac{m(r, 1/f^{(k)})}{T(r, f^{(k)})} = 1.$$

Hence there exists an increasing sequence $r_n \to \infty$ such that $\lim_{n\to\infty} m(r_n, 1/f^{(k)})/T(r_n, f^{(k)}) = 1$. By (6), we have

$$\lim_{n\to\infty} \frac{m(r_n, 1/f^{(k)})}{T(r_n, f)} = 1 + k - k\delta(\infty, f). \tag{11}$$

Hence by (7) and (11), we have

$$\begin{aligned} 2 - \delta(\infty, f) &= \lim_{r\to\infty} \frac{m_0(r, 1/f^k)}{T_0(r, f)} = \lim_{n\to\infty} \frac{m_0(r_n, 1/f^k)}{T_0(r_n, f)} \\ &= \lim_{n\to\infty} \frac{m(r_n, 1/f^k)}{T(r_n, f)} = 1 + k - k\delta(\infty, f). \end{aligned}$$

This yields $\delta(\infty, f) = 1$. Hence $\delta_0(\infty, f) = 1$ by Lemma 1. Thus we have

$$\begin{aligned} \delta_0(0, f^{(k)}) &= \liminf_{r\to\infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f^{(k)})} \\ &= \liminf_{r\to\infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f)} \lim_{r\to\infty} \frac{T_0(r, f)}{T_0(r, f^{(k)})} = \frac{2 - \delta_0(\infty, f)}{(k+1) - k\delta(\infty, f)} = 1 \end{aligned}$$

by $\delta(\infty, f) = \delta_0(\infty, f) = 1$. Then

$$1 = \delta_0(0, f^{(k)}) = \liminf_{r\to\infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f^{(k)})} \leq \limsup_{r\to\infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f^{(k)})} \leq 1.$$

Thus $\lim_{r\to\infty} m_0(r, 1/f^{(k)})/T_0(r, f^{(k)})$ exists. Therefore by using l'Hospital's rule, we obtain

$$\delta_0(0, f^{(k)}) = \lim_{r\to\infty} \frac{m_0(r, 1/f^{(k)})}{T_0(r, f^{(k)})} = \lim_{r\to\infty} \frac{m_0'(r, 1/f^{(k)})}{T_0'(r, f^{(k)})} = \lim_{r\to\infty} \frac{m(r, 1/f^{(k)})}{T(r, f^{(k)})} = \delta(0, f^{(k)}),$$

that is, $\delta(0, f^{(k)}) = 1$. Hence $\lim_{r\to\infty} N(r, 1/f^{(k)})/T(r, f^{(k)}) = 0$. This is a contradiction. Therefore we deduce that $\lim_{r\to\infty} N(r, 1/f^{(k)})/r^{\beta} = \infty$ or the conclusion of Theorem 1.3 holds in this case.

Similarly, we have $\lim_{r\to\infty} N(r, f^{(k)})/r^{\beta} = \infty$ or the conclusion of Theorem 1.3 holds.

(II) Let $F(x) = N_0(e^x, 1/f^{(k)})$ and $G(x) = A_0 T_0(e^x, f^{(k)})$. Then we have

$$F(x) = \int_1^{e^x} \frac{N(t, 1/f^{(k)})}{t}\, dt = \int_0^x N\left(e^r, \frac{1}{f^{(k)}}\right) dr, \tag{12}$$

$$G(x) = A_0 \int_1^{e^x} \frac{T(t, f^{(k)})}{t}\, dt = A_0 \int_0^x T(e^r, f^{(k)})\, dr. \tag{13}$$

Since $N(e^x, 1/f^{(k)})$ is increasing, from the result of (I) and Lemma 8, we get

$$\lim_{x\to\infty} F(x) \Big/ \left(\frac{N(e^x, 1/f^{(k)})^2}{N'(e^x, 1/f^{(k)}) e^x} \right) = 1. \tag{14}$$

Now we shall show that $F(x)$ and $G(x)$ satisfy the conditions of Lemma 9.

(i) From (12) and (13), we get $F'(x) = N(e^x, 1/f^{(k)})$, $G'(x) = A_0 T(e^x, f^{(k)})$. Obviously $F'(x)$ and $G'(x)$ are continuous fuctions.

(ii) Since $T(r, f^{(k)})$ is a convex function of $\log r$, $G'(x) > 0$. Thus $G(x)$ is an increasing convex function.

(iii) Since $F(x) > 0$ and

$$\frac{d^2}{dx^2}\left(\frac{1}{F(x)}\right) = \frac{1}{F^3(x)} \left\{ 2N^2\left(e^x, \frac{1}{f^{(k)}}\right) - F(x) N'\left(e^x, \frac{1}{f^{(k)}}\right) e^x \right\}.$$

From (14), if x is sufficiently large, we have

$$\frac{d^2}{dx^2}\left(\frac{1}{F(x)}\right) > 0.$$

Thus $F(x)$ is a convex function. From the result of (II) and Lemma 9, we obtain $\lim_{r\to\infty} N(r, 1/f^{(k)})/T(r, f^{(k)}) = A_0$. Similarly, we have $\lim_{r\to\infty} N(r, f^{(k)})/T(r, f^{(k)}) = B_0$. Thus we obtain

$$K(f^{(k)}) = \lim_{r\to\infty} \frac{N(r, 1/f^{(k)}) + N(r, f^{(k)})}{T(r, f^{(k)})} = A_0 + B_0 = \frac{2k(1 - \delta(\infty, f))}{1 + k - k\delta(\infty, f)}.$$

The proof of Theorem 1.3 is now complete.

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