G. NAKAMURA
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GENERIC FUNDAMENTAL POLYGONS FOR SURFACES OF GENUS THREE

Gou Nakamura

Abstract

In the study of extremal surfaces it is important to see the side-pairings of the generic fundamental polygons. For the regular 30-gon in the hyperbolic plane it is known that there are 1726 side-pairing patterns to make a compact surface of genus three. If we consider the mirror images of these patterns, then there are essentially 927 patterns. In the present paper we give the 927 side-pairing patterns completely.

1. Introduction

Let S be a compact Riemann surface of genus $g \geq 2$. Let $D(r)$ be a disk of radius r isometrically embedded in S. A surface S is said to be extremal if it admits an extremal disk $D(R_g)$, where R_g is determined by g as follows ([2]):

$$\cosh R_g = \frac{1}{2 \sin(\pi/(12g-6))}.$$

As a fundamental polygon of an extremal surface of genus g, we can take the regular and generic one, that is, the regular $(12g-6)$-gon with angles $2\pi/3$ in the hyperbolic plane ([2]). Then three vertices of the polygon are identified with a point in S. A typical extremal disk embedded in an extremal surface is the projection of the inscribed disk in the regular $(12g-6)$-gon ([2]). Then our concerns are the numbers of extremal disks that extremal surfaces can admit and the locations that extremal disks are embedded in. If $g \geq 4$, then extremal surfaces admit a unique extremal disk ([3]). If $g = 2$, then there are essentially 8 extremal surfaces and the locations of the extremal disks are precisely found out ([4], [7]). Thus our problem is to sudy the extremal surfaces of genus three. For each extremal surface of $g = 2$ we have the regular 18-gon as a fundamental polygon.

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The side-pairing patterns of the regular 18-gon to make surfaces of genus two play an important part in finding the locations of the extremal disks. For $g = 3$, therefore, it is important to obtain the possible side-pairing patterns of the regular 30-gon to make surfaces of genus three. It is known that there are 1726 side-pairing patterns for the regular 30-gon ([6], [1]) and that there are essentially 927 patterns if we consider mirror images. Hence, in the present paper we shall give all of the 927 side-pairing patterns for the study of extremal surfaces of $g = 3$. The methods we use here are similar to those of [5].

2. Trivalent graphs with 10 vertices and 15 edges

If we identify each pair of edges of the regular 30-gon to make a compact surface S of genus three, then two edges of the polygon become one curve in S and three vertices of the polygon become one point in S. Hence we have a figure in S with 10 points connected by 15 curves such that each point is the end point of the three curves. We consider it a trivalent graph with 10 vertices and 15 edges (see Figure 1).

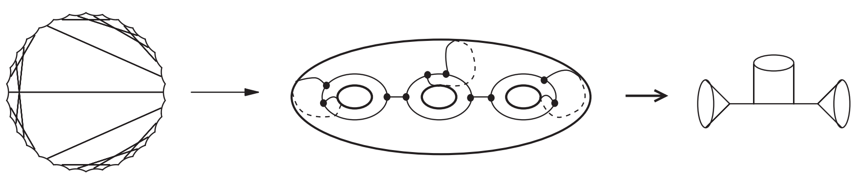


FIGURE 1. An example of a graph embedded in the surface of genus three

PROPOSITION. *Let $\mathcal{G}$ be the set of connected trivalent graphs G with* 10 *vertices and* 15 *edges such that G can be embedded in a compact surface of genus three and that $S \backslash G$ is simply connected. Then $\mathcal{G}$ consists of the following* 65 *graphs* (*see Figure* 2).

Proof. In order to obtain the graphs, we shall number the vertices from 1 to 10 and connect the 10 vertices by 15 edges such that each vertex has three edges. First, we have two cases such that (A) the vertex 1 is connected with three different vertices 2, 3, and 4 or (B) the vertex 1 is connected with two different vertices 2 and 3 (we may suppose that 1 and 2 are connected by two different edges then) (see Figure 3).

Next, we have three cases from (A) to connect the vertex 2 with three different vertices as follows: 2 is connected with 3 and 4, 2 is connected with 3 and 5, or 2 is connected with 5 and 6 (see Figure 4). In (A) it is sufficient to consider the graphs of which distinct vertices are not directly connected by two edges. In (B) we have two cases such that 2 is connected with either 3 or 4 (see Figure 5).

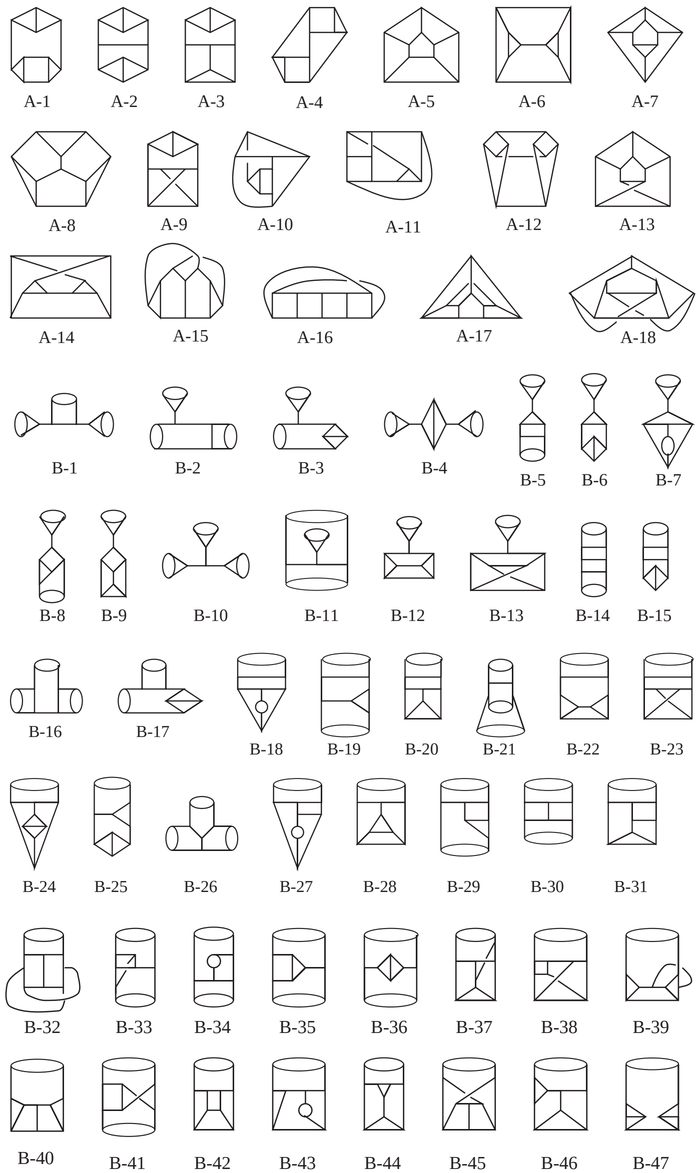


FIGURE 2. Graphs in $\mathscr{G}$

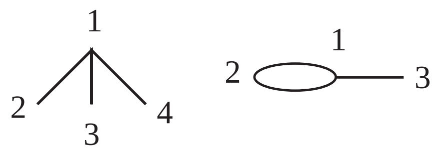


FIGURE 3. (A) and (B)

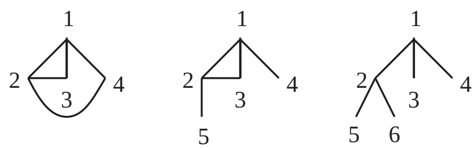


FIGURE 4. Three cases from (A)

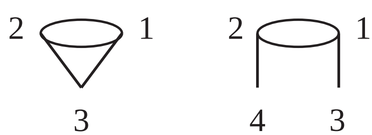


FIGURE 5. Two cases from (B)

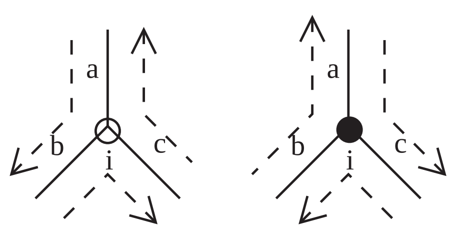


FIGURE 6. An empty circle and a shaded circle

If we repeat this process for the other vertices $3, 4, \ldots, 10$, then we have connected trivalent graphs with 10 vertices and with 15 edges. In order to examine whether each of these graphs G is embeddable in S, with connected complement, we shall consider whether there exist closed walks on G on condition that we walk on each edge of G exactly once in either direction and that we do not go back immediately on the same edge to which we just came ([5]

p. 452). For this purpose we shall assign each vertex an empty or a shaded circle called rotation ([5] p. 453 or [8] p. 16). That is, if we walk on the edge a toward the vertex i, then we have two different ways b or c (see Figure 6). If the empty (or shaded) circle is assigned to i, then we walk on the edge b (or c). Once a rotation is assigned to a vertex, then the ways through that vertex are determined (For example, suppose that an empty circle is assigned to the vertex i. After we walk to i along the edge b (or c), we walk on the edge c (or a) from i).

Assigning the rotations to the 10 vertices of each graph by computer, we see that the graphs from which our required closed walks arise are those in Figure 2. □

Remark. For example, our required closed walk does not arise from the following trivalent graphs (see Figure 7).

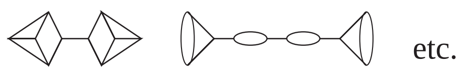


FIGURE 7. Examples of the graphs which are not in $\mathscr{G}$

3. Side-pairing patterns for the regular 30-gon

For each $G \in \mathscr{G}$, we obtain the side-pairing patterns of the regular 30-gon by the closed walks on G.

To depict the side-pairing patterns simply, we shall use a dot (•) in place of an edge of the regular 30-gon (see Figure 8).

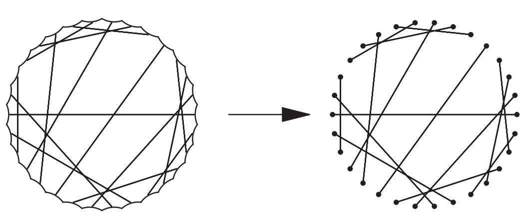
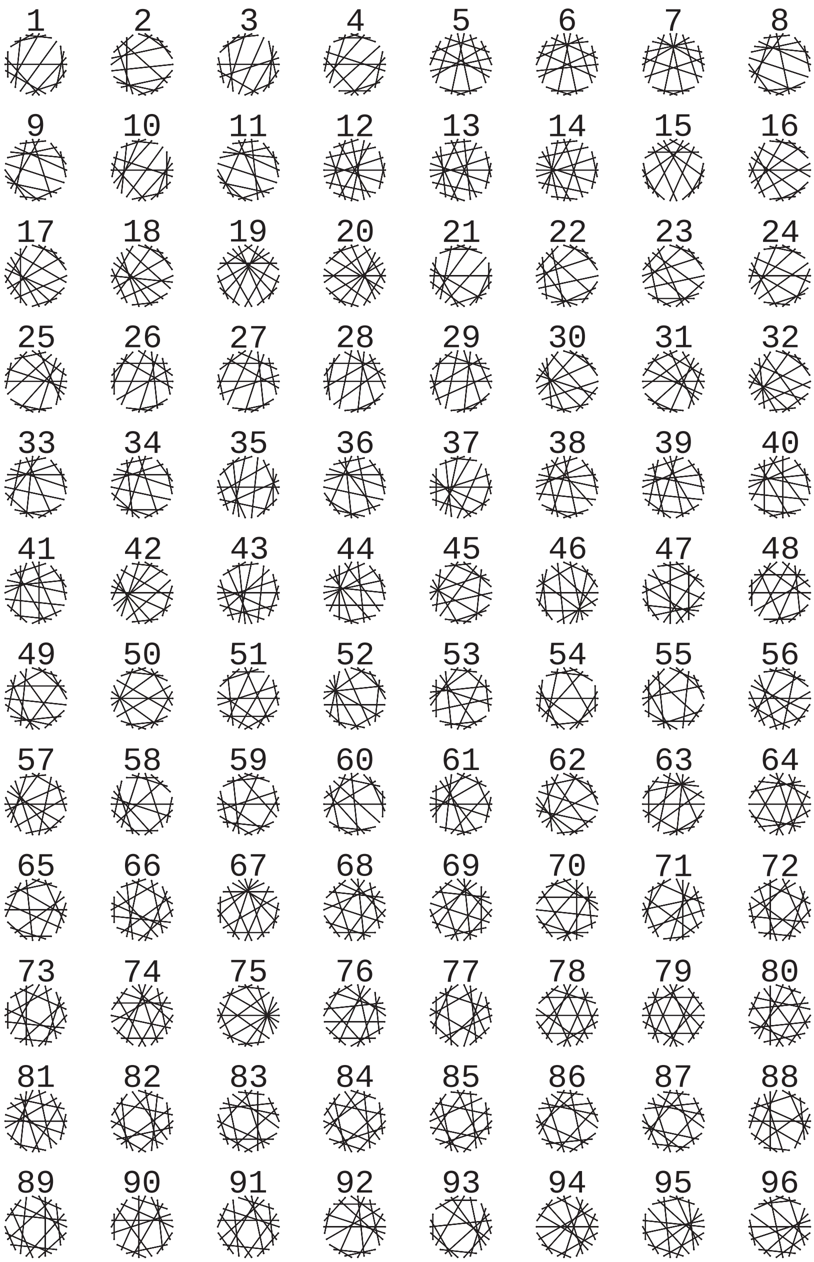


FIGURE 8. Dots denote edges.

THEOREM. *The* 927 *side-pairing patterns of the regular* 30-*gon up to mirror images are the following*: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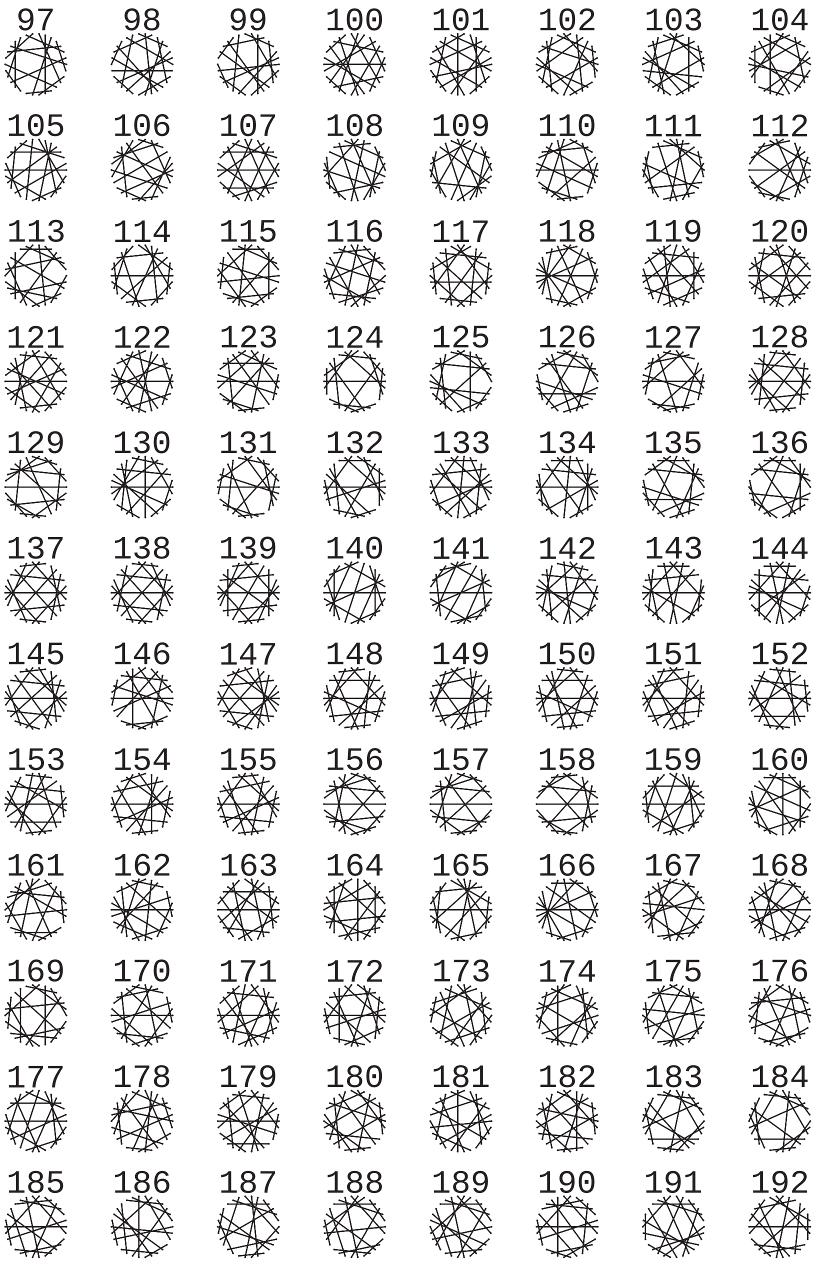
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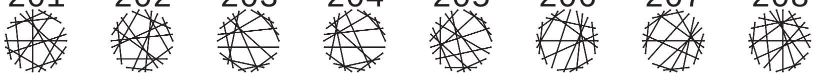

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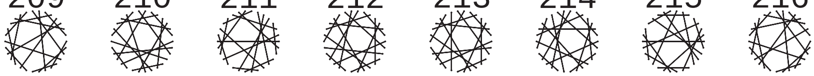

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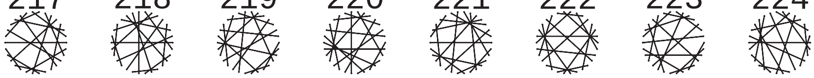

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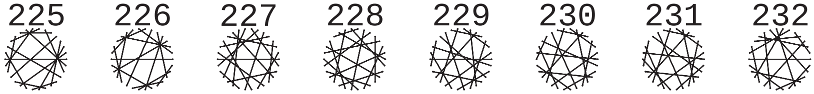

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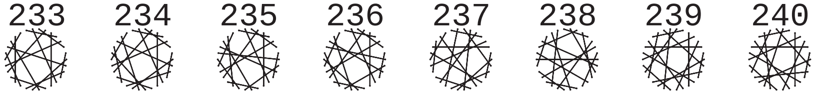

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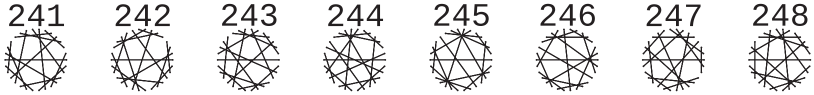

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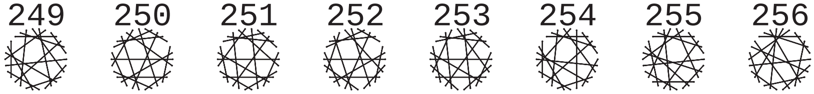

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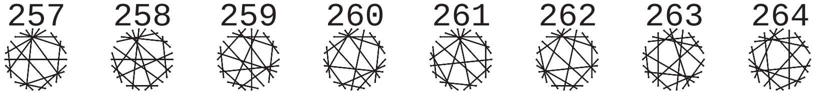

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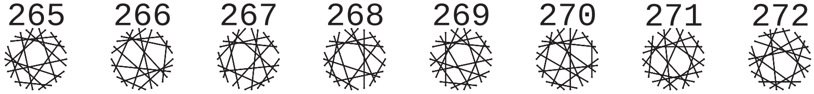

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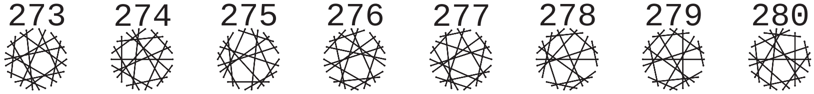

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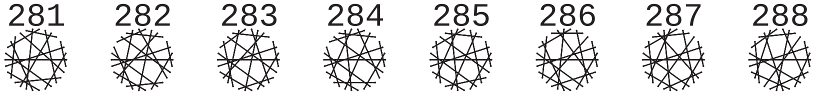

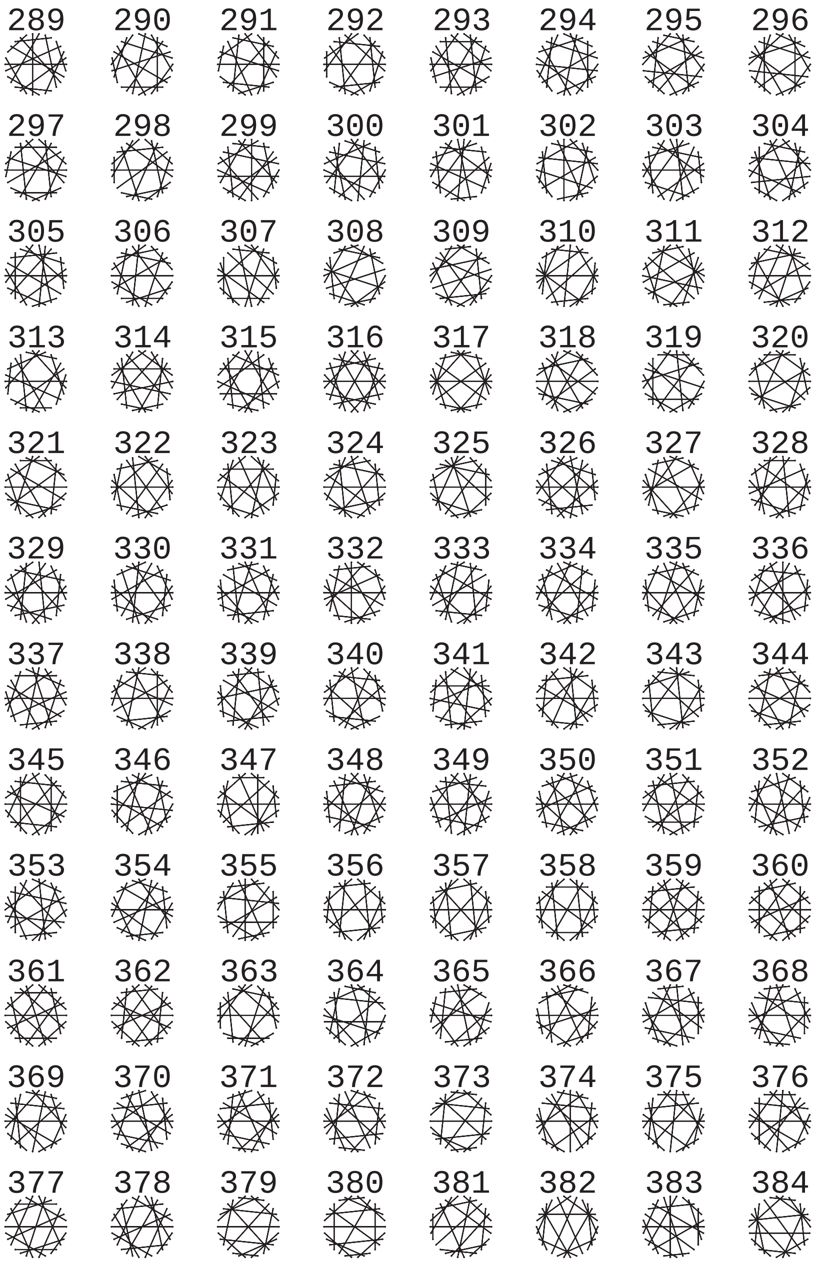
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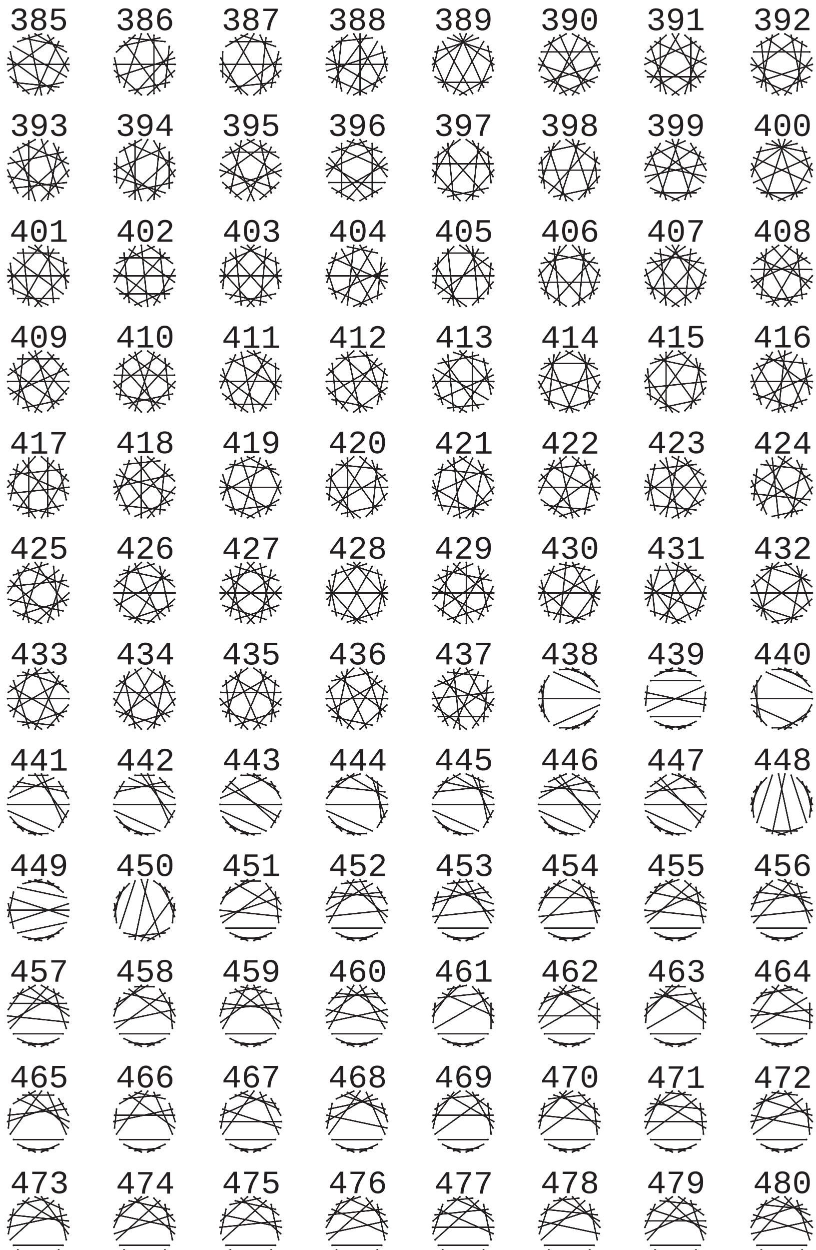
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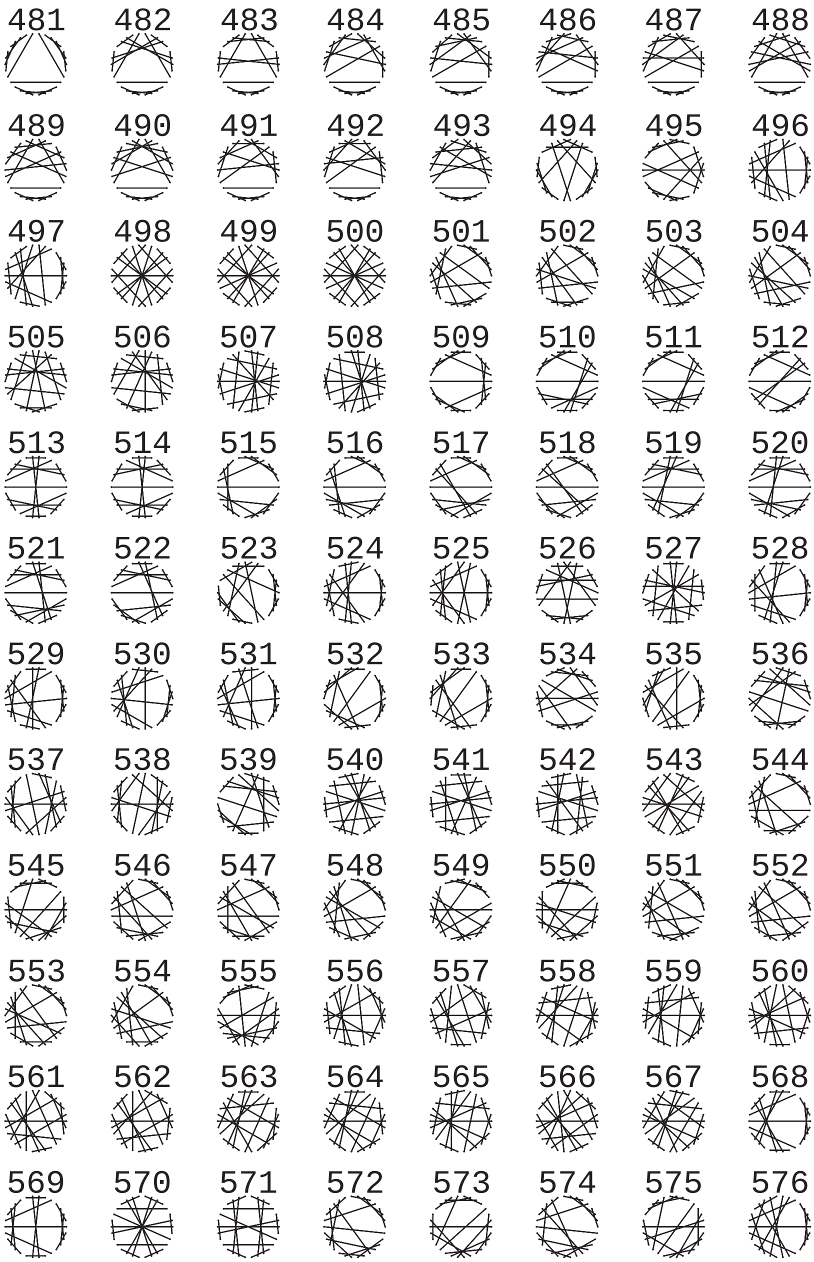
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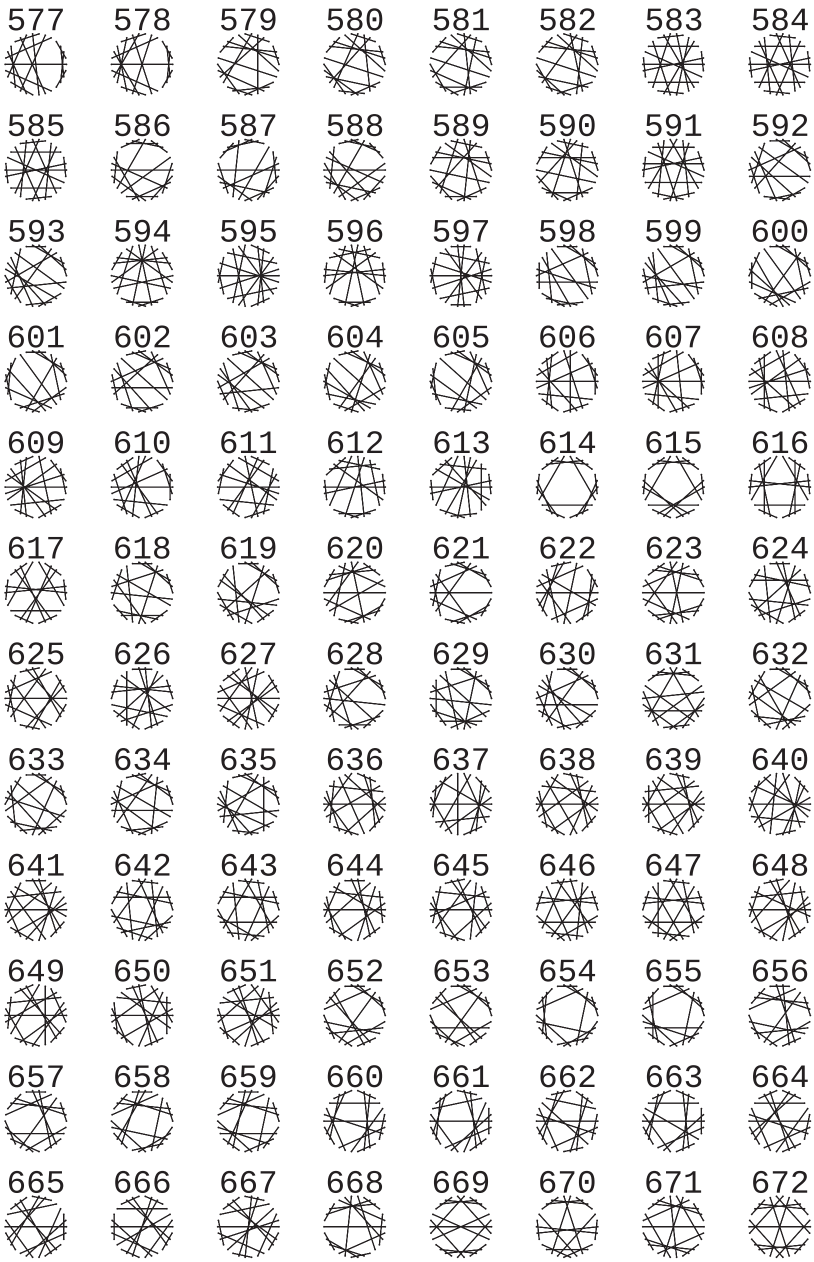
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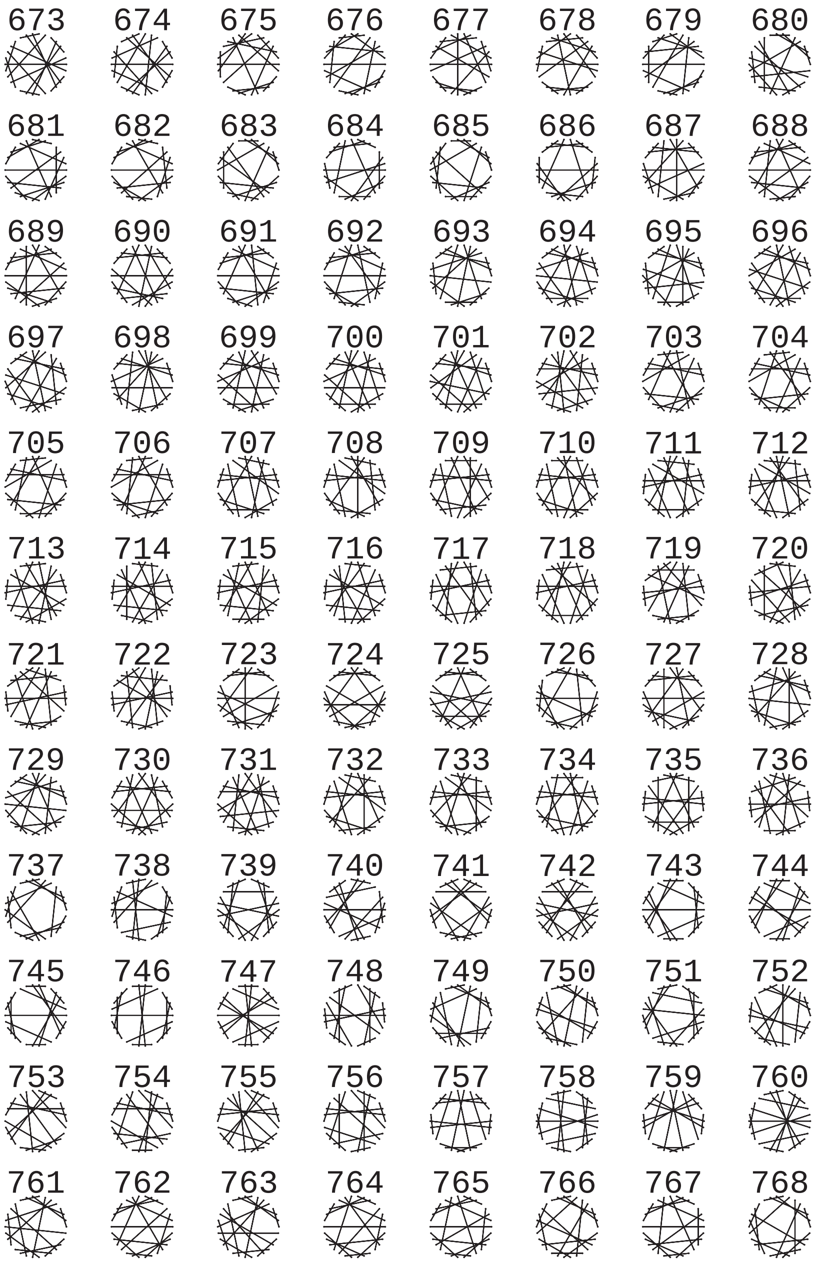
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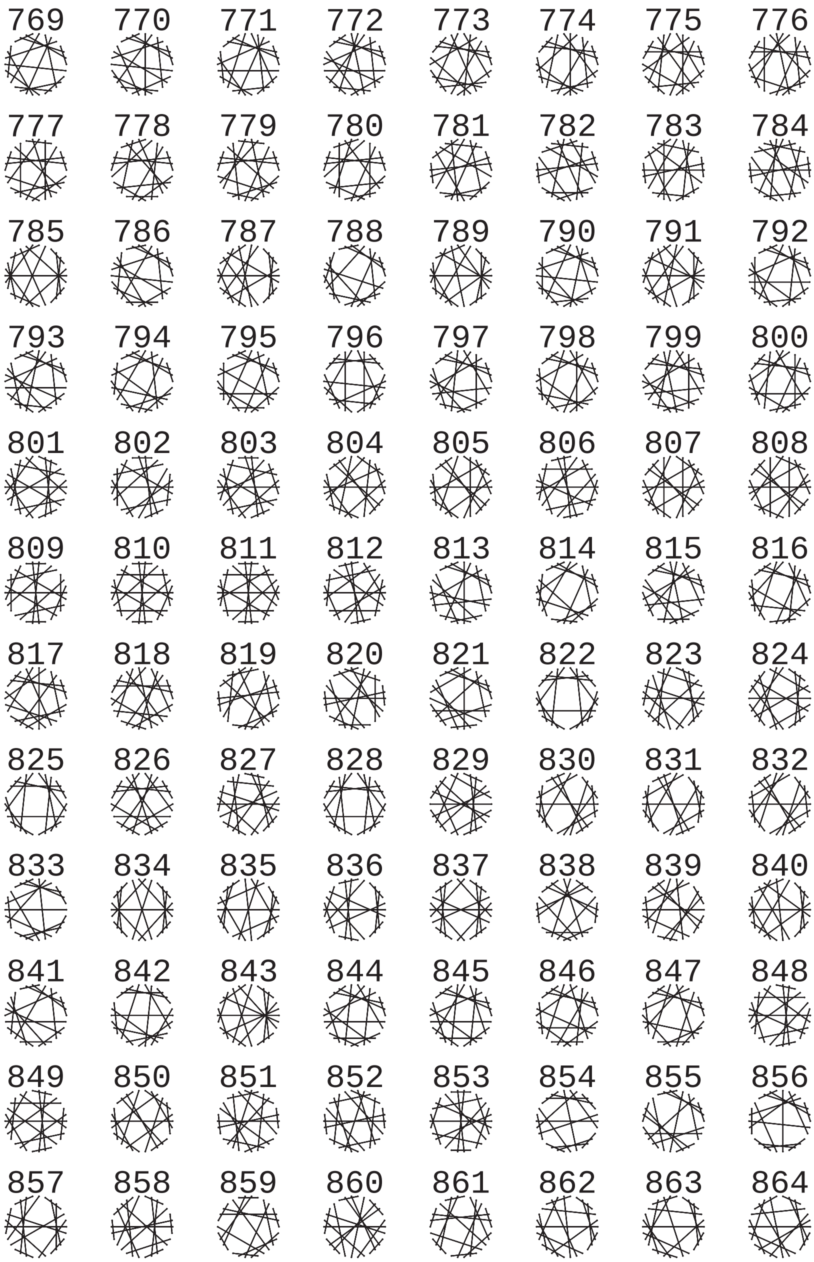
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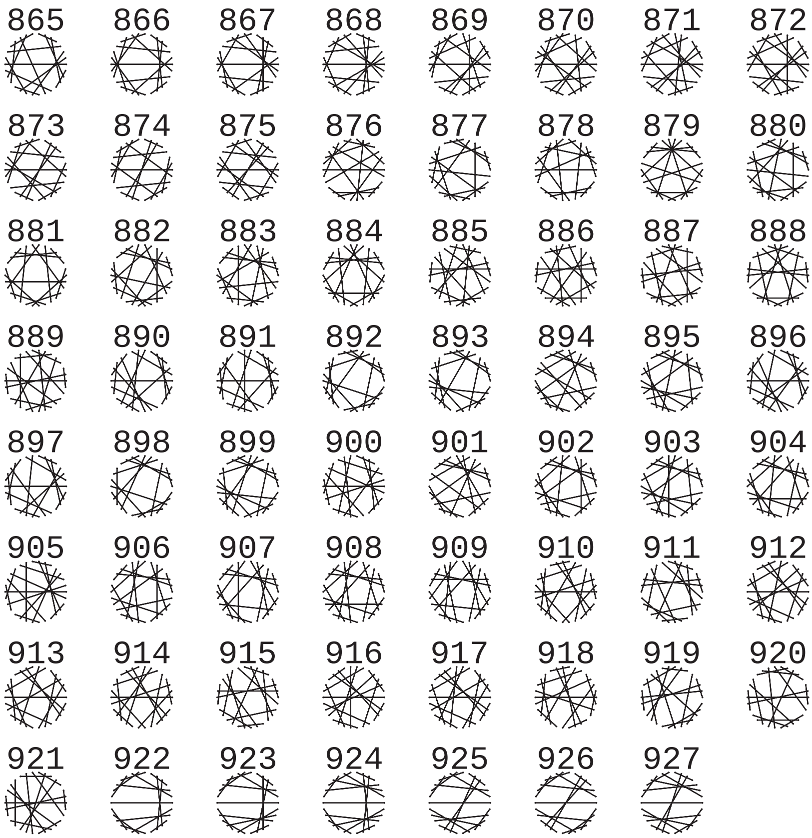

We shall show a table with respect to the side-pairing patterns arising from $G \in \mathscr{G}$. In the table below $P(G)$ denotes the set of all side-pairing patterns arising from G, $P^+(G)$ denotes the subset of $P(G)$ up to mirror images. The numbers in the column of $P^+(G)$ show the ones labeled in the figures of the side-pairing patterns in our theorem. Note that $\Sigma_{G \in \mathscr{G}} \# P(G) = 1726$ and $\Sigma_{G \in \mathscr{G}} \# P^+(G) = 927$.

G	$\#P^+(G)$	$\#P(G)$	$P^+(G)$	G	$\#P^+(G)$	$\#P(G)$	$P^+(G)$
A-1	14	24	1–14	A-6	36	68	123–158
A-2	6	8	15–20	A-7	24	48	159–182
A-3	24	48	21–44	A-8	20	40	183–202
A-4	68	128	45–112	A-9	6	12	203–208
A-5	10	16	113–122	A-10	32	64	209–240

A-11	72	144	241–312	B-21	4	6	568–571
A-12	6	8	313–318	B-22	14	26	572–585
A-13	44	88	319–362	B-23	6	12	586–591
A-14	18	36	363–380	B-24	6	8	592–597
A-15	16	24	381–396	B-25	16	32	598–613
A-16	14	20	397–410	B-26	4	4	614–617
A-17	25	42	411–435	B-27	10	16	618–627
A-18	2	4	436–437	B-28	24	48	628–651
B-1	2	3	438–439	B-29	16	32	652–667
B-2	4	8	440–443	B-30	7	10	668–674
B-3	4	8	444–447	B-31	48	96	675–722
B-4	2	2	448–449	B-32	14	24	723–736
B-5	4	8	450–453	B-33	6	8	737–742
B-6	4	8	454–457	B-34	6	10	743–748
B-7	3	4	458–460	B-35	8	16	749–756
B-8	8	16	461–468	B-36	4	4	757–760
B-9	12	24	469–480	B-37	24	48	761–784
B-10	1	1	481	B-38	28	52	785–812
B-11	2	2	482–483	B-39	8	16	813–820
B-12	7	12	484–490	B-40	12	18	821–832
B-13	3	6	491–493	B-41	7	10	833–839
B-14	7	10	494–500	B-42	14	26	840–853
B-15	8	16	501–508	B-43	8	16	854–861
B-16	6	10	509–514	B-44	14	26	862–875
B-17	8	16	515–522	B-45	14	24	876–889
B-18	5	8	523–527	B-46	32	64	890–921
B-19	16	32	528–543	B-47	6	10	922–927
B-20	24	48	544–567	Total	927	1726	

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Science Division
Center for General Education
Aichi Institute of Technology
Yakusa-cho, Toyota 470-0392
Japan
e-mail: gou@aitech.ac.jp