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THE GRADIENT OF A POLYNOMIAL AT INFINITY

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Abstract

We give a description of growth at infinity of the gradient of a polynomial in two complex variables near any of its fiber.

1. Introduction

Let $f : \boldsymbol{C}^n \to \boldsymbol{C}$ be a non-constant polynomial and let $\nabla f : \boldsymbol{C}^n \to \boldsymbol{C}^n$ be its gradient. There exists a finite set $B(f) \subset \boldsymbol{C}$ such that f is a locally trivial C^∞-bundle over $\boldsymbol{C} \backslash B(f)$ ([Ph], Appendix A1, [V], Corollary 5.1). The set $B(f)$ is the union of the set of critical values $C(f)$ of f and critical values $\Lambda(f)$ corresponding to the singularities of f at infinity. The set $\Lambda(f)$ is defined to be the set of all $\lambda \in \boldsymbol{C}$ for which there are no neighbourhood U of λ and a compact set $K \subset \boldsymbol{C}^n$ such that $f : f^{-1}(U) \backslash K \to U$ is a trivial C^∞-bundle. It turns out that for $\lambda \in \boldsymbol{C}$ the property of being in $\Lambda(f)$ depends on the behaviour of the gradient ∇f near the fiber $f^{-1}(\lambda)$.

Ha in [H2] defined the notion of the Łojasiewicz exponent $\tilde{\mathscr{L}}_{\infty, \lambda_0}(f)$ of the gradient ∇f at infinity near a fibre $f^{-1}(\lambda_0)$ in the following way

$$\tilde{\mathscr{L}}_{\infty, \lambda_0}(f) := \lim_{\delta \to 0^+} \mathscr{L}_\infty(\nabla f \mid f^{-1}(D_\delta)), \tag{1.1}$$

where $D_\delta := \{\lambda \in \boldsymbol{C} : |\lambda - \lambda_0| < \delta\}$ and $\mathscr{L}_\infty(\nabla f \mid f^{-1}(D_\delta))$ is the Łojasiewicz exponent at infinity of the mapping ∇f on the set $f^{-1}(D_\delta)$ (see the definition in Section 3) and gave, without proof, a characterization of $\Lambda(f)$ for $n = 2$ in terms of $\tilde{\mathscr{L}}_{\infty, \lambda_0}(f)$. Namely, $\lambda_0 \in \Lambda(f)$ if and only if $\tilde{\mathscr{L}}_{\infty, \lambda_0}(f) < 0$ (or equivalently $\tilde{\mathscr{L}}_{\infty, \lambda_0}(f) < -1$). A generalization of this result was given by Parusiński in [P]. Moreover, Ha also gave a formula for $\tilde{\mathscr{L}}_{\infty, \lambda_0}(f)$ in terms of Puiseux expansions of roots of the polynomial $f - \lambda_0$ at infinity for $\lambda_0 \in \Lambda(f)$ (this formula is analogous to the formula for the local Łojasiewicz exponent of the gradient ∇f, given in [KL]).

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The aim of this paper is to give in the case $n = 2$ a complete description of the behaviour of the gradient ∇f near any fibre $f^{-1}(\lambda)$ for $\lambda \in \boldsymbol{C}$. To achieve this we define a more convenient Łojasiewicz exponent at infinity of ∇f near a fibre $f^{-1}(\lambda)$ (equivalent to the above one, see Section 5 for $n = 2$ and [Sk] for arbitrary n) as the infimum of the Łojasiewicz exponents at infinity of ∇f on meromorphic curves "approximating" $f^{-1}(\lambda)$ at infinity. Precisely, for a non-constant polynomial $f : \boldsymbol{C}^n \to \boldsymbol{C}$ and $\lambda \in \boldsymbol{C}$ we define $\mathscr{L}_{\infty,\lambda}(f)$ by

$$\mathscr{L}_{\infty,\lambda}(f) := \inf_{\Phi} \frac{\deg \nabla f \circ \Phi}{\deg \Phi}, \tag{1.2}$$

where $\Phi = (\varphi_1, \dots, \varphi_n)$ is a meromorphic mapping at infinity (i.e. each φ_i is a meromorphic function defined in a neighbourhood of ∞ in $\overline{\boldsymbol{C}}$) such that $\deg \Phi := \max(\deg \varphi_1, \dots, \deg \varphi_n) > 0$ and $\deg (f - \lambda) \circ \Phi < 0$, where $\deg \varphi$ for φ meromorphic at infinity is defined as follows: if $\varphi(t) = \sum_{n=k}^{-\infty} a_k t^k$, $a_k \neq 0$, is the Laurent series of φ in a neighbourhood of ∞ then $\deg \varphi := k$; if $\varphi \equiv 0$ then $\deg \varphi := -\infty$. We shall also call such mappings meromorphic curves.

The main results of the paper are effective formulas for $\mathscr{L}_{\infty,\lambda}(f)$ for each $\lambda \in \boldsymbol{C}$ and properties of the function $\lambda \mapsto \mathscr{L}_{\infty,\lambda}(f)$ for $n = 2$. To describe them we outline the contents of the sections.

Section 2 has an auxiliary character and contains technical results on relations between roots of a polynomial and its derivatives.

In Section 3 we investigate $\mathscr{L}_{\infty,\lambda}(f)$ for $\lambda \in \Lambda(f)$. In particular we obtain the all results of Ha with complete proofs.

The main theorems are given in Section 4. They are Theorems 4.1, 4.5 and 4.6 which give effective formulas for $\mathscr{L}_{\infty,\lambda}(f)$ for each $\lambda \in \boldsymbol{C}$ in terms of the resultant $\operatorname{Res}_y(f(x,y) - \lambda, f_y'(x,y) - u)$, where λ, u are new variables, (x, y) is a generic system of coordinates in $\boldsymbol{C}^2$ and f_y' is the partial derivative of f with respect to y. As a consequence we obtain (Corollary 4.7) a basic property of the function $\lambda \mapsto \mathscr{L}_{\infty,\lambda}(f)$. Namely,

$$\mathscr{L}_{\infty,\lambda}(f) = \mathit{const.} \geq 0 \quad \text{for } \lambda \notin \Lambda(f),$$
$$\mathscr{L}_{\infty,\lambda}(f) \in [-\infty, -1) \quad \text{for } \lambda \in \Lambda(f).$$

The key role in the proof of Theorems 4.5 and 4.6 plays Proposition 4.4 which says that the function $\boldsymbol{C} \backslash \Lambda(f) \ni \lambda \mapsto \mathscr{L}_{\infty}(\nabla f \mid f^{-1}(\lambda))$ is constant.

In Section 5 we shall give a short proof of the equality

$$\tilde{\mathscr{L}}_{\infty,\lambda}(f) = \mathscr{L}_{\infty,\lambda}(f) \quad \text{for } \lambda \in \boldsymbol{C} \tag{1.3}$$

for $n = 2$. Recently Skalski in [Sk] proved (1.3) in n-dimensional case. His proof is based on an appropriate choice of a semi-algebraic set and the Curve Selection Lemma.

In Section 6 characterizations (in terms of the exponents $\mathscr{L}_{\infty,\lambda}(f)$ and $\tilde{\mathscr{L}}_{\infty,\lambda}(f)$) of sets for which the Malgrange and Fedorjuk conditions for f do not hold in n-dimensional case is given.

In the end of Introduction we explain some technical assumptions occured in Sections 2–5. Since one can easily show that the exponent $\mathscr{L}_{\infty,\lambda}(f)$ does not depend on linear change of coordinates in $\boldsymbol{C}^n$ we shall assume in Sections 2–5 that the polynomial $f \in \boldsymbol{C}[x, y]$ is monic with respect to y and $\deg f = \deg_y f$. Then we have a simple characterization of the set $\Lambda(f)$, which will be used in the paper. Namely, in [H1] and [K1] there was proved that

$$\Lambda(f) = \{\lambda \in \boldsymbol{C} : c_0(\lambda) = 0\}, \tag{1.4}$$

where the polynomial $c_0(\lambda)x^N + \cdots + c_N(\lambda)$, $c_0 \neq 0$, is the resultant of the polynomials $f(x, y) - \lambda$, $f'_y(x, y)$ with respect to the variable y.

2. Auxiliary results

Let f be a non-constant polynomial in two complex variables of the form

$$f(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x), \quad \deg a_i \leq i, \quad i = 1, \ldots, n. \tag{2.1}$$

It can be easily showed (see [CK1]).

Lemma 2.1. *If $n > 1$, then for every $\lambda_0 \in \boldsymbol{C}$ there exist $D \in \boldsymbol{N}$ and functions $\beta_1, \ldots, \beta_n$, $\gamma_1, \ldots, \gamma_{n-1}$, meromorphic at infinity, such that*

(a) $\deg \beta_i \leq D$, $\deg \gamma_j \leq D$,

(b) $f(t^D, y) - \lambda_0 = \prod_{i=1}^{n}(y - \beta_i(t))$,

(c) $f'_y(t^D, y) = n\prod_{j=1}^{n-1}(y - \gamma_j(t))$. □

We shall now give a lemma which directly follows from the property B.3 in [GP]. Local version of this lemma was proved in [KL].

Lemma 2.2. *Under notation and assumptions of Lemma 2.1 for every $i, j \in \{1, \ldots, n\}$, $i \neq j$, there exists $k \in \{1, \ldots, n-1\}$ such that*

$$\deg(\beta_i - \beta_j) = \deg(\beta_i - \gamma_k) \tag{2.2}$$

and conversely for every $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, n-1\}$ there exists $j \in \{1, \ldots, n\}$ such that (2.2) holds. □

Now we prove a proposition useful in the sequel. A local version of it is given in [P1] and [R1]. We put $\Psi_l(t) := (t^D, \gamma_l(t))$, $l \in \{1, \ldots, n-1\}$.

Proposition 2.3. *Under notations and assumptions of Lemma 2.1 we have*

$$\min_{i=1}^{n}\left(\sum_{j=1, j\neq i}^{n} \deg(\beta_i - \beta_j) + \min_{j=1, j\neq i}^{n} \deg(\beta_i - \beta_j)\right) = \min_{l=1}^{n-1}(\deg(f - \lambda_0) \circ \Psi_l). \tag{2.3}$$

Proof (after [R1]). There exists $i_0 \in \{1, \ldots, n\}$ such that the left hand side in (2.3) is equal to

$$\sum_{j=1, j\neq i_0}^{n} \deg(\beta_{i_0} - \beta_j) + \min_{j=1, j\neq i_0}^{n} \deg(\beta_{i_0} - \beta_j)$$

and $j_0 \in \{1, \dots, n\}$ such that

$$\min_{j=1, j\neq i_0}^{n} \deg(\beta_{i_0} - \beta_j) = \deg(\beta_{i_0} - \beta_{j_0}). \tag{2.4}$$

By Lemma 2.2 there exists $k_0 \in \{1, \dots, n-1\}$ such that

$$\deg(\beta_{i_0} - \beta_{j_0}) = \deg(\beta_{i_0} - \gamma_{k_0}). \tag{2.5}$$

We shall lead the further part of the proof in four steps.

A. We first show that for each $j \in \{1, \dots, n\}$ we have

$$\deg(\gamma_{k_0} - \beta_j) \geq \deg(\beta_{i_0} - \beta_{j_0}). \tag{2.6}$$

Take any $j \in \{1, \dots, n\}$ and consider two cases:

(a) $\deg(\beta_{i_0} - \beta_{j_0}) \leq \min_{s=1, s\neq j}^{n} \deg(\beta_s - \beta_j)$,

(b) $\deg(\beta_{i_0} - \beta_{j_0}) > \min_{s=1, s\neq j}^{n} \deg(\beta_s - \beta_j)$.

In case (a) by Lemma 2.2 there exists $p \in \{1, \dots, n\}$ such that

$$\deg(\gamma_{k_0} - \beta_j) = \deg(\beta_p - \beta_j) \geq \min_{s=1, s\neq j}^{n} \deg(\beta_s - \beta_j) \geq \deg(\beta_{i_0} - \beta_{j_0}),$$

which gives (2.6).

In case (b) by definition of i_0 and (2.4) we have

$$\begin{aligned}\sum_{s=1, s\neq i_0}^{n} &\deg(\beta_s - \beta_{i_0}) + \deg(\beta_{j_0} - \beta_{i_0}) \\ &\leq \sum_{s=1, s\neq j}^{n} \deg(\beta_s - \beta_j) + \min_{s=1, s\neq j}^{n} \deg(\beta_s - \beta_j).\end{aligned}$$

Hence and from (b) we get

$$\sum_{s=1, s\neq i_0}^{n} \deg(\beta_s - \beta_{i_0}) < \sum_{s=1, s\neq j}^{n} \deg(\beta_s - \beta_j).$$

Then there exists $s \neq i_0$, $s \neq j$ such that

$$\deg(\beta_s - \beta_{i_0}) < \deg(\beta_s - \beta_j).$$

Hence and from (2.5) we get

$$\begin{aligned}\deg(\beta_j - \beta_{i_0}) &= \deg(\beta_j - \beta_s + \beta_s - \beta_{i_0}) > \deg(\beta_s - \beta_{i_0}) \\ &\geq \deg(\beta_{j_0} - \beta_{i_0}) = \deg(\gamma_{k_0} - \beta_{i_0}).\end{aligned}$$

In consequence

$$\deg(\gamma_{k_0} - \beta_j) = \deg(\gamma_{k_0} - \beta_{i_0} + \beta_{i_0} - \beta_j) > \deg(\gamma_{k_0} - \beta_{i_0}) = \deg(\beta_{j_0} - \beta_{i_0}).$$

This gives (2.6) in case (b).

B. We shall now show that for each $j \in \{1, \ldots, n\}$, $j \neq i_0$, we have

$$\deg(\gamma_{k_0} - \beta_j) = \deg(\beta_{i_0} - \beta_j). \tag{2.7}$$

Take $j \in \{1, \ldots, n\}$, $j \neq i_0$, and consider two cases:

(a) $\deg(\gamma_{k_0} - \beta_j) > \deg(\gamma_{k_0} - \beta_{i_0})$,

(b) $\deg(\gamma_{k_0} - \beta_j) = \deg(\gamma_{k_0} - \beta_{i_0})$.

By (2.5) and (2.6) there are no more cases. In case (a) we have

$$\deg(\beta_{i_0} - \beta_j) = \deg(\beta_{i_0} - \gamma_{k_0} + \gamma_{k_0} - \beta_j) = \deg(\gamma_{k_0} - \beta_j),$$

which gives (2.7).

In case (b) by (2.4) and (2.5) we have

$$\begin{aligned}\deg(\beta_{i_0} - \beta_j) &= \deg(\beta_{i_0} - \gamma_{k_0} + \gamma_{k_0} - \beta_j) \leq \deg(\beta_{i_0} - \gamma_{k_0}) \\ &= \deg(\beta_{i_0} - \beta_{j_0}) \leq \deg(\beta_{i_0} - \beta_j),\end{aligned}$$

which gives (2.7) in case (b).

C. We notice that by Lemma 2.1 and equalities (2.4), (2.5) and (2.7) we have

$$\begin{aligned}\deg(f - \lambda_0) \circ \Psi_{k_0} &= \sum_{j=1}^{n} \deg(\gamma_{k_0} - \beta_j) \\ &= \sum_{j=1, j \neq i_0}^{n} \deg(\gamma_{k_0} - \beta_j) + \deg(\gamma_{k_0} - \beta_{i_0}) \\ &= \sum_{j=1, j \neq i_0}^{n} \deg(\beta_{i_0} - \beta_j) + \min_{j=1, j \neq i_0}^{n} \deg(\beta_{i_0} - \beta_j).\end{aligned}$$

Thus we have shown

$$\min_{i=1}^{n} \left(\sum_{j=1, j \neq i}^{n} \deg(\beta_i - \beta_j) + \min_{j=1, j \neq i}^{n} \deg(\beta_i - \beta_j) \right) \geq \min_{l=1}^{n-1} (\deg(f - \lambda_0) \circ \Psi_l). \tag{2.8}$$

D. We shall now show the inequality opposite to (2.8). There exist $l_0 \in \{1, \ldots, n-1\}$ and $j_0 \in \{1, \ldots, n\}$ such that

$$\min_{l=1}^{n-1} (\deg(f - \lambda_0) \circ \Psi_l) = \deg(f - \lambda_0) \circ \Psi_{l_0}, \tag{2.9}$$

$$\min_{j=1}^{n} \deg(\gamma_{l_0} - \beta_j) = \deg(\gamma_{l_0} - \beta_{j_0}). \tag{2.10}$$

Observe first that for any $j \in \{1, \ldots, n\}$, $j \neq j_0$, we have by (2.10)

$$\deg(\beta_j - \beta_{j_0}) = \deg(\beta_j - \gamma_{l_0} + \gamma_{l_0} - \beta_{j_0}) \le \deg(\beta_j - \gamma_{l_0}). \tag{2.11}$$

By Lemma 2.2 there exists $k_0 \in \{1, \dots, n-1\}$ such that $\deg(\gamma_{l_0} - \beta_{j_0}) = \deg(\beta_{k_0} - \beta_{j_0})$. Hence using Lemma 2.1 and (2.11) we get

$$\deg(f - \lambda_0) \circ \Psi_{l_0} = \sum_{j=1}^{n} \deg(\gamma_{l_0} - \beta_j) \ge \sum_{j=1, j \ne j_0}^{n} \deg(\beta_{j_0} - \beta_j) + \deg(\gamma_{l_0} - \beta_{j_0})$$

$$\ge \sum_{j=1, j \ne j_0}^{n} \deg(\beta_{j_0} - \beta_j) + \min_{j=1, j \ne j_0}^{n} \deg(\beta_j - \beta_{j_0}),$$

which gives the inequality opposite to (2.8).

This ends the proof. □

3. Critical values at infinity

Let $F : \boldsymbol{C}^n \to \boldsymbol{C}^m$, $n \ge 2$, be a polynomial mapping and let $S \subset \boldsymbol{C}^n$ be an unbounded set. We define

$$N(F|S) := \{\nu \in \boldsymbol{R} : \exists A, B > 0 \;\; \forall z \in S, (|z| > B \Rightarrow |F(z)| \ge A|z|^{\nu})\},$$

where $|\cdot|$ is the polycylindric norm. If $S = \boldsymbol{C}^n$ we put $N(F) := N(F|\boldsymbol{C}^n)$.

By the Łojasiewicz exponent at infinity of $F|S$ we shall mean $\mathscr{L}_\infty(F|S) := \sup N(F|S)$ when $N(F|S) \ne \emptyset$, and $-\infty$ when $N(F|S) = \emptyset$. Analogously $\mathscr{L}_\infty(F) := \sup N(F)$ when $N(F) \ne \emptyset$, and $-\infty$ when $N(F) = \emptyset$.

We give now a lemma needed in the sequel, which gives known formulas for the Łojasiewicz exponent at infinity of a polynomial on the zero set of another one. Let g, h be polynomials in two complex variables (x, y) and

$$0 < \deg h = \deg_y h.$$

Let $\tau \in \boldsymbol{C}$ and $R(x, \tau) := \operatorname{Res}_y(g(x, y) - \tau, h(x, y))$ be the resultant of $g(x, y) - \tau$ and $h(x, y)$ with respect to y. We put

$$R(x, \tau) = R_0(\tau)x^K + \cdots + R_K(\tau), \quad R_0 \ne 0,$$

$$T := h^{-1}(0).$$

Lemma 3.1 ([P2], Proposition 2.4). *Under above notation and assumptions there is*:

(i) $\mathscr{L}_\infty(g|T) > 0$ *if and only if* $R_0 = const.$,

(ii) $\mathscr{L}_\infty(g|T) = 0$ *if and only if* $R_0 \ne const.$ *and* $R_0(0) \ne 0$,

(iii) $-\infty < \mathscr{L}_\infty(g|T) < 0$ *if and only if there exists r such that* $R_0(0) = \cdots = R_r(0) = 0$ *and* $R_{r+1}(0) \ne 0$,

(iv) $\mathscr{L}_\infty(g|T) = -\infty$ *if and only if* $R_0(0) = \cdots = R_K(0) = 0$.

Moreover, in case (i)

$$\mathscr{L}_\infty(g|T) = \left[\max_{i=1}^{K} \frac{\deg R_i}{i}\right]^{-1}$$

and in case (iii)

$$\mathscr{L}_\infty(g|T) = -\left[\min_{i=0}^{r} \frac{\operatorname{ord}_0 R_i}{r+1-i}\right]^{-1}. \qquad \square$$

Let f be a polynomial in two complex variables of the form (2.1) and $\deg f > 1$. Fix $\lambda_0 \in \boldsymbol{C}$, denote $z := (x, y)$ and define

$$S_{\lambda_0} := \{z \in \boldsymbol{C}^2 : f(z) = \lambda_0\},$$

$$Y := \{z \in \boldsymbol{C}^2 : f_y'(z) = 0\}.$$

In notation of Lemma 2.1 we put $\Phi_i(t) := (t^D, \beta_i(t))$ for $i \in \{1, \dots, n\}$ and as previously $\Psi_j(t) := (t^D, \gamma_j(t))$ for $j \in \{1, \dots, n-1\}$.

Under these notation we give, without proof, a simple lemma which follows easily from Lemma 2.1.

Lemma 3.2. *We have*
(i) $\deg \Phi_i = D$, $i = 1, \dots, n$, $\deg \Psi_j = D$, $j = 1, \dots, n-1$,
(ii) $\mathscr{L}_\infty(f_y'|S_{\lambda_0}) = (1/D)\min_{i=1}^{n} \deg f_y' \circ \Phi_i$,
(iii) $\mathscr{L}_\infty(f - \lambda_0 \,|\, Y) = (1/D)\min_{j=1}^{n-1} \deg(f - \lambda_0) \circ \Psi_j$. $\square$

Now, we give a theorem important in the sequel.

Theorem 3.3. *If* $\mathscr{L}_\infty(f - \lambda_0, f_y') < 0$, *then*
(i) $\mathscr{L}_\infty(f - \lambda_0, f_y') = \mathscr{L}_\infty(f - \lambda_0 \,|\, Y)$,
(ii) $\lambda_0 \in \Lambda(f)$.
Moreover, if additionally $\mathscr{L}_\infty(f - \lambda_0, f_y') \neq -\infty$ *then*

$$\mathscr{L}_\infty(f - \lambda_0 \,|\, Y) < \mathscr{L}_\infty(f_y'|S_{\lambda_0}). \tag{3.1}$$

Proof. Let us start from (i). In the case $\mathscr{L}_\infty(f - \lambda_0, f_y') = -\infty$ we get easily (cf. [CK3], Theorem 3.1(iv)) that $\mathscr{L}_\infty(f - \lambda_0 \,|\, Y) = -\infty$, which gives (i) in this case.

Let us assume now that $\mathscr{L}_\infty(f - \lambda_0, f_y') \neq -\infty$. In this case by the Main Theorem in [CK1], (cf. [CK4], Theorem 1) we have

$$\mathscr{L}_\infty(f - \lambda_0, f_y') = \min(\mathscr{L}_\infty(f - \lambda_0 \,|\, Y), \mathscr{L}_\infty(f_y'|S_{\lambda_0})). \tag{3.2}$$

Hence to prove (i) in this case it suffices to show (3.1).

Assume to the contrary that (3.1) does not hold. Then by (3.2) and the assumption of the theorem we have $\mathscr{L}_\infty(f_y'|S_{\lambda_0}) < 0$. On the other hand, by Lemma 3.2(ii) there exists $i \in \{1, \dots, n\}$ such that

$$\deg f_y' \circ \Phi_i = D\mathscr{L}_\infty(f_y'|S_{\lambda_0}). \tag{3.3}$$

By the above we get $\deg f_y' \circ \Phi_i < 0$. Hence we have

$$\deg f_y' \circ \Phi_i = \sum_{j=1, j \neq i}^{n} \deg(\beta_i - \beta_j)$$
$$> \sum_{j=1, j \neq i}^{n} \deg(\beta_i - \beta_j) + \min_{j=1, j \neq i}^{n} \deg(\beta_i - \beta_j).$$

In consequence we get

$$\deg f_y' \circ \Phi_i > \min_{k=1}^{n} \left(\sum_{j=1, j \neq k}^{n} \deg(\beta_k - \beta_j) + \min_{j=1, j \neq k}^{n} \deg(\beta_k - \beta_j) \right).$$

Hence by Proposition 2.3, Lemma 3.2(iii) and (3.3)

$$\mathscr{L}_\infty(f - \lambda_0 \mid Y) < \mathscr{L}_\infty(f_y' | S_{\lambda_0}),$$

which gives a contradiction. Then (3.1) holds.

Assertion (ii) is a simple consequence of the facts $\mathscr{L}_\infty(f - \lambda_0 \mid Y) < 0$, Lemma 3.1 and (1.4).

This ends the proof. □

Let us fix the same notation as in Theorem 3.3.

THEOREM 3.4. *If* $\mathscr{L}_\infty(f - \lambda_0, f_y') < 0$, *then*

$$\mathscr{L}_{\infty, \lambda_0}(f) = \mathscr{L}_\infty(f - \lambda_0, f_y') - 1. \tag{3.4}$$

Proof. If $\mathscr{L}_\infty(f - \lambda_0, f_y') = -\infty$, then by Theorem 3.3(i) $\mathscr{L}_\infty(f - \lambda_0 \mid Y) = -\infty$. Hence by Lemma 3.2(iii) there exists $j \in \{1, \dots, n-1\}$ such that $(f - \lambda_0) \circ \Psi_j \equiv 0$. This implies $f_y' \circ \Psi_j \equiv f_x' \circ \Psi_j \equiv 0$. Hence $\deg \nabla f \circ \Psi_j = -\infty$ and in consequence $\mathscr{L}_{\infty, \lambda_0}(f) = -\infty$, which gives (3.4) in this case.

If $-\infty < \mathscr{L}_\infty(f - \lambda_0, f_y') < 0$ then again by Theorem 3.3(i) it suffices to show that

$$\mathscr{L}_{\infty, \lambda_0}(f) = \mathscr{L}_\infty(f - \lambda_0 \mid Y) - 1. \tag{3.5}$$

We shall first show the inequality

$$\mathscr{L}_{\infty, \lambda_0}(f) \geq \mathscr{L}_\infty(f - \lambda_0 \mid Y) - 1. \tag{3.6}$$

According to definition (1.2) of $\mathscr{L}_{\infty, \lambda_0}(f)$ it suffices to show that for any meromorphic curve $\Phi(t) = (\varphi_1(t), \varphi_2(t))$ satisfying

$$\deg \Phi > 0, \tag{3.7}$$

$$\deg(f - \lambda_0) \circ \Phi < 0, \tag{3.8}$$

we have

$$\frac{\deg \nabla f \circ \Phi}{\deg \Phi} \geq \mathscr{L}_\infty(f - \lambda_0 \mid Y) - 1. \tag{3.9}$$

From (3.7) and (3.8) it easily follows $\deg \varphi_1 > 0$. Superposing Φ, if necessary, with a meromorphic function at ∞ of degree 1, we may assume that $\Phi(t) = (t^{\deg \varphi_1}, \varphi(t))$. Then by (3.8) we also get easily that $\deg \Phi = \deg \varphi_1$. On the other hand, by Lemma 3.2(iii) it follows that there exists $l_* \in \{1, \dots, n-1\}$ such that

$$\mathscr{L}_\infty(f - \lambda_0 \mid Y) = \frac{\deg(f - \lambda_0) \circ \Psi_{l_*}}{\deg \Psi_{l_*}}. \tag{3.10}$$

Hence we get that inequality (3.9) can be replaced by the inequality

$$\frac{\deg \nabla f \circ \Phi}{\deg \Phi} \geq \frac{\deg(f - \lambda_0) \circ \Psi_{l_*}}{\deg \Psi_{l_*}} - 1. \tag{3.11}$$

At the cost of superpositions of Φ and Ψ_{l_*}, if necessary, with appropriate powers of t^α and t^β, which does not change the value of fraction in (3.11), we may assume that $\deg \Phi = \deg \Psi_{l_*}$. Moreover, increasing D in Lemma 2.1 we may also assume that $\deg \Phi = D$. Summing up, to show (3.6) it suffices to prove

$$\deg \nabla f \circ \Phi \geq \deg(f - \lambda_0) \circ \Psi_{l_*} - D. \tag{3.12}$$

Before the proof of this we notice that inequality (3.8) implies easily the following

$$\deg(f - \lambda_0) \circ \Phi \leq \deg \nabla f \circ \Phi + D. \tag{3.13}$$

Consider now two cases:

(a) there exists $l_0 \in \{1, \dots, n-1\}$ such that

$$\deg(\varphi - \gamma_{l_0}) < \min_{i=1}^{n} \deg(\varphi - \beta_i),$$

(b) for each $l \in \{1, \dots, n-1\}$

$$\deg(\varphi - \gamma_l) \geq \min_{i=1}^{n} \deg(\varphi - \beta_i).$$

In case (a) for each $j \in \{1, \dots, n\}$ we have

$$\deg(\gamma_{l_0} - \beta_j) = \deg(\gamma_{l_0} - \varphi + \varphi - \beta_j) = \deg(\varphi - \beta_j).$$

Then

$$\deg(f - \lambda_0) \circ \Psi_{l_0} = \deg(f - \lambda_0) \circ \Phi.$$

Hence, from (3.10) and Lemma 3.2(iii) we get

$$\deg(f - \lambda_0) \circ \Psi_{l_*} \leq \deg(f - \lambda_0) \circ \Phi. \tag{3.14}$$

By (3.13) and (3.14) we get (3.12) in case (a).

We shall now show (3.12) in case (b). Let $\min_{i=1}^{n} \deg(\varphi - \beta_i) = \deg(\varphi - \beta_{i_0})$ for some $i_0 \in \{1, \dots, n\}$. Then for each $l \in \{1, \dots, n-1\}$

$$\deg(\beta_{i_0} - \gamma_l) = \deg(\beta_{i_0} - \varphi + \varphi - \gamma_l) \leq \deg(\varphi - \gamma_l).$$

Hence

$$\deg f_y' \circ \Phi_{i_0} \leq \deg f_y' \circ \Phi. \tag{3.15}$$

On the other hand, by Proposition 2.3, Lemma 3.2(iii), and (3.10)

$$\begin{aligned}
\deg f_y' \circ \Phi_{i_0} &= \sum_{j=1, j\neq i_0}^{n} \deg(\beta_{i_0} - \beta_j) \\
&= \sum_{j=1, j\neq i_0}^{n} \deg(\beta_{i_0} - \beta_j) + \min_{j=1, j\neq i_0}^{n} \deg(\beta_{i_0} - \beta_j) - \min_{j=1, j\neq i_0}^{n} \deg(\beta_{i_0} - \beta_j) \\
&\geq \min_{k=1}^{n} \left(\sum_{j=1, j\neq k}^{n} \deg(\beta_k - \beta_j) + \min_{j=1, j\neq k}^{n} \deg(\beta_k - \beta_j) \right) - D \\
&= D\mathscr{L}_\infty(f - \lambda_0 \,|\, Y) - D = \deg(f - \lambda_0) \circ \Psi_{l_*} - D.
\end{aligned}$$

Hence and by (3.15) we get

$$\deg f_y' \circ \Phi \geq \deg(f - \lambda_0) \circ \Psi_{l_*} - D. \tag{3.16}$$

By (3.16) and the obvious inequality $\deg \nabla f \circ \Phi \geq \deg f_y' \circ \Phi$ we get inequality (3.12) in case (b). Then we have proved (3.12) and in consequence (3.6).

To finish the proof we have to show

$$\mathscr{L}_{\infty, \lambda_0}(f) \leq \mathscr{L}_\infty(f - \lambda_0 \,|\, Y) - 1. \tag{3.17}$$

By assumption, Theorem 3.3(i) and (3.10) we have

$$\deg(f - \lambda_0) \circ \Psi_{l_*} < 0. \tag{3.18}$$

Hence

$$\deg(f - \lambda_0) \circ \Psi_{l_*} = \deg \nabla f \circ \Psi_{l_*} + D. \tag{3.19}$$

Hence and from (3.10) we get

$$\mathscr{L}_\infty(f - \lambda_0 \,|\, Y) - 1 = \frac{\deg \nabla f \circ \Psi_{l_*}}{\deg \Psi_{l_*}}.$$

Hence taking into account (3.18) and (1.2) we obtain (3.17).

This ends the proof of the theorem. □

We shall now give three simple corollaries of Theorems 3.3 and 3.4.

COROLLARY 3.5. *The following conditions are equivalent*:
(i) $\mathscr{L}_\infty(f - \lambda_0, f_y') < 0$,
(ii) $\mathscr{L}_{\infty, \lambda_0}(f) < -1$,
(iii) $\mathscr{L}_{\infty, \lambda_0}(f) < 0$,
(iv) $\lambda_0 \in \Lambda(f)$,
(v) $\mathscr{L}_\infty(f - \lambda_0 \,|\, Y) < 0$.

Proof. (i) $\Rightarrow$ (ii). Theorem 3.4.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). By definition of $\mathcal{L}_{\infty,\lambda_0}(f)$ there exists a meromorphic curve Φ, $\deg \Phi > 0$, such that $\deg((f-\lambda_0)\circ\Phi, f'_y\circ\Phi) =: \alpha < 0$. Hence $\mathcal{L}_\infty(f-\lambda_0, f'_y) \le \alpha/\deg \Phi < 0$.

(i) $\Rightarrow$ (iv). Theorem 3.3(ii).

(iv) $\Rightarrow$ (i). [CK3], Theorem 3.1.

(i) $\Rightarrow$ (v). Theorem 3.3(i).

(v) $\Rightarrow$ (i). By Lemma 3.2 (iii) there exists a meromorphic curve Ψ, $\deg \Psi > 0$, such that $\deg((f-\lambda_0)\circ\Psi, f'_y\circ\Psi) =: \alpha < 0$. Hence $\mathcal{L}_\infty(f-\lambda_0, f'_y) \le \alpha/\deg \Psi < 0$.

This ends the proof. □

Corollary 3.6. *If* $\mathcal{L}_\infty(\nabla f) \le -1$, *then*

(i) *there exists* $\lambda_0 \in \boldsymbol{C}$ *such that* $\mathcal{L}_\infty(\nabla f) = \mathcal{L}_{\infty,\lambda_0}(f)$,

(ii) $\mathcal{L}_\infty(\nabla f) = \mathcal{L}_\infty(\nabla f \mid Y)$.

Proof. Let Φ, $\deg \Phi > 0$, be a meromorphic curve on which the Łojasiewicz exponent $\mathcal{L}_\infty(\nabla f)$ is attained. Then

$$\mathcal{L}_\infty(\nabla f) = \frac{\deg \nabla f\circ\Phi}{\deg \Phi}. \tag{3.20}$$

We shall show

$$\deg f\circ\Phi \le 0. \tag{3.21}$$

Indeed, it suffices to consider the case $\deg f\circ\Phi \ne 0$. Then

$$\frac{\deg f\circ\Phi}{\deg\Phi} \le \frac{\deg \nabla f\circ\Phi}{\deg\Phi} + 1 = \mathcal{L}_\infty(\nabla f) + 1 \le 0,$$

which gives (3.21).

Inequality (3.21) implies that there exists $\lambda_0 \in \boldsymbol{C}$ such that

$$\deg(f-\lambda_0)\circ\Phi < 0. \tag{3.22}$$

Then by (3.20), (3.22) and (1.2) we get $\mathcal{L}_{\infty,\lambda_0}(f) \le \mathcal{L}_\infty(\nabla f)$. The opposite inequality is obvious. This gives (i).

From (3.22), the assumption and (3.20) we get $\mathcal{L}_\infty(f-\lambda_0, f'_y) < 0$. Hence according to (i) and Theorems 3.4 and 3.3(i) we get

$$\mathcal{L}_\infty(\nabla f) = \mathcal{L}_\infty(f-\lambda_0 \mid Y) - 1.$$

Hence and from the obvious inequality

$$\mathcal{L}_\infty(f-\lambda_0 \mid Y) - 1 \ge \mathcal{L}_\infty(\nabla f \mid Y)$$

we obtain

$$\mathscr{L}_\infty(\nabla f) \geq \mathscr{L}_\infty(\nabla f \mid Y).$$

The opposite inequality is obvious, which gives (ii).
This ends the proof. □

Corollary 3.7. *If* $\mathscr{L}_\infty(f - \lambda_0, f'_y) < 0$ *and functions* $\beta_1, \dots, \beta_n$, *meromorphic at infinity, are as in Lemma* 2.1 *then*

$$\mathscr{L}_{\infty,\lambda_0}(f) + 1 = \frac{1}{D} \min_{i=1}^{n} \left(\sum_{j=1, j\neq i}^{n} \deg(\beta_i - \beta_j) + \min_{j=1, j\neq i}^{n} \deg(\beta_i - \beta_j) \right). \tag{3.23}$$

Proof. By Theorems 3.4 and 3.3 (i) we get

$$\mathscr{L}_{\infty,\lambda_0}(f) + 1 = \mathscr{L}_\infty(f - \lambda_0 \mid Y).$$

Hence, using Lemma 3.2 (iii) and Proposition 2.3 we obtain (3.23).
This ends the proof. □

At the end of this section we notice that from Corollary 3.5 it follows that all results of this section concern critical values of f at infinity. Indeed, by Corollary 3.5 one can always replace the assumption $\mathscr{L}_\infty(f - \lambda_0, f'_y) < 0$ with the assumption $\lambda_0 \in \Lambda(f)$.

We shall now discuss the relation of the above three corollaries with the results by Ha [H2]. It shall be shown in Section 5 that the above Łojasiewicz exponent $\mathscr{L}_{\infty,\lambda}(f)$, defined by (1.2), coincides with the Łojasiewicz exponent $\tilde{\mathscr{L}}_{\infty,\lambda}(f)$, defined by (1.1), introduced by Ha in [H2]. Thus Corollary 3.5 is a changed and extended version of Theorems 1.3.1 and 1.3.2 in [H2]. A proof of Theorem 1.3.2 in [H2] was also given by Kuo and Parusiński ([KP], Theorem 3.1). In turn, Corollaries 3.6(i) and 3.7 correspond exactly to Theorems 1.4.3 and 1.4.1 in [H2], respectively.

4. Effective formulas for $\mathscr{L}_{\infty,\lambda}(f)$

In this section f is a polynomial in two complex variables of the form (2.1). Let $(\lambda, u) \in \boldsymbol{C}^2$ and $Q(x, \lambda, u) := \mathrm{Res}_y(f - \lambda, f'_y - u)$ be the resultant of the polynomials $f - \lambda$ and $f'_y - u$ with respect to the variable y. By the definition of the resultant we get easily that $Q(0, \lambda, 0) = \pm n^n \lambda^{n-1} +$ terms of lower degrees. Hence $Q \neq 0$. We put

$$Q(x, \lambda, u) = Q_0(\lambda, u)x^N + \cdots + Q_N(\lambda, u), \quad Q_0 \neq 0. \tag{4.1}$$

Let us pass now to the effective calculations of $\mathscr{L}_{\infty,\lambda}(f)$. We start with the first main theorem concerning the case when λ_0 is a critical value of f at infinity.

Theorem 4.1. *A point* $\lambda_0 \in \boldsymbol{C}$ *is a critical value of* f *at infinity if and only if* $Q_0(\lambda_0, 0) = 0$. *Moreover*

(i) $\mathscr{L}_{\infty,\lambda_0}(f) = -\infty$ *if and only if* $Q_0(\lambda_0, 0) = \cdots = Q_N(\lambda_0, 0) = 0$,
(ii)

$$\mathscr{L}_{\infty,\lambda_0}(f) = -1 - \left[\min_{i=0}^{r} \frac{\operatorname{ord}_{(\lambda_0,0)} Q_i}{r+1-i}\right]^{-1}$$

if and only if there exists $r \in \{0, \ldots, N-1\}$ *such that* $Q_0(\lambda_0, 0) = \cdots = Q_r(\lambda_0, 0) = 0$, $Q_{r+1}(\lambda_0, 0) \neq 0$.

Proof. By Corollary 3.5 (iv) $\Leftrightarrow$ (i) and Theorem 3.1 in [CK3] we get the first assertion of the theorem. The second one follows from Theorems 3.1 and 3.3 in [CK3] and Theorem 3.4. □

The next considerations will be preceded by two lemmas. First we introduce notations.

Let $\mathscr{M}(t)$ be the field of germs of meromorphic functions at infinity i.e. the field of all Laurent series of the form $\sum_{n=-k}^{+\infty} a_{-n} t^{-n}$, $k \in \boldsymbol{Z}$, convergent in a neighbourhood of $\infty \in \overline{\boldsymbol{C}}$. Let $\mathscr{M}(t)^* := \bigcup_{k=1}^{\infty} \mathscr{M}(t^{1/k})$ be the field of convergent Puiseux series at infinity. Similarly as in the local case $\mathscr{M}(t)^*$ is an algebraically closed field. If $\varphi \in \mathscr{M}(t)^*$ and $\varphi(t) = \psi(t^{1/k})$ for $\psi \in \mathscr{M}(t)$, then we define $\deg \varphi := (1/k) \deg \psi$.

Using simple properties of the function deg and the Vieta formulae we obtain

Lemma 4.2. *Let*

$$P(x,t) = c_0(t)x^N + c_1(t)x^{N-1} + \cdots + c_N(t) = c_0(t)(x - \varphi_1(t)) \cdots (x - \varphi_N(t)),$$

where $c_0, c_1, \ldots, c_N \in \mathscr{M}(t)$, $c_0 \neq 0$, $\varphi_1, \ldots, \varphi_N \in \mathscr{M}(t)^*$. *Then*

$$\max_{i=1}^{N} \deg \varphi_i = \max_{i=1}^{N} \frac{\deg c_i - \deg c_0}{i}.$$ □

Let f be, as previously, a polynomial of the form (2.1). For every $\lambda \in \boldsymbol{C}$ we put, as before, $S_\lambda := f^{-1}(\lambda)$. Directly, by Lemmas 2.1 and 3.2, we get for every $\lambda \in \boldsymbol{C}$

$$\mathscr{L}_\infty(\nabla f \mid S_\lambda) = \mathscr{L}_\infty(f'_y | S_\lambda). \tag{4.2}$$

Lemma 4.3. *The function* $\boldsymbol{C} \setminus \Lambda(f) \ni \lambda \mapsto \mathscr{L}_\infty(\nabla f \mid S_\lambda)$ *is lower semicontinuous.*

Proof. Take $\lambda_0 \in \boldsymbol{C} \setminus \Lambda(f)$. Theorem 2 in [K2] gives that there exist a neighbourhood K of λ_0, a positive integer D, a vicinity U of infinity in $\boldsymbol{C}$ and holomorphic functions $K \times U \ni (\lambda, t) \mapsto \beta_i(\lambda, t)$, $i = 1, \ldots, n$, such that for every $\lambda \in K$ we have:

(a) functions $U \ni t \mapsto \beta_i(\lambda, t)$, $i = 1, \ldots, n$, are meromorphic at infinity,

(b) $\deg_t \beta_i(\lambda, t) \le D$, $i = 1, \dots, n$,

(c) $f(t^D, y) - \lambda = \prod_{i=1}^n (y - \beta_i(\lambda, t))$.

Put $\Phi_i(\lambda, t) := (t^D, \beta_i(\lambda, t))$. By (a), (b), (c) and Lemma 3.2 we have for every $\lambda \in K$

$$\mathscr{L}_\infty(f_y' | S_\lambda) = \frac{1}{D} \min_{i=1}^{n} \deg_t f_y' \circ \Phi_i(\lambda, t).$$

Since for every $i \in \{1, \dots, n\}$ the holomorphic function $K \times U \ni (\lambda, t) \mapsto f_y' \circ \Phi_i(\lambda, t)$ has an expansion in U in a Laurent series in variable t with coefficients holomorphic in K, then the function $K \ni \lambda \mapsto \min_{i=1}^n \deg_t f_y' \circ \Phi_i(\lambda, t)$ is constant in a vicinity $\tilde{K} \subset K$ of λ_0 and takes a value not greater than this constant at λ_0. In consequence $\boldsymbol{C} \setminus \Lambda(f) \ni \lambda \mapsto \mathscr{L}_\infty(f_y' | S_\lambda)$ is lower semicontinuous. Hence and by (4.2) we get the assertion of the theorem. □

Now, we shall prove an important proposition, which was indicated to us by A. Płoski. He obtained this result by studying the polar quotients. We shall give another direct proof of it.

Let

$$\delta := \left[\max_{i=1}^{N} \frac{\deg_u Q_i}{i} \right]^{-1}. \tag{4.3}$$

By an elementary property of the resultant Q it follows $\delta > 0$.

Proposition 4.4. *The function* $\boldsymbol{C} \setminus \Lambda(f) \ni \lambda \mapsto \mathscr{L}_\infty(\nabla f \mid S_\lambda)$ *is constant. Moreover,*

(i) *if* $\deg_u Q_0 = 0$, *then* $\mathscr{L}_\infty(\nabla f \mid S_\lambda) = \delta$ *for* $\lambda \in \boldsymbol{C} \setminus \Lambda(f)$,

(ii) *if* $\deg_u Q_0 > 0$, *then* $\mathscr{L}_\infty(\nabla f \mid S_\lambda) = 0$ *for* $\lambda \in \boldsymbol{C} \setminus \Lambda(f)$.

Proof. According to (4.2) it suffices to show the function $\boldsymbol{C} \setminus \Lambda(f) \ni \lambda \mapsto \mathscr{L}_\infty(f_y' | S_\lambda)$ is constant.

Assume first $\deg_u Q_0 = 0$. By the first assertion of Theorem 4.1 we have $Q_0(\lambda_0, 0) \neq 0$ for every $\lambda_0 \notin \Lambda(f)$. Then by Lemma 3.1

$$\mathscr{L}_\infty(f_y' | S_{\lambda_0}) = \left[\max_{i=1}^{N} \frac{\deg_u Q_i(\lambda_0, u)}{i} \right]^{-1}. \tag{4.4}$$

Hence and (4.3) it follows there exists a finite set $\Omega_1(f) \subset \boldsymbol{C} \setminus \Lambda(f)$ such that

$$\mathscr{L}_\infty(f_y' | S_\lambda) = \delta \quad \text{for } \lambda \notin (\Lambda(f) \cup \Omega_1(f))$$

and

$$\mathscr{L}_\infty(f_y' | S_\lambda) > \delta \quad \text{for } \lambda \in \Omega_1(f).$$

On the other hand, by Lemma 4.3, the function $\boldsymbol{C} \setminus \Lambda(f) \ni \lambda \mapsto \mathscr{L}_\infty(f_y' | S_\lambda)$ is lower semicontinuous. Hence $\Omega_1(f) = \emptyset$. This gives (i).

Assume now $\deg_u Q_0 > 0$. Let $\Omega_2(f) = \{\lambda_0 \in \mathbf{C} \setminus \Lambda(f) : \deg_u Q_0(\lambda_0, u) = 0\}$. Clearly, $\Omega_2(f)$ is a finite set. Then by Lemma 3.1 we have

$$\mathscr{L}_\infty(f_y' | S_\lambda) = 0 \quad \text{for } \lambda \notin (\Lambda(f) \cup \Omega_2(f))$$

and

$$\mathscr{L}_\infty(f_y' | S_\lambda) > 0 \quad \text{for } \lambda \in \Omega_2(f).$$

By Lemma 4.3 the function $\mathbf{C} \setminus \Lambda(f) \ni \lambda \mapsto \mathscr{L}_\infty(f_y' | S_\lambda)$ is lower semicontinuous. Hence $\Omega_2(f) = \emptyset$. This gives (ii).

This ends the proof. □

Now, we shall prove the second main theorem of the paper.

Theorem 4.5. *If* $\deg_u Q_0 = 0$ *then*

$$\mathscr{L}_{\infty,\lambda}(f) = \delta \quad \textit{for } \lambda \in \mathbf{C} \setminus \Lambda(f). \tag{4.5}$$

Proof. Let $\lambda_0 \notin \Lambda(f)$. We first show that

$$\delta \le \mathscr{L}_{\infty,\lambda_0}(f). \tag{4.6}$$

Take an arbitrary meromorphic curve $\Phi(t) = (x(t), y(t))$ such that $\deg \Phi > 0$ and $\deg(f - \lambda_0) \circ \Phi < 0$. To show (4.6) it suffices to prove

$$\frac{\deg f_y' \circ \Phi}{\deg \Phi} \ge \delta. \tag{4.7}$$

Notice that the inequality $\deg(f - \lambda_0) \circ \Phi < 0$ and (2.1) imply immediately $\deg \Phi = \deg x$. Put $\lambda(t) := f \circ \Phi(t)$, $u(t) := f_y' \circ \Phi(t)$. By a property of the resultant we have

$$Q(x(t), \lambda(t), u(t)) \equiv 0. \tag{4.8}$$

By the first assertion of Theorem 4.1 and the assumption of the theorem we have $Q_0(\lambda_0, 0) \ne 0$ and Q_0 does not depend on u. Since $\deg(\lambda(t) - \lambda_0) < 0$ then

$$\deg Q_0(\lambda(t), u(t)) = 0. \tag{4.9}$$

By (4.9) and (4.8) taking into account $\deg x > 0$ and $\deg \lambda \le 0$ we get easily

$$\deg u > 0. \tag{4.10}$$

Consider the polynomial in variable x

$$Q(x, \lambda(t), u(t)) = Q_0(\lambda(t), u(t)) x^N + \cdots + Q_N(\lambda(t), u(t))$$

with coefficients meromorphic at infinity. Identifying meromorphic functions at infinity with their germs in $\mathscr{M}(t)$ and using (4.8), (4.9) and Lemma 4.2 we get

$$\deg x(t) \le \max_{i=1}^{N} \frac{\deg Q_i(\lambda(t), u(t))}{i}.$$

Hence and from the inequalities (4.10) and $\deg \lambda(t) \leq 0$ we obtain

$$\deg x(t) \leq \deg u(t) \max_{i=1}^{N} \frac{\deg_u Q_i(\lambda, u)}{i} = \frac{1}{\delta} \deg u(t).$$

This gives (4.7) and then (4.6).

Now, we shall prove that

$$\mathscr{L}_{\infty,\lambda_0}(f) \leq \delta. \tag{4.11}$$

By Proposition 4.4 and (4.2) $\mathscr{L}_\infty(f'_y | S_{\lambda_0}) = \delta$. Hence and by Lemma 3.2 (ii) there exists $i \in \{1, \dots, n\}$ such that

$$\frac{\deg f'_y \circ \Phi_i}{\deg \Phi_i} = \delta.$$

On the other hand we have $\deg \nabla f \circ \Phi_i = \deg f'_y \circ \Phi_i$. Summing up, $\deg \Phi_i > 0$ and

$$\deg(f - \lambda_0) \circ \Phi_i = -\infty, \quad \frac{\deg \nabla f \circ \Phi_i}{\deg \Phi_i} = \delta.$$

Then by definition (1.2) of $\mathscr{L}_{\infty,\lambda_0}(f)$ we get $\mathscr{L}_{\infty,\lambda_0}(f) \leq \delta$.

This ends the proof. □

Now, we shall prove the third main theorem of the paper.

THEOREM 4.6. *If* $\deg_u Q_0 > 0$ *then*

$$\mathscr{L}_{\infty,\lambda}(f) = 0 \quad \textit{for } \lambda \in \boldsymbol{C} \setminus \Lambda(f). \tag{4.12}$$

Proof. Let $\lambda_0 \notin \Lambda(f)$.

If $\deg f = 1$ then we check easily that $\mathscr{L}_{\infty,\lambda_0}(f) = 0$.

Assume that $\deg f > 1$. Let us notice first that by $\lambda_0 \notin \Lambda(f)$ and Corollary 3.5 (iii) $\Rightarrow$ (iv)

$$\mathscr{L}_{\infty,\lambda_0}(f) \geq 0. \tag{4.13}$$

So, it suffices to show the inequality opposite to (4.13). By Proposition 4.4 and (4.2) $\mathscr{L}_\infty(f'_y | S_{\lambda_0}) = 0$. Hence and by Lemma 3.2 (ii) there exists $i \in \{1, \dots, n\}$ such that $\deg f'_y \circ \Phi_i = 0$. On the other hand we have $\deg \nabla f \circ \Phi_i = \deg f'_y \circ \Phi_i$. Summing up, $\deg \Phi_i > 0$ and

$$\deg(f - \lambda_0) \circ \Phi_i = -\infty, \quad \frac{\deg \nabla f \circ \Phi_i}{\deg \Phi_i} = 0.$$

Then by definition (1.2) of $\mathscr{L}_{\infty,\lambda_0}(f)$ we get $\mathscr{L}_{\infty,\lambda_0}(f) \leq 0$.

This ends the proof. □

From Theorems 4.1, 4.5, 4.6 we obtain

COROLLARY 4.7. *The function* $\boldsymbol{C} \ni \lambda \mapsto \mathscr{L}_{\infty,\lambda}(f)$ *takes values in* $[-\infty, -1)$ *if and only if* $\lambda \in \Lambda(f)$. *This function is constant and non-negative outside* $\Lambda(f)$. □

Now, we compare the functions $\lambda \mapsto \mathscr{L}_{\infty,\lambda}(f)$ and $\lambda \mapsto \mathscr{L}_{\infty}(\nabla f \mid S_\lambda)$.

Put $\Lambda_\infty(f) := \{\lambda \in \boldsymbol{C} : Q_0(\lambda, 0) = \cdots = Q_N(\lambda, 0) = 0\}$. By the first assertion of Theorem 4.1 we have $\Lambda_\infty(f) \subset \Lambda(f)$.

THEOREM 4.8. *The functions* $\boldsymbol{C} \ni \lambda \mapsto \mathscr{L}_{\infty,\lambda}(f)$ *and* $\boldsymbol{C} \ni \lambda \mapsto \mathscr{L}_{\infty}(\nabla f \mid S_\lambda)$ *are identical on the set* $(\boldsymbol{C} \setminus \Lambda(f)) \cup \Lambda_\infty(f)$. *Namely*,

(a) *if* $\deg_u Q_0 = 0$, *then* $\mathscr{L}_{\infty,\lambda}(f) = \mathscr{L}_{\infty}(\nabla f \mid S_\lambda) = \delta$ *for* $\lambda \notin \Lambda(f)$,

(b) *if* $\deg_u Q_0 > 0$, *then* $\mathscr{L}_{\infty,\lambda}(f) = \mathscr{L}_{\infty}(\nabla f \mid S_\lambda) = 0$ *for* $\lambda \notin \Lambda(f)$,

(c) $\mathscr{L}_{\infty,\lambda}(f) = \mathscr{L}_{\infty}(\nabla f \mid S_\lambda) = -\infty$ *for* $\lambda \in \Lambda_\infty(f)$.

For $\lambda \in \Lambda(f) \setminus \Lambda_\infty(f)$ *we have* $\mathscr{L}_{\infty,\lambda}(f) < \mathscr{L}_{\infty}(\nabla f \mid S_\lambda) - 1$.

Proof. Assertion (a) and (b) are simple consequences of Theorems 4.5, 4.6 and Proposition 4.4. We get assertion (c) from Theorem 4.1, Lemma 3.1 and (4.2).

If $\lambda_0 \in \Lambda(f) \setminus \Lambda_\infty(f)$ then there exists $r \in \{0, \ldots, N-1\}$ such that $Q_0(\lambda_0, 0) = \cdots = Q_r(\lambda_0, 0) = 0$, $Q_{r+1}(\lambda_0, 0) \neq 0$. Then by Theorem 4.1 $-\infty < \mathscr{L}_{\infty,\lambda_0}(f) < 0$. Hence and by Corollary 3.5, $\mathscr{L}_{\infty}(f - \lambda_0, f_y') < 0$. Then by Theorems 3.4, 3.3 and the formula (4.2) we obtain

$$\mathscr{L}_{\infty,\lambda_0}(f) + 1 = \mathscr{L}_{\infty}(f - \lambda_0, f_y') = \mathscr{L}_{\infty}(f - \lambda_0 \mid Y) < \mathscr{L}_{\infty}(f_y' | S_{\lambda_0}) = \mathscr{L}_{\infty}(\nabla f \mid S_{\lambda_0}).$$

This ends the proof. □

We illustrate the above corollary and theorem with two simple examples

Example 4.9. (a) For $f(x, y) := y^{n+1} + xy^n + y$, $n > 1$, we have $\mathscr{L}_{\infty,\lambda}(f) = \mathscr{L}_{\infty}(\nabla f \mid S_\lambda) = 1/n$ for $\lambda \neq 0$ and $\mathscr{L}_{\infty,0}(f) = -1 - 1/(n-1)$, $\mathscr{L}_{\infty}(\nabla f \mid S_0) = 0$.

(b) For $f(x, y) := y^2$ we have $\mathscr{L}_{\infty,\lambda}(f) = \mathscr{L}_{\infty}(\nabla f \mid S_\lambda) = 0$ for $\lambda \neq 0$ and $\mathscr{L}_{\infty,0}(f) = \mathscr{L}_{\infty}(\nabla f \mid S_0) = -\infty$. □

At the end of this section we shall give a theorem that the exponent $\mathscr{L}_{\infty,\lambda}(f)$ is attained on a meromorphic curve.

Under notation of Lemma 3.2 we have

THEOREM 4.10. *If* $\lambda_0 \in (\boldsymbol{C} \setminus \Lambda(f)) \cup \Lambda_\infty(f)$ *then there exists* $i \in \{1, \ldots, n\}$ *such that*

$$\mathscr{L}_{\infty,\lambda_0}(f) = \frac{\deg \nabla f \circ \Phi_i}{\deg \Phi_i}. \tag{4.14}$$

If $\lambda_0 \in \Lambda(f)$ *then there exists* $j \in \{1, \ldots, n-1\}$ *such that*

$$\mathscr{L}_{\infty,\lambda_0}(f) = \frac{\deg \nabla f \circ \Psi_j}{\deg \Psi_j}. \tag{4.15}$$

Proof. Equality (4.14) is a simple consequence of Theorem 4.8 and Lemma 3.2.

For $\lambda_0 \in \Lambda(f)$ by Theorems 3.3 and 3.4 we obtain

$$\mathscr{L}_{\infty,\lambda_0}(f) = \mathscr{L}_\infty(f - \lambda_0 \mid Y) - 1. \tag{4.16}$$

By Lemma 3.2 there exists $j \in \{1, \dots, n-1\}$ such that

$$\mathscr{L}_\infty(f - \lambda_0 \mid Y) = \frac{\deg(f - \lambda_0) \circ \Psi_j}{\deg \Psi_j}.$$

From Corollary 3.5 $\deg(f - \lambda_0) \circ \Psi_j < 0$. Hence by a simple calculation we get

$$\mathscr{L}_\infty(f - \lambda_0 \mid Y) - 1 = \frac{\deg \nabla f \circ \Psi_j}{\deg \Psi_j}.$$

Then, using (4.16) we obtain (4.15).

This ends the proof. □

5. Equivalence of the definitions of $\tilde{\mathscr{L}}_{\infty,\lambda}(f)$ and $\mathscr{L}_{\infty,\lambda}(f)$

In the Introduction we have defined $\tilde{\mathscr{L}}_{\infty,\lambda}(f)$ and $\mathscr{L}_{\infty,\lambda}(f)$ by formulas (1.1) and (1.2), respectively. We notice that the limit in (1.1) always exists (it may happen to be $-\infty$) because by definition of $\mathscr{L}_\infty(\nabla f \mid f^{-1}(D_\delta))$ the function $\delta \mapsto \mathscr{L}_\infty(\nabla f \mid f^{-1}(D_\delta))$ is non-increasing.

We now prove (1.3) for $n = 2$.

THEOREM 5.1. *Let $f : \boldsymbol{C}^2 \to \boldsymbol{C}$ be a non-constant polynomial and $\lambda_0 \in \boldsymbol{C}$. Then*

$$\tilde{\mathscr{L}}_{\infty,\lambda_0}(f) = \mathscr{L}_{\infty,\lambda_0}(f)$$

holds.

Proof. Obviously

$$\tilde{\mathscr{L}}_{\infty,\lambda_0}(f) \le \mathscr{L}_{\infty,\lambda_0}(f).$$

We shall now prove the opposite inequality. Since the set $\Lambda(f)$ is finite then there is a $\delta > 0$ such that $(\bar{D}_\delta \setminus \{\lambda_0\}) \cap \Lambda(f) = \emptyset$, where $\bar{D}_\delta = \{\lambda \in \boldsymbol{C} : |\lambda - \lambda_0| \le \delta\}$. According to Corollary 4.7 we have

$$\mathscr{L}_{\infty,\lambda}(f) \ge \mathscr{L}_{\infty,\lambda_0}(f) \quad \text{for } \lambda \in \bar{D}_\delta. \tag{5.1}$$

Since the set $f^{-1}(\bar{D}_\delta)$ is semi-algebraic and closed in $\boldsymbol{C}^2$, by the Curve Selection Lemma the exponent $\mathscr{L}_\infty(\nabla f \mid f^{-1}(\bar{D}_\delta))$ is attained on a meromorphic curve Φ_δ, $\deg \Phi_\delta > 0$, lying in this set (see [CK4], Proposition 1). It is easy to see that there exists $\tilde{\lambda} \in \bar{D}_\delta$ such that $\deg(f - \tilde{\lambda}) \circ \Phi_\delta < 0$. By definition of $\mathscr{L}_{\infty,\tilde{\lambda}}(f)$ and (5.1) we get $\mathscr{L}_\infty(\nabla f \mid f^{-1}(\bar{D}_\delta)) \ge \mathscr{L}_{\infty,\tilde{\lambda}}(f) \ge \mathscr{L}_{\infty,\lambda_0}(f)$. Hence

$$\lim_{\delta\to 0^+} \mathscr{L}_\infty(\nabla f \mid f^{-1}(D_\delta)) = \lim_{\delta\to 0^+} \mathscr{L}_\infty(\nabla f \mid f^{-1}(\bar{D}_\delta)) \ge \mathscr{L}_{\infty,\lambda_0}(f).$$

This ends the proof. □

6. *n*-dimensional case

Let $f : \boldsymbol{C}^n \to \boldsymbol{C}$, $n \ge 2$, be a non-constant polynomial. In Section 3 we have described the set $\Lambda(f)$ of critical values of f at infinity for $n = 2$ in terms of the exponent $\mathscr{L}_{\infty,\lambda}(f)$. In this section we shall characterize two another sets also connected to behaviour of the gradient of f at infinity in terms of $\mathscr{L}_{\infty,\lambda}(f)$ and $\tilde{\mathscr{L}}_{\infty,\lambda}(f)$.

Let's start with definitions.

A polynomial f is said to satisfy the Malgrange condition for a value $\lambda_0 \in \boldsymbol{C}$ if

$$\text{(6.1)}\quad \exists \eta_0, \delta_0, R_0 > 0 \quad \forall p \in \boldsymbol{C}^n, \quad (|p| > R_0 \wedge |f(p) - \lambda_0| < \delta_0 \Rightarrow |p|\,|\nabla f(p)| > \eta_0).$$

By $K_\infty(f)$ we denote the set of $\lambda \in \boldsymbol{C}$ for which the Malgrange condition does not hold. It is easy to check that $\lambda \in K_\infty(f)$ if and only if there exists a sequence $\{p_k\} \subset \boldsymbol{C}^n$ such that

$$\text{(6.2)}\qquad \lim_{k\to\infty} |p_k| = \infty, \quad \lim_{k\to\infty} f(p_k) = \lambda, \quad \text{and} \quad \lim_{k\to\infty} |p_k|\,|\nabla f(p_k)| = 0.$$

A polynomial f is said to satisfy the Fedorjuk condition for a value $\lambda_0 \in \boldsymbol{C}$ if

$$\text{(6.3)}\quad \exists \eta_0, \delta_0, R_0 > 0 \quad \forall p \in \boldsymbol{C}^n, \quad (|p| > R_0 \wedge |f(p) - \lambda_0| < \delta_0 \Rightarrow |\nabla f(p)| > \eta_0).$$

By $\tilde{K}_\infty(f)$ we denote the set of $\lambda \in \boldsymbol{C}$ for which the Fedorjuk condition does not hold. It is easy to check that $\lambda \in \tilde{K}_\infty(f)$ if and only if there exists a sequence $\{p_k\} \subset \boldsymbol{C}^n$ such that

$$\text{(6.4)}\qquad \lim_{k\to\infty} |p_k| = \infty, \quad \lim_{k\to\infty} f(p_k) = \lambda, \quad \text{and} \quad \lim_{k\to\infty} |\nabla f(p_k)| = 0.$$

The known properties of the sets $\Lambda(f)$, $K_\infty(f)$ and $\tilde{K}_\infty(f)$ are collected in the following proposition.

Proposition 6.1 (cf. [JK], [P], [S]). *We have*
(a) *the set* $K_\infty(f)$ *is finite,*
(b) *the set* $\tilde{K}_\infty(f)$ *is either finite or equal to* $\boldsymbol{C}$,
(c) $\Lambda(f) \subset K_\infty(f) \subset \tilde{K}_\infty(f)$,
(d) $\Lambda(f) = K_\infty(f) = \tilde{K}_\infty(f)$ *for* $n = 2$. □

We shall show (see Remark 6.5) that the inclusions in (c) can be proper for $n > 2$.

Let us pass to characterizations of the sets $K_\infty(f)$ and $\tilde{K}_\infty(f)$ in terms of $\mathscr{L}_{\infty,\lambda}(f)$ and $\tilde{\mathscr{L}}_{\infty,\lambda}(f)$.

Proposition 6.2. *For $\lambda_0 \in \boldsymbol{C}$ the following conditions are equivalent*:
(i) $\lambda_0 \in K_\infty(f)$,
(ii) $\tilde{\mathscr{L}}_{\infty,\lambda_0}(f) < -1$,
(iii) $\mathscr{L}_{\infty,\lambda_0}(f) < -1$.

Proof. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Take $\lambda_0 \notin K_\infty(f)$. Then by (6.1) we obtain $\mathscr{L}_\infty(\nabla f \mid f^{-1}(D_{\delta_0})) \geq -1$. Hence by definition (1.1) we get $\tilde{\mathscr{L}}_{\infty,\lambda_0}(f) \geq -1$. From the obvious inequality

$$\mathscr{L}_{\infty,\lambda}(f) \geq \tilde{\mathscr{L}}_{\infty,\lambda}(f) \quad \text{for } \lambda \in \boldsymbol{C} \tag{6.5}$$

we also get $\mathscr{L}_{\infty,\lambda_0}(f) \geq -1$. This gives the required sequence of implications.

We now show the implication (i) $\Rightarrow$ (iii). Let $\lambda_0 \in K_\infty(f)$ and $\{p_k\} \subset \boldsymbol{C}^n$ be a sequence satisfying (6.2). Since $K_\infty(f)$ is finite there exists a closed disc $\bar{D}_\delta := \{\lambda \in \boldsymbol{C} : |\lambda - \lambda_0| \leq \delta\}$ such that $\bar{D}_\delta \cap K_\infty(f) = \{\lambda_0\}$. Since $f^{-1}(\bar{D}_\delta)$ is a semi-algebraic and closed set in $\boldsymbol{C}^n$, then by the Curve Selection Lemma the exponent $\mathscr{L}_\infty(\nabla f \mid f^{-1}(\bar{D}_\delta))$ is attained on a meromorphic curve Φ, $\deg \Phi > 0$, lying in this set (cf. [CK4], Proposition 1). Thus there exists a $\tilde{\lambda} \in \bar{D}_\delta$ such that $\deg(f - \tilde{\lambda}) \circ \Phi < 0$. On the other hand almost all elements of the sequence $\{p_k\}$ lie in $f^{-1}(\bar{D}_\delta)$. Then (6.2) implies $\mathscr{L}_\infty(\nabla f \mid f^{-1}(\bar{D}_\delta)) < -1$. In consequence $\deg \nabla f \circ \Phi / \deg \Phi = \mathscr{L}_\infty(\nabla f \mid f^{-1}(\bar{D}_\delta)) < -1$. Hence we get $\tilde{\lambda} \in K_\infty(f)$ and thus $\tilde{\lambda} = \lambda_0$. Summing up, there exists a meromorphic curve Φ, $\deg \Phi > 0$, such that $\deg(f - \lambda_0) \circ \Phi < 0$ and $\deg \nabla f \circ \Phi / \deg \Phi < -1$. Then by definition (1.2) we have

$$\mathscr{L}_{\infty,\lambda_0}(f) < -1.$$

This gives the desired implication and ends the proof. □

Proposition 6.3. *For $\lambda_0 \in \boldsymbol{C}$ the following conditions are equivalent*:
(i) $\lambda_0 \in \tilde{K}_\infty(f)$,
(ii) $\tilde{\mathscr{L}}_{\infty,\lambda_0}(f) < 0$,
(iii) $\mathscr{L}_{\infty,\lambda_0}(f) < 0$.

Proof. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). This follows, analogously as in the previous proposition, directly from (6.3). The implication (i) $\Rightarrow$ (iii) is given in [R2]. □

Now, we show an example how with the help of $\mathscr{L}_{\infty,\lambda}(f)$ one can find the sets $K_\infty(f)$ and $\tilde{K}_\infty(f)$. We consider the Rabier's polynomial (see [R], Remark 9.1).

Proposition 6.4. *Let $f^R : \boldsymbol{C}^3 \to \boldsymbol{C}$, $f^R(x, y, z) := (xy - 1)yz$. Then*
(a) $K_\infty(f^R) = \{0\}$,
(b) $\mathscr{L}_{\infty,\lambda}(f^R) = -1$ *for* $\lambda \neq 0$ *and* $\mathscr{L}_{\infty,0}(f^R) = -\infty$,
(c) $\tilde{K}_\infty(f^R) = \boldsymbol{C}$.

Proof. (a) We first show $0 \in K_\infty(f^R)$. Taking $\tilde{\Phi}(t) := (t, 1/t, 0)$, we have $\deg \tilde{\Phi} > 0$, $f^R \circ \tilde{\Phi}(t) \equiv 0$ and $\deg \nabla f^R \circ \tilde{\Phi} = -\infty$. Hence according to (1.2) we get $\mathscr{L}_{\infty,0}(f^R) = -\infty$ and thus $0 \in K_\infty(f^R)$. To prove the opposite inclusion assume that there exists $\lambda \neq 0$ such that $\lambda \in K_\infty(f^R)$. Then by Proposition 6.2 $\mathscr{L}_{\infty,\lambda}(f^R) < -1$. Then there exists a meromorphic curve $\Phi = (\varphi_1, \varphi_2, \varphi_3)$ such that $\deg \Phi > 0$ and

$$\deg(f^R - \lambda) \circ \Phi < 0, \tag{6.6}$$

$$\deg \nabla f^R \circ \Phi < -\deg \Phi. \tag{6.7}$$

From (6.6)

$$\deg((\varphi_1\varphi_2 - 1)\varphi_2\varphi_3) = 0, \tag{6.8}$$

whereas from (6.7) we get $\deg f'_z \circ \Phi < -\deg \Phi$ and thus

$$\deg((\varphi_1\varphi_2 - 1)\varphi_2) < -\deg \Phi. \tag{6.9}$$

By (6.8) and (6.9) we get $-\deg \varphi_3 < -\deg \Phi$, which is impossible.

(b) For every $\lambda \in \boldsymbol{C}$ and $\Phi_\lambda(t) := (t, 1/2t, -4\lambda t)$ we have $f^R \circ \Phi_\lambda \equiv \lambda$ and $\deg \nabla f^R \circ \Phi_\lambda = -1$. Hence

$$\mathscr{L}_{\infty,\lambda}(f^R) \leq \frac{\deg \nabla f^R \circ \Phi_\lambda}{\deg \Phi_\lambda} = -1.$$

From (a), we have $\lambda \notin K_\infty(f^R)$ if $\lambda \neq 0$. Hence by Proposition 6.2 $\mathscr{L}_{\infty,\lambda}(f^R) \geq -1$. In consequence $\mathscr{L}_{\infty,\lambda}(f^R) = -1$ for $\lambda \neq 0$.

The equality $\mathscr{L}_{\infty,0}(f^R) = -\infty$ has been proved in (a).

(c) It follows from (b) and Proposition 6.3. □

Remark 6.5. By Propositon 6.4 we have

$$K_\infty(f^R) \subsetneq \tilde{K}_\infty(f^R). \tag{6.10}$$

One can show that for the polynomial $f^{PZ}(x,y,z) := x - 3x^5y^2 + 2x^7y^3 + yz$ (see [PZ]) we have

$$\emptyset = \Lambda(f^{PZ}) \quad \text{and} \quad K_\infty(f^{PZ}) \neq \emptyset.$$

We shall show now a relation between $\mathscr{L}_\infty(\nabla f)$ and $\mathscr{L}_{\infty,\lambda}(f)$ for $n \geq 2$. Analogously as Corollary 3.6 (i) we prove

Proposition 6.6. *Let $f : \boldsymbol{C}^n \to \boldsymbol{C}$, $n \geq 2$, be a non-constant polynomial. If $\mathscr{L}_\infty(\nabla f) \leq -1$, then there exists $\lambda_0 \in \boldsymbol{C}$ such that*

$$\mathscr{L}_\infty(\nabla f) = \mathscr{L}_{\infty,\lambda_0}(f). \tag{6.11}$$ □

Directly from the above proposition we obtain

Corollary 6.7. *Let $f : \boldsymbol{C}^n \to \boldsymbol{C}$, $n \geq 2$, be a non-constant polynomial. The following conditions are equivalent*:

(i) $K_\infty(f) \neq \emptyset$,

(ii) $\mathscr{L}_\infty(\nabla f) < -1$.

Proof. (i) $\Rightarrow$ (ii). Take $\lambda_0 \in K_\infty(f)$. Then by Proposition 6.2 we have $\mathscr{L}_{\infty,\lambda_0}(f) < -1$. Then $\mathscr{L}_\infty(\nabla f) < -1$.

(ii) $\Rightarrow$ (i). By Proposition 6.6 there exists $\lambda_0 \in \boldsymbol{C}$ such that $\mathscr{L}_{\infty,\lambda_0}(f) = \mathscr{L}_\infty(\nabla f) < -1$. Hence by Proposition 6.2 $\lambda_0 \in K_\infty(f)$.

This ends the proof. □

At the end we pose one question.

Question 6.8. For a non-constant polynomial $f : \boldsymbol{C}^n \to \boldsymbol{C}$, $n > 2$, does there exist a number $c_f \in [-1, +\infty)$ such that

$$\mathscr{L}_{\infty,\lambda}(f) = \tilde{\mathscr{L}}_{\infty,\lambda}(f) = c_f \quad \text{for any } \lambda \notin K_\infty(f)$$

(cf. Corollary 4.7 and Proposition 6.4)?

References

[CK1] Chądzyński, J. and Krasiński, T.: Exponent of growth of polynomial mappings of $\boldsymbol{C}^2$ into $\boldsymbol{C}^2$. In: Singularities. Banach Center Publ. 20, 147–160, PWN, Warszawa 1988.

[CK2] Chądzyński, J. and Krasiński, T.: Sur l'exposant de Łojasiewicz à l'infini pour les applications polynomiales de $\boldsymbol{C}^2$ dans $\boldsymbol{C}^2$ et les composantes des automorphismes polynomiaux de $\boldsymbol{C}^2$. C. R. Acad. Sci. Paris, Série I, **315** (1992), 1399–1402.

[CK3] Chądzyński, J. and Krasiński, T.: On the Łojasiewicz exponent at infinity for polynomial mappings of $\boldsymbol{C}^2$ into $\boldsymbol{C}^2$ and components of polynomial automorphisms of $\boldsymbol{C}^2$. Ann. Polon. Math. **57** (1992), 291–302.

[CK4] Chądzyński, J. and Krasiński, T.: A set on which the Łojasiewicz exponent at infinity is attained. Ann. Polon. Math. **67** (1997), 191–197.

[F] Fedorjuk, M. V.: The asymptotics of the Fourier transform of the exponential function of a polynomial. Dokl. Akad. Nauk **227** (1976), 580–583 (in Russian); english transl. in Soviet Math. Dokl. (2) **17** (1976), 486–490.

[GP] Gwoździewicz, J. and Płoski, A.: Formulae for the singularities at infinity of plane algebraic curves. Univ. Iagel. Acta Math. **39** (2001), 109–133.

[H1] Ha, H. V.: Sur la fibration globale des polynômes de deux variables complexes. C. R. Acad. Sci. Paris, Série I, **309** (1989), 231–234.

[H2] Ha, H. V.: Nombres de Łojasiewicz et singularitiés à l'infini des polynômes de deux variables complexes. C. R. Acad. Sci. Paris, Série I, **311** (1990), 429–432.

[JK] Jelonek, Z. and Kurdyka, K.: On asymptotic critical values of a complex polynomial. J. Reine Angew. Math. (to appear). Preprint nr 614b (2001), Institute of Mathematics of the Polish Academy of Sciences.

[K1] Krasiński, T.: The level sets of polynomials in two variables and jacobian hypothesis. Wydawnictwo Uniwersytetu Łódzkiego, Łódź 1991. (in Polish).

[K2] Krasiński, T.: The branches at infinity of a pencil of polynomials in two complex variables. Ann. Polon. Math. **55** (1991), 213–220.

[KL] Kuo, T. C. and Lu, Y. C.: On analytic function germs of two complex variables. Topology **16** (1977), 299–310.

[KP] Kuo, T. C. and Parusiński, A.: Newton polygon to an arc. Real and Complex Singularities. Eds. J. W. Bruce and F. Tari. 1998, 76–93.

[P] Parusiński, A.: On the bifurcation set of complex polynomial with isolated singularities at infinity. Compos. Math. **97** (1995), 369–384.

[PZ] Păunescu, L. and Zaharia, A.: On the Łojasiewicz exponent at infinity for polynomial functions. Kodai Math. J. **20** (1997), 269–274.

[Ph] Pham, F.: Vanishing homologies and the *n*-variables saddlepoint method. Proc. Symposia Pure Math. 40, Part 2 (1983), 319–333.

[P1] Płoski, A.: On the maximal polar quotient of an analytic plane curve. Kodai Math. J. **24** (2001), 120–133.

[P2] Płoski, A.: Polar quotients and singularities at infinity of polynomials in two complex variables. Ann. Polon. Math. **78** (2002), 49–58.

[R] Rabier, P. J.: Ehresmann's fibrations and Palais-Smale conditions for morphisms of Finsler manifolds. Annals of Math. **146** (1997), 647–691.

[R1] Rodak, T.: Łojasiewicz exponent for formal mappings. Preprint no 2002/04. Faculty of Mathematics of the University of Łódź (http://imul.uni.lodz.pl/preprints).

[R2] Rodak, T.: The Fedorjuk condition and the Łojasiewicz exponent near a fibre of a polynomial. Preprint no 2002/16. Faculty of Mathematics of the University of Łódź (http://imul.uni.lodz.pl/preprints).

[Sk] Skalski, G.: On the equivalence of definitions of the Łojasiewicz exponents near the fibre of polynomials. Preprint no 2002/15. Faculty of Mathematics of the University of Łódź (http://imul.uni.lodz.pl/preprints).

[S] Spodzieja, S.: Łojasiewicz inequalities at infinity for the gradient of a polynomial. Bull. Polish Acad. Sci. Math. **50** (2002), 273–281.

[V] Verdier, J. L.: Stratifications de Whitney et théorème de Bertini-Sard. Invent. Math. **36** (1976), 295–312.

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