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SUBORDINATIONS BY ALPHA-CONVEX FUNCTIONS

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Abstract

Let $H(\mathrm{U})$ be the space of analytic functions in the unit disk U and let $\mathscr{D} = \{\varphi \in H(\mathrm{U}) : \varphi(0) = 1, \varphi(z) \neq 0, z \in \mathrm{U}\}$. For the functions $\phi, \varphi \in \mathscr{D}$ we will determine simple sufficient conditions such that

$$\left[\frac{\varphi(z)}{\phi(z) + (1/\gamma)z\phi'(z)}\right]^{1/\beta} f(z) \prec k(z) \Rightarrow \mathrm{I}_{\phi,\varphi;\beta,\gamma}[f](z) \prec k(z),$$

for all $k \in \mathscr{M}'_{1/\beta}$, where

$$\mathrm{I}_{\phi,\varphi;\beta,\gamma}[f](z) = \left[\frac{\gamma}{z^{\gamma}\phi(z)}\int_0^z f^{\beta}(t)t^{\gamma-1}\varphi(t)\,\mathrm{d}t\right]^{1/\beta}$$

and $\mathscr{M}'_{1/\beta}$ represents the class of $1/\beta$-convex functions (not necessarily normalized).

In particular, we will give sufficient conditions on ϕ and φ so that the operators $\mathrm{I}_{\phi,\varphi;\beta,\gamma}$ are averaging operators on certain subsets of $H(\mathrm{U})$. In addition, some particular cases of the main result, obtained for appropriate choices of the ϕ and φ functions, will also be given.

1. Introduction

Let $H(\mathrm{U})$ be the space of analytic functions in the unit disk $\mathrm{U} = \{z \in \boldsymbol{C} : |z| < 1\}$ and let $\mathscr{D} = \{\varphi \in H(\mathrm{U}) : \varphi(0) = 1, \varphi(z) \neq 0, z \in \mathrm{U}\}$. We denote by $\mathscr{A}$ the class of analytic functions in U and usually normalized, i.e.

$$\mathscr{A} = \{f \in H(\mathrm{U}) : f(0) = 0, f'(0) = 1\}.$$

If $\beta, \gamma \in \boldsymbol{C}$ with $\beta \neq 0$, for the functions $\phi, \varphi \in \mathscr{D}$ we will define the integral operator $\mathrm{I}_{\phi,\varphi;\beta,\gamma} : \mathscr{K}_{\varphi;\beta,\gamma} \to H(\mathrm{U})$ of the form

$$\mathrm{I}_{\phi,\varphi;\beta,\gamma}[f](z) = \left[\frac{\gamma}{z^{\gamma}\phi(z)}\int_0^z f^{\beta}(t)t^{\gamma-1}\varphi(t)\,\mathrm{d}t\right]^{1/\beta}, \tag{1.1}$$

where $\mathscr{K}_{\varphi;\beta,\gamma} \subset H(\mathrm{U})$ will be determined in Lemma 3.1, such that the integral operator (1.1) is well defined.

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The first major result concerning this operator was given in [8], and some particular cases were previously studied in a large number of papers.

For a set $E \subset \boldsymbol{C}$ let denote by $\operatorname{co} E$ the *convex hull* of E. In [10] and [4] the authors introduced the concept of *averaging* (*or mean-value*) *operator* on an arbitrary set $K \subset H(\mathrm{U})$, like an operator $\mathrm{I} : K \to H(\mathrm{U})$ that satisfies

$$\mathrm{I}[f](0) = f(0) \quad \text{and} \quad \mathrm{I}[f](\mathrm{U}) \subset \operatorname{co} f(\mathrm{U}), \quad \text{for all } f \in K.$$

For $f, g \in H(\mathrm{U})$ we say that the function f is *subordinate* to g, written $f(z) \prec g(z)$, if g is univalent in U, $f(0) = g(0)$ and $f(\mathrm{U}) \subseteq g(\mathrm{U})$.

By using several results involving differential subordinations and subordination chains techniques, we will determine simple sufficient conditions on $\phi, \varphi \in \mathscr{D}$ such that

$$\left[\frac{\varphi(z)}{\phi(z) + (1/\gamma)z\phi'(z)}\right]^{1/\beta} f(z) \prec k(z) \Rightarrow \mathrm{I}_{\phi,\varphi;\beta,\gamma}[f](z) \prec k(z),$$

for all $k \in \mathscr{M}'_{1/\beta}$, where $\mathscr{M}'_{1/\beta}$ represents the class of $1/\beta$-convex functions (not necessarily normalized) and is given by (2.2). In particular, we will give conditions on ϕ and φ so that the operators $\mathrm{I}_{\phi,\varphi;\beta,\gamma}$ are averaging operators on certain subsets of $H(\mathrm{U})$, and in addition, some special cases of the main result obtained for appropriate choices of the ϕ and φ functions will also be presented.

2. Preliminaries

In order to prove our main results, we will need the following definitions and lemmas presented in this section.

Let denote by $\mathscr{Q}$ the set of functions q that are analytic and injective on $\overline{\mathrm{U}} \backslash E(q)$, where

$$E(q) = \left\{\zeta \in \partial\mathrm{U} : \lim_{z \to \zeta} q(z) = \infty\right\},$$

and such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathrm{U} \backslash E(q)$.

Lemma 2.1 [1]. *Let $q \in \mathscr{Q}$, with $q(0) = a$, and let $p(z) = a + a_n z^n + \cdots$ be analytic in* U *with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exist points $z_0 \in \mathrm{U}$ and $\zeta_0 \in \partial\mathrm{U} \backslash E(q)$, and an $m \geq n \geq 1$ for which $p(|z| < |z_0|) \subset q(\mathrm{U})$, and*

(i) $p(z_0) = q(\zeta_0)$,

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$,

(iii) $\operatorname{Re} \dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \operatorname{Re}\left[\dfrac{z_0 q''(z_0)}{q'(z_0)} + 1\right]$.

For $\alpha \in \boldsymbol{R}$, a function $f \in H(\mathrm{U})$ with $f(0) = 0$ and $f'(0) \neq 0$ is called to be an *α-convex function* (not necessarily normalized), if

$$\operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(\frac{zf''(z)}{f'(z)}+1\right)\right]>0,\quad z\in \mathrm{U},$$

and we denote this class by $\mathscr{M}_\alpha$. The class of α-convex functions was introduced by P. T. Mocanu in [9]. Note that all α-convex functions are univalent and starlike, and moreover [7],

$$\mathscr{M}_\alpha \subset \mathscr{M}_\beta \subset \mathscr{M}_0,\quad \text{for } 0\le \frac{\beta}{\alpha}\le 1. \tag{2.1}$$

For $\alpha\in \boldsymbol{R}$ we denote by

(2.2)

$$\mathscr{M}'_\alpha=\left\{f\in H(\mathrm{U}): f'(0)\neq 0, \operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(\frac{zf''(z)}{f'(z)}+1\right)\right]>0, z\in \mathrm{U}\right\},$$

and then

$$\mathscr{K}'\equiv \mathscr{M}'_1=\left\{f\in H(\mathrm{U}): f'(0)\neq 0, \operatorname{Re}\frac{zf''(z)}{f'(z)}+1>0, z\in \mathrm{U}\right\}$$

represents the class of *convex functions* (not necessarily normalized) in U.

The next lemma gives us a necessary and sufficient condition for an operator to be an averaging operator.

Lemma 2.2 [10], [4, Lemma 2]. *Let $K\subset H(\mathrm{U})$ and let an operator $\mathrm{I}: K\to H(\mathrm{U})$ that satisfies $\mathrm{I}[f](0)=f(0)$ for all $f\in K$. A necessary and sufficient condition for* I *to be an averaging operator on K is that*

$$f\in K,\ k \text{ convex and } f(z)\prec k(z)\Rightarrow \mathrm{I}[f](z)\prec k(z).$$

Let $c\in \boldsymbol{C}$ with $\operatorname{Re} c>0$, and let $N=N(c)=(|c|\sqrt{1+2\operatorname{Re} c}+\operatorname{Im} c)/\operatorname{Re} c$. If χ is the univalent function $\chi(z)=2Nz/(1-z^2)$, then we define the *open door function* R_c by

$$R_c(z)=\chi\left(\frac{z+b}{1+\bar{b}z}\right),\quad z\in U, \tag{2.3}$$

where $b=\chi^{-1}(c)$.

Remark that R_c is univalent in U, $R_c(0)=c$ and $R_c(U)=\chi(U)$ is the complex plane slit along the half-lines $\operatorname{Re} w=0$, $\operatorname{Im} w\ge N$ and $\operatorname{Re} w=0$, $\operatorname{Im} w\le -N$.

Lemma 2.3 [3, Theorem 1]. *Let $\phi,\varphi\in \mathscr{D}$ and let α, β, γ and δ be complex numbers with $\beta\neq 0$, $\alpha+\delta=\beta+\gamma$ and $\operatorname{Re}(\alpha+\delta)>0$. If $f\in \mathscr{A}$ satisfies*

$$\alpha\frac{zf'(z)}{f(z)}+\frac{z\varphi'(z)}{\varphi(z)}+\delta\prec R_{\alpha+\delta}(z),$$

and the function F is defined by

(2.4)
$$F(z) = \left[\frac{\beta+\gamma}{z^{\gamma}\phi(z)}\int_0^z f^{\alpha}(t)t^{\delta-1}\varphi(t)\,\mathrm{d}t\right]^{1\beta} = z + \cdots,$$

then $F \in \mathcal{A}$, $F(z)/z \neq 0$, $\forall z \in \mathrm{U}$ *and*

$$\mathrm{Re}\left[\beta\frac{zF'(z)}{F(z)} + \frac{z\phi'(z)}{\phi(z)} + \gamma\right] > 0, \quad z \in \mathrm{U}.$$

(*All powers in* (2.4) *are principal ones.*)

Lemma 2.4 [6, Lemma 1.2c]. *Let* $n \geq 0$ *be an integer and let* $\gamma \in \boldsymbol{C}$, *with* $\mathrm{Re}\,\gamma > -n$. *If* $f(z) = \sum_{m\geq n} a_m z^m$ *is analytic in* U *and* F *is defined by*

$$F(z) = \frac{1}{z^{\gamma}}\int_0^z f(t)t^{\gamma-1}\,\mathrm{d}t,$$

then $F(z) = \sum_{m\geq n} a_m z^m/(m+\gamma)$ *is analytic in* U.

Lemma 2.5 [2, Theorem 2], [4, Theorem 2]. *Let k be convex* (*univalent*) *in* U *and let* $A \geq 0$. *Suppose* $M > 4/|h'(0)|$ *and that* B *and* D *are analytic in* U, *with* $D(0) = 0$ *and*

$$\mathrm{Re}\,B(z) \geq A + M|D(z)|, \quad z \in \mathrm{U}.$$

If p *is analytic in* U *with* $p(0) = k(0)$, *and if* p *satisfies*

$$Az^2p''(z) + B(z)zp'(z) + p(z) + D(z) \prec k(z),$$

then $p(z) \prec k(z)$.

A function $L : \mathrm{U} \times [0, +\infty) \to \boldsymbol{C}$ is called a *subordination* (or a *Loewner*) *chain* if $L(\cdot\,; t)$ is analytic and univalent in U for all $t \geq 0$, and $L(z; s) \prec L(z; t)$, when $0 \leq s \leq t$.

Lemma 2.6 [11, p. 159]. *The function* $L(z; t) = a_1(t)z + a_2(t)z^2 + \cdots$, *with* $a_1(t) \neq 0$ *for* $t \geq 0$, *and* $\lim_{t\to\infty}|a_1(t)| = \infty$, *is a subordination chain if and only if there exist constants* $r \in (0, 1]$ *and* $M > 0$ *such that*

(i) $L(z; t)$ *is analytic in* $|z| < r$ *for each* $t \geq 0$, *locally absolutely continuous in* $[0, \infty)$ *for each* $|z| < r$, *and satisfies*

$$|L(z; t)| \leq M|a_1(t)|, \quad \textit{for } |z| < r \textit{ and } t \geq 0$$

(ii) *there exists a function* $p(z, t)$ *analytic in* U *for all* $t \in [0, \infty)$ *and measurable in* $[0, \infty)$ *for each* $z \in \mathrm{U}$, *such that* $\mathrm{Re}\,p(z, t) > 0$ *for* $z \in \mathrm{U}$, $t \in [0, \infty)$, *and*

$$\frac{\partial L(z; t)}{\partial t} = z\frac{\partial L(z; t)}{\partial z}p(z, t), \quad \textit{for } |z| < r \textit{ and for almost all } t \in [0, \infty).$$

3. Main results

First we need to determine sufficient conditions on the ϕ and φ functions such that the integral operators $\mathrm{I}_{\phi,\varphi;\beta,\gamma}$ are well defined.

From the fact that

$$\mathrm{I}_{\phi,\varphi;\beta,\gamma}[f](z)=\left(\frac{\gamma}{\beta+\gamma}\right)^{1/\beta}\tilde{F}(z),\quad \text{if } \gamma\neq 0, \tag{3.1}$$

where

$$\tilde{F}(z)=\left[\frac{\beta+\gamma}{z^{\gamma}\phi(z)}\int_0^z f^{\beta}(t)t^{\gamma-1}\varphi(t)\,\mathrm{d}t\right]^{1/\beta},$$

in order to determine the subset $\mathscr{K}_{\varphi;\beta,\gamma}\subset H(\mathrm{U})$ so that the operator $\mathrm{I}_{\phi,\varphi;\beta,\gamma}$ given by (1.1) is well defined, we need to find the set $\mathscr{K}_{\varphi;\beta,\gamma}$ such that $\tilde{F}\in H(\mathrm{U})$ for all $f\in\mathscr{K}_{\varphi;\beta,\gamma}$.

Lemma 3.1. *Let* $\beta,\gamma\in\boldsymbol{C}$ *with* $\beta\neq 0$, $\mathrm{Re}(\beta+\gamma)>0$ *and let* $\phi,\varphi\in\mathscr{D}$. *If* $R_{\beta+\gamma}$ *represents the* open door *function defined by* (2.3) *and if*

$$\mathscr{K}_{\varphi;\beta,\gamma}=\left\{f\in\mathscr{A}:\beta\frac{zf'(z)}{f(z)}+\frac{z\varphi'(z)}{\varphi(z)}+\gamma\prec R_{\beta+\gamma}(z)\right\},\quad \textit{for } \beta\neq 1,$$

and

$$\mathscr{K}_{\varphi;1,\gamma}=H(\mathrm{U}),\quad \textit{for } \beta=1,\ \textit{if in addition } \mathrm{Re}\,\gamma>0,$$

then the integral operator $\mathrm{I}_{\phi,\varphi;\beta,\gamma}$ *is well-defined.*

Proof. If $\beta\neq 1$, from (3.1) by using Lemma 2.3 for the case $\alpha=\beta$ and $\delta=\gamma$ we deduce the first part of the result.

If $\beta=1$, denoting $t=wz$ we have

$$\tilde{F}(z)=\frac{\gamma+1}{\phi(z)}\int_0^1 f(wz)\varphi(wz)w^{\gamma-1}\,\mathrm{d}w,$$

and from (3.1) according to Lemma 2.4 we obtain the second part of our result. □

Using Lemma 2.3, the previous result and the relation (3.1), respectively Lemma 2.4 and the relation (3.1) we deduce the next two remarks:

Remark 3.1. Under the assumptions of Lemma 3.1, for $\beta\neq 1$, we have

$$\tilde{F}\in\mathscr{A},\quad \frac{\tilde{F}(z)}{z}\neq 0,\quad z\in U\qquad\text{and}\qquad \mathrm{Re}\left[\beta\frac{z\tilde{F}'(z)}{\tilde{F}(z)}+\frac{z\phi'(z)}{\phi(z)}+\gamma\right]>0,\quad z\in\mathrm{U},$$

hence,

$$\mathrm{I}_{\phi,\varphi;\beta,\gamma}[f](z)=\left(\frac{\gamma}{\beta+\gamma}\right)^{1/\beta}z+\cdots\in H(\mathrm{U}),\quad \forall f\in\mathscr{K}_{\varphi;\beta,\gamma}\ \text{and}\ \beta\neq 1.$$

Remark 3.2. Under the assumptions of Lemma 3.1, *we have*

$$\mathrm{I}_{\phi,\varphi;\beta,\gamma}[f](0)=f(0),\quad \forall f\in\mathscr{K}_{\varphi;\beta,\gamma}.$$

THEOREM 3.1. *Let $\beta, \gamma \in \boldsymbol{C}$ with $\beta > 0$ and* $\operatorname{Re}\gamma > 0$, *let $\phi, \varphi \in \mathscr{D}$ and suppose that*

$$\text{(i)} \quad \operatorname{Re}\left[\frac{z\phi'(z)}{\phi(z)} + \gamma\right] > 0, \quad z \in \mathrm{U}.$$

Let $k \in \mathscr{M}'_{1/\beta}$ and $f \in \mathscr{K}_{\varphi;\beta,\gamma}$. Then

$$\left[\frac{\varphi(z)}{\phi(z) + (1/\gamma)z\phi'(z)}\right]^{1/\beta} f(z) \prec k(z) \Rightarrow \mathrm{I}_{\phi,\varphi;\beta,\gamma}[f](z) \prec k(z).$$

Proof. Since $\beta \neq 0$ and $\operatorname{Re}(\beta + \gamma) > 0$, according to Lemma 3.1, the operator $\mathrm{I}_{\phi,\varphi;\beta,\gamma}$ is well-defined on the set $\mathscr{K}_{\varphi;\beta,\gamma}$.

From the assumption $[\varphi(z)/(\phi(z) + (1/\gamma)z\phi'(z))]^{1/\beta} f(z) \prec k(z)$ we have $f(0) = k(0)$. If $\beta \neq 1$ then $f \in \mathscr{A}$, hence $k(0) = f(0) = 0$ i.e. $k \in \mathscr{M}_{1/\beta}$, so it follows that k is univalent in U. If $\beta = 1$ then $k \in \mathscr{K}' \equiv \mathscr{M}'_1$, hence k is a convex (and univalent) function in U.

If we denote by $F(z) = \mathrm{I}_{\phi,\varphi;\beta,\gamma}[f](z)$, then by Remark 3.2 we have $F(0) = f(0)$ and

$$\left[\frac{\varphi(z)}{\phi(z) + (1/\gamma)z\phi'(z)}\right]^{1/\beta} f(z) = F(z)\left[\frac{\beta}{\gamma}\frac{zF'(z)}{F(z)}\frac{1}{H(z)} + 1\right]^{1/\beta}, \tag{3.2}$$

where

$$H(z) = 1 + \frac{1}{\gamma}\frac{z\phi'(z)}{\phi(z)}.$$

Remark that the assumption (i) implies $H(z) \neq 0$ for all $z \in \mathrm{U}$.

Thus, we need to prove the next implication:

$$F(z)\left[\frac{\beta}{\gamma}\frac{zF'(z)}{F(z)}\frac{1}{H(z)} + 1\right]^{1/\beta} \prec k(z) \Rightarrow F(z) \prec k(z). \tag{3.3}$$

For the particular case $\beta = 1$, the implication (3.3) becomes

$$F(z) + \frac{1}{\gamma H(z)} zF'(z) \prec k(z) \Rightarrow F(z) \prec k(z).$$

According to Lemma 2.5 for $A = 0$ and $D(z) \equiv 0$, and by using the inequality (i) we deduce that the above implication holds.

Now we will prove our result for the case $\beta \neq 1$. Without loss of generality we can assume that k satisfies the conditions of the theorem on the closed disk $\overline{\mathrm{U}}$ and $k'(\zeta) \neq 0$ for $|\zeta| = 1$. If not, then we replace f, k, ϕ and φ with $f_r(z) = f(rz)$, $k_r(z) = k(rz)$, $\phi_r(z) = \phi(rz)$ and $\varphi_r(z) = \varphi(rz)$ where $0 < r < 1$, and then k_r is univalent on $\overline{\mathrm{U}}$. Since

$$\left[\frac{\varphi_r(z)}{\phi_r(z) + (1/\gamma)z\phi_r'(z)}\right]^{1/\beta} f_r(z) \prec k_r(z),$$

we would then prove that

$$F_r(z) = F(rz) = \mathrm{I}_{\phi_r,\varphi_r;\beta,\gamma}[f_r](z) \prec k_r(z), \quad \text{for } 0 < r < 1,$$

and by letting $r \to 1^-$ we obtain $F(z) \prec k(z)$.

If we suppose that the implication (3.3) is not true, i.e. $F(z) \not\prec k(z)$, then from Lemma 2.1 there exist points $z_0 \in \mathrm{U}$ and $\zeta_0 \in \partial\mathrm{U}$, and a number $m \geq 1$, such that

$$F(z_0) = k(\zeta_0) \tag{3.4}$$

and

$$z_0 F'(z_0) = m\zeta_0 k'(\zeta_0). \tag{3.5}$$

To prove the implication (3.3) we define the function $L : \mathrm{U} \times [0,\infty) \to \boldsymbol{C}$ by

$$L(z;t) = k(z)\left[\frac{\beta}{\gamma} t \frac{zk'(z)}{k(z)} \frac{1}{H(z_0)} + 1\right]^{1/\beta} = a_1(t)z + \cdots, \tag{3.6}$$

and we will show that $L(z;t)$ is a subordination chain.

From the fact that $zk'(z)/k(z)|_{z=0} = 1$ and the assumptions (i) and $\beta > 0$, we have

$$\operatorname{Re} \frac{\beta}{\gamma} \frac{zk'(z)}{k(z)}\bigg|_{z=0} \frac{1}{H(z_0)} > 0,$$

hence $L(z;t)$ is analytic in $|z| < r < 1$, for sufficient small $r > 0$ and for all $t \geq 0$. We also have that $L(z;t)$ is continuously differentiable on $[0,\infty)$ for each $|z| < r < 1$.

A simple calculus shows that

$$a_1(t) = \frac{\partial L(0;t)}{\partial z} = k'(0)\left[\frac{\beta}{\gamma} t \frac{1}{H(z_0)} + 1\right]^{1/\beta},$$

and because $k'(0) \neq 0$, from (i) and $\beta > 0$ we deduce

$$\operatorname{Re}\left[\frac{\beta}{\gamma} t \frac{1}{H(z_0)} + 1\right] \geq 1 > 0, \quad \forall t \geq 0,$$

hence $a_1(t) \neq 0$, $\forall t \geq 0$. From (i) we have $(\beta/\gamma)t(1/H(z_0)) \neq 0$, $\forall z_0 \in \mathrm{U}$ and $\forall t > 0$, so we obtain that $\lim_{t\to\infty} |a_1(t)| = \infty$.

Using the definition (3.6), by a directly computation we obtain

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] = t\beta \operatorname{Re}\left[\left(1 - \frac{1}{\beta}\right)\frac{zk'(z)}{k(z)} + \frac{1}{\beta}\left(1 + \frac{zk''(z)}{k'(z)}\right)\right] + \operatorname{Re}[\gamma H(z_0)].$$

From the above relation, by using the fact that $k \in \mathscr{M}'_{1/\beta}$ and the assumption (i), we deduce that

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] > 0, \quad \forall z \in \mathrm{U}, \forall t \geq 0,$$

and according to Lemma 2.6 we conclude that $L(z;t)$ is a subordination chain. This implies in particular:

$$k(z) = L(z;0) \prec L(z;t), \quad \forall t \geq 0. \tag{3.7}$$

Using the equality (3.2) and the relations (3.4) and (3.5) we obtain

$$\begin{aligned}\left[\frac{\varphi(z_0)}{\phi(z_0) + (1/\gamma)z_0\phi'(z_0)}\right]^{1/\beta} f(z_0) &= F(z_0)\left[\frac{\beta}{\gamma}\frac{z_0F'(z_0)}{F(z_0)}\frac{1}{H(z_0)} + 1\right]^{1/\beta} \\ &= k(\zeta_0)\left[\frac{\beta}{\gamma}m\frac{\zeta_0k'(\zeta_0)}{k(\zeta_0)}\frac{1}{H(z_0)} + 1\right]^{1/\beta} \\ &= L(\zeta_0;m), \quad m \geq 1,\end{aligned}$$

and then, according to (3.7) we deduce that

$$\left[\frac{\varphi(z_0)}{\phi(z_0) + (1/\gamma)z_0\phi'(z_0)}\right]^{1/\beta} f(z_0) = L(\zeta_0;m) \notin k(\mathrm{U}).$$

This last relation contradicts the assumption $[\varphi(z)/(\phi(z) + (1/\gamma)z\phi'(z))]^{1/\beta}f(z) \prec k(z)$, then we finally conclude that $F(z) \prec k(z)$. □

4. Particular cases

In this section we will discuss several particular cases of Theorem 3.1 obtained for appropriate choices of the ϕ and φ functions.

1. For a given function $\phi \in \mathscr{D}$, taking $\varphi(z) = \phi(z) + (1/\gamma)z\phi'(z)$ in Theorem 3.1, then $\varphi(0) = 1$ and the assumption (i) of the theorem is equivalent to

$$\operatorname{Re}\left[\gamma\frac{\varphi(z)}{\phi(z)}\right] > 0, \quad z \in \mathrm{U}.$$

Since $\phi(z) \neq 0$, $\forall z \in \mathrm{U}$, from the above inequality it follows that $\varphi \in \mathscr{D}$ and then we obtain:

Corollary 4.1. *Let* $\beta, \gamma \in \boldsymbol{C}$ *with* $\beta > 0$ *and* $\operatorname{Re}\gamma > 0$, *let* $\phi \in \mathscr{D}$ *and suppose that*

$$\text{(i)} \quad \operatorname{Re}\left[\frac{z\phi'(z)}{\phi(z)} + \gamma\right] > 0, \quad z \in \mathrm{U}.$$

Let $k \in \mathscr{M}'_{1/\beta}$ *and* $f \in \mathscr{K}_{\phi+(1/\gamma)z\phi';\beta,\gamma}$. *Then*

$$f(z) \prec k(z) \Rightarrow \mathrm{I}_{\phi,\phi+(1/\gamma)z\phi';\beta,\gamma}[f](z) \prec k(z),$$

where

$$\mathrm{I}_{\phi,\phi+(1/\gamma)z\phi';\beta,\gamma}[f](z) = \left[\frac{\gamma}{z^\gamma\phi(z)}\int_0^z f^\beta(t)t^{\gamma-1}\left(\phi(t) + \frac{1}{\gamma}t\phi'(t)\right)\,\mathrm{d}t\right]^{1/\beta}. \tag{4.1}$$

Using this corollary in the special case $k(z) = f(z)$ we obtain the next example:

Example 4.1. Let $\beta, \gamma \in \boldsymbol{C}$ with $\beta > 0$ and $\operatorname{Re}\gamma > 0$, let $\phi \in \mathscr{D}$ and suppose that

$$\text{(i)}\quad \operatorname{Re}\left[\frac{z\phi'(z)}{\phi(z)} + \gamma\right] > 0, \quad z \in \mathrm{U}.$$

If $f \in \mathscr{K}_{\phi+(1/\gamma)z\phi';\beta,\gamma} \cap \mathscr{M}'_{1/\beta}$, then

$$\mathrm{I}_{\phi,\phi+(1/\gamma)z\phi';\beta,\gamma}[f](z) \prec f(z),$$

where $\mathrm{I}_{\phi,\phi+(1/\gamma)z\phi';\beta,\gamma}[f]$ is given by (4.1).

Our next result deals with a general class of averaging integral operators.

Corollary 4.2. *Let* $\beta, \gamma \in \boldsymbol{C}$ *with* $\beta \geq 1$ *and* $\operatorname{Re}\gamma > 0$, *let* $\phi \in \mathscr{D}$ *and suppose that*

$$\text{(i)}\quad \operatorname{Re}\left[\frac{z\phi'(z)}{\phi(z)} + \gamma\right] > 0, \quad z \in \mathrm{U}.$$

Then the integral operator $\mathrm{I}_{\phi,\phi+(1/\gamma)z\phi';\beta,\gamma}$ *given by* (4.1) *is an averaging operator on* $\mathscr{K}_{\phi+(1/\gamma)z\phi';\beta,\gamma}$.

Proof. If $f \in \mathscr{K}_{\phi+(1/\gamma)z\phi';\beta,\gamma}$, then from Remark 3.2 we have $\mathrm{I}_{\phi,\phi+(1/\gamma)z\phi';\beta,\gamma}[f](0) = f(0)$. Let consider an arbitrary convex function k such that $f(z) \prec k(z)$.

For the case $\beta = 1$ we have $k \in \mathscr{K}' \equiv \mathscr{M}'_1$ and, according to Corollary 4.1 we deduce that $\mathrm{I}_{\phi,\phi+(1/\gamma)z\phi';1,\gamma}[f](z) \prec k(z)$.

For the case $\beta > 1$, since $f(z) \prec k(z)$ and $f \in \mathscr{K}_{\phi+(1/\gamma)z\phi';\beta,\gamma}$ then $k(0) = f(0) = 0$. From (2.1) we have $k \in \mathscr{M}_1 \subset \mathscr{M}_{1/\beta} \subset \mathscr{M}'_{1/\beta}$ for $\beta > 1$, and using Corollary 4.1 we obtain that $\mathrm{I}_{\phi,\phi+(1/\gamma)z\phi';\beta,\gamma}[f](z) \prec k(z)$ for $\beta > 1$.

Now, from Lemma 2.2 we conclude that in the both two cases the integral operator $\mathrm{I}_{\phi,\phi+(1/\gamma)z\phi';\beta,\gamma}$ is an averaging operator on $\mathscr{K}_{\phi+(1/\gamma)z\phi';\beta,\gamma}$. □

Remark 4.1. Remark that this corollary generalizes Theorem 1 of [5] that may be obtained for the particular case $\phi(z) = 1$.

2. Taking $\phi(z) = \varphi(z) = e^{\lambda z}$ with $\lambda \in \boldsymbol{C}$ in Theorem 3.1, then $\phi \equiv \varphi \in \mathscr{D}$ and the condition (i) of the theorem reduces to $\operatorname{Re}(\lambda z + \gamma) > 0$, $\forall z \in \mathrm{U}$. Since this inequality holds whenever $\operatorname{Re}\gamma \geq |\lambda|$, we obtain:

Corollary 4.3. *Let* $\beta, \gamma \in \boldsymbol{C}$ *with* $\beta > 0$ *and* $\operatorname{Re}\gamma > 0$. *For a number* $\lambda \in \boldsymbol{C}$, *suppose in addition that* $\operatorname{Re}\gamma \geq |\lambda|$. *Let* $k \in \mathscr{M}'_{1/\beta}$ *and* $f \in \mathscr{K}_{e^{\lambda z};\beta,\gamma}$. *Then*

$$\left(\frac{\gamma}{\lambda z+\gamma}\right)^{1/\beta} f(z) \prec k(z) \Rightarrow \mathrm{I}_{e^{\lambda z}, e^{\lambda z}; \beta, \gamma}[f](z) \prec k(z),$$

where

$$\mathrm{I}_{e^{\lambda z}, e^{\lambda z}; \beta, \gamma}[f](z) = \left[\frac{\gamma}{z^{\gamma} e^{\lambda z}} \int_0^z f^{\beta}(t) t^{\gamma-1} e^{\lambda t}\, \mathrm{d}t\right]^{1/\beta}.$$

By the same reasons, if we take $\phi(z) = \varphi(z) = e^{\lambda z}$ with $\lambda \in \boldsymbol{C}$ in Corollary 4.2, we have:

COROLLARY 4.4. *Let* $\beta, \gamma \in \boldsymbol{C}$ *with* $\beta > 0$ *and* $\operatorname{Re}\gamma > 0$. *For a number* $\lambda \in \boldsymbol{C}$, *suppose in addition that* $\operatorname{Re}\gamma \geq |\lambda|$. *Then the integral operator* $\mathrm{I}_{e^{\lambda z}, e^{\lambda z}; \beta, \gamma}$ *given by*

$$\mathrm{I}_{e^{\lambda z}, e^{\lambda z} + (\lambda z/\gamma) e^{\lambda z}; \beta, \gamma}[f](z) = \left[\frac{\gamma}{z^{\gamma} e^{\lambda z}} \int_0^z f^{\beta}(t) t^{\gamma-1} \left(e^{\lambda z} + \frac{\lambda z}{\gamma} e^{\lambda z}\right) \mathrm{d}t\right]^{1/\beta}$$

is an averaging operator on $\mathscr{K}_{e^{\lambda z}; \beta, \gamma}$.

Remark 4.2. Note that this corollary also extends Theorem 1 of [5] that may be obtained for the special case $\lambda = 0$.

3. Considering $\phi(z) = 1 + \lambda z$, $\lambda \in \boldsymbol{C}$, then $\phi(0) = 1$ and for $|\lambda| \leq 1$ we have $\phi(z) \neq 0$, $\forall z \in \mathrm{U}$, i.e. $\phi \in \mathscr{D}$.

It is easy to check that the condition (i) of Theorem 3.1 becomes

$$\operatorname{Re}\left[\frac{\lambda z}{1+\lambda z} + \gamma\right] > 0, \quad z \in \mathrm{U}. \tag{4.2}$$

Letting $\chi(\zeta) = \zeta/(1+\zeta)$, since $\chi'(0) \neq 0$ and

$$\operatorname{Re}\frac{\zeta \chi''(\zeta)}{\chi'(\zeta)} + 1 = \operatorname{Re}\frac{1-\zeta}{1+\zeta} > 0, \quad |\zeta| < 1,$$

the function χ is a convex function in $D = \{\zeta \in \boldsymbol{C} : |\zeta| < |\lambda|\}$, if $|\lambda| \leq 1$. From the fact that $\chi(\bar{\zeta}) = \bar{\chi}(\zeta)$, $\zeta \in D$, we deduce that the function χ maps the disk D onto the convex domain $\chi(D)$ that is symmetric with respect the real axis. Hence

$$\inf\left\{\frac{\lambda z}{1+\lambda z} : z \in \mathrm{U}\right\} = t(-|\lambda|) = \frac{|\lambda|}{|\lambda| - 1}, \quad \text{for } |\lambda| < 1,$$

which shows that the condition (4.2) holds whenever $\operatorname{Re}\gamma \geq |\lambda|/(1-|\lambda|)$, for $|\lambda| < 1$. From here, by using Theorem 3.1 we have:

COROLLARY 4.5. *Let* $\beta, \gamma \in \boldsymbol{C}$ *with* $\beta > 0$ *and* $\operatorname{Re}\gamma > 0$. *Let* $\varphi \in \mathscr{D}$ *and for a number* $\lambda \in \boldsymbol{C}$ *with* $|\lambda| < 1$, *suppose in addition that*

$$\operatorname{Re}\gamma \geq \frac{|\lambda|}{1-|\lambda|}.$$

Let $k \in \mathcal{M}'_{1/\beta}$ *and* $f \in \mathcal{K}_{\varphi;\beta,\gamma}$. *Then*

$$\left[\frac{\varphi(z)}{1+((\gamma+1)/\gamma)\lambda z}\right]^{1/\beta} f(z) \prec k(z) \Rightarrow \mathrm{I}_{1+\lambda z,\varphi;\beta,\gamma}[f](z) \prec k(z).$$

By the same reasons, if we take $\phi(z) = 1 + \lambda z$ with $|\lambda| < 1$ in Corollary 4.2, we have:

Corollary 4.6. *Let* $\beta, \gamma \in \boldsymbol{C}$ *with* $\beta > 0$ *and* $\operatorname{Re}\gamma > 0$. *For a number* $\lambda \in \boldsymbol{C}$ *with* $|\lambda| < 1$, *suppose in addition that*

$$\operatorname{Re}\gamma \geq \frac{|\lambda|}{1-|\lambda|}.$$

Then the integral operator $\mathrm{I}_{1+\lambda z,1+((\gamma+1)/\gamma)\lambda z;\beta,\gamma}$ *given by*

$$\mathrm{I}_{1+\lambda z,1+((\gamma+1)/\gamma)\lambda z;\beta,\gamma}[f](z) = \left[\frac{\gamma}{z^{\gamma}(1+\lambda z)}\int_0^z f^{\beta}(t)t^{\gamma-1}\left(1+\frac{\gamma+1}{\gamma}\lambda t\right)\mathrm{d}t\right]^{1/\beta}$$

is an averaging operator on $\mathcal{K}_{1+((\gamma+1)/\gamma)\lambda z;\beta,\gamma}$.

Remark 4.3. This corollary also extends Theorem 1 of [5] that can be obtained for the particular case $\lambda = 0$.

4. If we take $\phi(z) = (1+z)^{2a}$ with $a \leq 0$ in Theorem 3.1, then the condition (i) of the theorem reduces to

$$\operatorname{Re}\left[\frac{2az}{1+z}+\gamma\right] > 0, \quad z \in \mathrm{U}.$$

If $a < 0$, the function $\chi(z) = 2az/(1+z)$ maps the unit disk U onto the half-plane $\Delta = \{w \in \boldsymbol{C} : \operatorname{Re} w > a\}$, and we deduce that the above inequality holds if and only if $\operatorname{Re}\gamma \geq -a$.

If $a = 0$, the same inequality holds for all $\gamma \in \boldsymbol{C}$ with $\operatorname{Re}\gamma > 0$, hence we obtain the next result:

Corollary 4.7. *Let* $\beta, \gamma \in \boldsymbol{C}$ *with* $\beta > 0$ *and* $\operatorname{Re}\gamma > 0$. *Let* $\varphi \in \mathcal{D}$ *and for a number* $a \leq 0$ *suppose in addition that*

$$\operatorname{Re}\gamma \geq -a.$$

Let $k \in \mathcal{M}'_{1/\beta}$ *and* $f \in \mathcal{K}_{\varphi;\beta,\gamma}$. *Then*

$$\left[\frac{\varphi(z)}{(1+z)^{2a-1}(1+((2a+\gamma)/\gamma)z)}\right]^{1/\beta} f(z) \prec k(z) \Rightarrow \mathrm{I}_{(1+z)^{2a},\varphi;\beta,\gamma}[f](z) \prec k(z).$$

Similarly, by taking $\phi(z) = (1+z)^{2a}$ with $a \le 0$ in Corollary 4.2 we have:

COROLLARY 4.8. *Let* $\beta, \gamma \in \boldsymbol{C}$ *with* $\beta > 0$ *and* $\operatorname{Re}\gamma > 0$. *For a number* $a \le 0$ *suppose in addition that*

$$\operatorname{Re}\gamma \ge -a.$$

Then the integral operator $\mathrm{I}_{(1+z)^{2a},\,(1+z)^{2a-1}(1+((2a+\gamma)/\gamma)z);\beta,\gamma}$ *given by*

$$\mathrm{I}_{(1+z)^{2a},\,(1+z)^{2a-1}(1+((2a+\gamma)/\gamma)z);\beta,\gamma}[f](z) = \left[\frac{\gamma}{z^{\gamma}(1+z)^{2a}}\int_0^z f^{\beta}(t)t^{\gamma-1}(1+t)^{2a-1}\left(1+\frac{2a+\gamma}{\gamma}t\right)\mathrm{d}t\right]^{1/\beta}$$

is an averaging operator on $\mathscr{K}_{(1+z)^{2a-1}(1+((2a+\gamma)/\gamma)z);\beta,\gamma}$.

Remark 4.4. For the particular case $a = 0$ we remark that this corollary represents Theorem 1 of [5].

REFERENCES

[1] S. S. MILLER AND P. T. MOCANU, Differential subordinations and univalent functions, Michigan Math. J., **28** (1981), 157–171.

[2] ———, Mean-value theorems in the complex plane, Babeş-Bolyai Univ. Fac. of Math. Res. Sem., Seminar on Geometric Function Theory, Preprint **5** (1986), 63–67.

[3] ———, Classes of univalent integral operators, J. Math. Anal. Appl., **157**, 1 (1991), 147–167.

[4] ———, Averaging operators and generalized Robinson differential inequality, J. Math. Anal. Appl., **13**, 2 (1993), 459–469.

[5] ———, A class of nonlinear averaging operators, J. Math. Anal. Appl., **197** (1996), 318–323.

[6] ———, Differential Subordinations. Theory and Applications, Marcel Dekker, New York, 1999.

[7] ——— AND M. O. READE, Bazilevič functions and generalized convexity, Rev. Roumaine Math. Pures Appl., **19** (1974), 213–224.

[8] ———, Starlike integral operators, Pacific J. Math., **79** (1978), 157–168.

[9] P. T. MOCANU, Une propriété de conexité generalisée dans la théorie de la representation conforme, Mathematica (Cluj), **11**, 34 (1969), 127–133.

[10] ———, Second order averaging operators for analytic functions, Rev. Roumaine Math. Pures Appl., **33**, 10 (1988), 875–881.

[11] Ch. POMMERENKE, Univalent Functions, Vanderhoeck and Ruprecht, Göttingen, 1975.

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