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ON TWO MEROMORPHIC FUNCTIONS THAT SHARE ONE VALUE CM

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Abstract

In this paper, we deal with the problem of uniqueness of meromorphic functions that share one finite value CM and obtain some results which improve some theorems given by R. Nevanlinna and R. Brück and are related to a result of H. X. Yi. Examples are provided to show that our results are sharp.

1. Introduction and results

In this paper the term "meromorphic" will always mean meromorphic in the complex plane. We use the standard notations and results of the Nevanlinna theory (See [1] or [2], for example). In particular, $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ except possibly for a set E of r of finite linear measure. We say that two non-constant meromorphic functions f and g share the value a CM (counting multiplicities), if f and g have the same a-points with the same multiplicity. Let k be a positive integer, we denote by $N_{k)}(r, 1/(f-a))$ the counting function of a-points of f with multiplicity $\le k$, by $N_{(k+1}(r, 1/(f-a))$ the counting function of a-points of f with multiplicity $> k$, and by $E_{1)}(a, f)$ the set of simple a-points of f.

In [3] R. Nevanlinna proved the following theorem:

Theorem A. *Let f and g be two non-constant entire functions satisfying $N(r, 1/f) = N(r, 1/g) = 0$. If f and g share the value 1 CM, then either $f = g$ or $fg = 1$.*

H. X. Yi [4] improved Theorem A and proved the following theorem:

Theorem B. *Let f and g be two non-constant meromorphic functions satisfying $\overline{N}(r, 1/f) + \overline{N}(r, f) = S(r, f)$ and $\overline{N}(r, 1/g) + \overline{N}(r, g) = S(r, g)$. If $E_{1)}(1, f) = E_{1)}(1, g)$, then either $f = g$ or $fg = 1$.*

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On the other hand R. Brück [5] proved the following theorem:

THEOREM C. *Let f be a non-constant entire function satisfying $N(r,1/f')=S(r,f)$. If f and f' share the value 1 CM, then $f-1=c(f'-1)$, where c is a nonzero constant.*

In this paper we improve the Theorem A and Theorem C and obtain the following results:

THEOREM 1. *Let f and g be two non-constant meromorphic functions satisfying $\overline{N}(r,1/g)+\overline{N}(r,g)=S(r,g)$ and $\overline{N}(r,f)=S(r,f)$. If f and g share the value 1 CM, then f and g satisfy one of the following*:

(i) $f-1=c(g-1)$, *where c is a nonzero constant. In particular, when* $c=1$, $f=g$;

(ii) $(f-b)g=1-b$, *where $b(\neq 1)$ is a constant. In particular, when $b=0$,* $fg=1$;

(iii) $T(r,f)=N_{2)}(r,1/f)+S(r,f)$ *and* $T(r,g)=N_{1)}(r,1/f')+S(r,f)$.

From Theorem 1, we deduce the following corollaries:

COROLLARY 1. *Let f and g be two non-constant meromorphic functions satisfying $\overline{N}(r,1/g)+\overline{N}(r,g)=S(r,g)$ and $\overline{N}(r,f)=S(r,f)$. If f and g share the value 1 CM, and if $T(r,f)\neq N_{2)}(r,1/f)+S(r,f)$, then either $f=g$ or $fg=1$.*

This improves a result of Theorem A and is related to Theorem B (see Remark 1).

COROLLARY 2. *Let f be a non-constant meromorphic function satisfying $\overline{N}(r,1/f')+\overline{N}(r,f)=S(r,f)$. If f and $f^{(k)}$ $(k\geq 1)$ share the value 1 CM, then $f-1=c(f^{(k)}-1)$, where c is a nonzero constant.*

It is obvious that Theorem C is a special case of Corollary 2.

Remark 1. The following example shows that the condition f and g share the value 1 CM in Theorem 1 or Corollary 1 cannot be replaced by the condition $E_{1)}(1,f)=E_{1)}(1,g)$:

Example 1. Let $f(z)=(e^z-1)(e^z+1)^2+1$ and $g(z)=e^z$. Obviously, $E_{1)}(1,f)=E_{1)}(1,g)$, $\overline{N}(r,1/g)+\overline{N}(r,g)=0$ and $\overline{N}(r,f)=0$. But the conclusion of Theorem 1 or Corollary 1 is not valid.

Remark 2. The following examples show that each of the above cases in Theorem 1 definitely happens:

Example 2. Let $f(z) = 1 + 2(e^z - 1)$ and $g(z) = e^z$. Then f and g satisfy the hypotheses of Theorem 1 and $f - 1 = 2(g - 1)$. Hence case (i) occurs.

Example 3. Let $f(z) = e^z + 5$ and $g(z) = -4e^{-z}$. Then f and g satisfy the hypotheses of Theorem 1 and $(f - 5)g = -4$. Hence case (ii) occurs.

Example 4. Let $f(z) = e^z(e^z - 1) + 1$ and $g(z) = e^z$. Then f and g satisfy the hypotheses of Theorem 1 and $T(r, f) = N_{1)}(r, 1/f) + S(r, f)$, $T(r, g) = N_{1)}(r, 1/f') + S(r, f)$. Hence case (iii) occurs.

Example 5. Let $f(z) = (e^z + 1)^2$ and $g(z) = -(1/2)e^z$. Then f and g satisfy the hypotheses of Theorem 1 and $T(r, f) = N_{2)}(r, 1/f) + S(r, f)$, $T(r, g) = N_{1)}(r, 1/f') + S(r, f)$. This also belongs to case (iii).

2. Some lemmas

For the proof of our results we need the following lemmas:

Lemma 1 [4]. *Let f be a non-constant meromorphic function. If $\overline{N}(r, 1/f) + \overline{N}(r, f) = S(r, f)$, then $T(r, f) = N_{1)}(r, 1/(f - 1)) + S(r, f)$.*

Lemma 2 [6]. *Let f and g be two non-constant meromorphic functions. If f and g share the value 1 CM, and if $A(f'/g') \neq ((f - 1)/(g - 1))^2$, for every nonzero constant A, then*

$$N_{1)}\left(r, \frac{1}{f - 1}\right) \leq \overline{N}\left(r, \frac{1}{f'}\right) + \overline{N}\left(r, \frac{1}{g'}\right) + \overline{N}(r, f) + \overline{N}(r, g) + S(r, f) + S(r, g).$$

Lemma 3 [7]. *Let f be a non-constant meromorphic function. If $\overline{N}(r, 1/f) + \overline{N}(r, f) = S(r, f)$, then $\overline{N}(r, 1/f') = S(r, f)$.*

3. The proofs

3.1 Proof of Theorem 1

Since f and g share the value 1 CM, we get by using the Lemma 1 that

$$\begin{aligned} T(r, g) &= N_{1)}\left(r, \frac{1}{g - 1}\right) + S(r, g) = N_{1)}\left(r, \frac{1}{f - 1}\right) + S(r, g) \\ &\leq T(r, f) + S(r, g). \end{aligned} \tag{3.1}$$

It follows that every $S(r, g)$ is also an $S(r, f)$. We consider the following meromorphic function:

$$h = \frac{f - 1}{g - 1}. \tag{3.2}$$

Since f and g share the value 1 CM, we may obtain from (3.2)

$$\overline{N}\left(r,\frac{1}{h}\right)+\overline{N}(r,h)\le\overline{N}(r,f)+\overline{N}(r,g),$$

by $\overline{N}(r,f)=S(r,f)$ and $\overline{N}(r,g)=S(r,g)$, we deduce that

$$\overline{N}\left(r,\frac{1}{h}\right)+\overline{N}(r,h)=S(r,f). \tag{3.3}$$

Differentiating (3.2) we obtain

$$f'=h'(g-1)+hg'. \tag{3.4}$$

If $f'(z_0)=0$ and $g(z_0)\neq 1$, then from (3.4), $0=h'(z_0)(g(z_0)-1)+h(z_0)g'(z_0)$. Then from (3.3) and (3.4) we deduce that

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f'}\right)&\le\overline{N}\left(r,\frac{1}{h'/h+g'/(g-1)}\right)+\overline{N}_{(2}\left(r,\frac{1}{g-1}\right)+S(r,f)\\
&\le T\left(r,\frac{h'}{h}+\frac{g'}{g-1}\right)+O(1)+\overline{N}_{(2}\left(r,\frac{1}{g-1}\right)+S(r,f)\\
&\le N\left(r,\frac{h'}{h}\right)+N\left(r,\frac{g'}{g-1}\right)+S(r,h)+S(r,g)\\
&\quad+\overline{N}_{(2}\left(r,\frac{1}{g-1}\right)+S(r,f)\\
&\le\overline{N}\left(r,\frac{1}{h}\right)+\overline{N}(r,h)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g-1}\right)\\
&\quad+\overline{N}_{(2}\left(r,\frac{1}{g-1}\right)+S(r,f)\\
&\le N\left(r,\frac{1}{g-1}\right)+S(r,f)\\
&\le T(r,g)+S(r,f).\end{aligned} \tag{3.5}$$

On the other hand, by Lemma 2, there are two cases that we need to observe separately.

Case I. $A(f'/g')=((f-1)/(g-1))^2$, for some nonzero constant A.

By integration, we get $A/(f-1)=1/(g-1)+B$, where B is a constant, which may be written in the form

$$f(1-B+Bg)=(A+B)(g-1)+1. \tag{3.6}$$

We consider three subcases.

Case I.1: $B = 0$. From (3.6), we see that $f - 1 = A(g - 1)$, which is (i).

Case I.2: $B = 1$. From (3.6), we find that $g[f - (1 + A)] = 1 - (1 + A)$, which is (ii).

Case I.3: $A = -B$. From (3.6), it follows that $f = -1/A[g - (1 + 1/A)]$. Hence $\overline{N}(r, f) = \overline{N}(r, 1/(g - (1 + 1/A))) = S(r, f)$. Then by the second fundamental theorem for g we deduce that if $A \neq -1$,

$$T(r, g) \leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g - (1 + 1/A)}\right) + \overline{N}(r, g) + S(r, g) \tag{3.7}$$
$$\leq S(r, g) + S(r, f) = S(r, f).$$

From (3.6), it is easy to see that $S(r, f) = S(r, g)$. Thus, by (3.7) we see that $T(r, g) = S(r, g)$, a contradiction. Hence we obtain that $A = -1$. Then by (3.6) we get $fg = 1$, which is (ii).

Now suppose $B \neq 0, 1$ and $A \neq -B$. Thus (3.6) reads

$$f = \frac{(A + B)[g - (1 - 1/(A + B))]}{B[g - (1 - 1/B)]}.$$

Hence $\overline{N}(r, f) = \overline{N}(r, 1/(g - (1 - 1/B))) = S(r, f)$. This contradicts the second fundamental theorem for g.

Case II. $A(f'/g') \neq ((f - 1)/(g - 1))^2$, for every nonzero constant A.

From Lemma 2, the hypotheses of Theorem 1 and Lemma 3 it follows that

$$N_{1)}\left(r, \frac{1}{g - 1}\right) \leq \overline{N}\left(r, \frac{1}{f'}\right) + S(r, f),$$

and so, from Lemma 1,

$$T(r, g) \leq \overline{N}\left(r, \frac{1}{f'}\right) + S(r, f). \tag{3.8}$$

The combination of (3.5) and (3.8) yields

$$T(r, g) = \overline{N}\left(r, \frac{1}{f'}\right) + S(r, f). \tag{3.9}$$

Since f and g share 1 CM, we see from (3.9) and Lemma 1 that

$$N\left(r, \frac{1}{f - 1}\right) = \overline{N}\left(r, \frac{1}{f'}\right) + S(r, f). \tag{3.10}$$

From (3.10) and the second fundamental theorem for f, we find that

$$T(r,f) \le N\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + N\left(r,\frac{1}{f-1}\right) - N\left(r,\frac{1}{f'}\right) + S(r,f)$$
$$\le N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f'}\right) - N\left(r,\frac{1}{f'}\right) + S(r,f).$$

It follows that

$$T(r,f) = N\left(r,\frac{1}{f}\right) + S(r,f) \quad \text{and} \quad N\left(r,\frac{1}{f'}\right) = \overline{N}\left(r,\frac{1}{f'}\right) + S(r,f).$$

From this we conclude that

$$T(r,f) = N_{2)}\left(r,\frac{1}{f}\right) + S(r,f) \quad \text{and} \quad N\left(r,\frac{1}{f'}\right) = N_{1)}\left(r,\frac{1}{f'}\right) + S(r,f).$$

From this and (3.9) we arrive at the conclusion (iii). ■

3.2 Proof of Corollary 1

By Theorem 1, we divide into the following two cases:

Case 1. $f - 1 = c(g-1)$, where c is a nonzero constant. This implies that $\overline{N}(r, 1/(f-(1-c))) = \overline{N}(r,1/g) = S(r,f)$. If $c \ne 1$, we get by using the second fundamental theorem for f

$$T(r,f) \le \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-(1-c)}\right) + \overline{N}(r,f) + S(r,f)$$
$$\le \overline{N}\left(r,\frac{1}{f}\right) + S(r,f).$$

It follows that $T(r,f) = N_{1)}(r,1/f) + S(r,f)$, which contradicts our hypotheses. Therefore, $c = 1$ and so $f = g$.

Case 2. $(f-b)g = 1-b$, where $b(\ne 1)$ is a constant. Similar to the proof of Case 1, we obtain $b = 0$. Hence $fg = 1$. ■

3.3 Proof of Corollary 2

Let $F = f$ and $G = f^{(k)}$. Then it is clear that $\overline{N}(r,F) = \overline{N}(r,f)$ and $\overline{N}(r,1/G) + \overline{N}(r,G) = \overline{N}(r,1/f^{(k)}) + \overline{N}(r,f)$. Note that

$$\overline{N}\left(r,\frac{1}{f^{(k)}}\right) = \overline{N}\left(r,\frac{f'}{f^{(k)}}\cdot\frac{1}{f'}\right) \le \overline{N}\left(r,\frac{f'}{f^{(k)}}\right) + \overline{N}\left(r,\frac{1}{f'}\right)$$
$$\le N\left(r,\frac{f^{(k)}}{f'}\right) + \overline{N}\left(r,\frac{1}{f'}\right) + S(r,f)$$
$$\le k\overline{N}\left(r,\frac{1}{f'}\right) + (k-1)\overline{N}(r,f) + S(r,f) = S(r,f).$$

Thus from the hypotheses of Corollary 2, it follows readily that F and G share the value 1 CM, $\overline{N}(r,1/G)+\overline{N}(r,G)=S(r,F)$ and $\overline{N}(r,F)=S(r,F)$. From this, it is not difficult to see that $S(r,F)=S(r,G)$. Applying Theorem 1 to F and G, we divide into the following three cases:

Case 1. $F-1=c(G-1)$, where c is a nonzero constant, so that $f-1=c(f^{(k)}-1)$, is what we wanted.

Case 2. $G(F-b)=1-b$, where $b(\neq 1)$ is a constant, so that

$$f^{(k)}(f-b)=1-b. \tag{3.11}$$

From this a short calculation with Laurent series shows that

$$N(r,f)=S(r,f). \tag{3.12}$$

Using (3.11) again we obtain $\overline{N}(r,1/(f-b))=S(r,f)$, from this and the second fundamental theorem for f, we have

$$m\left(r,\frac{1}{f-1}\right)=S(r,f). \tag{3.13}$$

Set

$$\Delta=\frac{f^{(k)}-1}{f-1}. \tag{3.14}$$

Keeping in mind that f and $f^{(k)}$ share 1 CM, we see that (3.12), (3.13) and (3.14) imply that

$$T(r,\Delta)=S(r,f). \tag{3.15}$$

Then from (3.11) and (3.14) it follows that

$$\Delta f^2-(\Delta+b\Delta-1)f-(1+b\Delta)=0.$$

From this, (3.15) we conclude $T(r,f)=S(r,f)$ which is a contradiction.

Case 3. $T(r,f^{(k)})=N_{1)}(r,1/f')+S(r,f)$. Since $\overline{N}(r,1/f')=S(r,f)$, we have $T(r,f^{(k)})=S(r,f)=S(r,f^{(k)})$ which is a contradiction. ■

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