AFFINE SUBMANIFOLDS AND THE THEOREM OF CARTAN-AMBROSE-HICKS

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Abstract

In this article E. Cartan's theorem about the existence of (local) totally geodesic submanifolds with a prescribed tangent plane is generalized to manifolds which are equipped only with a linear connection; there is also given a global version of the theorem. The results are used for a short and geometric proof of the Theorem of Cartan-Ambrose-Hicks, and more generally of a generalization which is concerned with the existence of affine maps of arbitrary rank.

Manifolds with a linear connection can be regarded as generalizations of affine spaces (in the sense of linear algebra). Therefore, we will shortly call them *affine manifolds*. The morphisms of the category of affine manifolds are the *affine maps*, as they are defined in [KN] I, p. 225; and in this spirit it is convenient to speak of *affine submanifolds* instead of auto-parallel ones; see [KN] II.

As affine submanifolds and affine maps essentially are determined by their 1-jet at one point, it is a natural question to ask for criteria which guarantee the local and global existence of affine submanifolds resp. affine maps corresponding to given linear initial data at one point. In riemannian geometry these local questions where partially already treated by E. Cartan in [C]. His work was continued by W. Ambrose, N. J. Hicks and R. Hermann, see [A, Hi1, Hi2, He]. The most important result in this context is known as the Theorem of Cartan-Ambrose-Hicks which states the global equivalence of simply connected, geodesically complete affine manifolds under suitable hypotheses; see [Hi1, W]. It is a special case of Theorem 4 of this article, in which the dimension of the manifolds may be different, and the global affine map we look for may be neither an immersion nor a submersion. Theorem 2 is a local version of this result. In

*This article is dedicated to Professor Peter Dombrowski. The second author had a fruitful collaboration over 20 years with him. After his retirement Prof. Dombrowski continued to inspire the authors by his interest in their work. Furthermore, this article proves again the great advantage of using covariant differentiation for vector fields *along maps*: This technique he developped (extending J. L. Koszul's method) when giving a course on differential geometry at MIT in 1961 and which, under his influence, was published later in [GKM]. At last we thank him for communicating to us the two examples of special affine manifolds $(\boldsymbol{R}^3, \nabla)$ described in Sections 1 and 2.

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both cases the affine maps, we are looking for, are constructed via their graphs which have to be affine submanifolds of the affine product space of the manifolds being involved. In other terms, we derive these results from the local resp. global existence theorem for affine submanifolds (Theorem 1 and Theorem 3). As is well known, global theorems of this kind are in general obtained by gluing together local solutions, mostly a tedious work. We avoid it by using an idea of K. Tsukada from his paper [T] on totally geodesic submanifolds of riemannian spaces, where these submanifolds are obtained via integral manifolds of a distribution $\mathscr{D}$ which is defined on the Graßmann bundle of the ambient space and which in general is not completely integrable. Theorem 5 shed more light upon this distribution by characterizing those "points" where $\mathscr{D}$ is involutive. By this result we get a second very short proof of the "smooth and handy" version of Theorem 1 which, in fact, is the basis for Theorems 2–4.

It should be mentioned that Theorems 1–4 can be found also in [Hi2]. But Hicks' existence theorems for affine maps rely on additional assumptions which are shown here to be unnecessary; moreover, his proof of the local existence theorem for affine submanifolds remains incomplete as an example below shows. The essential tool which fills the latter gap is Lemma 2 of this article. In addition, it opens the possibility for this very geometric treatment.

1. Preliminaries and notations

In this article (M, ∇) and $(\tilde{M}, \tilde{\nabla})$ denote two connected, affine manifolds with torsion and curvature tensor T and R resp. $\tilde{T}$ and $\tilde{R}$ and with the exponential map exp resp. $\widetilde{\exp}$. We put $m := \dim M$. Furthermore, let $p_0 \in M$ and $\tilde{p}_0 \in \tilde{M}$ be two fixed points. We are interested in the existence of an affine submanifold N through p_0 with a prescribed tangent space $T_{p_0}N$ and the existence of an affine map f from a neighbourhood $U = U(p_0)$ into $\tilde{M}$ mapping p_0 to $\tilde{p}_0$ with a prescribed tangent map $T_{p_0}f : T_{p_0}M \to T_{\tilde{p}_0}\tilde{M}$. For the formulation of the results the following notions are important.

Definition 1. Let points $p \in M$, $\tilde{p} \in \tilde{M}$, a linear subspace $V \subset T_pM$ and a linear map $A : T_pM \to T_{\tilde{p}}\tilde{M}$ be given. We say that V is *torsion and curvature invariant* iff V satisfies $T(V, V) \subset V$ and $R(V, V)V \subset V$, and that A *preserves the torsion and curvature tensor fields* iff $A(T(u, v)) = \tilde{T}(Au, Av)$ and $A(R(u, v)w) = \tilde{R}(Au, Av)Aw$ for all $u, v, w \in T_pM$. Furthermore, we put $V(A) := \{(v, Av) \mid v \in T_pM\} \subset T_{(p, \tilde{p})}(M \times \tilde{M})$.

Let us recall the definition of an affine map $f : M \to \tilde{M}$: It is a C^∞ map whose differential commutes with covariant differentiation[1], i.e., $\tilde{\nabla}_X f_* Y = f_* \nabla_X Y$. Consequently, the image $f \circ \gamma$ of each geodesic γ of M is a geodesic of $\tilde{M}$, and the image f_*Z of a parallel vector field Z along a curve $\alpha : J \to M$ is a parallel

[1] For the covariant differentiation of vector fields along maps see [P] p. 36.

vector field along $f \circ \alpha$. Therefore, the parallel translation and the differential of f commute in the following sense:

$$(1) \qquad (T_{\alpha(t_2)}f) \circ \left(\overset{t_2}{\underset{t_1}{\|}} \alpha \right) = \left(\overset{t_2}{\underset{t_1}{\|}} f \circ \alpha \right) \circ (T_{\alpha(t_1)}f) \quad \text{for all } t_1, t_2 \in J.$$

Furthermore, for each point $p \in M$ the differential $T_p f$ preserves the torsion and curvature tensor fields. It should be mentioned that, conversely, the affinity of f follows from (1).

We say that an (immersed) submanifold N of (M, ∇) is *affine*, if it can be equipped with a covariant derivative ∇^N such that the inclusion map $(N, \nabla^N) \hookrightarrow (M, \nabla)$ becomes affine, in other terms, if $\nabla_X Y \in \mathfrak{X}(N)$ for all $X, Y \in \mathfrak{X}(N)$. Because of (1), a submanifold N is affine iff its tangent bundle TN is invariant under parallel translation in M along curves $\alpha : J \to N$. If N is an affine submanifold, then each geodesic γ of N is also a geodesic of M (in other words, affine submanifolds are totally geodesic), and each tangent space $T_p N$ is torsion and curvature invariant. As mentioned in the introduction, in [KN] II affine submanifolds are said to be *auto-parallel*.

Obviously the image $f(N)$ of every injective affine immersion $f : N \to M$ is an affine submanifold of M. Therefore, affine immersions represent "affine submanifolds with self-intersections". In a riemannian manifold the affine submanifolds are exactly the totally geodesic ones, but in affine manifolds this equivalence is not true as is demonstrated by the following example due to E. Cartan: Let ∇^0 be the canonical covariant derivative in $\boldsymbol{R}^3$ and denote by $v \times w$ the cross product in this space. Then $\nabla_X Y = \nabla^0_X Y + X \times Y$ defines another linear connection on $\boldsymbol{R}^3$, and in the affine manifold $(\boldsymbol{R}^3, \nabla)$ the (usual) planes are totally geodesic, but not affine submanifolds; notice that the geodesics of $(\boldsymbol{R}^3, \nabla)$ are again the straight lines.

As we have seen, affine maps and affine submanifolds are related to each other very closely. In the following Proposition 1 we will describe a further relation; for that we remind the reader to the affine product $M^\times := M \times \tilde{M}$ of the affine manifolds M and $\tilde{M}$: It is the C^∞ product manifold with the unique covariant derivative $\nabla^\times$ such that the canonical projections $\mathrm{pr} : M^\times \to M$ and $\widetilde{\mathrm{pr}} : M^\times \to \tilde{M}$ are affine maps. The geodesics of $(M^\times, \nabla^\times)$ are the curves $(\gamma, \tilde{\gamma}) : J \to M^\times$ where $\gamma : J \to M$ and $\tilde{\gamma} : J \to \tilde{M}$ are geodesics of M resp. $\tilde{M}$; and its torsion and curvature tensors $T^\times$ resp. $R^\times$ are described by $T^\times((u, \tilde{u}), (v, \tilde{v})) = (T(u, v), \tilde{T}(\tilde{u}, \tilde{v}))$ and $R^\times((u, \tilde{u}), (v, \tilde{v}))(w, \tilde{w}) = (R(u, v)w, \tilde{R}(\tilde{u}, \tilde{v})\tilde{w})$.

Proposition 1.

(a) *A linear map $A : T_p M \to T_{\tilde{p}} \tilde{M}$ preserves the torsion and curvature tensor fields if and only if $V(A)$ is torsion and curvature invariant with respect to $\nabla^\times$.*

(b) *A C^∞-map $f : U \to \tilde{M}$ from an open subset $U \subset M$ into $\tilde{M}$ is affine if and only if its graph is an affine submanifold of the affine product $M^\times$.*

The simple proof is left to the reader.

2. Local Theorems

At first we are interested in the existence of a (local) affine submanifold fitting to given initial data (p_0, V), where p_0 is a point of M and V a linear subspace of $T_{p_0}M$. For its construction we choose a normal neighbourhood $\exp_{p_0} : U^T \to U$ of p_0 in M; here U^T is a star shaped neighbourhood of 0 in $T_{p_0}M$ on which the exponential map is a diffeomorphism into M.

Let us fix the following notations: For every $u \in T_{p_0}M$ let J_u denote the interval $\{t \in \boldsymbol{R} \mid tu \in U^T\}$ and $\gamma_u : J_u \to M$ the geodesic $t \mapsto \exp_{p_0}(tu)$. Furthermore, for every $u \in U^T$ let us abbreviate $p(u) := \exp_{p_0}(u)$, $\tau_u := \|_0^1 \gamma_u$ the parallel translation along γ_u from $T_{p_0}M$ to $T_{p(u)}M$ and $V_u := \tau_u(V) \subset T_{p(u)}M$. As $\exp_{p_0}(V \cap U^T)$ is the union $\bigcup_{u \in V} \gamma_u(J_u)$, we call the regular submanifold $\exp_{p_0}(V \cap U^T)$ the *geodesic umbrella* associated to the data (p_0, V).

THEOREM 1. *In the above situation the geodesic umbrella* $N := \exp_{p_0}(V \cap U^T)$ *is an affine submanifold of* M *if and only if for every* $u \in V \cap U^T$ *one has*[2]

$$T(\dot\gamma_u(1), V_u) \subset V_u \quad \textit{and} \quad R(\dot\gamma_u(1), V_u)V_u \subset V_u. \tag{2}$$

Of course, condition (2) *is satisfied, if* V_u *is torsion and curvature invariant.*

The riemannian version of the theorem is attributed to E. Cartan. The above version can be found in [Hi2, Theorem 9], but his proof is incomplete as we indicate after the proof of Theorem 1.

If the geodesic umbrella N is affine, its construction implies $V_u = T_{p(u)}N$ for every $u \in V \cap U^T$ and therefore, the linear subspaces V_u satisfy (2). Thus, it remains to prove that (2) implies the affinity of the umbrella N. This step is based on two lemmas, the first of which is also used by Hicks (see [Hi2, Proposition 2]).

LEMMA 1. *For every* $u \in U^T \backslash \{0\}$ *and* $v \in T_{p_0}M$ *the infinitesimal variation* $Y^v : t \mapsto F_*(\partial/\partial s)|_{(0,t)}$ *of the geodesic variation* $F : (s,t) \mapsto \exp_{p_0}(t \cdot (u + sv)) = \gamma_{u+sv}(t)$ *is the Jacobi field along the geodesic* γ_u *satisfying the initial data* $Y^v(0) = 0$ *and* $(\nabla_\partial Y^v)(0) = v$.

Remember that in (M, ∇) the Jacobi fields along a geodesic γ are the solutions of the differential equation

$$\nabla_\partial \nabla_\partial Y = R(\dot\gamma, Y)\dot\gamma + \nabla_\partial(T(\dot\gamma, Y)), \tag{3}$$

see [KN] II, p. 63. It should also be mentioned that ∂ denotes the canonical unit vector field of $\boldsymbol{R}$.

LEMMA 2. *Let be given a vector* $u \in U^T \backslash \{0\}$, *a Jacobi field* Y *along* γ_u

[2] Notice that the single vectors in (2) form the radial vector field $p(u) \mapsto \dot\gamma_u(1)$.

satisfying $Y(0) = 0$ *and a vector field* $X \in \mathfrak{X}(U)$, *which is parallel along every geodesic* γ_v $(v \in T_{p_0}M)$. *Then the vector field* $\nabla_Y X : t \mapsto \nabla_{Y(t)} X$ *satisfies the differential equation*

$$\nabla_\partial \nabla_Y X = R(\dot{\gamma}_u, Y)(X \circ \gamma_u).$$

Proof. If we put $v := (\nabla_\partial Y)(0)$ and define F and Y^v as in Lemma 1, then we have $Y = Y^v$, hence

$$\nabla_{Y(t)} X = (\nabla_{\partial/\partial s}(X \circ F))|_{(0,t)}.$$

Because of $X \circ F(s,t) = X \circ \gamma_{u+sv}(t)$ the parallelity of X along the geodesic rays implies

$$\nabla_{\partial/\partial t}(X \circ F)|_{(s,t)} = \nabla_{\partial_t}(t \mapsto X \circ F(s,t)) = 0.$$

Therefore, the structure equation for the curvature tensor (see [P] p. 83) gives

$$\begin{aligned}(\nabla_\partial \nabla_Y X)(t) &= (\nabla_{\partial/\partial t} \nabla_{\partial/\partial s}(X \circ F))|_{(0,t)} \\ &= \left(R\left(F_* \frac{\partial}{\partial t}, F_* \frac{\partial}{\partial s} \right)(X \circ F) + \nabla_{\partial/\partial s} \nabla_{\partial/\partial t}(X \circ F) \right)\Big|_{(0,t)} \\ &= R(\dot{\gamma}_u(t), Y(t))(X \circ \gamma_u(t)). \qquad \square\end{aligned}$$

Proof for Theorem 1 "$\Leftarrow$". At the moment we fix three vectors $u, v, w \in V$ with $u \in U^T \backslash \{0\}$, use the Jacobi field Y^v and the map F from Lemma 1. Because of $u + sv \in V$ the image of the map F lies in the geodesic umbrella N. Therefore, Lemma 1 shows

$$\forall t \in J_u\colon\ Y^v(t) \in T_{\gamma_u(t)} N. \tag{4}$$

In the following we use the development $z : J_u \to T_{p_0}M$ of vector fields Z along γ_u in the sense of Cartan given by $z(t) := \tau_{tu}^{-1} Z(t)$, which satisfies

$$\forall t \in J_u\colon\ (\nabla_\partial Z)(t) = \tau_{tu}(z'(t)). \tag{5}$$

Besides, for $t \in J_u$ we also define the tensors $\hat{T}_u(t)$ and $\hat{R}_u(t)$ on $T_{p_0}M$ by

$$\hat{T}_u(t)(x) := \tau_{tu}^{-1} T(\dot{\gamma}_{tu}(1), \tau_{tu}(x)) \quad \text{and} \quad \hat{R}_u(t)(x,y) := \tau_{tu}^{-1} R(\dot{\gamma}_{tu}(1), \tau_{tu}(x)) \tau_{tu}(y)$$

for all $x, y \in T_{p_0}M$. According to condition (2) the linear subspace V is invariant with respect to these tensors, i.e.:

$$\hat{T}_u(t)(V) \subset V \quad \text{and} \quad \hat{R}_u(t)(V,V) \subset V. \tag{6}$$

If now y denotes the development of the Jacobi field Y^v, then the Jacobi equation (3) implies that y is a solution of the linear differential equation

$$y''(t) = \hat{R}_u(t)(y(t), u) + (\hat{T}_u(y))'(t)$$

with the initial values $y(0) = 0$, $y'(0) = v \in V$. Combining this with (6) we obtain

$$y(J_u) \subset V, \quad \text{hence } \forall t \in J_u\colon\ Y^v(t) \in V_{tu}. \tag{7}$$

Now let us vary the vector v: As our consideration takes place in the normal neighbourhood U, there are no conjugate points along $\gamma_u : J_u \to U$; combining (4) and (7) we therefore obtain

$$V_u = T_{p(u)}N \quad \text{for every } u \in V \cap U^T. \tag{8}$$

We continue the argumentation with the Jacobi field Y^v and apply formula (5) to the development z of the vector field $Z : t \mapsto \nabla_{Y^v(t)} X^w$, where X^w denotes that radially parallel vector field with $X^w(p_0) = w$. According to Lemma 2 we get

$$z'(t) = \hat{R}_u(t)(y(t), w).$$

Since $z(0) = 0 \in V$ (because $Y^v(0) = 0$ and therefore $Z(0) = 0$), we obtain from (7) and (8)

$$z(J_u) \subset V, \quad \text{hence in particular } \nabla_{Y^v(1)} X^w \in V_u = T_{p(u)}N.$$

Repeating the argument of the absense of conjugate points, we find $\nabla_X(X^w|N) \in \mathfrak{X}(N)$ for every $X \in \mathfrak{X}(N)$. If now $(w_1, \ldots, w_r)$ is a basis of V, then $(X^{w_1}|N, \ldots, X^{w_r}|N)$ is a frame field of the tangent bundle TN, and therefore, we finally get $\nabla_X Y \in \mathfrak{X}(N)$ for all $X, Y \in \mathfrak{X}(N)$; thereby the affinity of the geodesic umbrella N is proved. $\square$

It should be mentioned that the proof of Theorem 9 in [Hi2] runs exactly along the above lines (with some other notation), but it ends at formula (8). Thereby Hicks has only proved $\nabla_v Y \in TN$ for vectors $v \in TN$ in *radial directions* and arbitrary vector fields $Y \in \mathfrak{X}(N)$. The following example shows that his proof is really incomplete: Let $E \in \mathfrak{X}(\boldsymbol{R}^3)$, $\langle \cdot , \cdot \rangle$ and ∇^0 denote the radial vector field defined by $E_p \cong p$, the canonical riemannian metric of $\boldsymbol{R}^3$ and its Levi-Civita connection, respectively. With respect to the covariant derivative $\nabla_X Y = \nabla^0_X Y + \langle Y, E \rangle \cdot X \times E$ every 2-dimensional linear subspace $N \subset \boldsymbol{R}^3$ is a geodesic umbrella with center $p_0 = 0$ satisfying formula (8) with $V := T_0 N$; nevertheless, N is no affine submanifold of $(\boldsymbol{R}^3, \nabla)$.

From Theorem 1 we will now derive a criterion on the existence of a local affine map for which at one point the differential is prescribed by a linear map $A : T_{p_0} M \to T_{\tilde{p}_0} \tilde{M}$. For that we choose normal neighbourhoods $\exp_{p_0} : U^T \to U$ and $\widetilde{\exp}_{\tilde{p}_0} : \tilde{U}^T \to \tilde{U}$ of p_0 in M resp. of $\tilde{p}_0$ in $\tilde{M}$ with $A(U^T) \subset \tilde{U}^T$.

As at the beginning of this section, for every $\tilde{u} \in T_{\tilde{p}_0} \tilde{M}$ we define the interval $\tilde{J}_{\tilde{u}}$ and the geodesic $\tilde{\gamma}_{\tilde{u}} : \tilde{J}_{\tilde{u}} \to \tilde{U}$. Because we have $J_u \subset \tilde{J}_{Au}$, we can consider the geodesic $(\gamma_u, \tilde{\gamma}_{Au}) : J_u \to M^\times$ for every $u \in T_{p_0} M$. In addition for every $u \in U^T$ we define the linear map $A_u := (\|_0^1 \tilde{\gamma}_{Au}) \circ A \circ (\|_1^0 \gamma_u) : T_{\gamma_u(1)} M \to T_{\tilde{\gamma}_{Au}(1)} \tilde{M}$. Notice, if there exists an affine map $f : U \to \tilde{M}$ with $T_{p_0} f = A$, then we have $\tilde{\gamma}_{Au} = f \circ \gamma_u$ for every u, hence $A_u = T_{\gamma_u(1)} f$; thus in this case the maps A_u preserve the torsion and curvature tensor fields. Therefore, the assumption of the following theorem is necessary.

Theorem 2. *If in the above situation for every $u \in U^T$ the linear map A_u preserves the torsion and curvature tensor fields, then*

$$f := \widetilde{\exp}_{\tilde{p}_0} \circ A \circ (\exp_{p_0}|U^T)^{-1} : U \to \tilde{M}$$

is an affine map satisfying $f(p_0) = \tilde{p}_0$ and $T_{p_0}f = A$.

This result is a generalized local version of the Theorem of Cartan-Ambrose-Hicks in which the map A is assumed to be an isomorphism (see [KN] I, p. 257). The general version was also proved in [Hi2], but under additional unpleasant assumptions. The idea to prove this theorem via the graph of f was born, when we read a remark of R. Hermann in [He].

Proof. Of course, f is a C^∞ map satisfying $f(p_0) = \tilde{p}_0$ and $T_{p_0}f = A$; its graph is the geodesic umbrella $N := \exp^\times_{(p_0,\tilde{p}_0)}(V(A) \cap (U^T \times \tilde{U}^T))$ in the affine product $M^\times = M \times \tilde{M}$ (for $V(A)$ see Definition 1). According to Proposition 1(b) it remains to prove the affinity of N in $M^\times$. Starting from the linear subspace $V(A) = T_{(p_0,\tilde{p}_0)}N \subset T_{(p_0,\tilde{p}_0)}M^\times$ we define the linear subspaces $V(A)_{(u,Au)}$ analogously to the construction at the beginning of this section. It is easy to see that $V(A)_{(u,Au)}$ coincides with $V(A_u)$. As every map A_u preserves the torsion and curvature tensor fields (because of the assumptions), we know from Proposition 1(a) that the subspaces $V(A)_{(u,Au)}$ are torsion and curvature invariant. Therefore, Theorem 1 implies the affinity of the submanifold N. □

3. Graßmann bundle

For each point $p \in M$ let L_pM be the set of frames of T_pM, which we describe by isomorphisms $u : \boldsymbol{R}^m \to T_pM$ as in [KN] I, p. 56; by $\pi : LM \to M$ we denote the entire frame bundle.

For some fixed number $r \in \{1, \dots, m-1\}$ let $\tau : G_r(TM) \to M$ denote the Graßmann bundle; its fibre over p is the Graßmann manifold $G_r(T_pM)$ of the r-dimensional subspaces $V \subset T_pM$. This bundle is associated to the frame bundle via the map

$$\varrho : LM \times G_r(\boldsymbol{R}^m) \to G_r(TM), \quad (u, V) \mapsto u(V).$$

For each $u \in L_pM$ the map $\varrho_u : G_r(\boldsymbol{R}^m) \to G_r(T_pM), V \mapsto u(V)$ is a diffeomorphism; and if $\boldsymbol{V} \in G_r(\boldsymbol{R}^m)$ denotes the subspace which is spanned by the first canonical unit vectors $e_1, \dots, e_r \in \boldsymbol{R}^m$, then the fibre bundle morphism

$$\varrho^{\boldsymbol{V}} : LM \to G_r(TM), \quad u \mapsto u(\boldsymbol{V}) = \operatorname{span}\{u(e_1), \dots, u(e_r)\} \tag{9}$$

is a surjective submersion (even a principal fibre bundle). The linear connection $\mathscr{H}(LM)$ of LM corresponding to ∇ induces a connection $\mathscr{H}(G_r(TM))$ (in the sense of Ehresmann) on the Graßmann bundle (see [KN] I, p. 87); it is given by

$$\mathscr{H}_{\varrho(u,\boldsymbol{V})}(G_r(TM)) = \varrho^{\boldsymbol{V}}_* \mathscr{H}_u(LM) \subset T_{\varrho(u,\boldsymbol{V})}(G_r(TM)) \quad (u \in LM). \tag{10}$$

This vector bundle $\mathscr{H}(\mathrm{G}_r(TM))$ contains a canonical subbundle $\mathscr{D}$ characterized by

$$\tau_*\mathscr{D}_V = V \quad (V \in \mathrm{G}_r(TM)); \tag{11}$$

see [T] p. 400; of course, $\mathscr{D}$ is a subbundle of $T(\mathrm{G}_r(TM))$ of rank r; in general it is not completely integrable.

Proposition 2.

(a) *If $\alpha : J \to M$ is a broken C^∞ curve and $V \in \mathrm{G}^r(T_pM)$ a subspace with $0 \in J$ and $p = \alpha(0)$, then the horizontal lift V_α of α in $\mathrm{G}_r(TM)$ with the initial value $V_\alpha(0) = V$ is given by the parallel displacement of V in (M, ∇) along α, i.e., $V_\alpha(t) = (\|_0^t \alpha)(V) \subset T_{\alpha(t)}M$.*

(b) *In the situation* (a) *the curve V_α is tangential to $\mathscr{D}$ iff $\dot\alpha(t) \in V_\alpha(t)$ for all $t \in J$.*

(c) *If in the situation* (a) *α is a geodesic with $\dot\alpha(0) \in V$, then V_α is already tangential to $\mathscr{D}$.*

(d) *For every two sections $X, Y \in \Gamma(\mathscr{D})$ and every element $V \in \mathrm{G}_r(TM)$ the relation $\nabla_{X(V)}\tau_* Y \in V$ holds. Therefore, every integral manifold S of $\mathscr{D}$ can be equipped uniquely with a covariant derivative ∇^S such that $\tau|S : (S, \nabla^S) \to (M, \nabla)$ becomes an affine immersion.*

(e) *If $f : N \to M$ is an affine map of rank r, then its Gauß map $g^f : N \to \mathrm{G}_r(TM)$, $p \mapsto f_* T_pN$ is tangential to $\mathscr{D}$; therefore, $S := g^f(N)$ is an integral manifold of $\mathscr{D}$ and g^f is a submersion onto S, which is affine with respect to the covariant derivative ∇^S described in* (e).

Remark 1. The statements (d) and (e) are generalizations of Tsukada's Theorem 3.1 in [T].

Proof. *For* (a)–(c): Because of (9) we can write $V = u(\boldsymbol{V})$ with some $u \in \mathrm{L}_{\alpha(0)}M$. As $\tilde\alpha : t \mapsto (\|_0^t \alpha) \circ u$ is the horizontal lift of α in $\mathrm{L}M$ with $\tilde\alpha(0) = u$, the horizontal lift V_α is $\varrho^{\boldsymbol{V}} \circ \tilde\alpha : t \mapsto (\|_0^t \alpha)(u(\boldsymbol{V})) = (\|_0^t \alpha)(V)$.—Since $\tau_* \mid \mathscr{H}_{V_\alpha(t)}(\mathrm{G}_r(TM))$ is an isomorphism onto $T_{\alpha(t)}M$, the statement (b) follows from $\tau \circ V_\alpha = \alpha$ and (11). The statement (c) is now clear, because the tangent vector field of a geodesic is parallel.

For (d): If $X, Y \in \Gamma(\mathscr{D})$ and $V \in \mathrm{G}_r(TM)$ are given, let c be the maximal integral curve of X with $c(0) = V$. Then c is a horizontal curve over $\alpha := \tau \circ c$, hence $c = V_\alpha$ because of (a); therefore for every $t \in J$ we obtain $\tau_* Y \circ c(t) \in \tau_*\mathscr{D}_{c(t)} = c(t) = V_\alpha(t) = (\|_0^t \alpha)(V)$. Therefrom we derive $\nabla^M_{X(V)}\tau_* Y = (\nabla^M_\partial(\tau_* Y \circ c))(0) \in V$.—If now S is an integral manifold of $\mathscr{D}$, then (according to the statement just proved) for all $X, Y \in \mathfrak{X}(S)$ and $V \in S$ there exists a vector $Z(V) \in \mathscr{D}_V = T_V S$ such that $\nabla_{X(V)}\tau_* Y = \tau_* Z(V)$. In this way a vector field $Z \in \mathfrak{X}(S)$ is defined. By $\nabla^S_X Y := Z$ we get the covariant derivative in question.

For (e): If $p \in N$ and $v \in T_pN$ are given, we choose a curve $\alpha :]-1, 1[\to N$ with $\dot\alpha(0) = v$ and put $V := g^f(p)$. From (1) we get $g^f \circ \alpha(t) = (\|_0^t(f \circ \alpha))(V) = V_{f\circ\alpha}(t)$ and additionally $(\mathrm{d}/\mathrm{d}t)(f \circ \alpha(t)) \in g^f(\alpha(t)) = V_{f\circ\alpha}(t)$. Applying (b) we find that the curve $g^f \circ \alpha$ is tangential to $\mathscr{D}$, in particular $g^f_* v = (\dot{\mathrm{d}}/\mathrm{d}t)|_{t=0} \cdot (g^f \circ \alpha(t)) \in \mathscr{D}_{g^f(p)}$. Hence, the map g^f is tangential to $\mathscr{D}$.—According to the rank theorem, for every point $p \in N$ we can find a neighbourhood $U = U(p)$ such that $L := f(U)$ is a submanifold of M and $s := f|U : U \to L$ a submersion onto L. Obviously we have $g^f|U = g^i \circ s$ where i denotes the inclusion map $L \hookrightarrow M$. Now, g^i is an injective immersion, which can be considered as a parametrization of $g^f(U)$; because g^f is tangential to $\mathscr{D}$, $g^f(U)$ therefore is an integral manifold of $\mathscr{D}$. Consequently, the entire set $S := g^f(N)$ is an integral manifold of $\mathscr{D}$. Since $f = (\tau|S) \circ g^f$ and $\tau|S$ are affine maps, it is easily seen that also $g^f : N \to S$ is an affine submersion. $\square$

Remark 2. By the way we have proved that every affine map can globally be written as the composition $i \circ s$ of an affine submersion s and an affine immersion i; this result is already known from [LR, Theorem 1].

4. Global Theorems

We will now prove a globalization of Theorem 1. For that we introduce a further notation. Let again a point $p_0 \in M$ and a linear subspace $V \in \mathrm{G}_r(T_{p_0}M)$ be fixed. If $\gamma : [0, b_\gamma] \to M$ is a broken geodesic with $\gamma(0) = p_0$ and with the "break points" $0 < t_1 < \cdots < t_n < b_\gamma$, let $V_\gamma : [0, b_\gamma] \to \mathrm{G}_r(TM)$ denote the horizontal lift of γ as in Proposition 2(a). Furthermore, put $t_0 := 0$, $t_{n+1} := b_\gamma$ and $\gamma_i := \gamma \,|\, [t_i, t_{i+1}]$. By $\Gamma(M, V)$ we denote the set of all such broken geodesics such that $\dot{\gamma}_i(t_i) \in V_\gamma(t_i)$ holds for every i; then automatically one has $\dot{\gamma}_i(t) \in V_\gamma(t)$ for every $t \in [t_i, t_{i+1}]$. In other terms, the elements of $\Gamma(M, V)$ are those broken geodesics $\gamma : [0, b_\gamma] \to M$ emanating from p_0 for which V_γ is tangential to the subbundle $\mathscr{D}$ everywhere (see Proposition 2(b)).

Theorem 3. *If in the above situation for every geodesic* $(\gamma : [0, b_\gamma] \to M) \in \Gamma(M, V)$ *the linear subspace* $V_\gamma(b_\gamma)$ *is torsion and curvature invariant, then there exists one* (*and up to an affine diffeomorphism exactly one*) *geodesically closed, affine immersion* $F : N \to M$ *from a simply connected affine manifold* N *and a point* $q_0 \in N$ *such that* $F(q_0) = p_0$ *and* $F_* T_{q_0} N = V$.

The attribute *geodesically closed* means: For every maximal geodesic $\tilde{c} : J \to N$ the image $F \circ \tilde{c} : J \to M$ is a maximal geodesic, too. *If* M *is geodesically complete, then* F *is geodesically closed if and only if* N *is geodesically complete.*

In the special case of a complete riemannian manifold M Theorem 3 is due to R. Hermann [He]. He used strongly the Theorem of Hopf-Rinow and remarked that therefore he could not see how to generalize the theorem to affine manifolds. Theorem 3 proves that even the geodesical completeness of M is not needed.

Proof. We start with an arbitrary normal neighbourhood $\exp_{p_0}$:

$U^T \to U$ of p_0 in M. Then we can apply Theorem 1 and find that the geodesical umbrella $N_0 = \exp_{p_0}(V \cap U^T)$ is an affine submanifold of M. According to Proposition 2(e) the image $S_0 := g^{N_0}(N_0)$ of its Gauß map $g^{N_0} : p \mapsto T_p N_0$ is an integral manifold of $\mathscr{D}$ with $V \in S_0$. Let S be the maximal connected integral manifold of $\mathscr{D}$ which contains S_0 (see [N, Theorem 4] or [BH, Theorem 1.3 and 1.4], a proof of the paracompactness of S can be found in [LR] p. 94). Because of Proposition 2(d) $\tau|S$ is an affine immersion with respect to a suitable covariant derivative of S. Of course, we have $\tau(V) = p_0$ and $\tau_* T_V S = \tau_* \mathscr{D}_V = V$.

Now the crucial point is to prove that $\tau|S$ is geodesically closed. For that, let a maximal geodesic $\tilde{c} : \tilde{J} \to S$ be given. As $\tau \circ \tilde{c}$ is a geodesic in M, it can be extended to a maximal geodesic $c : J \to M$. Let us assume $\delta := \sup \tilde{J} < \sup J$. Then we choose some broken geodesic $\tilde{\gamma} : [0, d] \to S$ starting from $\tilde{\gamma}(0) = V$ with $\tilde{\gamma}(d) \in \tilde{c}(\tilde{J})$. We may assume $\tilde{\gamma}(d) = \tilde{c}(d)$. Then the horizontal lift V_γ of the broken geodesic

$$\gamma : [0, \delta] \to M, \quad t \mapsto \begin{cases} \tau \circ \tilde{\gamma}(t) & \text{for } t \in [0, d] \\ c(t) & \text{for } t \in]d, \delta] \end{cases}$$

(with initial point p_0) is given by $V_\gamma(t) = \tilde{\gamma}(t)$ for $t \in [0, d]$ and $V_\gamma(t) = \tilde{c}(t)$ for $t \in]d, \delta[$. In particular, we have $\dot{c}(d) = \tau_* \dot{\tilde{c}}(d) \in \tau_* T_{\tilde{c}(d)} S = \tau_* \mathscr{D}_{\tilde{c}(d)} = \tilde{c}(d) = V_\gamma(d)$. As the analogous argument holds for the other break points of γ, we find $\gamma \in \Gamma(M, V)$. Therefore $V_\gamma : [0, \delta] \to \mathrm{G}_r(TM)$ is tangential to $\mathscr{D}$, i.e., $V_\gamma(\delta)$ is a good candidate in order to continue $\tilde{c}$. For realizing this idea we choose some normal neighbourhood $\exp_{p_1} : U_1^T \to U_1$ of $p_1 := c(\delta)$. Because of the hypothesis of Theorem 3, we can again apply Theorem 1 replacing V by $V_1 := V_\gamma(\delta)$; hence the geodesic umbrella $N_1 = \exp_{p_1}(V_1 \cap U_1^T)$ is an affine submanifold of M and its "Gauß image" $S_1 := g^{N_1}(N_1)$ is a further integral manifold of $\mathscr{D}$ containing the "point" $V_1 \in S_1$. If we now choose some $\varepsilon > 0$ such that $J(\delta) :=]\delta - \varepsilon, \delta + \varepsilon[\subset]d, \sup J[$ and $c(J(\delta)) \subset U_1$, then $c(J(\delta))$ lies in N_1 because $\dot{c}(\delta) \in V_1$. Furthermore, since N_1 is an affine submanifold we get for all $t \in]\delta - \varepsilon, \delta[$

$$g^{N_1} \circ c(t) = T_{c(t)} N_1 = \left(\overset{t}{\underset{\delta}{\|}} c \right)(T_{p_1} N_1) = \left(\overset{t}{\underset{\delta}{\|}} c \right)(V_1) = V_\gamma(t) = \tilde{c}(t), \tag{12}$$

hence $\tilde{c}(t) \in S \cap S_1$. Therefore, S_1 is a subset of S and g^{N_1} an affine diffeomorphism into S. Therefrom we conclude that $g^{N_1} \circ c \,|\, J(\delta)$ is a geodesic in S continuing $\tilde{c}$ beyond δ (because of (12)) in contradiction to the maximality of $\tilde{c}$. Thus we have proved $\sup \tilde{J} = \sup J$. In the same way we get $\inf \tilde{J} = \inf J$, hence $\tilde{J} = J$.

In order to define the affine immersion $F : N \to M$ of Theorem 3 we use the universal covering $\varphi : N \to S$ of S and put $F := (\tau|S) \circ \varphi$. Let us now prove the uniqueness of F. For that let $\tilde{F} : \tilde{N} \to M$ be another affine immersion and $\tilde{q}_0 \in \tilde{N}$ a point which have the same properties as F and q_0. According to Proposition 2(e) its Gauß map $g^{\tilde{F}}$ is a local affine diffeomorphism into S satisfying $\tau \circ g^{\tilde{F}} = \tilde{F}$. Since $\tilde{N}$ is simply connected and we have $g^{\tilde{F}}(\tilde{q}_0) = V =$

$\varphi(q_0)$, there exists a local affine diffeomorphism $f : \tilde{N} \to N$ such that $\varphi \circ f = g^{\tilde{F}}$ and $f(\tilde{q}_0) = q_0$. From the construction we get $F \circ f = \tilde{F}$. As $\tilde{F}$ is geodesically closed, f is geodesically closed, too. Therefore, according to the following Lemma f is a covering map, in fact even an affine diffeomorphism because of the simple connectedness of N. □

Lemma 3. *If N and $\tilde{N}$ are connected affine manifolds of the same dimension, then each geodesically closed, affine local diffeomorphism $f : \tilde{N} \to N$ is a covering map.*

This lemma is a generalization of Hicks' Theorem 3 in [Hi1], in which N is assumed to be geodesically complete. One can follow Hicks' proof; where he uses the geodesically completeness of $\tilde{N}$ the argumentation keeps valid if instead of that we use that f is geodesically closed.

Now we will also derive a global version of Theorem 2. Let again a linear map $A : T_{p_0}M \to T_{\tilde{p}_0}\tilde{M}$ be given. If we suppose the affine manifold $\tilde{M}$ to be geodesically complete, then for every broken geodesic $\gamma : [0, b_\gamma] \to M$ with $\gamma(0) = p_0$ there exists a unique broken geodesic $\tilde{\gamma} : [0, b_\gamma] \to \tilde{M}$ with $\tilde{\gamma}(0) = \tilde{p}_0$ such that the following is true: If $0 < t_1 < \cdots < t_n < b_\gamma$ are the "break points" of γ, $t_0 := 0$, $t_{n+1} := b_\gamma$, $\gamma_i := \gamma \,|\, [t_i, t_{i+1}]$ and $\tilde{\gamma}_i := \tilde{\gamma} \,|\, [t_i, t_{i+1}]$ and if we define

$$A_\gamma(t) := \left(\overset{t}{\underset{0}{\|}}\,\tilde{\gamma}\right) \circ A \circ \left(\overset{0}{\underset{t}{\|}}\,\gamma\right) : T_{\gamma(t)}M \to T_{\tilde{\gamma}(t)}\tilde{M}$$

for every $t \in [0, b_\gamma]$, then $\tilde{\gamma}$ has no other "break points" than $t_1, \ldots, t_n$ and

$$\dot{\tilde{\gamma}}_i(t_i) = A_\gamma(t_i)(\dot{\gamma}_i(t_i)) \tag{13}$$

holds for every $i = 0, \ldots, n$. Another characteriziation of $\tilde{\gamma}$ is the following: If $C : [0, b_\gamma] \to T_{p_0}M$ denotes the development of γ in $T_{p_0}M$ (see [KN] I), then $\tilde{\gamma}$ is the broken geodesic with the development $A \circ C$.

Theorem 4. *Let us assume that M is simply connected and $\tilde{M}$ geodesically complete and that the linear map $A : T_{p_0}M \to T_{\tilde{p}_0}\tilde{M}$ has the following property: For every broken geodesic $\gamma : [0, b_\gamma] \to M$ emanating from p_0 the linear map $A_\gamma(b_\gamma)$ (defined above) preserves the torsion and curvature tensors. Then there exists one and only one affine map $f : M \to \tilde{M}$ with $f(p_0) = \tilde{p}_0$ and $T_{p_0}f = A$.*

This result generalizes the Theorem of Cartan-Ambrose-Hicks, in which A is supposed to be an isomorphism. It should be mentioned that under this condition f is a covering map if also M is geodesically complete, and a diffeomorphism if in addition also $\tilde{M}$ is simply connected. It should also be noticed that Hicks has treated this general case in [Hi2] but under additional unpleasant assumptions.

Proof. As in the proof of Theorem 2 one proves

$$V(A_\gamma(t)) = V(A)_{(\gamma,\tilde{\gamma})}(t) \tag{14}$$

in the situation described before Theorem 4; here $V(A)_{(\gamma,\tilde{\gamma})}(t)$ is constructed in the affine product $M^\times = M \times \tilde{M}$ by starting from the linear subspace $V(A)$ (see Definition 1); it means it is the corresponding horizontal lift of the geodesic $(\gamma,\tilde{\gamma}) : [0, b_\gamma] \to M^\times$ in the Graßmann bundle $\mathrm{G}_m(TM^\times)$. Condition (13) implies $(\gamma,\tilde{\gamma}) \in \Gamma(M^\times, V(A))$ (see the beginning of this section); and every geodesic of the latter set is obtained in this way. Furthermore, Proposition 1(a) and (14) show that $V(A)_{(\gamma,\tilde{\gamma})}(b_\gamma)$ is torsion and curvature invariant, because $A_\gamma(b_\gamma)$ preserves the torsion and curvature tensors (as supposed in Theorem 4). Therefore, we can apply Theorem 3 and obtain: There exists an affine, geodesically closed immersion $F : N \to M^\times$ from a simply connected affine manifold N into $M^\times$ and a point $q_0 \in N$ such that $F(q_0) = (p_0, \tilde{p}_0)$ and $F_* T_{q_0} N = V(A)$. The map $g := \mathrm{pr} \circ F : N \to M$ is then an affine map between m-dimensional manifolds; in particular, it has constant rang $\mathrm{rk}\, g$. Because $g_* T_{q_0} N = \mathrm{pr}_*(F_* T_{q_0} N) = \mathrm{pr}_*\, V(A) = T_{p_0} M$ we get $\mathrm{rk}\, g = m$, that means, g is an affine local diffeomorphism. Since F is geodesically closed, it is easily seen that also g is geodesically closed. Therefore we can apply Lemma 3 and find that g is an affine covering map, in fact even an affine diffeomorphism because of the simple connectedness of M. Now $f := \tilde{\mathrm{pr}} \circ F \circ g^{-1} : M \to \tilde{M}$ is "the" affine map possessing all properties which are stated in Theorem 4. □

Remark 3. If the affine manifolds (M, ∇) and $(\tilde{M}, \tilde{\nabla})$ have parallel torsion and curvature tensors, then the parallel translation along curves in these manifolds are isomorphisms preserving the torsion and curvature tensors. Therefore, in this situation it is easier to fulfill the hypotheses of Theorem 1–4: If V is torsion and curvature invariant (resp. if A preserves the torsion and curvature tensors), then automatically the subspaces V_u and $V_\gamma(b_\gamma)$ of Theorem 1 resp. 2 are torsion and curvature invariant (resp. the linear maps A_u and $A_\gamma(b_\gamma)$ of Theorem 2 resp. 4 preserve the torsion and curvature tensors). Important examples of such manifolds are the *reductive homogeneous spaces*; see [KN] II. For them we immediately deduce the following corollary from Theorem 3, which generalizes the well known result about the 1:1 correspondence between totally geodesic submanifolds of symmetric spaces and Lie triple systems.

Corollary. *Let $M = G/H$ be a reductive homogeneous space with origin $o \in M$ and the* $\mathrm{Ad}(H)$*-invariant splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G; we equip M with the canonical linear connection* (*see* [KN] II, *p.* 192). *Furthermore, let $\mathfrak{n}$ be a linear subspace of $\mathfrak{m}$ and $V \subset T_o M$ its image under the canonical isomorphism $\mathfrak{m} \to T_o M$. In this situation there exists a affine submanifold in M corresponding to the initial data (o, V) if and only if*

$$\forall X, Y, Z \in \mathfrak{n}\colon\ ([X, Y]_{\mathfrak{m}} \in \mathfrak{n} \text{ and } [[X, Y]_{\mathfrak{h}}, Z] \in \mathfrak{n}). \tag{15}$$

Furthermore, if (15) *is satisfied, then there exists an affine immersion $F : N \to M$ with the properties described in Theorem* 3 (*replacing p_0 by o*).

Remark 4. The manifold N of the corollary can also be given the structure of a homogeneous reductive space whose canonical covariant derivative coincides with the original one: As the torsion and the curvature tensor of the latter covariant derivative are parallel again, the Theorem of Cartan-Ambrose-Hicks can be used to show that the Lie group $\mathfrak{A}(N)$ of all affine diffeomorphisms $N \to N$ acts transitively on N, and its Lie algebra $\mathfrak{a}(N)$ can be identified with a Lie subalgebra of $\mathfrak{X}(N)$ (see [KN] I, p. 232). The Lie algebra of the isotropy group $\mathfrak{A}(N)_{q_0}$ with respect to some point $q_0 \in F^{-1}(\{o\})$ is $\mathfrak{h}_{q_0} := \{X \in \mathfrak{a}(N) \mid X_{q_0} = 0\}$; and $\mathfrak{m}_{q_0} := \{X \in \mathfrak{a}(N) \mid \forall v \in T_{q_0}N : \nabla_v X = T(v, X_{q_0})\}$ can be shown to be an $\mathrm{Ad}(\mathfrak{A}(N)_{q_0})$-invariant subspace being complementary to $\mathfrak{h}_{q_0}$; in this way $N \cong \mathfrak{A}(N)/\mathfrak{A}(N)_{q_0}$ is given the structure of a reductive homogeneous space. The idea for the construction of $\mathfrak{m}_{q_0}$ is the following: For every $u \in T_{q_0}N$ and $s \in \boldsymbol{R}$ there exists a unique element $f_s \in \mathfrak{A}(N)$ with initial value $f_s(q_0) = \gamma_u(s)$ and with differential $T_{q_0}f_s = \|_0^s \gamma_u$. (In the theory of symmetric spaces f_s is called a *transvection* along the geodesic γ_u, see [C] p. 266.) Then (f_s) is a 1-parameter subgroup of $\mathfrak{A}(N)$ with some generator A. The corresponding fundamental vector field $X := A^*$ is an element of the Lie algebra $\mathfrak{a}(N)$ satisfying $X_{q_0} = u$ and $\nabla_v X = T(v, u)$ for all $v \in T_{q_0}N$. With this insight it is easy to show that the above subspace $\mathfrak{m}_{q_0}$ has the stated properties.

Moreover, if $\Phi : \mathfrak{m}_{q_0} \to \mathfrak{m}$ denotes the linear map induced by the differential $T_{q_0}F : T_{q_0}N \to T_oM$ of the immersion F of the Corollary, then for every $X \in \mathfrak{m}_{q_0}$ and for the 1-parameter subgroups $\gamma_X : \boldsymbol{R} \to \mathfrak{A}(N)$ resp. $\gamma_{\Phi(X)} : \boldsymbol{R} \to \mathfrak{A}(M)$ induced by X resp. $\Phi(X)$ we have $F \circ \gamma_X(t) = \gamma_{\Phi(X)}(t) \circ F$ for all $t \in \boldsymbol{R}$. Hence, at least "partially" F behaves like a morphism of homogeneous spaces.

5. Involutivity of the subbundle $\mathscr{D}$

Let us recall that by definition the subbundle $\mathscr{D} \subset T(\mathrm{G}_r(TM))$ is *involutive* at some "point" $V \in \mathrm{G}_r(TM)$ iff $[X, Y](V) \in \mathscr{D}_V$ for all $X, Y \in \Gamma(\mathscr{D})$, where $[X, Y](V)$ denotes the value of the Lie bracket $[X, Y]$ at the point V.

THEOREM 5. *The subbundle $\mathscr{D}$ is involutive at the point $V \in \mathrm{G}_r(TM)$ if and only if the subspace V is torsion and curvature invariant.*

Proof. At first a general remark: If a fibre bundle $\tau : E \to M$ is equipped with an Ehresmann connection $\mathscr{H} \subset TE$, then we assign to it the tensor field Ω of type $(1, 2)$ on E characterized by the following equation:

$$\forall X, Y \in \mathfrak{X}(E) \colon\ \Omega(X, Y) = -[X_{\mathscr{H}}, Y_{\mathscr{H}}]_{\mathscr{V}};$$

here $\mathscr{V}$ denotes the vertical subbundle kern τ_* and the indices $\mathscr{H}$ and $\mathscr{V}$ mean that one has to regard the horizontal resp. vertical part of the respective vector field. This tensor field Ω is called the *curvature form* of $\mathscr{H}$. For every $X, Y \in \Gamma(\mathscr{H})$ and $e \in E$ we have

$$[X, Y](e) \in \mathscr{H}_e \Leftrightarrow \Omega(X(e), Y(e)) = 0. \tag{16}$$

Applying this construction on the connections $\mathscr{H}(\mathrm{L}M)$ and $\mathscr{H}(\mathrm{G}_r(TM))$ of Section 3 we come to curvature forms which we denote by $\Omega_{\mathrm{L}M}$ and $\Omega_{\mathrm{G}_r(TM)}$. They are related to each other by

$$
(17)\qquad \varrho_*^V(\Omega_{\mathrm{L}M}(\xi_1,\xi_2)) \\ = \Omega_{\mathrm{G}_r(TM)}(\varrho_*^V\xi_1,\varrho_*^V\xi_2) \quad \text{for all } u\in \mathrm{L}M \text{ and } \xi_1,\xi_2\in\mathscr{H}_u(\mathrm{L}M).
$$

For the proof of (17) one takes notice of (9) and (10) and uses similar arguments as for the proof of the structure equation for the curvature tensor (e.g. see [P] p. 83).

The curvature form $\Omega_{\mathrm{G}_r(TM)}$ is of interest for us because of formula (16) which in combination with (11) implies: $\mathscr{D}$ is involutive at $V\in\mathrm{G}_r(TM)$ if and only if

$$
(18)\qquad \forall X_1,X_2\in\Gamma(\mathscr{D})\colon\ (\Omega_{\mathrm{G}_r(TM)}(X_1(V),X_2(V))=0 \text{ and } \tau_*[X_1,X_2](V)\in V).
$$

Now we show that for all $p\in M$, $V\in\mathrm{G}_r(T_pM)$ and $w_1,w_2\in\mathscr{H}_V(\mathrm{G}_r(TM))$ we have

$$
(19)\qquad \Omega_{\mathrm{G}_r(TM)}(w_1,w_2)=\frac{\dot{\mathrm{d}}}{\mathrm{d}t}\bigg|_{t=0}(\exp(t\cdot R(v_1,v_2))(V)) \quad \text{with } v_i:=\tau_*w_i.
$$

Here $R(v_1,v_2)$ is considered as an endomorphism of T_pM; hence, $t\mapsto\exp(t\cdot R(v_1,v_2))$ is a 1-parameter subgroup of $\mathrm{GL}(T_pM)$ and $t\mapsto \exp(t\cdot R(v_1,v_2))(V)$ a curve in $\mathrm{G}_r(T_pM)$.

For the proof of (19) we use (9) in order to choose a $u\in\mathrm{L}_pM$ with $u(\boldsymbol{V})=V$. Furthermore, let $\xi_i\in\mathscr{H}_u(\mathrm{L}M)$ be the horizontal lift of v_i; then we have $\varrho_*^V\xi_i=w_i$. If now $\omega: T(\mathrm{L}M)\to\mathrm{End}(\boldsymbol{R}^m)$ denotes the connection form of $\mathscr{H}(\mathrm{L}M)$, then $\omega\circ\Omega_{\mathrm{L}M}$ is the usual corresponding curvature form with values in $\mathrm{End}(\boldsymbol{R}^m)$ satisfying

$$
\omega(\Omega_{\mathrm{L}M}(\xi_1,\xi_2))=u^{-1}\circ R(v_1,v_2)\circ u \quad \text{because } \pi_*\xi_i=v_i
$$

(see [P] p. 282/286 or [KN] I, p. 133). Calling the definition of ω in our mind we get

$$
\begin{aligned}
\Omega_{\mathrm{L}M}(\xi_1,\xi_2)&=\frac{\dot{\mathrm{d}}}{\mathrm{d}t}\bigg|_{t=0}(u\circ\exp(t\cdot(u^{-1}\circ R(v_1,v_2)\circ u)))\\
&=\frac{\dot{\mathrm{d}}}{\mathrm{d}t}\bigg|_{t=0}(\exp(t\cdot R(v_1,v_2))\circ u),
\end{aligned}
$$

and therefore (because of (17))

$$
\begin{aligned}
\Omega_{\mathrm{G}_r(TM)}(w_1,w_2)=\varrho_*^V(\Omega_{\mathrm{L}M}(\xi_1,\xi_2))&=\frac{\dot{\mathrm{d}}}{\mathrm{d}t}\bigg|_{t=0}(\varrho^V(\exp(t\cdot R(v_1,v_2))\circ u))\\
&=\frac{\dot{\mathrm{d}}}{\mathrm{d}t}\bigg|_{t=0}(\exp(t\cdot R(v_1,v_2))(V));
\end{aligned}
$$

thus (19) is verified. Now, it is easy to prove:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\exp(t\cdot R(v_1,v_2))(V))=0 \Leftrightarrow \exp(t\cdot R(v_1,v_2))(V)\equiv V \Leftrightarrow R(v_1,v_2)(V)\subset V.$$

Hence, using (10) we get

$$(20)\qquad (\forall X_1, X_2 \in \Gamma(\mathscr{D}) : \Omega_{\mathrm{G}_r(TM)}(X_1(V), X_2(V))=0) \Leftrightarrow R(V,V)V\subset V.$$

On the other hand we obtain from Proposition 2(d) and the structure equation for the torsion

$$\forall X_1, X_2 \in \Gamma(\mathscr{D})\colon\ (\tau_*[X_1,X_2](V)\in V \Leftrightarrow T(\tau_*X_1(V),\tau_*X_2(V))\in V),$$

hence,

$$(\forall X_1, X_2 \in \Gamma(\mathscr{D}) : \tau_*[X_1,X_2](V)\in V) \Leftrightarrow T(V,V)\subset V.$$

Combining this result with (20) we finish the proof because of (18). □

Theorem 5 enables us to give an alternative proof of a "smooth and handy" version of Theorem 1. For that we use the following generalization of a result of F. Nübel; [N, Theorem 1].

Proposition 3. *Let M be a C^∞ manifold and $\mathscr{D}$ a vector subbundle of TM. Furthermore, let V be a linear space, $U\subset V$ a star shaped neighbourhood of* 0 *and $\varphi : U\to M$ a C^∞ map. For every $u\in V$ put $J_u := \{t\in \boldsymbol{R} \mid tu\in U\}$ and define $\beta_u : J_u\to V$, $t\mapsto tu$. If then $\mathscr{D}$ is involutive at all points $p\in\varphi(U)$ and if all curves $\varphi\circ\beta_u$ are tangential to $\mathscr{D}$, then the entire map φ is tangential to $\mathscr{D}$.*

Remark 5. Because in this proposition there are made no special assumptions on the subbundle $\mathscr{D}$, this result can be applied in many situations. For instance, we will use it in a forthcoming paper to prove an analogue of Theorem 1 for spherically bent submanifolds of a riemannian space.

The proof of Proposition 3 is based on the following result of Blumenthal and Hebda (see [BH] p. 165).

Lemma 4. *In the situation of Proposition* 3 *let I and J denote open intervals containing* 0, *$F : I\times J\to M$ a C^∞ map and Y the vector filed $t\mapsto F_*(\partial/\partial s)|_{(0,t)}$ along the curve $\alpha : J\to M$, $t\mapsto F(0,t)$. We suppose that for every $s\in I$ the curve $\alpha_s : t\mapsto F(s,t)$ is tangential to $\mathscr{D}$, the subbundle $\mathscr{D}$ is involutive at the points of $F(I\times J)$ and that $Y(0)\in\mathscr{D}_{\alpha(0)}$. Then $Y(t)$ lies in $\mathscr{D}_{\alpha(t)}$ for all $t\in J$.*

Proof of Proposition 3. For $p\in U$ and $v\in T_pV\cong V$ we apply Lemma 4 on the map $F : (s,t)\mapsto\varphi(t(p+sv))$. Because of $Y(0)=0\in\mathscr{D}_{\varphi(0)}$ we get $\varphi_*v = (\mathrm{d}/\mathrm{d}s)|_{s=0}\varphi(p+sv) = Y(1)\in\mathscr{D}_{\varphi(p)}$. □

As announced, we give now an alternative proof of Theorem 1 under the natural, but slightly stronger hypothesis that for every $u \in V \cap U^T$ the subspace V_u is torsion and curvature invariant.

We put $r := \dim V$. Then $\varphi : V \cap U^T \to G_r(TM)$, $u \mapsto V_u$ is an injective C^∞ immersion. The proof of the differentiability of φ runs along the lines of the construction of bundle charts in the proof of Ehresmann's fibre bundle theorem in [E]. If we define β_u as in Proposition 3, we obtain $\varphi \circ \beta_u : t \mapsto V_{tu}$, which is the horizontal lift of the geodesic γ_u (see Proposition 2(a)). According to Proposition 2(c) this curve is tangential to the subbundle $\mathscr{D}$ which is involutive at every "point" of the r-dimensional submanifold $S := \varphi(V \cap U^T) \subset G_r(TM)$ because of the hypothesis and Theorem 5. Proposition 3, therefore, shows that S is an integral manifold of $\mathscr{D}$. As $\tau \circ \varphi = \exp_{p_0} | (V \cap U^T)$, the map $\tau | S$ is an affine immersion onto the geodesic umbrella $N = \exp_{p_0}(V \cap U^T)$ according to Proposition 2(d). Consequently N is an affine submanifold of M.

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