A. PARMEGGIANI AND M. WAKAYAMA
KODAI MATH. J.
**25** (2002), 254–277

# A REMARK ON SYSTEMS OF DIFFERENTIAL EQUATIONS ASSOCIATED WITH REPRESENTATIONS OF $\mathfrak{sl}_2(\boldsymbol{R})$ AND THEIR PERTURBATIONS

Alberto Parmeggiani and Masato Wakayama

**Abstract**

We give here a number of examples of non-commutative harmonic oscillators "in disguise" that can be exactly solved by using the tensor product of the oscillator representation and the finite dimensional representation of $\mathfrak{sl}_2(\boldsymbol{R})$, and its perturbations.

## 1. Introduction

The aim of this paper[1] is to produce a variety of examples of systems, similar to those introduced in [4], [5] and [6], that can be exactly solved by a (quasi/perturbed) *creation-annihilation* procedure, in the sense we can determine the spectrum, eigenfunctions and give deformations that produce (or destroy) multiplicity (that, as we well see, is bounded depending on the dimension of the associated finite dimensional representation) in an *invariant* way, that is depending **only** on the representation and **not** on the particular realization of the system, by using a suitable tensor product representation of $\mathfrak{sl}_2(\boldsymbol{R})$. Due to the particular nature of the systems considered here, we remark that we will **not** give any new results from the ODE (or PDE) viewpoint, but we will give a different interpretation of the spectral quantities associated with such systems.

We now make ideas more precise by considering an example.

Let $\{X^+, H, X^-\}$ be the standard basis of $\mathfrak{sl}_2(\boldsymbol{R})$. Let $(\omega, \mathscr{S}(\boldsymbol{R}))$ be the *oscillator representation*, let $\psi$, resp. $\psi^\dagger$, be the annihilation, resp. creation, operators (see (3) below), and let $\varphi_n := (\psi^\dagger)^n e^{-x^2/2}$ (essentially, the Hermite functions). Let $I_2$ be the $2 \times 2$ identity matrix of $\boldsymbol{C}^2$. With $D_x = -i\partial_x$, let us consider

$$L(x, D_x) = I_2\left(\frac{D_x^2 + x^2}{2}\right) + A, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

[1] This paper was referred to in the bibliography of [6] with the title *On Certain Systems of Differential Equations Associated with Lie-Algebra Representations and Their Perturbations.*

Received January 9, 2002.

as an unbounded self-adjoint operator in $L^2(\boldsymbol{R};\boldsymbol{C}^2)$ with domain $B^2(\boldsymbol{R};\boldsymbol{C}^2)$ (it then has discrete spectrum). Of course, the spectrum of $I_2(D_x^2+x^2)/2$ is given by the numbers $\{n+1/2; n\in\boldsymbol{Z}_+\}$, with multiplicity 2, and eigenfunctions $\varphi_n\otimes e_j$, $j=1,2$, where $e_1, e_2$ is the canonical basis of $\boldsymbol{C}^2$. How does the presence of $A$ affect the spectrum of $I_2(D_x^2+x^2)/2$? One may easily compute the spectrum of $L(x,D_x)$ as follows. Since $e_1, e_2$ are also eigenvectors of $A$, one gets that the spectrum of $L(x,D_x)$ is given by the numbers $n+1/2\pm 1$, $n\in\boldsymbol{Z}_+$, and a "natural" basis of eigenfunctions by the functions $\varphi_n\otimes e_k$, $n\in\boldsymbol{Z}_+$, $k=1,2$, as above. From the representation viewpoint, we are dealing with the $\mathfrak{sl}_2(\boldsymbol{R})$-representation $\rho_{\text{trivial}}:\mathfrak{sl}_2(\boldsymbol{R})\to \mathrm{End}_{\boldsymbol{C}}(\mathscr{S}(\boldsymbol{R};\boldsymbol{C}^2))$, defined by

$$\rho_{\text{trivial}}(X^{\pm})=\omega(X^{\pm})\otimes I_2,\quad \rho_{\text{trivial}}(H)=\omega(H)\otimes I_2.$$

In other words, we are thinking of $A$ as a "0th-order" perturbation of the "principal part" $I_2(D_x^2+x^2)/2$.

On the other hand, we may think of $A$ as $\pi_{\text{vect}}(H)$, where $(\pi_{\text{vect}},\boldsymbol{C}^2)$ is the *vector* representation of $\mathfrak{sl}_2(\boldsymbol{R})$ (see (4) below). Then the map $\rho_{\text{vect}}=\omega\otimes\pi_{\text{vect}}:\mathfrak{sl}_2(\boldsymbol{R})\to\mathrm{End}_{\boldsymbol{C}}(\mathscr{S}(\boldsymbol{R};\boldsymbol{C}^2))$, defined by

$$\rho_{\text{vect}}(X^{\pm})=\omega(X^{\pm})\otimes I_2+1\otimes\pi_{\text{vect}}(X^{\pm}),\quad \rho_{\text{vect}}(H)=\omega(H)\otimes I_2+1\otimes\pi_{\text{vect}}(H),$$

is a representation, and $L(x,D_x)=\rho_{\text{vect}}(H)$. The spectrum is of course given again by the numbers $n+1/2\pm 1$, $n\in\boldsymbol{Z}_+$, which are now written in terms of the representation as $n+1/2+(2j-2+1)$, $j=0,2-1$, where 2 is the partition (see below) associated with $\pi_{\text{vect}}$. A "natural" basis of eigenfunctions is now given by

$$\varphi_{j,n}^{\pm}:=\rho_{\text{vect}}(X^+)^n\varphi_j^{\pm},\quad n\in\boldsymbol{Z}_+,\ j=0,1,$$

where

$$\varphi_0^+:=\varphi_0\otimes\begin{bmatrix}0\\1\end{bmatrix},\quad \varphi_0^-:=\psi^\dagger\varphi_0\otimes\begin{bmatrix}0\\1\end{bmatrix},$$

$$\varphi_1^+:=(\psi^\dagger)^2\varphi_0\otimes\begin{bmatrix}0\\1\end{bmatrix}+\varphi_0\otimes\begin{bmatrix}1\\0\end{bmatrix},\quad \varphi_1^-:=(\psi^\dagger)^3\varphi_0\otimes\begin{bmatrix}0\\1\end{bmatrix}+3\psi^\dagger\varphi_0\otimes\begin{bmatrix}1\\0\end{bmatrix}.$$

Notice that the $\varphi_0^{\pm}$ have weights $\mp 1/2$, resp., the $\varphi_1^{\pm}$ weights $2j\mp 1/2$, resp., and that the $\varphi_{j,n}^{\pm}$ have in general weights $2n+2j\mp 1/2$. We may hence think of $A$ as being as "heavy" as $I_2(D_x^2+x^2)/2$. In this case, the spectrum has a "creation/annihilation" structure, for $\rho_{\text{vect}}(X^-)\varphi_j^{\pm}=0$, $j=0,1$.

One may now intertwine system $\rho_{\text{vect}}(H)$ by the unitary operators $e^{\pm x^2J/2}$, $J$ being the matrix $\begin{bmatrix}0&-1\\1&0\end{bmatrix}$, obtaining the operator

$$\begin{aligned}Q_{\text{vect}}(x,D_x)&=e^{x^2J/2}\rho_{\text{vect}}(H)e^{-x^2J/2}\\&=\frac{-\partial_x^2+2x^2}{2}I+\left(x\partial_x+\frac{1}{2}\right)J+\begin{bmatrix}\cos(x^2)&\sin(x^2)\\\sin(x^2)&-\cos(x^2)\end{bmatrix},\end{aligned}$$

whose spectral problem reads as

$$\begin{cases} \left(\dfrac{-\partial_x^2+2x^2}{2}+\cos(x^2)\right)u_1-\left(x\partial_x+\dfrac{1}{2}-\sin(x^2)\right)u_2=\lambda u_1 \\ \left(\dfrac{-\partial_x^2+2x^2}{2}-\cos(x^2)\right)u_2+\left(x\partial_x+\dfrac{1}{2}+\sin(x^2)\right)u_1=\lambda u_2. \end{cases}$$

At a first glance, it seems quite non-trivial to obtain the eigenvalues (and eigenfunctions) of $Q_{\mathrm{vect}}(x,D_x)$, but of course we already know everything about the spectral resolution of $Q_{\mathrm{vect}}(x,D_x)$. More importantly, we can actually "invariantly" solve the spectral problem, *regardless* the choice of the vector space $\boldsymbol{C}^2$.

Recall that in the above mentioned papers, we considered systems of the kind

$$Q_{\mathrm{h}}(x,D_x)=\frac{-\partial_x^2+2x^2}{2}I_2+\left(x\partial_x+\frac{1}{2}\right)J=\Psi(1)\Psi^\dagger(1)-\frac{1}{2}I_2,$$

where

$$\Psi^\dagger(1)=\frac{1}{\sqrt{2}}(xI_2+\partial_xJ-xJ),\quad \text{and}\quad \Psi(1)=\frac{1}{\sqrt{2}}(xI_2+\partial_xJ+xJ).$$

Notice that $Q_{\mathrm{h}}(x,D_x)=e^{x^2J/2}\rho_{\mathrm{trivial}}(H)e^{-x^2J/2}$, and that $Q_{\mathrm{h}}(x,D_x)$ is unitarily equivalent (through a symplectic scaling) to

$$Q_{(\sqrt{2},\sqrt{2})}(x,D_x)=\begin{bmatrix}\sqrt{2}&0\\0&\sqrt{2}\end{bmatrix}\left(-\frac{\partial_x^2}{2}+\frac{x^2}{2}\right)+J\left(x\partial_x+\frac{1}{2}\right),$$

that was the starting system of the analysis carried out in [4] and [5] of systems such as

$$Q_{(\alpha,\beta)}(x;D_x)=I(\alpha,\beta)\left(-\frac{\partial_x^2}{2}+\frac{x^2}{2}\right)+J\left(x\partial_x+\frac{1}{2}\right),$$

$I(\alpha,\beta)$ being the diagonal matrix $\begin{bmatrix}\alpha&0\\0&\beta\end{bmatrix}$, with $\alpha,\beta\in\boldsymbol{R}$ and $\alpha\beta>1$ (which is equivalent to an ellipticity assumption).

Since the irreducible finite-dimensional representations of $\mathfrak{sl}_2(\boldsymbol{R})$ are classified by the dimension (the vector-representation is therefore classified by the dimension 2), we may (invariantly) summarize our discussion so far in the following structure of the spectrum of $Q_{\mathrm{vect}}(x,D_x)$.

*Example* 1.1. Let $(\pi_{\mathrm{vect}},V)$ be the vector representation (see (4) below). There exists an $L^2(\boldsymbol{R};V)$-complete system of Schwartz eigenfunctions $\{\xi_{0,N}^+,\xi_{0,N}^-,\xi_{1,N}^+,\xi_{1,N}^-\}_{N\in\boldsymbol{Z}_+}\subset\mathscr{S}(\boldsymbol{R};V)$, with the $+$-functions being even and the $-$-ones being odd, such that

$$Q_{\mathrm{vect}}(x, D_x)\xi_{0,N}^{\pm} = \left(2N \mp \frac{1}{2}\right)\xi_{0,N}^{\pm}, \quad Q_{\mathrm{vect}}(x, D_x)\xi_{1,N}^{\pm} = \left(2N + 2 \mp \frac{1}{2}\right)\xi_{1,N}^{\pm}.$$

Hence, as a consequence, the $L^2$-structure of $\mathrm{Spec}(Q_{\mathrm{vect}}(x, D_x))$ is given by

| **eigenvalue** | $-1/2$ | $2N - 1/2 \ (N \geq 1)$ | $1/2$ | $2N + 1/2 \ (N \geq 1)$ |
|---|---|---|---|---|
| **eigenvector** | $\xi_0^+$ | $\xi_{0,N}^+, \xi_{1,N-1}^+$ | $\xi_0^-$ | $\xi_{0,N}^-, \xi_{1,N-1}^-$ |
| **multiplicity** | 1 | 2 | 1 | 2 |

One may further introduce a parameter $\varepsilon \in \boldsymbol{C}$, and consider the system $L_\varepsilon(x, D_x) = \rho_{\mathrm{trivial}}(H) + \varepsilon 1 \otimes A$. This "deformation" of the considered "harmonic oscillators in disguise", that amounts to perturbing the principal part of the system by "lower order terms", gives the existence of "quasi-creation/annihilation" operators. By means of these "quasi-creation/annihilation" operators, can one find a basis **independent of** $\varepsilon$, constructed by using $\rho_{\mathrm{vect}}$, that "deforms" $\rho_{\mathrm{trivial}}(H)$ into $\rho_{\mathrm{vect}}(H)$? And if that is the case, what is then a "lower order term" for $\rho_{\mathrm{vect}}$, in the sense that the spectrum may still be **explicitly** computed? A partial answer is Theorem 3.4 below, that represents a sort of "rigidity theorem". Such a basis indeed exists, but it is the "trivial" one, that is the one given by the functions $\varphi_n \otimes e_j$, $n \in \boldsymbol{Z}_+$, $j = 1, 2$.

*Example* 1.2. Let

$$Q_{\mathrm{vect},\varepsilon}(x, D_x) := \frac{-\partial_x^2 + 2x^2}{2} I + \left(x\partial_x + \frac{1}{2}\right) J + \varepsilon \begin{bmatrix} \cos(x^2) & \sin(x^2) \\ \sin(x^2) & -\cos(x^2) \end{bmatrix}, \quad \varepsilon \in \boldsymbol{R}.$$

The system $Q_{\mathrm{vect},\varepsilon}(x, D_x)$ interpolates systems $Q_{\mathrm{h}}(x, D_x)$ and $Q_{\mathrm{vect}}(x, D_x)$, it has spectrum given by the numbers $2N + 1/2 \pm \varepsilon$ (even eigenfunctions) and $2N + 3/2 \pm \varepsilon$ (odd eigenfunctions), where $N \in \boldsymbol{Z}_+$, with **multiplicity one** for any $N \geq 0$ when $\varepsilon \notin (1/2)\boldsymbol{Z}$.

Notice that $Q_{\mathrm{vect},\varepsilon}(x, D_x)$ is equivalent in $\mathscr{S}'(\boldsymbol{R}; V)$ (through a transformation that is also unitary in $L^2(\boldsymbol{R}; V)$) to $\rho_\varepsilon(H) = \omega(H) \otimes I_V + \varepsilon 1 \otimes \pi_{\mathrm{vect}}(H)$, whence we have the right of thinking of $\rho_\varepsilon : \mathfrak{sl}_2(\boldsymbol{R}) \to \mathrm{End}_{\boldsymbol{C}}(\mathscr{S}'(\boldsymbol{R}; V))$ as a deformation of $\rho_0 = \rho_{\mathrm{trivial}}$ into $\rho_1 = \rho_{\mathrm{vect}}$. The maps $\rho_\varepsilon$, with $\varepsilon \neq 0, 1$, are not representations, but associated with them, one may find the "quasi-creation/annihilation" operators referred to earlier.

*Remark* 1.3. We remark that considering the $n$-fold tensor product of the oscillator representation of $\mathfrak{sl}_2(\boldsymbol{R})$, that is the $\mathfrak{sl}_2(\boldsymbol{R})$-module $(\omega_1 \otimes \cdots \otimes \omega_n, \mathscr{S}(\boldsymbol{R}^n))$, yields naturally the same kind of results for systems in the multivariable case (the PDE case).

Let us put things in the following perspective. Recall that, given $N \in \boldsymbol{Z}_+$, an $m$-tuple $\nu = (\nu_1, \dots, \nu_m) \in \boldsymbol{Z}_+^m$ with $\nu_1 + \cdots + \nu_m = N$ and $\nu_1 \geq \cdots \geq \nu_m \geq 1$, is called a *partition* of $N$. Moreover, to each partition $\nu$ of $N$, there corresponds a (unique, up to equivalence) decomposition into irreducibles of the $\mathfrak{sl}_2(\boldsymbol{R})$-module $(\pi_1 \oplus \cdots \oplus \pi_m, V)$, where $\dim_{\boldsymbol{C}} V = N$ and $\pi_j$ corresponds to $\nu_j$, $1 \leq j \leq m$ (see Section 2 below).

Let hence $N = 2$, $\nu_{\text{vect}} = 2$ be the partition associated with the vector-representation, and let $\nu_{\text{trivial}} = (1, 1)$ be the one associated with the direct sum of two trivial representations. We hence have the correspondences

$$\nu_{\text{vect}} \mapsto \rho_{\text{vect}}(H) = L(x, D_x), \quad \text{and} \quad \nu_{\text{trivial}} \mapsto \rho_{\text{trivial}}(H) + A = L(x, D_x).$$

Hence $\rho_{\text{vect}}(H)$ and $\rho_{\text{trivial}}(H) + A$ are isospectral (even is $\mathscr{S}'(\boldsymbol{R}; V)$). The matrix $A$ in the $\nu_{\text{trivial}}$ case, is thought of as a suitable two-parameter perturbation ($m = 2$) written in terms of $\pi_{\text{trivial}} \oplus \pi_{\text{trivial}}$ (see Section 2 below). More generally, given $N \in \boldsymbol{Z}_+$, let $\nu$ be a partition of $N$, $(\pi_{(\nu)}, V)$ the corresponding (up to isomorphisms) $\mathfrak{sl}_2(\boldsymbol{R})$-module, with $\dim V = N$, and let $\rho_{(\nu)}(H) = \omega \otimes \pi_{(\nu)}(H)$ be the corresponding system. So, if $\nu \in \boldsymbol{Z}_+^m$ and $\nu' \in \boldsymbol{Z}_+^{m'}$ are partitions of $N$, we will see (see Remark 4.1 below) that the systems $\rho_\nu(H) +$ ($m$-parameter suitable perturbation) and $\rho_{\nu'}(H) +$ ($m'$-parameter suitable perturbation) are **isospectral** (even in $\mathscr{S}'(\boldsymbol{R}; V)$). In some sense, the partition $\nu$ does not distinguish among different, but equivalent, forms of the same system, the only "invariant" in the spectrum being the parity of $N$: *when N is even, every system associated with a partition ν of N is isospectral to a perturbation of one associated with the partition* $\nu_{\text{can}}^{(N/2)} = (2, 2, \dots, 2) \in \boldsymbol{Z}_+^{N/2}$, *whereas when N is odd, every system associated with a partition ν of N is isospectral to a perturbation of one associated with the partition* $\nu_{\text{can}}^{((N-1)/2)} = (2, 2, \dots, 2, 1) \in \boldsymbol{Z}_+^{(N-1)/2+1}$. Loosely speaking, *in the system case one has as primary symmetries the parity in the variable* $x \in \boldsymbol{R}$ *and that in the size of the system.*

We will hence study $N \times N$ systems in $L^2(\boldsymbol{R}; \boldsymbol{C}^N)$ such as

$$(1) \quad L(x, D_x) = \mu I_N \left( \frac{D_x^2 + x^2}{2} \right) + A, \quad \mu > 0, \; A \in \mathrm{Mat}_N(\boldsymbol{C}) \text{ diagonalizable},$$

and suitable perturbations parametrized by the partitions of $N$, and also "genuine" infinite dimensional perturbations (see Section 5 below). Of course, the discussion extends to cases such as $L(x, D_x) = B(D_x^2 + x^2) + A$, with $B = B^* > 0$ and $[A, B] = 0$. Since

$$\mathrm{Spec}(L(x, D_x)) = \mathrm{Spec}(e^{-B(x)} L(x, D_x) e^{B(x)}), \quad B \in C^\infty(\boldsymbol{R}; \mathfrak{u}(N)),$$

one is able to treat by the same methods, seemingly more difficult systems.

*Example* 1.4. Let $\alpha : \boldsymbol{R} \to \mathsf{U}(N)$ be a smooth map valued in the unitary group, with **temperate** growth on $\boldsymbol{R}$. Put

$$A(x) = (\partial_x \alpha(x)) \alpha(x)^{-1}, \quad B(x) = \frac{1}{2}(\partial_x A(x) - A(x)^2) + 2\alpha(x) A \alpha(x)^*.$$

Then the system defined by

$$P(x, D_x) = I_N\left(\frac{-\partial_x^2 + x^2}{2}\right) + A(x)\partial_x + B(x),$$

is unitarily equivalent to the system $I_N(-\partial_x^2 + x^2)/2 + A$.

We hence aim at making this approach, when possible, as much "invariant" (and algebraic) as possible, with respect to the matrix $A$ and the representation space $V$. The main idea will be writing the matrix $A$ in terms of $\mathfrak{sl}_2(\boldsymbol{R})$-symmetries. One of the advantages of our approach is to describe the spectrum in terms of "pure $\mathfrak{sl}_2(\boldsymbol{R})$-symmetries" of $A$. Though dealing with a number of special examples, our method finally gives the possibility of studying (through a three-step recurrence equation) the spectrum of more "asymmetric" cases following the lines of [4] and [5] (see also [6] and [3]), as in Section 5, and to naturally deal with more geometric systems, such as the one in Section 6 below.

## 2. Tensor products

Let us recall a few elementary facts about $\mathfrak{sl}_2(\boldsymbol{R})$. Let us consider the *standard basis* $\{H, X^+, X^-\}$ of $\mathfrak{sl}_2(\boldsymbol{R})$, that is to say, the following commutation-relations of $\mathfrak{sl}_2(\boldsymbol{R})$ are fulfilled:

$$[X^+, X^-] = H, \quad [H, X^\pm] = \pm 2X^\pm. \tag{2}$$

Let us set

$$\psi := \frac{x + \partial_x}{\sqrt{2}}, \quad \psi^\dagger := \frac{x - \partial_x}{\sqrt{2}}.$$

Then $[\psi, \psi^\dagger] = 1$, whence the map $\omega : \mathfrak{sl}_2(\boldsymbol{R}) \to \mathrm{End}_{\boldsymbol{C}}(\mathscr{S}(\boldsymbol{R}))$ defined by

$$\omega(H) = \psi\psi^\dagger - \frac{1}{2}, \quad \omega(X^+) = \frac{(\psi^\dagger)^2}{2}, \quad \omega(X^-) = -\frac{(\psi)^2}{2}, \tag{3}$$

gives the *oscillator representation* of $\mathfrak{sl}_2(\boldsymbol{R})$ on $\mathscr{S}(\boldsymbol{R})$ (and on $\mathscr{S}'(\boldsymbol{R})$). Because the action of $\mathfrak{sl}_2(\boldsymbol{R})$ leaves the parity invariant, we have the *irreducible* decomposition of $\omega$:

$$\mathscr{S}(\boldsymbol{R}) = \mathscr{S}_{\mathrm{even}}(\boldsymbol{R}) \oplus \mathscr{S}_{\mathrm{odd}}(\boldsymbol{R}) =: \mathscr{S}_+(\boldsymbol{R}) \oplus \mathscr{S}_-(\boldsymbol{R}).$$

Put $\omega^\pm := \omega|_{\mathscr{S}_\pm(\boldsymbol{R})}$. Then $\varphi_0 = e^{-x^2/2}$ (resp. $\psi^\dagger\varphi_0$) gives the lowest weight vector of the irreducible representation of $(\omega^+, \mathscr{S}_+(\boldsymbol{R}))$ (resp. of $(\omega^-, \mathscr{S}_-(\boldsymbol{R}))$) (see [2]).

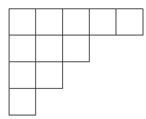
Let now $(\pi, V)$ be an $N$-dimensional ($N \geq 1$) representation of $\mathfrak{sl}_2(\boldsymbol{R})$, where we take $V \simeq \boldsymbol{C}^N$. For example, when $V = \boldsymbol{C}^2$ is irreducible, one has the *vector*-representation

$$\pi_{\mathrm{vect}}(H) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \pi_{\mathrm{vect}}(X^+) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \pi_{\mathrm{vect}}(X^-) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{4}$$

As it is well-known, the $\mathfrak{sl}_2(\boldsymbol{R})$-module $(\pi, V)$ admits a decomposition into irreducible components (as $\mathfrak{sl}_2(\boldsymbol{R})$-modules), that we will write as

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_m, \quad \pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_m,$$

the $(\pi_j, V_j)$ being irreducible $\mathfrak{sl}_2(\boldsymbol{R})$-modules, where $\dim V_j = v_j$, $1 \le j \le m$, and $v_1 + \cdots + v_m = N$. By Proposition 2.2 below, by virtue of the fact that an isomorphism class is determined by a dimension, we may assume that $(v_1, v_2, \dots, v_m) \in \boldsymbol{Z}_+^m$ be a **partition** of $N$, that is $v_1 \ge v_2 \ge \cdots \ge v_m \ge 1$ and $v_1 + \cdots + v_m = N$. Remark that giving a partition $(v_1, v_2, \dots, v_m)$ of $N$ is equivalent to giving a *Young-diagram* with $N$ boxes, where the number of cells in the first row is $v_1$, the one in the second is $v_2$ etc., the one in the last row is $v_m$. For example, the partition $(5, 3, 2, 1)$ of 11 is represented by the Young diagram



It is also convenient to recall the following facts about commutation relations (among generators) of $\mathfrak{sl}_2(\boldsymbol{R})$, and the list of finite dimensional irreducible $\mathfrak{sl}_2(\boldsymbol{R})$-modules (see [2]).

Lemma 2.1 ([2], p. 52). *For all $X, Y \in \mathfrak{sl}_2(\boldsymbol{R})$ and for any given $n \in \boldsymbol{Z}_+ \setminus \{0\}$ one has the formulas*

- $$[X^n, Y] = \sum_{k=0}^{n-1} X^k [X, Y] X^{n-k-1},$$
- $$[H, (X^\pm)^n] = \pm 2n (X^\pm)^n,$$
- $$[(X^\pm)^n, X^\mp] = n (X^\pm)^{n-1} (H \pm (n-1)).$$

Proposition 2.2 ([2], p. 55). *Let $(\pi, V)$ be an irreducible $\mathfrak{sl}_2(\boldsymbol{R})$-module of dimension $v$, for some $v \ge 1$. Then $V$ has a basis $\{v_0, \dots, v_{v-1}\}$, such that*

$$\pi(H) v_j = (2j - v + 1) v_j, \quad 0 \le j \le v - 1,$$
$$\pi(X^+) v_j = v_{j+1}, \quad 0 \le j < v - 1, \quad \pi(X^+) v_{v-1} = 0,$$
$$\pi(X^-) v_0 = 0, \quad \pi(X^-) v_j = j(v - j) v_{j-1}, \quad 1 \le j \le v - 1.$$

*Furthermore, with $C$ denoting the Casimir operator of $\mathfrak{sl}_2(\boldsymbol{R})$, that is the generator of the center of the enveloping algebra $\mathscr{U}(\mathfrak{sl}_2(\boldsymbol{R}))$ of $\mathfrak{sl}_2(\boldsymbol{R})$; $C = H^2 + 2(X^+X^- + X^-X^+)$,*

$$\pi(C) v = (v^2 - 1) v, \quad \forall v \in V.$$

*In particular, $V$ is determined up to isomorphism by its dimension.*

Hence, since in our case we are dealing with $V = V_1 \oplus \cdots \oplus V_m$, $V_j$ being irreducible $\mathfrak{sl}_2(\boldsymbol{R})$-modules, we have for each $(\pi_j, V_j)$ a basis $\{v_0^{(j)}, \ldots, v_{\nu_j-1}^{(j)}\}$, $1 \le j \le m$, with the above properties. At this point, it is convenient to **fix** a Hermitean product $\langle \cdot, \cdot \rangle$ in $V$ that makes the basis $\{v_k^{(j)}; k = 0, \ldots, \nu_j - 1, 1 \le j \le m\}$ **orthogonal** (whence the $\oplus$ in the decomposition of $V$ becomes orthogonal). Then there exists a positive-definite Hermitean matrix $\Omega$ such that $\langle v, w \rangle = \langle \Omega\tilde{v}, \tilde{w} \rangle_{\mathrm{can}}$, where $\langle \cdot, \cdot \rangle_{\mathrm{can}}$ denotes the canonical Hermitean product of $\boldsymbol{C}^N$ and $\tilde{v}, \tilde{w}$, respectively, are the vectors of $\boldsymbol{C}^N$ corresponding to $v, w$, respectively. We will then denote by $\mathsf{U}_\Omega(N)$ the Lie group of the *unitary* matrices on $V$ with respect to $\langle\,,\rangle$, and $A^*$ will stand for the adjoint of $A$ with respect to the aforedefined Hermitean product. We will finally write $I_V$ for the identity map of $V$, and $I_N$ for the identity map of $\boldsymbol{C}^N$.

Definition 2.3. Let $N$ be an integer $\ge 1$. It is well known that the Young diagrams with $N$ boxes "parametrize" bijectively the partitions of $N$ and the $N$-dimensional $\mathfrak{sl}_2(\boldsymbol{R})$-modules $(\pi, V)$ (through the irreducible decomposition of $\pi$), up to equivalence. We hence get a correspondence

$$\nu \mapsto \pi_{(\nu)}, \quad \pi \mapsto \nu_\pi.$$

In our case, fixing $\nu \in \mathscr{Y}_N =$ the set of partitions of $N$, will therefore fix an $N \times N$-system.

In the sequel, given a finite dimensional complex vector space $V$, we will systematically identify $L^2(\boldsymbol{R}; V)$ with $L^2(\boldsymbol{R}) \otimes V$. Likewise for $\mathscr{S}'(\boldsymbol{R}; V)$ and $\mathscr{S}(\boldsymbol{R}; V)$. Having fixed the Hermitean structure on $V$, from now on we will take on $L^2(\boldsymbol{R}; V)$ the following inner-product:

$$(f, g)_{L^2(\boldsymbol{R}; V)} := \int_{\boldsymbol{R}} \langle f(x), g(x) \rangle \, dx, \quad \forall f, g \in L^2(\boldsymbol{R}; V).$$

Definition 2.4. Let $A \in \mathsf{U}_\Omega(N)$, and $B \in C^\infty(\boldsymbol{R}; \mathsf{U}_\Omega(N))$ with temperate growth in $x \in \boldsymbol{R}$. Define, for $\nu \in \mathscr{Y}_N$,

$$Q(\nu) = Q(\nu; x, D_x) := \omega(H) \otimes I_V + 1 \otimes (\pi_1(H) \oplus \cdots \oplus \pi_m(H)) = \omega \otimes \pi(H),$$

where $\pi = \pi_{(\nu)}$ and, recall, $\omega \otimes \pi(Y) = \omega(Y) \otimes I_V + 1 \otimes \pi(Y)$, for any given $Y \in \mathfrak{sl}_2(\boldsymbol{R})$. Define also

$$Q_{A,B}(\nu) = Q_{A,B}(\nu; x, D_x) := B(x) A Q(\nu; x, D_x) A^{-1} B(x)^{-1}.$$

Note that

$$T_{A,B} := B(\cdot)A : L^2(\boldsymbol{R}; V) \xrightarrow{\text{isometry}} L^2(\boldsymbol{R}; V),$$

$$T_{A,B} \in \mathrm{Aut}_{\boldsymbol{C}}(\mathscr{S}(\boldsymbol{R}; V)) \cap \mathrm{Aut}_{\boldsymbol{C}}(\mathscr{S}'(\boldsymbol{R}; V)).$$

Moreover, $Q(v)$ is a globally elliptic self-adjoint (with respect to the scalar-product defined above) unbounded operator in $L^2(\boldsymbol{R}; V)$, with domain $B^2(\boldsymbol{R}; V)$ and compact resolvent. Recall that

$$B^2(\boldsymbol{R}; V) = \{f \in \mathscr{S}'(\boldsymbol{R}; V); x^\alpha \partial_x^\beta f \in L^2(\boldsymbol{R}; V), \forall \alpha, \beta \in \boldsymbol{Z}_+, |\alpha| + |\beta| \le 2\}.$$

*Example* 2.5. Let $V = \boldsymbol{C}^N$. One has

$$Q_{\mathrm{h}}(x, D_x) = Q_{I_N, \exp(x^2 J/2)}(1, 1) \quad Q_{\mathrm{vect}}(x, D_x) = Q_{I, \exp(x^2 J/2)}(2).$$

The usual $1 \times 1$-harmonic oscillator is

$$\frac{1}{2}(-\partial_x^2 + x^2) = Q_{1,1}(1),$$

whereas the scalar $N \times N$ one is

$$\frac{1}{2}(-\partial_x^2 + x^2) I_N = Q_{I_N, I_N}(\underbrace{1, 1, \dots, 1}_{N}).$$

The next step is now to establish the spectral resolution of $Q_{A,B}(v)$. One has the following proposition.

Proposition 2.6. *Consider the* $\mathfrak{sl}_2(\boldsymbol{R})$*-module* $(\omega \otimes \pi, \mathscr{S}(\boldsymbol{R}; V))$. *Put*

$$\rho(H) := Q_{A,B}(v_\pi; x, D_x), \quad \rho(X^\pm) := T_{A,B}(\omega \otimes \pi(X^\pm)) T_{A,B}^*.$$

*Then* $(\rho, \mathscr{S}(\boldsymbol{R}; V))$ *defines a representation of* $\mathfrak{sl}_2(\boldsymbol{R})$. *Furthermore*, $\rho$ *is equivalent to the tensor product representation* $(\omega \otimes \pi, \mathscr{S}(\boldsymbol{R}; V))$. *In fact, the operator* $T_{A,B}$ *defines the intertwining operator between these representations*:

$$\rho(X) T_{A,B} = T_{A,B} \omega \otimes \pi(X), \quad \forall X \in \mathfrak{sl}_2(\boldsymbol{R}).$$

*In particular, the system defined by the operator* $Q_{A,B}(v_\pi; x, D_x)$ *is unitarily equivalent to the system defined by the operator* $Q(v_\pi; x, D_x)$.

*Proof.* One just recalls that

$$\begin{aligned}
&[T_{A,B}\omega \otimes \pi(Y) T_{A,B}^*, T_{A,B}\omega \otimes \pi(Z) T_{A,B}^*] \\
&= T_{A,B}([\omega(Y), \omega(Z)] \otimes I_V + 1 \otimes [\pi(Y), \pi(Z)]) T_{A,B}^* \\
&= T_{A,B}(\omega([Y,Z]) \otimes I_V + 1 \otimes \pi([Y,Z])) T_{A,B}^* = T_{A,B}(\omega \otimes \pi([Y,Z])) T_{A,B}^*,
\end{aligned}$$

for all $Y, Z \in \mathfrak{sl}_2(\boldsymbol{R})$. The proof then follows by direct check. □

Using the irreducible decomposition of the tensor product representation $\omega \otimes \pi$, we now have the following theorem.

THEOREM 2.7. *Let* $1 \le k \le m$ *and let* $0 \le j \le \nu_k - 1$. *Pick constants* $c^{(k)\pm}_{\ell,j}$, $0 \le l \le j$,

$$c^{(k)\pm}_{0,0} = 1, \quad c^{(k)\pm}_{\ell,j} = \alpha \binom{j}{\ell} \frac{\Gamma(\nu_k - \ell)}{\Gamma(j - \ell \mp 1/2)} \frac{\Gamma(j \mp 1/2)}{\Gamma(\nu_k)}, \quad 1 \le j,\ 1 \le \ell \le j,\ \alpha > 0,$$

*where* $\Gamma$ *is the Gamma-function. In particular,* $c^{(k)\pm}_{\ell,j} > 0$, *for all* $\ell = 0, \dots, j$, *all* $j = 0, \dots, \nu_k - 1$, *and all* $k = 1, \dots, m$. *Define the Schwartz functions*

$$\varphi^{(k)+}_j := \sum_{\ell=0}^{j} c^{(k)+}_{\ell,j} \omega(X^+)^{j-\ell} \varphi_0 \otimes v^{(k)}_\ell, \quad \varphi^{(k)-}_j := \sum_{\ell=0}^{j} c^{(k)-}_{\ell,j} \omega(X^+)^{j-\ell} \psi^\dagger \varphi_0 \otimes v^{(k)}_\ell,$$

*for* $0 \le j \le \nu_k - 1$ *and* $k = 1, \dots, m$. *Put*

$$h^{(k)\pm}_j = h^{(k)\pm}_{j,0} := T_{A,B} \varphi^{(k)\pm}_j, \quad \text{and} \quad h^{(k)\pm}_{j,n} := \rho(X^+)^n h^{(k)\pm}_j, \tag{5}$$

*for* $0 \le j \le \nu_k - 1$, $k = 1, \dots, m$, *and* $n \ge 0$. *Then*

$$\rho(X^-) h^{(k)\pm}_{j,0} = 0, \quad 0 \le j \le \nu_k - 1,\ 1 \le k \le m,$$

*and the system of functions* $\{h^{(k)\pm}_{j,n}\}^{0 \le j \le \nu_k - 1, 1 \le k \le m}_{n \in \boldsymbol{Z}_+} \subset \mathscr{S}(\boldsymbol{R}; V)$, *is an* **orthogonal basis** *of* $L^2(\boldsymbol{R}; V)$, *dense in both* $\mathscr{S}(\boldsymbol{R}; V)$ *and* $\mathscr{S}'(\boldsymbol{R}; V)$, *made of the* **eigenfunctions** *of* $Q_{A,B}(\nu_\pi)$:

$$Q_{A,B}(\nu_\pi; x, D_x) h^{(k)\pm}_{j,n} = \left(2n + 2j + 2 \mp \frac{1}{2} - \nu_k\right) h^{(k)\pm}_{j,n},$$

*for any* $0 \le j \le \nu_k - 1$, *any* $k = 1, \dots, m$, *and any* $n \in \boldsymbol{Z}_+$. *In particular, the lowest eigenvalue depends on the partition* $\nu$ *in the following way*:

$$\min \operatorname{Spec}(Q_{A,B}(\nu_\pi; x, D_x)) = \frac{3}{2} - \nu_1.$$

*Proof.* We start off by proving that $\rho(X^-) h^{(k)\pm}_{j,0} = 0$. We will consider only the $+$-case, the other being similar. It clearly suffices to prove that

$$(\omega(X^-) \otimes I_V + 1 \otimes \pi(X^-)) \varphi^{(k)+}_j = 0.$$

A simple computation shows that it holds by virtue of the choice of the constants $c^{(k)+}_{\ell,j}$, for they satisfy the recurrence

$$c^{(k)\pm}_{0,j} = \alpha, \quad c^{(k)\pm}_{\ell,j} = \frac{(j - \ell + 1)(j - \ell + 1 \mp 1/2)}{\ell(\nu_k - \ell)} c^{(k)\pm}_{\ell-1,\,j}, \quad 1 \le j,\ 1 \le \ell \le j.$$

We now compute the action of $Q_{A,B}(\nu_\pi; x, D_x)$ on the $h^{(k)\pm}_{j,n}$'s. Since from Lemma 2.1 one has

$$[Q_{A,B}(\nu_\pi; x, D_x), \rho(X^+)^n] h^{(k)\pm}_j = 2n \rho(X^+)^n h^{(k)\pm}_j, \quad \forall n \in \boldsymbol{Z}_+ \backslash \{0\},$$

it is possible to reduce matters to the case $n = 0$, for which it suffices to show that

$$\omega\otimes\pi(H)(\omega(X^+)^{j-\ell}\varphi_0\otimes v_\ell^{(k)})=\left(2j+\frac{3}{2}-v_k\right)\omega(X^+)^{j-\ell}\varphi_0\otimes v_\ell^{(k)}.$$

But this is an immediate consequence of the fact that the weight of $\omega(X^+)^{j-\ell}\varphi_0$ is $2(j-\ell)+1/2$, and the one of $v_\ell^{(k)}$ is $2\ell-v_k+1$. (In fact, each function $\omega(X^+)^{j-\ell}\varphi_0\otimes v_\ell^{(k)}$ is a weight vector of $\omega\otimes\pi(H)$.)

Next we prove that $\{h_{j,n}^{(k)\pm}, n\in \boldsymbol{Z}_+, 0\le j\le v_k-1, 1\le k\le m\}$ is an orthogonal basis of $L^2(\boldsymbol{R};V)$. By virtue of the orthogonal decomposition

$$L^2(\boldsymbol{R};V)=L^2_+(\boldsymbol{R};V)\oplus L^2_-(\boldsymbol{R};V)=\left(\bigoplus_{k=1}^m L^2_+(\boldsymbol{R};V_k)\right)\oplus\left(\bigoplus_{k=1}^m L^2_-(\boldsymbol{R};V_k)\right),$$

we may suppose $m=1$ and consider only the +-case (so, in the following claim we will drop the index $k$). Since $\omega(X^+)^n\varphi_0=$ (**even** polynomial of degree $2n)\times\varphi_0$, it suffices to prove the following elementary lemma.

LEMMA 2.8. *Let $k\in\boldsymbol{Z}_+$. Then, upon defining $N_v=\{0,1,\dots,v-1\}$,*

$$\begin{aligned}W_k^+ &:= \mathrm{Span}_{\boldsymbol{C}}\{h_{\ell,n}^+; n\in\boldsymbol{Z}_+, \ell\in N_v, n+\ell\le k\}\\ &= T_{A,B}(\mathrm{Span}_{\boldsymbol{C}}\{\omega(X^+)^n\varphi_0\otimes v_\ell; n\in\boldsymbol{Z}_+,\ell\in N_v, n+\ell\le k\})=: S_k^+.\end{aligned}$$

*In particular, for any given* **even** *polynomial $p\in\boldsymbol{C}[x]$ and $0\le j\le v-1$, one has that $T_{A,B}(p\varphi_0\otimes v_j)$ belongs to some $W_k^+$.*

*Proof of Lemma* 2.8. We may suppose $A=B=I_V$, so that $T_{A,B}$ is the identity operator and $h_{j,n}^+=(\omega\otimes\pi(X^+))^n\varphi_j^+=:\varphi_{j,n}^+$. One has the following elementary facts. Let $n,n'\in\boldsymbol{Z}_+$, $0\le j,j'\le v-1$. Then

(i) $\quad(\omega(X^+)^n\varphi_0\otimes v_j,\omega(X^+)^{n'}\varphi_0\otimes v_{j'})=0,\quad$ whenever $n\ne n'$ or $j\ne j'$;

(ii) $\quad(\varphi_j^+,\varphi_{j'}^+)=0,\quad$ whenever $j\ne j'$;

(iii) $\quad(\varphi_{j,n}^+,\varphi_{j',n'}^+)=0,\quad$ whenever $n\ne n'$ or $j\ne j'$.

In particular, the $\omega(X^+)^n\varphi_0\otimes v_\ell$'s, resp. the $\varphi_{\ell,n}^+$'s, are linearly independent. The problem is therefore equivalent to proving that $S_k^+\subset W_k^+$ for all $k\in\boldsymbol{Z}_+$, for it is clear that

$$\dim S_k^+=\dim W_k^+=\begin{cases}\dfrac{(k+1)(k+2)}{2}, & \text{if } 0\le k\le v-1,\\[2mm] \dfrac{v(v+1)}{2}+v(k-v+1), & \text{if } k\ge v.\end{cases}$$

Let us proceed by induction on $k$. When $k=0$, one obviously has the claim, for $\varphi_{0,0}^+=\varphi_0^+=\varphi_0\otimes v_0$. Suppose next we have the result up to $k$. We prove it for $k+1$. Being $(\omega\otimes\pi(X^+))^n=\sum_{s=0}^n\binom{n}{s}\omega(X^+)^{n-s}\otimes\pi(X^+)^s$, we get that

$$\begin{aligned}&\mathrm{Span}_{\boldsymbol{C}}\{\varphi^{+}_{j,n}; n\in \boldsymbol{Z}_{+}, j\in N_{\nu}, n+j=k+1\}\\ &\qquad\subset \mathrm{Span}_{\boldsymbol{C}}\{\omega(X^{+})^{n}\varphi_0\otimes v_j; n\in \boldsymbol{Z}_{+}, j\in N_{\nu}, n+j=k+1\},\end{aligned}$$

whence the equality

$$\begin{aligned}&\mathrm{Span}_{\boldsymbol{C}}\{\varphi^{+}_{j,n}; n\in \boldsymbol{Z}_{+}, j\in N_{\nu}, n+j=k+1\}\\ &\qquad= \mathrm{Span}_{\boldsymbol{C}}\{\omega(X^{+})^{n}\varphi_0\otimes v_j; n\in \boldsymbol{Z}_{+}, j\in N_{\nu}, n+j=k+1\},\end{aligned}$$

by virtue of the fact that the two vector spaces have the same dimension $\min\{k+2,\nu\}$. Since $W_k^{+}\subset W_{k+1}^{+}$ and by the induction hypothesis $S_k^{+}=W_k^{+}$, we get

$$\begin{aligned}W_{k+1}^{+}&=W_k^{+}\oplus \mathrm{Span}_{\boldsymbol{C}}\{\varphi^{+}_{j,n}; n\in \boldsymbol{Z}_{+}, j\in N_{\nu}, n+j=k+1\}\\ &=S_k^{+}\oplus \mathrm{Span}_{\boldsymbol{C}}\{\omega(X^{+})^{n}\varphi_0\otimes v_j; n\in \boldsymbol{Z}_{+}, j\in N_{\nu}, n+j=k+1\}=S_{k+1}^{+}.\end{aligned}$$

This ends the proof of the lemma.

Hence, if $(T^{*}_{A,B}f, h^{(k),\pm}_{j,n})_{L^2(\boldsymbol{R};V)}=0$ for any $n\in\boldsymbol{Z}_{+}$, any $j=0,\dots,\nu_k-1$, and any $k=1,\dots,m$, then also $(f_j^{(k)}, p(x)\varphi_0)_{L^2(\boldsymbol{R};\boldsymbol{C})}=0$ for any polynomial $p$, $f_j^{(k)}$ being the $v_j^{(k)}$-component of $f$. Then by usual arguments $f_j^{(k)}=0$, whence $f=0$. The result follows. □

*Remark* 2.9. It follows that as an $\mathfrak{sl}_2(\boldsymbol{R})$-module, one has the irreducible decomposition of $(\rho,\mathscr{S}(\boldsymbol{R};V))$:

$$\mathscr{S}(\boldsymbol{R};V)\simeq \underbrace{\bigoplus_{k=1}^{m}\bigoplus_{j=0}^{\nu_k-1}\overline{\mathrm{Span}\{h^{(k)+}_{j,n}\}_{n\in\boldsymbol{Z}_{+}}}}_{\text{even}}\oplus \underbrace{\bigoplus_{k=1}^{m}\bigoplus_{j=0}^{\nu_k-1}\overline{\mathrm{Span}\{h^{(k)-}_{j,n}\}_{n\in\boldsymbol{Z}_{+}}}}_{\text{odd}},$$

where the closure refers to the $\mathscr{S}$-topology. The same decomposition holds for $L^2(\boldsymbol{R};V)$, with closure in the $L^2$-topology, and orthogonal $\oplus$-sum. In particular, the functions $h^{(k)+}_{j,0}$ and $h^{(k)-}_{j,0}$, $0\le j\le \nu_k-1$, $k=1,\dots,m$, give the lowest-weight vectors of the irreducible summands, respectively.

*Remark* 2.10. Let $\nu\in\mathscr{Y}_N$ and let $V\simeq\boldsymbol{C}^N$. Fixing the another inner-product in $L^2(\boldsymbol{R};V)$ and taking $A\in\mathsf{U}(N)$ and $B\in C^\infty(\boldsymbol{R};\mathsf{U}(N))$ with temperate growth (where $\mathsf{U}(N)$ denotes the unitary group with respect to the canonical Hermitian structure, rather than the structures tailored to the basis given by $\pi_{(\nu)}$ as we did above) yields that in general $Q_{A,B}(\nu)$ is no-longer self-adjoint. In this case Theorem 2.7 holds true, the only difference being that the $\oplus$'s are no longer orthogonal sums.

Example 1.1 is now a consequence of Theorem 2.7, upon considering the $\mathfrak{sl}_2(\boldsymbol{R})$-module $(\pi_{\mathrm{vect}},\boldsymbol{C}^2)$ given by the *vector representation*, and defining

$$\xi_{j,n}^{\pm} = \rho(X^+)^n e^{x^2 J/2} \varphi_j^{\pm}, \quad j = 0, 1,$$

where (one picks $\alpha = 2$ to determine the basis)

$$\varphi_0^+ := \varphi_0 \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \varphi_0^- := \psi^\dagger \varphi_0 \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\varphi_1^+ := 2\omega(X^+)\varphi_0 \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \varphi_0 \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \varphi_1^- := 2\omega(X^+)\psi^\dagger \varphi_0 \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\psi^\dagger \varphi_0 \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Corollary 2.11. *Same hypotheses of Theorem* 2.7, *with* $v = (v_1, \dots, v_m) \in \mathcal{Y}_N$. *Let* $\mathbf{1} \in \mathcal{U}(\mathfrak{sl}_2(\boldsymbol{R}))$ *be the unit element of the enveloping algebra. Take* $\delta \in \boldsymbol{C}^m$, *and let us set*

$$Q_{A,B,\delta}(v) := T_{A,B}(Q(v) + 1 \otimes \pi_\delta(\mathbf{1}))T_{A,B}^*,$$

*where we put* $\pi_k(\mathbf{1}) = I_{V_k}$. (*Recall*, $\pi_\delta(\mathbf{1}) = \delta_1 \pi_1(\mathbf{1}) \oplus \cdots \oplus \delta_m \pi_m(\mathbf{1})$.) *Then*

$$\mathrm{Spec}(Q_{A,B,\delta}(v)) = \left\{ n + \frac{1}{2} + (2j - v_k + 1) + \delta_k; n \in \boldsymbol{Z}_+, 0 \le j \le v_k - 1, 1 \le k \le m \right\},$$

*with the* **same** *basis* $\{h_{\ell,n}^{(k)\pm}\}_{n \in \boldsymbol{Z}_+, 0 \le \ell \le v_k - 1}^{1 \le k \le m}$ (*independent of* $\delta$).

*Proof of Corollary* 2.11. We immediately have that $\pi|_{V_k}(\mathbf{1}) = I_{V_k}$, whence

$$T_{A,B}(1 \otimes \pi_\delta(\mathbf{1}))T_{A,B}^* h_{\ell,n}^{(k)\pm} = \delta_k h_{\ell,n}^{(k)\pm}, \ \forall n \in \boldsymbol{Z}_+, \ 0 \le \ell \le v_k - 1, \ 1 \le k \le m. \ \square$$

## 3. Perturbing the representation

We now define "quasi-creation/annihilation" operators, suited for deforming the system defined by $Q_{A,B}(v_\pi; x, D_x)$. Of course, once the $\mathfrak{sl}_2(\boldsymbol{R})$-module is fixed (that is, once a partition of $N$ is fixed), it suffices to "deform" the operator $Q(v_\pi; x, D_x)$.

Definition 3.1. Let $v = (v_1, v_2, \dots, v_m) \in \mathcal{Y}_N$. Put $\pi = \pi_{(v)}$ (so that also $v = v_\pi$). Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m) \in \boldsymbol{R}^m$, and let us define $\pi_\varepsilon = \varepsilon_1 \pi_1 \oplus \cdots \oplus \varepsilon_m \pi_m$. Define the following operators (the "quasi-creation" operator and the "quasi-annihilation" operator, respectively):

$$\mathsf{A}^+(v; \varepsilon) := \omega(X^+) \otimes I_V + 1 \otimes \pi_\varepsilon(X^+),$$

$$\mathsf{A}^-(v; \varepsilon) := \omega(X^-) \otimes I_V + 1 \otimes \pi_\varepsilon(X^-),$$

and finally, the following "twisted" harmonic oscillator:

$$Q_\varepsilon(v; x, D_x) = \omega(H) \otimes I_V + 1 \otimes \pi_\varepsilon(H).$$

Notice that $Q_{(1,1,\dots,1)}(v; x, D_x) = Q(v; x, D_x)$ then.

One has the following crucial property.

Lemma 3.2. *One has*

$$[\omega \otimes \pi(X^-), \mathsf{A}^-(v;\varepsilon)] = [\omega \otimes \pi(X^+), \mathsf{A}^+(v;\varepsilon)] = 0, \tag{6}$$

$$[Q_\varepsilon(v), (\omega \otimes \pi(X^+))^n] = 2n(\omega \otimes \pi(X^+))^{n-1}\mathsf{A}^+(v;\varepsilon), \quad \forall n \geq 1. \tag{7}$$

*Formula* (7) *holds also in the case* $\omega \otimes \pi(X^-)$ *and* $\mathsf{A}^-(v;\varepsilon)$, *upon replacing* $2n$ *by* $-2n$. *In particular* (*with either* $+$ *or* $-$ *throughout*)

$$\begin{aligned}[Q_\varepsilon(v), (\omega \otimes \pi(X^\pm))^n]& \\ = \pm 2n(\omega \otimes \pi(X^\pm))^n &\pm 2n(\omega \otimes \pi(X^\pm))^{n-1}(1 \otimes (\pi_\varepsilon - \pi)(X^\pm)),\end{aligned} \tag{8}$$

*for all* $n \geq 1$. *Finally*

$$[Q_\varepsilon(v), Q(v)] = 0. \tag{9}$$

(*All of the formulas above are understood on* $\mathscr{S}'(\boldsymbol{R}; V)$.)

*Proof.* Formula (6) and formula (9) are clear. As regards formula (7), the proof obviously goes by induction. It suffices to consider only the case relative to $\mathsf{A}^+(v;\varepsilon)$. The case relative to $\mathsf{A}^-(v;\varepsilon)$ is similar. The step $n = 1$ is obvious. So, suppose it true up to $n$, and consider the case $n+1$ (for simplicity we consider only the $+$-case): since $[a, bc] = [a, b]c + b[a, c]$, we have with $a = Q_\varepsilon(v)$, $b = (\omega \otimes \pi(X^+))^n$ and $c = \omega \otimes \pi(X^+)$,

$$\begin{aligned}&[Q_\varepsilon(v), (\omega \otimes \pi(X^+))^{n+1}] \\ &= [Q_\varepsilon(v), (\omega \otimes \pi(X^+))^n]\omega \otimes \pi(X^+) + (\omega \otimes \pi(X^+))^n[Q_\varepsilon(v), \omega \otimes \pi(X^+)] \\ &= 2n(\omega \otimes \pi(X^+))^{n-1}\mathsf{A}^+(v;\varepsilon)\omega \otimes \pi(X^+) + 2(\omega \otimes \pi(X^+))^n\mathsf{A}^+(v;\varepsilon) \\ &= 2(n+1)(\omega \otimes \pi(X^+))^n\mathsf{A}^+(v;\varepsilon),\end{aligned}$$

by virtue of the induction hypothesis and equation (6). Formula (8) is an immediate consequence of formula (7) and the definitions. This concludes the proof of the lemma. □

*Remark* 3.3. It is worth noting that, with $\varepsilon^2 := (\varepsilon_1^2, \ldots, \varepsilon_m^2) \in \boldsymbol{R}^m$, the operators $\mathsf{A}^+(v;\varepsilon)$, $\mathsf{A}^-(v;\varepsilon)$ and $Q_\varepsilon(v)$ satisfy the following relations:

$$\begin{gathered}[\mathsf{A}^+(v;\varepsilon), \mathsf{A}^-(v;\varepsilon)] = Q_\varepsilon(v) + 1 \otimes (\pi_{\varepsilon^2} - \pi_\varepsilon)(H) = Q_{\varepsilon^2}(v), \\ [Q_\varepsilon(v), \mathsf{A}^\pm(v;\varepsilon)] = \pm 2(\mathsf{A}^\pm(v;\varepsilon) + 1 \otimes (\pi_{\varepsilon^2} - \pi_\varepsilon)(X^\pm)) = \pm 2\mathsf{A}^\pm(v;\varepsilon^2).\end{gathered}$$

Hence the map $\omega \otimes \pi_\varepsilon : \mathfrak{sl}_2(\boldsymbol{R}) \to \mathrm{End}_{\boldsymbol{C}}(\mathscr{S}'(\boldsymbol{R}; V))$ is a representation iff $\varepsilon \in \{0,1\}^m$. Notice moreover that the functions

$$\varphi_j^{(k)+}(\varepsilon) := \sum_{\ell=0}^{j} c_{\ell,j}^{(k)+}(\varepsilon)\omega(X^+)^{j-\ell}\varphi_0 \otimes v_\ell^{(k)},$$

$$\varphi_j^{(k)-}(\varepsilon) := \sum_{\ell=0}^{j} c_{\ell,j}^{(k)-}(\varepsilon)\omega(X^+)^{j-\ell}\psi^\dagger\varphi_0 \otimes v_\ell^{(k)},$$

where one chooses $c_{0,0}^{(k)\pm}(\varepsilon) = 1$ and

$$c_{\ell,j}^{(k)\pm}(\varepsilon) = \frac{c_{0,j}^{(k)\pm}(\varepsilon)}{\varepsilon_k^\ell}\binom{j}{\ell}\frac{\Gamma(\nu_k-\ell)}{\Gamma(j-\ell\mp 1/2)}\frac{\Gamma(j\mp 1/2)}{\Gamma(\nu_k)}, \quad 1 \le \ell \le j \le \nu_k - 1,$$

are solutions to

$$\mathsf{A}^-(\nu;\varepsilon)\varphi_j^{(k)\pm}(\varepsilon) = 0,$$

such that, upon choosing $c_{0,j}^{(k)\pm}(\varepsilon) = \varepsilon_k^j$,

$$\varphi_j^{(k)+}(\varepsilon) \to \frac{\Gamma(\nu_k-j)}{\Gamma(\mp 1/2)}\frac{\Gamma(j\mp 1/2)}{\Gamma(\nu_k)}\varphi_0 \otimes v_j^{(k)},$$

$$\varphi_j^{(k)-}(\varepsilon) \to \frac{\Gamma(\nu_k-j)}{\Gamma(\mp 1/2)}\frac{\Gamma(j\mp 1/2)}{\Gamma(\nu_k)}\psi^\dagger\varphi_0 \otimes v_j^{(k)},$$

as $\varepsilon \to 0$ in $\boldsymbol{R}^m$.

We are now ready to study the spectrum of $Q_{\varepsilon,A,B}(\nu)$ in terms of $\omega \otimes \pi$, by using Lemma 3.2.

THEOREM 3.4. *Let* $\nu = (\nu_1, \dots, \nu_m) \in \mathscr{Y}_N$, *and pick* $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \boldsymbol{R}^m$. *Let us define* $Q_{\varepsilon,A,B}(\nu; x, D_x) := T_{A,B}Q_\varepsilon(\nu; x, D_x)T_{A,B}^*$. *Then the spectrum of* $Q_{\varepsilon,A,B}(\nu)$ *in* $L^2(\boldsymbol{R}; V)$ (*but also in* $\mathscr{S}'(\boldsymbol{R}; V)$, *and in* $\mathscr{S}(\boldsymbol{R}; V)$) *is given in terms of* $\omega \otimes \pi$ *by*

$$\mathrm{Spec}(Q_{\varepsilon,A,B}(\nu)) = \left\{2n+1 \mp \frac{1}{2} + \varepsilon_k(2j-\nu_k+1); n \in \boldsymbol{Z}_+, 0 \le j \le \nu_k - 1, 1 \le k \le m\right\},$$

*where to* $-$ *there correspond the* **even** *eigenfunctions* $g_{\ell,n}^{(k)+} := \omega(X^+)^n\varphi_0 \otimes v_\ell^{(k)}$, *and to* $+$ *there correspond the* **odd** *eigenfunctions* $g_{\ell,n}^{(k)-} := \omega(X^+)^n\psi^\dagger\varphi_0 \otimes v_\ell^{(k)}$.

*The description holds true also in case* $\varepsilon \in \boldsymbol{C}^m$, *and upon addition of* $T_{A,B}(1 \otimes \pi_\delta(\mathbf{1}))T_{A,B}^*$, $\delta \in \boldsymbol{C}^m$, *the k-part of the spectrum being then shifted by a factor* $\delta_k$ (*with the same eigenfunctions*).

*Remark* 3.5. It follows that

$$\mathrm{Spec}(Q_{(\pm\varepsilon_1,\dots,\pm\varepsilon_m),A,B}(\nu)) = \mathrm{Spec}(Q_{(\varepsilon_1,\dots,\varepsilon_m),A,B}(\nu))$$

(any choices of $\pm$). This follows from the change of index, for an arbitrary $k$, given by $v_k - 1 - j = j'$.

*Proof of Theorem* 3.4. We may suppose $A = B = I_V$, so that $\rho = \omega \otimes \pi$. Let us define for $1 \le k \le m$, $0 \le j \le v_k - 1$ and $n \in \boldsymbol{Z}_+$, the following functions (belonging to $\mathscr{S}(\boldsymbol{R}; V)$):

$$f_j^{(k)+} = f_{j,0}^{(k)+} = \varphi_0 \otimes v_j^{(k)}, \quad f_{j,n}^{(k)+} = \rho(X^+)^n f_j^{(k)+},$$
$$f_j^{(k)-} = f_{j,0}^{(k)-} = \psi^\dagger \varphi_0 \otimes v_j^{(k)}, \quad f_{j,n}^{(k)-} = \rho(X^+)^n f_j^{(k)-}.$$

Recall that $\pi(X^+)v_{v_k-1}^{(k)} =: v_{v_k}^{(k)} = 0$, for any given $k = 1, \dots, m$ (i.e. $v_{v_k-1}^{(k)}$ is the highest weight vector of $\pi_{v_k}$). In particular $f_{v_k}^{(k)\pm} = 0$, whence also $f_{v_k,n}^{(k)\pm} = 0$ for any given $n \in \boldsymbol{Z}_+$. It will be also convenient to put $f_{n-j}^{(k)\pm} = 0$ whenever $j > n$. One has, by the same arguments of Lemma 2.8, that

(10) $\{f_{j,n}^{(k)\pm}; n \in \boldsymbol{Z}_+, 0 \le j \le v_k - 1, 1 \le k \le m\}$ is a basis of $L^2(\boldsymbol{R}; V)$.

One immediately has, for $0 \le j \le v_k - 1$,

$$Q_{\varepsilon,A,B}(v; x, D_x) f_{j,0}^{(k)\pm} = \left(1 \mp \frac{1}{2} + \varepsilon_k(2j - v_k + 1)\right) f_{j,0}^{(k)\pm}.$$

Let us consider for simplicity throughout the sequel of the proof the $+$-case.

As a consequence of formula (8) above, one has, for $n \ge 1$,

$$Q_\varepsilon(v; x, D_x) f_{j,n}^{(k)+} = \left(2n + \frac{1}{2} + \varepsilon_k(2j - v_k + 1)\right) f_{j,n}^{(k)+} + 2n(\varepsilon_k - 1) f_{j+1,n-1}^{(k)+}, \tag{11}$$

and, in particular, being $f_{v_k,n-1}^{(k)+} = 0$,

$$Q_\varepsilon(v; x, D_x) f_{v_k-1,n}^{(k)+} = \left(2n + \frac{1}{2} + \varepsilon_k(v_k - 1)\right) f_{v_k-1,n}^{(k)+}, \quad n \in \boldsymbol{Z}_+. \tag{12}$$

Let us now set, for $0 \le \ell \le v_k - 2$ and $k = 1, \dots, m$,

$$g_{\ell,n}^{(k)+} := f_{\ell,n}^{(k)+} + \sum_{j=1}^{v_k-1-\ell} \alpha_{j,n}^{(k)+} f_{\ell+j,n-j}^{(k)+}, \quad \text{and} \quad g_{v_k-1,n}^{(k)+} := f_{v_k-1,n}^{(k)+},$$

where, recall, $f_{s,n-j}^{(k)+} = 0$ when $n < j$, for any given $0 \le s \le v_k - 1$, and the coefficients $\{\alpha_{j,n}^{(k)+}\}_{1 \le j \le v_k-1-\ell}$ are picked to be $\alpha_{j,n}^{(k)+} = (-1)^j \binom{n}{j}$, $1 \le j \le v_k - \ell$. Since

$$\begin{bmatrix} g_{v_k-1,n}^{(k)+} \\ g_{v_k-2,n+1}^{(k)+} \\ \vdots \\ g_{0,n+v_k-1}^{(k)+} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ * & * & \cdots & 1 \end{bmatrix} \begin{bmatrix} f_{v_k-1,n}^{(k)+} \\ f_{v_k-2,n+1}^{(k)+} \\ \vdots \\ f_{0,n+v_k-1}^{(k)+} \end{bmatrix}, \quad \forall n \in \boldsymbol{Z}_+, \ \forall k = 1, \dots, m, \tag{13}$$

one has that $\{g_{\ell,n}^{(k)\pm}\}$ is a complete system of $L^2(\boldsymbol{R}; V)$, and simple computations show that, by virtue of the choice of the coefficients $\alpha_{j,n}^{(k)\pm}$,

$$Q_\varepsilon(v; x, D_x) g_{\ell,n}^{(k)+} = \left(2n + \frac{1}{2} + \varepsilon_k(2\ell - v_k + 1)\right) g_{\ell,n}^{(k)+}.$$

To see that the functions $g_{\ell,n}^{(k)\pm}$ are the usual vector-valued Hermite ones, it suffices to notice that

$$\sum_{s=1}^{n} \binom{n}{s} \omega(X^+)^{n-s} \varphi_0 \otimes v_{\ell+s}^{(k)} + \sum_{j=1}^{v_k-1-\ell} (-1)^j \binom{n}{j} \sum_{s'=0}^{n-j} \binom{n-j}{s'} \omega(X^+)^{n-j-s'} \varphi_0 \otimes v_{\ell+j+s'}^{(k)} = 0,$$

whence one immediately gets

$$g_{\ell,n}^{(k)+} = \omega(X^+)^n \varphi_0 \otimes v_\ell^{(k)}, \quad g_{\ell,n}^{(k)-} = \omega(X^+)^n \psi^\dagger \varphi_0 \otimes v_\ell^{(k)}.$$

This concludes the proof of the theorem. □

*Remark* 3.6. One could interpret matters also as follows. Define the bounded operators $T_j : L^2(\boldsymbol{R}; V) \to L^2(\boldsymbol{R}; V)$, $j = 1, 2$, by

$T_1 : f_{j,n}^{(k)\pm} \mapsto$ above finite linear combination of the (Hermite functions $\otimes\, v_\ell^{(k)}$),

and

$$T_2 : g_{\ell,n}^{(k)\pm} \mapsto \text{above finite linear combination of the } (f_{j,n}^{(k)\pm}).$$

Then

$$T_1 = T_2^{-1}.$$

Notice that $T_1$ operates in terms of the trivial representation, whereas $T_2$ operates in terms of the tensor representation.

## 4. The system $L(x, D_x)$ of the introduction (see (1))

In this case, let us take $V = \boldsymbol{C}^N$, $\mu = 1$, and consider a complex $N \times N$ **diagonalizable** matrix $A$. We distinguish two cases: $N$ is even, $N$ is odd.

In the first case, let $\alpha_1, \ldots, \alpha_N \in \boldsymbol{C}$ be the eigenvalues of $A$, possibly repeated according to their multiplicity. Organize them in pairs like, say, $(\alpha_j, \alpha_{N/2+j})$, $1 \le j \le N/2$, and associate with this pattern the partition $v_{\text{can}}^{(N/2)} := (\underbrace{2, 2, \ldots, 2}_{N/2})$ of $N$, and the representation $\pi_{\text{can}}^{(N/2)} := \underbrace{\pi_{\text{vect}} \oplus \pi_{\text{vect}} \oplus \cdots \oplus \pi_{\text{vect}}}_{N/2}$. Then, with $\varepsilon = ((\alpha_1 - \alpha_{N/2+1})/2, \ldots, (\alpha_{N/2} - \alpha_N)/2) \in \boldsymbol{C}^{N/2}$ and $\delta = ((\alpha_1 + \alpha_{N/2+1})/2, \ldots, (\alpha_{N/2} + \alpha_N)/2) \in \boldsymbol{C}^{N/2}$,

$$U_A^{-1} A U_A = \pi_{\mathrm{can},\varepsilon}^{(N/2)}(H) + \pi_{\mathrm{can},\delta}^{(N/2)}(\mathbf{1}),$$

and

$$L(x, D_x) = U_A(Q_\varepsilon(\nu_{\mathrm{can}}^{(N/2)}) + 1 \otimes \pi_{\mathrm{can},\delta}^{(N/2)}(\mathbf{1}))U_A^{-1}. \tag{14}$$

In the second case, we proceed as before: we group the eigenvalues of $A$, let them be $\alpha_1, \dots, \alpha_{N-1}, \alpha_N \in \boldsymbol{C}$ (possibily repeated according to their multiplicity), group them two-by-two in pairs such as $(\alpha_j, \alpha_{(N-1)/2+j})$, $1 \le j \le (N-1)/2$, leaving out the eigenvalue $\alpha_N$, and consider the partition $\nu_{\mathrm{can}}^{((N-1)/2)} := (\underbrace{2, 2, \dots, 2}_{(N-1)/2}, 1)$ and the associated representation $\pi_{\mathrm{can}}^{((N-1)/2)} := \underbrace{\pi_{\mathrm{vect}} \oplus \cdots \oplus \pi_{\mathrm{vect}}}_{(N-1)/2} \oplus \pi_{\mathrm{trivial}}$. Notice that $\pi_{\mathrm{trivial}}(\mathbf{1}) = 1$. Let then $\varepsilon = ((\alpha_1 - \alpha_{(N-1)/2+1})/2, \dots, (\alpha_{(N-1)/2} - \alpha_{N-1})/2, 0) \in \boldsymbol{C}^{(N-1)/2+1}$ and $\delta = ((\alpha_1 + \alpha_{(N-1)/2+1})/2, \dots, (\alpha_{(N-1)/2} + \alpha_{N-1})/2, \alpha_N) \in \boldsymbol{C}^{(N-1)/2+1}$. As before,

$$U_A^{-1} A U_A = \pi_{\mathrm{can},\varepsilon}^{((N-1)/2)}(H) + \pi_{\mathrm{can},\delta}^{((N-1)/2)}(\mathbf{1}),$$

and

$$L(x, D_x) = U_A(Q_\varepsilon(\nu_{\mathrm{can}}^{((N-1)/2)}) + 1 \otimes \pi_{\mathrm{can},\delta}^{((N-1)/2)}(\mathbf{1}))U_A^{-1}. \tag{15}$$

Of course, $L(x, D_x)$ may also be thought of as

$$L(x, D_x) = U_A\left(Q(\underbrace{1, 1, \dots, 1}_{N}) + 1 \otimes (\underbrace{\pi_{\mathrm{trivial},\alpha_1} \oplus \cdots \oplus \pi_{\mathrm{trivial},\alpha_N}}_{N})(\mathbf{1})\right)U_A^{-1}, \tag{16}$$

where $\alpha_1, \dots, \alpha_N$ are the eigenvalues of $A$. The interest in writing $L(x, D_x)$ in the forms given by (14) or (15) (according to whether the dimension is even or odd, respectively), resides exactly in writing the matrix $A$, through a representation whose irreducible summands are **not** all *trivial*, as a combination of suitable perturbations of images of elements of $\mathscr{U}(\mathfrak{sl}_2(\boldsymbol{R}))$ that commute with $H$.

However, notice that the number of independent parameters in all cases is exactly the **same**, that is $N$, for in case of equation (14) (resp. (15)), one considers perturbations of the form $U_A(1 \otimes \pi_t(H) + 1 \otimes \pi_s(\mathbf{1}))U_A^{-1}$, with $(t, s) \in \boldsymbol{C}^{N/2} \times \boldsymbol{C}^{N/2}$ (resp. $(t, s) \in \boldsymbol{C}^{(N-1)/2} \times \boldsymbol{C}^{(N-1)/2+1}$), whereas in case of equation (16), one considers perturbations of the form $U_A(1 \otimes \pi_z(\mathbf{1}))U_A^{-1}$, with $z \in \boldsymbol{C}^N$.

The above argument shows that the spectrum of the family of systems $Q(\nu)$ (and the kind of perturbations considered here) is actually parametrized by the **parity** of $N$, and **not** by the partition $\nu$. More precisely, one has the following remark.

*Remark* 4.1. With $\pi_{\mathrm{can}} = \pi_{\mathrm{can}}^{(N/2)}$ when $N$ is even, $\pi_{\mathrm{can}} = \pi_{\mathrm{can}}^{((N-1)/2)}$ when $N$ is odd, respectively, let $\pi = \pi_{(\nu)} = \pi_1 \oplus \cdots \oplus \pi_m$, for some $\nu \in \mathscr{Y}_N$. We think of $\pi(H)$ as an $N \times N$-matrix $A \in \mathrm{Mat}_N(\boldsymbol{C})$ whose eigenvalues are then given by the

$\alpha_j^{(k)} = 2j - \nu_k + 1$, and eigenvectors by the $v_j^{(k)}$, $0 \le j \le \nu_k - 1$, $k = 1, \dots, m$, hence fixing an isomorphism $V \simeq \boldsymbol{C}^N$. Let $\varepsilon, \delta$ be the complex perturbation parameters constructed as above by suitably grouping the eigenvalues $\alpha_j^{(k)}$. Then

$$\pi(H) = A = \pi_{\mathrm{can},\varepsilon}(H) + \pi_{\mathrm{can},\delta}(\mathbf{1}),$$

whence

$$Q(\nu; x, D_x) = Q_\varepsilon(\nu_{\mathrm{can}}; x, D_x) + 1 \otimes \pi_{\mathrm{can},\delta}(\mathbf{1}).$$

It is hence possible to change $\pi$ into $\pi_{\mathrm{can}}$ by perturbation, and obtain that the eigenvalues of the $Q(\nu)$ are functions **only** of the perturbation parameters, once the **parity** of the size $N$ is fixed. This makes the basis $\{\varphi_n \otimes v_j^{(k)}\}$, the $\varphi_n$ being the Hermite functions, the **universal** one to study the map

$$\mathscr{Y}_N \times \boldsymbol{C}^\ell \times \boldsymbol{C}^\ell \ni (\nu, \varepsilon, \delta) \mapsto \mathrm{Spec}_{\mathscr{S}'(\boldsymbol{R};\, V)}(Q_\varepsilon(\nu) + \pi_{(\nu),\delta}(\mathbf{1})) \subset \boldsymbol{C}, \tag{17}$$

where $\ell = N/2$ when $N$ is even, $\ell = (N-1)/2 + 1$ when $N$ is odd.

One naturally generalizes Theorem 2.7 and the above considerations to systems of the kind

$$\mathscr{L}(x, D_x) = Q_\varepsilon(\nu; x, D_x) \otimes \pi_{(\nu'),\mu}(\mathbf{1}) + A \otimes B$$

(considered as unbounded operators on $L^2(\boldsymbol{R}; V \otimes V')$ with domain $B^2(\boldsymbol{R}; V \otimes V')$). Here we take $N, N' \in \boldsymbol{N}$, $\nu = (\nu_1, \dots, \nu_m) \in \mathscr{Y}_N$, $\nu' = (\nu'_1, \dots, \nu'_{m'}) \in \mathscr{Y}_{N'}$, $\varepsilon, \delta \in \boldsymbol{C}^m$, $\mu \in \boldsymbol{R}_+^{m'}$, where $(\pi_{(\nu)}, V)$ and $(\pi_{(\nu')}, V')$ are two $\mathfrak{sl}_2(\boldsymbol{R})$-modules, of (complex) dimension $N$ and $N'$, respectively, where $A = 1 \otimes (\pi_{(\nu),\varepsilon}(H) + \pi_{(\nu),\delta}(\mathbf{1}))$ and $B = \pi_{(\nu'),\alpha}(H) + \pi_{(\nu'),\beta}(\mathbf{1})$, with $\alpha, \beta \in \boldsymbol{C}^{m'}$.

## 5. A "genuine" infinite-dimensional perturbation

The following example deals with an infinite-dimensional perturbation of the system. Let us consider an $\mathfrak{sl}_2(\boldsymbol{R})$-module $(\pi, V)$, $\dim_{\boldsymbol{C}} V = N$, and let us define, $V = V_1 \oplus \cdots \oplus V_m$ being the decomposition into irreducible summands of $V$, the bounded operators (in the $+$-case and $-$-case, respectively) $\mathsf{M}_k^\pm : L_\pm^2(\boldsymbol{R}; V_k) \to L_\pm^2(\boldsymbol{R}; V_k)$,

$$\mathsf{M}_k^\pm(h_{\ell,n}^{(k)\pm}) = A_n^\pm(\ell,k) h_{\ell,n+1}^{(k)\pm} + B_n^\pm(\ell,k) h_{\ell,n-1}^{(k)\pm} + C_n^\pm(\ell,k) h_{\ell,n}^{(k)\pm},$$

for $n \in \boldsymbol{Z}_+$, $0 \le \ell \le \nu_k - 1$, $k = 1, \dots, m$, where $\{A_n^\pm(\ell,k)\}_{n \in \boldsymbol{Z}_+}$, $\{B_n^\pm(\ell,k)\}_{n \in \boldsymbol{Z}_+}$, $\{C_n^\pm(\ell,k)\}_{n \in \boldsymbol{Z}_+} \in \ell^\infty(\boldsymbol{Z}_+; \boldsymbol{C})$. Then, upon setting

$$\mathsf{M} := \left(\bigoplus_{k=1}^m \mathsf{M}_k^+\right) \oplus \left(\bigoplus_{k=1}^m \mathsf{M}_k^-\right) : L^2(\boldsymbol{R}; V) \xrightarrow{\text{bounded}} L^2(\boldsymbol{R}; V),$$

the spectrum of the operator (and of all the unitary equivalents of it)

$$Q(\nu_\pi; x, D_x) + \mathsf{M} = \omega(H) \otimes I_V + 1 \otimes \pi(H) + \mathsf{M} : L^2(\boldsymbol{R}; V) \xrightarrow{\text{unbounded}} L^2(\boldsymbol{R}; V),$$

with domain $B^2(\boldsymbol{R}; V)$, may be explicitly computed in terms of $\omega \otimes \pi$, upon studying a three-term recurrence system. (Of course, one may consider cases with infinitely many steps, and cases in which the coefficients $A_n^{\pm}, B_n^{\pm}$ have temperate growth in $n \in \boldsymbol{Z}_+$, the resulting operator $Q(v_\pi) + \mathsf{M} : \mathscr{S}'(\boldsymbol{R}; V) \to \mathscr{S}'(\boldsymbol{R}; V)$ being then continuous, and defined on a suitable domain when considered as an unbounded operator in $L^2(\boldsymbol{R}; V)$.) In fact, to get the recurrence that produces the spectrum, write (in the +-case and −-case, respectively)

$$W_\ell^{(k)\pm} := \overline{\mathrm{Span}\{h_{\ell,n}^{(k)\pm}\}_{n\in\boldsymbol{Z}_+}}, \quad \text{so that } L^2_\pm(\boldsymbol{R}; V) = \bigoplus_{k=1}^{m} \bigoplus_{\ell=0}^{v_k - 1} W_\ell^{(k)\pm}.$$

Then

$$\mathsf{M}|_{W_\ell^{(k)\pm}} = \mathsf{M}_k^{\pm}|_{W_\ell^{(k)\pm}} : W_\ell^{(k)\pm} \xrightarrow{\text{bounded}} W_\ell^{(k)\pm},$$
$$Q(v_\pi)|_{W_\ell^{(k)\pm}\cap\mathscr{S}'(\boldsymbol{R}; V)} : W_\ell^{(k)\pm} \cap \mathscr{S}'(\boldsymbol{R}; V) \to W_\ell^{(k)\pm} \cap \mathscr{S}'(\boldsymbol{R}; V),$$

and the eigenvalue equations (for $k = 1, \ldots, m$ and $0 \le \ell \le v_k - 1$)

$$(Q(v_\pi) + \mathsf{M})u = \lambda u, \quad u \in W_\ell^{(k)\pm} \cap \mathscr{S}'(\boldsymbol{R}; V),$$

is solved for **non-zero** $u = \sum_{n=0}^{+\infty} \zeta_{\ell,n}^{(k)\pm}(\lambda) h_{\ell,n}^{(k)\pm} \in \mathscr{S}(\boldsymbol{R}; V)$ (because of ellipticity), where $\{\zeta_{\ell,n}^{(k)\pm}(\lambda)\}_{n\in\boldsymbol{Z}_+} \subset \boldsymbol{C}$ and $\lambda \in \boldsymbol{C}$ are determined by the recurrence equations

$$(\lambda_{\ell,n}^{(k)\pm} + C_n^{\pm}(\ell,k) - \lambda)\zeta_{\ell,n}^{(k)\pm}(\lambda) + A_{n-1}^{\pm}(\ell,k)\zeta_{\ell,n-1}^{(k)\pm}(\lambda) + B_{n+1}^{\pm}(\ell,k)\zeta_{\ell,n+1}^{(k)\pm}(\lambda) = 0,$$

where $\zeta_{\ell,-1}^{(k)\pm}(\lambda) := 0$, $\zeta_{\ell,0}^{(k)\pm}(\lambda) \in \boldsymbol{C}\backslash\{0\}$, the $\lambda_{\ell,n}^{(k)\pm}$'s are the eigenvalues of the operator $Q(v_\pi)$, and $|\zeta_{\ell,n}^{(k)\pm}(\lambda)| \to 0$ as $n \to +\infty$ faster than $n^{-d}$, for any $d \in \boldsymbol{Z}_+$, when $\lambda$ is an eigenvalue of $Q(v_\pi) + \mathsf{M}$ (see [4]).

## 6. A geometrical example

Let us denote by $\Omega^*_{\mathscr{S}}(\boldsymbol{R}^n) := \mathscr{S}(\boldsymbol{R}^n; \wedge^*_{\boldsymbol{C}}(\boldsymbol{R}^n))$, where $\wedge^*_{\boldsymbol{C}}(\boldsymbol{R}^n) := \wedge^*(\boldsymbol{R}^n) \otimes_{\boldsymbol{R}} \boldsymbol{C}$, the (complex-valued) Schwartz space sections of differential forms on $\boldsymbol{R}^n$, and consider the differential complex

$$0 \to \Omega^0_{\mathscr{S}}(\boldsymbol{R}^n) \xrightarrow{D} \Omega^1_{\mathscr{S}}(\boldsymbol{R}^n) \xrightarrow{D} \cdots \xrightarrow{D} \Omega^n_{\mathscr{S}}(\boldsymbol{R}^n) \to 0,$$

where the operator $D$ is defined by

$$D : \Omega^k_{\mathscr{S}}(\boldsymbol{R}^n) \to \Omega^{k+1}_{\mathscr{S}}(\boldsymbol{R}^n), \quad D\alpha = \frac{1}{\sqrt{2}}\left(d\alpha + \sum_{j=1}^{n} x_j\, dx_j \wedge \alpha\right), \quad \forall \alpha \in \Omega^k_{\mathscr{S}}(\boldsymbol{R}^n),$$

$d$ being the usual exterior differentiation. Let $*_e$ denote the Hodge-star, with respect to the usual Euclidean structure of $\boldsymbol{R}^n$, and let $\Omega^*_{L^2}(\boldsymbol{R}^n) := L^2(\boldsymbol{R}^n; \wedge^*_{\boldsymbol{C}}(\boldsymbol{R}^n))$ be the Hilbert space of the $L^2$ complex-valued differential forms. Then $D$ extends to a unbounded operator on $\Omega^*_{L^2}(\boldsymbol{R}^n)$ and its $L^2$-adjoint is the complex

$$0 \to \Omega^n_{\mathscr{S}}(\boldsymbol{R}^n) \xrightarrow{D^*} \Omega^{n-1}_{\mathscr{S}}(\boldsymbol{R}^n) \xrightarrow{D^*} \cdots \xrightarrow{D^*} \Omega^0_{\mathscr{S}}(\boldsymbol{R}^n) \to 0,$$

defined by (recall that $d^* = (-1)^{n(k+1)+1} *_e d *_e$)

$$D^* : \Omega^k_{\mathscr{S}}(\boldsymbol{R}^n) \to \Omega^{k-1}_{\mathscr{S}}(\boldsymbol{R}^n), \quad D^*\alpha = \frac{1}{\sqrt{2}}\left(d^*\alpha + \sum_{j=1}^n x_j \frac{\partial}{\partial x_j} \rfloor \alpha\right), \quad \forall \alpha \in \Omega^k_{\mathscr{S}}(\boldsymbol{R}^n).$$

Then $D^2 = (D^*)^2 = 0$. It is elementary to check that on $k$-forms

$$DD^* + D^*D = \frac{1}{2}\sum_{j=1}^n (-\partial^2_{x_j} + x_j^2) + k - \frac{n}{2}.$$

Hence for $\varepsilon \in \boldsymbol{R}$, on $\Omega^*_{\mathscr{S}}(\boldsymbol{R}^n)$ we may consider the following "deformation" of $DD^* + D^*D$,

$$(DD^* + D^*D)_\varepsilon := \left(\frac{1}{2}\sum_{j=1}^n (-\partial^2_{x_j} + x_j^2)\right) \mathrm{Id} + \varepsilon \sum_{k=0}^n \left(k - \frac{n}{2}\right) \Pi^{(n)}_k,$$

where $\Pi^{(n)}_k : \Omega^*_{\mathscr{S}}(\boldsymbol{R}^n) \xrightarrow{\text{projection}} \Omega^k_{\mathscr{S}}(\boldsymbol{R}^n)$. Let now $(M, \sigma)$ be a compact symplectic manifold of dimension $2m$, $m \geq 1$. Let $w_\sigma$ be the skew-symmetric bivector field dual to $\sigma$ (see, e.g., [1] and references therein; for example, when $\sigma = d\xi \wedge dx$, then $w_\sigma = \partial/\partial x \wedge \partial/\partial \xi$), and let $w^{(k)}_\sigma : \Omega^k(M) \times \Omega^k(M) \to C^\infty(M)$ be the $(-1)^k$-symmetric pairing induced by $w_\sigma$, where $\Omega^k(M)$ is the space of smooth real-valued differential $k$-forms on $M$. (In the sequel $\Omega^*(M)$ will denote the space of smooth real-valued differential forms on $M$.) Choose $v_M = \sigma^m/m!$ as volume form on $M$, and define the symplectic Hodge's star

$$*_\sigma : \Omega^k(M) \to \Omega^{2m-k}(M), \quad \text{by the condition } \beta \wedge (*_\sigma \alpha) = (w^{(k)}_\sigma(\beta, \alpha)) v_M.$$

Then $*_\sigma *_\sigma = \mathrm{Id}$. Following [1], one puts

$$\mathsf{L} : \Omega^k(M) \ni \alpha \mapsto \sigma \wedge \alpha \in \Omega^{k+2}(M),$$
$$\mathsf{L}^* = -*_\sigma \mathsf{L} *_\sigma : \Omega^k(M) \ni \alpha \mapsto w_\sigma \rfloor \alpha \in \Omega^{k-2},$$

and

$$\mathsf{A} = \sum_{k=0}^{2m} (k-m)\Pi^M_k : \Omega^*(M) \to \Omega^*(M), \quad \text{where } \Pi^M_k : \Omega^*(M) \xrightarrow{\text{projection}} \Omega^k(M).$$

For $p \in M$, we will denote by $\mathsf{A}_p$ (resp. $\mathsf{L}_p$) the map induced by $\mathsf{A}$ (resp. $\mathsf{L}$) on $\wedge^*(T^*_p M)$.

One has the following important relations

$$[\mathsf{L}, \mathsf{L}^*] = \mathsf{A}, \quad [\mathsf{A}, \mathsf{L}] = 2\mathsf{L}, \quad [\mathsf{A}, \mathsf{L}^*] = -2\mathsf{L}^*. \tag{18}$$

Since $\sigma$ and $w_\sigma$ are **real** operators, one is allowed to think of the operators $\mathsf{L}$, $\mathsf{L}^*$ and $\mathsf{A}$ as acting on $\Omega^*_{\boldsymbol{C}}(M)$, resp. on $\Omega^k_{\boldsymbol{C}}(M)$, just by "declaring" them complex-linear (here $\Omega^*_{\boldsymbol{C}}(M)$, resp. $\Omega^k_{\boldsymbol{C}}(M)$, stands for the space of smooth complex-valued differential forms, resp. $k$-forms, on $M$). Relations (18) hold true also in this case. The map $w_\sigma^{(k)}$ extends to a $(-1)^k$-symmetric pairing (we keep the notation) $w_\sigma^{(k)} : \Omega^k_{\boldsymbol{C}}(M) \times \Omega^k_{\boldsymbol{C}}(M) \to C^\infty(M; \boldsymbol{C})$, by $w_\sigma^{(k)}(\beta \otimes b, \alpha \otimes a) := abw_\sigma^{(k)}(\beta, \alpha)$, where $a, b \in \boldsymbol{C}$ and $\alpha, \beta \in \Omega^k(M)$. It follows that $*_\sigma(\alpha \otimes a) = a(*_\sigma\alpha)$ for all $a \in \boldsymbol{C}$ and $\alpha \in \Omega^k(M)$, whence $*_\sigma$ extends to a $\boldsymbol{C}$-linear map $\Omega^k_{\boldsymbol{C}}(M) \to \Omega^{2m-k}_{\boldsymbol{C}}(M)$, and $\mathsf{L}^*$ is the (formal) adjoint of $\mathsf{L}$ with respect to the Hermitian form on $\Omega^*_{\boldsymbol{C}}(M)$ given by

$$(\beta, \alpha)_M := \int_M \beta \wedge (*_\sigma \bar{\alpha}) = \int_M (w_\sigma^{(k)}(\beta, \bar{\alpha}))v_M, \quad \alpha, \beta \in \Omega^k_{\boldsymbol{C}}(M).$$

Definition 6.1 (see [1]). Let $(\pi, V)$ be an $\mathfrak{sl}_2(\boldsymbol{R})$-module of dimension $\le +\infty$. One says that $V$ is an **$\mathfrak{sl}_2(\boldsymbol{R})$-module of finite $H$-spectrum** if

- $V$ can be decomposed as the direct sum of eigenspaces of $H$;
- $H$ has only finitely many distinct eigenvalues.

In view of relations (18), one has the following proposition (see [1]).

Proposition 6.2. *The map* $\pi_{\text{form}} : \mathfrak{sl}_2(\boldsymbol{R}) \to \text{End}_{\boldsymbol{C}}(\Omega^*_{\boldsymbol{C}}(M))$, *defined by*

$$\pi_{\text{form}}(X^+) = \mathsf{L}, \quad \pi_{\text{form}}(X^-) = \mathsf{L}^*, \quad \pi_{\text{form}}(H) = \mathsf{A},$$

*defines* $(\pi_{\text{form}}, \Omega^*_{\boldsymbol{C}}(M))$ *as an* $\mathfrak{sl}_2(\boldsymbol{R})$*-module* **of finite $H$-spectrum**. *Note that* $\Omega^k_{\boldsymbol{C}}(M)$ *is the eigenspace of* $\mathsf{A}$ *relative to the eigenvalue* $k - m$.

*Remark* 6.3. In [1], the author considered $\mathfrak{sl}_2(\boldsymbol{C})$ instead of $\mathfrak{sl}_2(\boldsymbol{R})$. This gives no problem here.

Let us now fix an arbitrary $p \in M$, and consider the Hilbert space $\mathsf{H}_p = \Omega^*_{L^2}(\boldsymbol{R}^n) \otimes \wedge^*_{\boldsymbol{C}}(T^*_pM) = L^2(\boldsymbol{R}^n; \wedge^*_{\boldsymbol{C}}(\boldsymbol{R}^n) \otimes \wedge^*_{\boldsymbol{C}}(T^*_pM))$. The elements of $\mathsf{H}_p$ are of the form (with standard notation) $\sum_{|I|=j, |J|=k} f_I(x) g_J(p)\, dx_I \otimes dy_J$, with $f \in L^2(\boldsymbol{R}^n; \boldsymbol{C})$ and $g_J(p) \in \boldsymbol{C}$, and with $x \in \boldsymbol{R}^n$ and $y = (y_1, \dots, y_{2m})$ local symplectic coordinates at $p$. We may fix an inner product in $\mathsf{H}_p$ by requiring that the $\{dx_I \otimes dy_J\}_{I,J}$ be an orthogonal basis. We may then introduce the following natural "Laplacian"

$$Q_p = (DD^* + D^*D)_\varepsilon \otimes I_{\bigwedge^*_{\boldsymbol{C}}(T^*_pM)} + I_{\Omega_{L^2}(\boldsymbol{R}^n)} \otimes \mathsf{A}_p : \mathsf{H}_p \to \mathsf{H}_p$$

Hence, the spectral properties of $Q_p$ in terms of the tensor product representation, are obtained by Theorem 2.7 (and its corollaries), once a decomposition into irreducibles of $(\pi_{\text{form}}, \wedge^*_{\boldsymbol{C}}(T^*_pM))$ is found, that is to say, once a partition of $\dim_{\boldsymbol{C}} \wedge^*_{\boldsymbol{C}}(T^*_pM) = 2^{2m}$ is fixed (hereafter, $\dim = \dim_{\boldsymbol{C}}$). Notice that the basis $\{h^{(k)\pm}_{j,n}\}$ of Theorem 2.7 **diagonalizes** $Q_p$ for all $\varepsilon \in \boldsymbol{R}$.

*Examples.* • When $m = 1$, we have $\dim \wedge^0_{\boldsymbol{C}}(T_p^* M) = \dim \wedge^2_{\boldsymbol{C}}(T_p^* M) = 1$ and $\dim \wedge^1_{\boldsymbol{C}}(T_p^* M) = 2$, whence the pattern

$$\wedge^0_{\boldsymbol{C}}(T_p^* M) \xrightarrow{\mathsf{L}_p} \wedge^2_{\boldsymbol{C}}(T_p^* M) \xrightarrow{\mathsf{L}_p} 0, \quad \wedge^1_{\boldsymbol{C}}(T_p^* M) \xrightarrow{\mathsf{L}_p} 0.$$

Hence the partition of $2^2$ we are looking for is $\nu = (2, 1, 1)$.

• When $m = 2$, we have $\dim \wedge^0_{\boldsymbol{C}}(T_p^* M) = \dim \wedge^4_{\boldsymbol{C}}(T_p^* M) = 1$,

$$\dim \wedge^2_{\boldsymbol{C}} (T_p^* M) = 6, \ \dim \wedge^1_{\boldsymbol{C}}(T_p^* M) = \dim \wedge^3_{\boldsymbol{C}}(T_p^* M) = 4,$$

whence the pattern

$$\wedge^0_{\boldsymbol{C}}(T_p^* M) \xrightarrow{\mathsf{L}_p} \wedge^2_{\boldsymbol{C}}(T_p^* M) = \mathrm{Im}\, \mathsf{L}_p|_{\wedge^0_{\boldsymbol{C}}(T_p^* M)} \oplus \mathrm{Ker}\, \mathsf{L}_p|_{\wedge^2_{\boldsymbol{C}}(T_p^* M)} \xrightarrow{\mathsf{L}_p} \wedge^4_{\boldsymbol{C}}(T_p^* M) \xrightarrow{\mathsf{L}_p} 0,$$

$$\wedge^1_{\boldsymbol{C}}(T_p^* M) \xrightarrow{\mathsf{L}_p} \wedge^3_{\boldsymbol{C}}(T_p^* M) \xrightarrow{\mathsf{L}_p} 0, \quad \text{with } \dim_{\boldsymbol{C}} \mathrm{Ker}\, \mathsf{L}_p|_{\wedge^2_{\boldsymbol{C}}(T_p^* M)} = 5.$$

Hence, the partition of $2^4$ we are looking for is $\nu = (3, 2, 2, 2, 2, 1, 1, 1, 1, 1)$.

Since $M$ is compact we may regard the point $p \in M$ as a parameter, and consider

$$Q = (DD^* + D^* D)_\varepsilon \otimes I_{\Omega^*_{\boldsymbol{C}}(M)} + I_{\Omega_{L^2}(\boldsymbol{R}^n)} \otimes \mathsf{A},$$

as an operator acting on $L^2(\boldsymbol{R}^n; \wedge^*_{\boldsymbol{C}}(\boldsymbol{R}^n)) \otimes \Omega^*_{\boldsymbol{C}}(M)$, that is the space of **finite** linear combinations of tensor-products of elements of $L^2$ and of $\Omega^*_{\boldsymbol{C}}(M)$, and hence we may think of the $\Omega^*_{\boldsymbol{C}}(M)$-part as parameters, separately from the $\boldsymbol{R}^n$-part. This gives no problem for $M$ is compact and by virtue of the Lebesgue dominated convergence theorem. (When $M$ is not compact, one may use $\Omega^*_{\mathscr{S}}(M)$.) Again, we can then use Theorem 2.7 (and its corollaries) to write down the spectrum of $Q$. Notice, moreover, that we may solve by explicit Hilbert-space methods, equations such as

$$Q\alpha = \beta, \quad \beta \in \mathscr{S}(\boldsymbol{R}^n; \wedge^*_{\boldsymbol{C}}(\boldsymbol{R}^n)) \otimes \Omega^*_{\boldsymbol{C}}(M)),$$

where (with standard notation and with $(x; y) \in \boldsymbol{R}^n \times M$)

$$\beta(x; y) \overset{\text{locally}}{=} \sum_{|I|=j, |J|=k} f_I(x) g_J(y)\, dx_I \otimes dy_J, \quad f_I \in L^2(\boldsymbol{R}^n; \boldsymbol{C}), \ \ g_J \in C^\infty(M; \boldsymbol{C}).$$

One may finally proceed further to consider $Q$ as an (unbounded) operator on the space of sections $L^2(\boldsymbol{R}^n \times M; \wedge^*_{\boldsymbol{C}}(T^* \boldsymbol{R}^n) \otimes \wedge^*_{\boldsymbol{C}}(T^* M))$. However, in this case things are more delicate, for $Q$ is no longer elliptic, and the spectrum is no longer discrete.

*Acknowledgements.* The first author wishes to thank the Graduate School of Mathematics of Kyushu University, Fukuoka (Japan) for the warm hospitality.

Work in part supported by Grant-in-Aid for Scientific Research (B) No. 11440010, the Ministry of Education, Science and Culture of Japan, and by the Italian Ministry of University and Scientific Research.

## References

[1] D. Yan, Hodge structure on symplectic manifolds, Adv. Math., **120** (1996), 143–154.
[2] R. Howe and E. C. Tan, Non-Abelian Harmonic Analysis: Applications of $SL(2,\boldsymbol{R})$, Universitext, Springer-Verlag, New York, 1992.
[3] H. Ochiai, Non-commutative harmonic oscillators and Fuchsian ordinary differential operators, Comm. Math. Phys., **217** (2001), 357–373.
[4] A. Parmeggiani and M. Wakayama, Non-commutative harmonic oscillators-I, Forum Math., **14** (2002), 539–604.
[5] A. Parmeggiani and M. Wakayama, Non-commutative harmonic oscillators-II, Forum Math., **14** (2002), 669–690.
[6] A. Parmeggiani and M. Wakayama, Oscillator representations and systems of ordinary differential equations, Proc. Natl. Acad. Sci. USA, **98** (2001), 26–30.

Department of Mathematics
University of Bologna
Piazza di Porta S. Donato 5
40127 Bologna, Italy
e-mail: parmeggi@dm.unibo.it

Faculty of Mathematics
Kyushu University
Hakozaki Higashi-ku
Fukuoka 812-8581, Japan
e-mail: wakayama@math.kyushu-u.ac.jp