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A NOTE ON A UNICITY THEOREM OF K. TOHGE

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Abstract

In this paper, we deal with the problem of uniqueness of meromorphic functions sharing three values CM, and get rid of the restriction on the hyper-orders in a unicity theorem of K. Tohge. An example is provided to show that the result in this paper is best possible.

1. Introduction and main results

Let f and g be two non-constant meromorphic functions in the complex plane. It is assumed that the reader is familiar with the standard notations of Nevanlinna's theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$ and so on, which can be found in [1]. We use E to denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. The notation $S(r,f)$ denotes any quantity satisfying $S(r,f)=o(T(r,f))$ $(r\to\infty, r\notin E)$.

Let a be a complex number, we say that f and g share the value a CM provided $f-a$ and $g-a$ have the same zeros counting multiplicities (see [2]). We say that f and g share ∞ CM provided that $1/f$ and $1/g$ share 0 CM. In this paper, we also need the following definition.

Definition. Let f be a non-constant meromorphic function. The hyper-order of f, denoted $\nu(f)$, is defined by

$$\nu(f)=\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}.$$

In 1988, K. Tohge [3] proved the following theorems:

Theorem A. *Let f and g be two distinct transcendental meromorphic functions sharing* $0,1$ *and* ∞ CM. *If f' and g' share* 0 CM, *then f and g satisfy one of the following relations*:

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(i) $f \cdot g \equiv 1$,
(ii) $(f-1)(g-1) \equiv 1$,
(iii) $f+g \equiv 1$,
(iv) $f \equiv cg$,
(v) $f-1 \equiv c(g-1)$,
(vi) $[(c-1)f+1] \cdot [(c-1)g-c] \equiv -c$,
where c $(\neq 0, 1)$ *is a constant.*

Theorem B. *Let* f *and* g *be two distinct transcendental meromorphic functions sharing* $0, 1$ *and* ∞ CM, *and let* a $(\neq 0)$ *be a finite complex number. If* f' *and* g' *share* a CM *and* $\max\{\nu(f), \nu(g)\} < 1$, *then* f *and* g *satisfy one of the following relations*:
(i) $f \cdot g \equiv 1$,
(ii) $(f-1)(g-1) \equiv 1$,
(iii) $[(c-1)f+1] \cdot [(c-1)g-c] \equiv -c$, *where* c $(\neq 0, 1)$ *is a constant.*

Now it is natural to ask the following question:

Question 1. What can be said if we get rid of the condition "$\max\{\nu(f), \nu(g)\} < 1$" in Theorem B?

In this paper, we shall answer Question 1 and obtain a new result. Indeed, we shall prove the following theorem:

Theorem 1. *Let* f *and* g *be two distinct transcendental meromorphic functions sharing* $0, 1$ *and* ∞ CM, *and let* a $(\neq 0)$ *be a finite complex number. If* f' *and* g' *share* a CM, *then* f *and* g *satisfy one of the following relations*:
(i) $f = Ae^{a\omega z}$, $g = (1/A)e^{-a\omega z}$, *where* ω *satisfying* $\omega^2 = -1$, *and* A $(\neq 0)$ *are constants*;
(ii) $f = 1 + Ae^{a\omega z}$, $g = 1 + (1/A)e^{-a\omega z}$, *where* ω *satisfying* $\omega^2 = -1$, *and* A $(\neq 0)$ *are constants*;
(iii) $f(z) = 1/(c-1)(Ae^{a(c-1)\omega z} - 1)$, $g(z) = c/(c-1)(1 - (1/A)e^{-a(c-1)\omega z})$, *where* A, c *and* ω *are constants satisfying* $A \neq 0$, $c \neq 0, 1$ *and* $\omega^2 = 1/c$.

It is obvious that if f and g satisfy the relations (i), (ii) and (iii) of Theorem 1, then the order of f is equal to 1. By Theorem 1 we immediately deduce the following uniqueness theorem of meromorphic functions.

Theorem 2. *Let* f *and* g *be two transcendental meromorphic functions sharing* $0, 1$ *and* ∞ CM, *and suppose that* f' *and* g' *share* a CM, *where* a $(\neq 0)$ *is a finite complex number. If the order of* f *is not equal to* 1, *then* $f \equiv g$.

2. Some lemmas

The following notations are used throughout this paper.

Let h be a non-constant meromorphic function, and let k be a positive integer. We denote by $N_{k)}(r, 1/(h-a))$ the counting function of a-points of h with multiplicity $\leq k$, and denote by $N_{(k}(r, 1/(h-a))$ the counting function of a-points of h with multiplicity $\geq k$ (see [2]).

Let f and g share $0, 1$ and ∞ CM, we denote by $N_0(r)$ the counting function of the zeros of $f-g$ not containing the zeros of $f, 1/f$ and $f-1$ (see [4] or [5]).

Lemma 1 (see [2, Lemma 9.1]). *Let f and g be two non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If*

$$\delta_{1)}(0, f) + \delta_{1)}(1, f) > \frac{3}{2},$$

where

$$\delta_{1)}(0, f) = 1 - \limsup_{r\to\infty} \frac{N_{1)}(r, 1/f)}{T(r, f)}, \quad \delta_{1)}(1, f) = 1 - \limsup_{r\to\infty} \frac{N_{1)}(r, 1/(f-1))}{T(r, f)},$$

then $f + g \equiv 1$.

Lemma 2 (see [2, p. 369]). *Let F and G be two non-constant meromorphic functions, and let*

$$\phi \equiv \frac{F''}{F'} - \frac{G''}{G'}.$$

If z_∞ is a common simple pole of F and G, then $\phi(z_\infty) = 0$.

Lemma 3 (see [4, Lemma 4]). *Let f and g be meromorphic functions sharing $0, 1, \infty$ CM. If $f \not\equiv g$, then*

$$N_{(2}\left(r, \frac{1}{f}\right) + N_{(2}\left(r, \frac{1}{f-1}\right) + N_{(2}(r, f) = S(r, f).$$

Lemma 4 (see [5, Lemma 7] or [6, Lemma 3]). *Let f and g be two distinct non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If f is a Möbius transformation of g, then f and g satisfy one of the following relations*:

(i) $f \cdot g \equiv 1$,
(ii) $(f-1)(g-1) \equiv 1$,
(iii) $f + g \equiv 1$,
(iv) $f \equiv cg$,
(v) $f - 1 \equiv c(g-1)$,
(vi) $[(c-1)f + 1] \cdot [(c-1)g - c] \equiv -c$,

where c $(\neq 0, 1)$ is a constant.

Lemma 5 (see [8, p. 120]). *Let $f_1, f_2, \ldots, f_n$ be meromorphic functions linearly independent over the complex number field C such that*

$$\sum_{i=1}^{n} f_i \equiv 1.$$

Then

$$T(r, f_j) < \sum_{i=1}^{n} N\left(r, \frac{1}{f_i}\right) + (n-1) \sum_{\substack{i=1 \\ i \neq j}}^{n} \overline{N}(r, f_i) + S(r) \quad (1 \leq j \leq n),$$

where $T(r) = \max_{1 \leq i \leq n}\{T(r, f_i)\}$ *and* $S(r) = o(T(r))$ $(r \to \infty, r \notin E)$.

LEMMA 6 (see [2, Theorem 1.62] or [10, Theorem 1]). *Let* $f_1, f_2, \dots, f_n$ *be non-constant meromorphic functions, and let* f_{n+1} $(\not\equiv 0)$ *be a meromorphic function such that*

$$\sum_{i=1}^{n+1} f_i \equiv 1. \tag{2.1}$$

If there exists a subset $I \subseteq R^+$ *satisfying* $mesI = \infty$ *such that*

$$\sum_{i=1}^{n+1} N\left(r, \frac{1}{f_i}\right) + n \sum_{\substack{i=1 \\ i \neq j}}^{n+1} \overline{N}(r, f_i) < (\lambda + o(1)) T(r, f_j) \quad (r \to \infty, r \in I, j = 1, 2, \dots, n), \tag{2.2}$$

where $\lambda < 1$. *Then* $f_{n+1} \equiv 1$.

Remark. Lemma 6 plays an important role for the proof of Theorem 1. Now we give a simple proof of Lemma 6. Suppose that

$$\sum_{j=1}^{n} f_j \not\equiv 0. \tag{2.3}$$

Without loss of generality, let

$$\sum_{j=1}^{n} f_j \equiv \sum_{i=1}^{k} a_i f_i, \tag{2.4}$$

where $f_1, f_2, \dots, f_k$ $(1 \leq k \leq n)$ are linearly independent over the complex number field C, and $a_1, a_2, \dots, a_k$ are nonzero constants. By (2.1) and (2.4), we have

$$\sum_{i=1}^{k} a_i f_i + f_{n+1} \equiv 1. \tag{2.5}$$

By Lemma 5, (2.2) and (2.5) we can easily verify that $f_1, f_2, \dots, f_k, f_{n+1}$ are linearly dependent over the complex number field C, hence we have

$$c_1 f_1 + c_2 f_2 + \cdots + c_k f_k + c_{k+1} f_{n+1} = 0, \tag{2.6}$$

where $c_1, c_2, \dots, c_k, c_{k+1}$ are constants not all equal to zero. Noting that

$f_1, f_2, \dots, f_k$ are linearly independent over the complex number field C, we can see that $c_{k+1} \neq 0$. From (2.6) we have

$$f_{n+1} = -\frac{c_1}{c_{k+1}} f_1 - \cdots - \frac{c_k}{c_{k+1}} f_k, \tag{2.7}$$

substituting (2.7) into (2.5) we get

$$\sum_{i=1}^{k} \left(a_i - \frac{c_i}{c_{k+1}} \right) f_i \equiv 1. \tag{2.8}$$

From (2.8) we can see that $a_i - \dfrac{c_i}{c_{k+1}}$ $(i = 1, 2, \dots, k)$ are not all equal to zero. By Lemma 5, (2.2) and (2.8) we can have a contradiction. Thus, $\sum_{j=1}^{n} f_j \equiv 0$, and $f_{n+1} \equiv 1$, which proves Lemma 6.

In 1999, H. X. Yi [11] proved the following result, which is an extension of Lemma 6: Let $f_1, f_2, \dots, f_n$ be non-constant meromorphic functions, and let $f_{n+1}, f_{n+2}, \dots, f_{n+m}$ be meromorphic functions such that

$$f_k \not\equiv 0 \quad (k = n+1, n+2, \dots, n+m)$$

and

$$\sum_{i=1}^{n+m} f_i \equiv A,$$

where A is a nonzero constant. If there exists a subset $I \subseteq R^+$ satisfying $mesI = \infty$ such that

$$\sum_{i=1}^{n+m} N\left(r, \frac{1}{f_i}\right) + (n+m-1) \sum_{\substack{i=1 \\ i \neq j}}^{n+m} \overline{N}(r, f_i) < (\lambda + o(1)) T(r, f_j)$$
$$(r \to \infty, r \in I, j = 1, 2, \dots, n),$$

where $\lambda < 1$. Then there exist $t_i \in \{0, 1\}$ $(i = 1, 2, \dots, m)$ such that

$$\sum_{i=1}^{m} t_i f_{n+i} \equiv A.$$

Lemma 7 (see [5, Theorem 1]). *Let f and g be two distinct non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If*

$$\limsup_{\substack{r \to \infty \\ r \notin E}} \frac{N_0(r)}{T(r, f)} > \frac{1}{2},$$

then f is a Möbius transformation of g.

Lemma 8 (see [5, Theorem 2]). *Let f and g be two non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If*

$$0 < \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r,f)} \le \frac{1}{2},$$

then f is not any Möbius transformation of g, and f and g satisfy one of the following relations:

(i) $f \equiv \dfrac{e^{s\gamma}-1}{e^{(k+1)\gamma}-1}, \; g \equiv \dfrac{e^{-s\gamma}-1}{e^{-(k+1)\gamma}-1},$

(ii) $f \equiv \dfrac{e^{(k+1)\gamma}-1}{e^{(k+1-s)\gamma}-1}, \; g \equiv \dfrac{e^{-(k+1)\gamma}-1}{e^{-(k+1-s)\gamma}-1},$

(iii) $f \equiv \dfrac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1}, \; g \equiv \dfrac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1},$

where s and k (≥ 2) are positive integers such that $1 \le s \le k$, and s and $k+1$ are relatively prime, and γ is a non-constant entire function.

Lemma 9 (see [4, Lemma 1]). *Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM, then there exist two entire functions α and β such that*

$$f \equiv \frac{e^{\alpha}-1}{e^{\beta}-1}, \quad g \equiv \frac{e^{-\alpha}-1}{e^{-\beta}-1}, \tag{2.9}$$

where $e^{\beta} \not\equiv 1$, $e^{\alpha} \not\equiv 1$ and $e^{\beta-\alpha} \not\equiv 1$, and

$$T(r,g) + T(r,e^{\alpha}) + T(r,e^{\beta}) = O(T(r,f)) \quad (r \notin E). \tag{2.10}$$

Lemma 10 (see [12, Lemma 2.4]). *Let h be a non-constant meromorphic function and let α, β, γ be meromorphic functions such that $T(r,\alpha) + T(r,\beta) + T(r,\gamma) = S(r,h)$, and $\alpha \not\equiv 0$ or $\gamma \not\equiv 0$. Furthermore, let*

$$H = \alpha h^2 + \beta h + \gamma.$$

If $\overline{N}(r,h) = S(r,h)$, $\overline{N}(r,1/h) = S(r,h)$ and $N_{1)}(r,1/H) = S(r,h)$, then $\beta^2 - 4\alpha\gamma \equiv 0$.

3. Proof of Theorem 1

By the assumptions of Theorem 1, we have

$$\frac{f'-a}{g'-a} = e^{\delta}, \tag{3.1}$$

where δ is an entire function. Suppose that $e^{\delta} \equiv A$, where A is a nonzero constant. From (3.1) we get

$$f - Ag = (1-A)az + C, \tag{3.2}$$

where C is a constant. Since $f \not\equiv g$, from (3.2) we know that

$$(1 - A)az + C \not\equiv 0. \tag{3.3}$$

By (3.2) and (3.3) we get

$$\delta_{1)}(0, f) + \delta_{1)}(1, f) = 2.$$

By Lemma 1, we have $f + g \equiv 1$ and $f' + g' \equiv 0$, which implies that a is a Picard value of f' and g'. This contradicts Hayman's inequality (see [1, Theorem 3.5]). Thus e^{δ} is not a constant, and hence

$$\delta' \not\equiv 0. \tag{3.4}$$

By logarithmic differentiation, from (3.1) we obtain

$$\delta' = \frac{f''}{f' - a} - \frac{g''}{g' - a}. \tag{3.5}$$

By Lemma 2, (3.4) and (3.5), we get

$$N_{1)}(r, f) \le N\left(r, \frac{1}{\delta'}\right) \le T(r, \delta') + O(1) = S(r, f). \tag{3.6}$$

By Lemma 3, we have

$$N_{(2}(r, f) = S(r, f). \tag{3.7}$$

By (3.6) and (3.7), we obtain

$$N(r, f) = S(r, f). \tag{3.8}$$

We discuss the following two cases.

Case 1. Suppose that f is a Möbius transformation of g. By Lemma 4, we know that f and g satisfy one of the six relations in Lemma 4.

Assume that f and g satisfy the relation (i) in Lemma 4. Let $f = e^{\alpha}$, where α is a non-constant entire function. Then $g = e^{-\alpha}$. Substituting f and g into (3.1) we get

$$\frac{\alpha' e^{2\alpha} - a e^{\alpha}}{-\alpha' - a e^{\alpha}} = e^{\delta}. \tag{3.9}$$

By (3.9) we have

$$T(r, e^{\delta}) \ge T(r, e^{\alpha}) + S(r, f) \tag{3.10}$$

and

$$\frac{\alpha'}{a} e^{\alpha} + e^{\delta} + \frac{\alpha'}{a} e^{\delta - \alpha} \equiv 1. \tag{3.11}$$

By Lemma 6, (3.10) and (3.11) we obtain

$$\frac{\alpha'}{a}e^{\delta-\alpha} \equiv 1, \quad \frac{\alpha'}{a}e^{\alpha} + e^{\delta} \equiv 0. \tag{3.12}$$

From (3.12) we get $\alpha(z) = a\omega z + C$, where ω satisfying $\omega^2 = -1$, and C are constants. Thus $f(z) = Ae^{a\omega z}$ and $g(z) = (1/A)e^{-a\omega z}$, where A is a nonzero constant. From this we have the relation (i) in Theorem 1.

Assume that f and g satisfy the relation (ii) in Lemma 4. In the same manner as above, we can obtain $f(z) = 1 + Ae^{a\omega z}$ and $g(z) = 1 + (1/A)e^{-a\omega z}$, where ω satisfying $\omega^2 = -1$, and A ($\neq 0$) are constants. From this we have the relation (ii) in Theorem 1.

Assume that f and g satisfy the relation (vi) in Lemma 4. In the same manner as above, we can obtain $f(z) = 1/(c-1)(Ae^{a(c-1)\omega z} - 1)$ and $g(z) = c/(c-1)(1 - (1/A)e^{-a(c-1)\omega z})$, where ω satisfying $\omega^2 = 1/c$, and A ($\neq 0$) are constants. From this we have the relation (iii) in Theorem 1.

Assume that f and g satisfy the relation (iii) in Lemma 4. Since f and g share $0, 1$ and ∞ CM, from the relation (iii) in Lemma 4, we know that 0 and 1 are Picard values of f. Thus $N(r,f) = T(r,f) + S(r,f)$, which contradicts (3.8).

Assume that f and g satisfy the relations (iv) and (v) in Lemma 4. In the same manner as above, we can obtain contradictions.

Case 2. Suppose that f is not any Möbius transformation of g. By Lemma 7, we consider the following two subcases.

Subcase 2.1. Assume that

$$0 < \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r,f)} \le \frac{1}{2}.$$

By Lemma 8, we know that f and g satisfy one of the three relations in Lemma 8.

Assume that f and g satisfy the relation (i) in Lemma 8. Then we have $N(r,f) = T(r,f) + S(r,f)$, which contradicts (3.8).

Assume that f and g satisfy the relation (ii) in Lemma 8. By (3.8) we know that $k = s$. Thus,

$$f = e^{k\gamma} + e^{(k-1)\gamma} + \cdots + 1, \quad g = e^{-k\gamma} + e^{-(k-1)\gamma} + \cdots + 1. \tag{3.13}$$

By (3.13) we obtain

$$T(r,f) = kT(r,e^{\gamma}) + S(r,f), \quad T(r,g) = kT(r,e^{\gamma}) + S(r,f). \tag{3.14}$$

Substituting (3.13) into (3.1) we get

$$\frac{k\gamma' e^{2k\gamma} + (k-1)\gamma' e^{(2k-1)\gamma} + \cdots + \gamma' e^{(k+1)\gamma} - ae^{k\gamma}}{-k\gamma' - (k-1)\gamma' e^{\gamma} - \cdots - \gamma' e^{(k-1)\gamma} - ae^{k\gamma}} = e^{\delta}. \tag{3.15}$$

By (3.14) and (3.15) we have

$$T(r, e^{\delta}) \geq kT(r, e^{\gamma}) + S(r, f) \tag{3.16}$$

and

$$\frac{k\gamma'}{a}e^{k\gamma} + \frac{(k-1)\gamma'}{a}e^{(k-1)\gamma} + \cdots + \frac{\gamma'}{a}e^{\gamma} + e^{\delta} + \frac{\gamma'}{a}e^{\delta-\gamma} + \frac{2\gamma'}{a}e^{\delta-2\gamma} + \cdots + \frac{k\gamma'}{a}e^{\delta-k\gamma} \equiv 1. \tag{3.17}$$

By Lemma 6, (3.14), (3.16) and (3.17) we obtain

$$\frac{k\gamma'}{a}e^{\delta-k\gamma} \equiv 1$$

and hence

$$e^{\delta} \equiv \frac{a}{k\gamma'}e^{k\gamma}. \tag{3.18}$$

Substituting (3.18) into (3.17) we get

$$\left(\frac{k\gamma'}{a} + \frac{a}{k\gamma'}\right)e^{k\gamma} + \left(\frac{(k-1)\gamma'}{a} + \frac{1}{k}\right)e^{(k-1)\gamma} + \cdots + \left(\frac{\gamma'}{a} + \frac{k-1}{k}\right)e^{\gamma} \equiv 0. \tag{3.19}$$

From (3.19) we obtain

$$\frac{k\gamma'}{a} + \frac{a}{k\gamma'} \equiv 0, \quad \frac{(k-1)\gamma'}{a} + \frac{1}{k} \equiv 0, \quad \frac{\gamma'}{a} + \frac{k-1}{k} \equiv 0. \tag{3.20}$$

From (3.20) we have a contradiction.

Assume that f and g satisfy the relation (iii) in Lemma 8. By (3.8) we know that $s = k$. Thus,

$$f = -e^{k\gamma} - e^{(k-1)\gamma} - \cdots - e^{\gamma}, \quad g = -e^{-k\gamma} - e^{-(k-1)\gamma} - \cdots - e^{-\gamma}.$$

In the same manner as above, we can obtain a contradiction.

SUBCASE 2.2. Assume that

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r, f)} = 0.$$

Thus,

$$N_0(r) = S(r, f). \tag{3.21}$$

Noting that f and g share $0, 1$ and ∞ CM, by Lemma 9 we have (2.9) and (2.10). If $e^{\beta} \equiv C$, where C is a nonzero constant. From (2.9) we obtain

$$\frac{(f-1)g}{f(g-1)} \equiv C.$$

From this we get that f is a Möbius transformation of g, which is a contradiction. Thus, e^{β} is not a constant. From (2.9) we obtain

$$f - g = \frac{(e^{\alpha} - 1)(1 - e^{\beta-\alpha})}{e^{\beta} - 1}. \tag{3.22}$$

We use $N_0^*(r)$ to denote the counting function of the common zeros of $e^{\alpha} - 1$ and $e^{\beta} - 1$. From (3.22), the following formula is obviously

$$N_0(r) = N_0^*(r) + S(r, f).$$

From this and (3.21) we have

$$N_0^*(r) = S(r, f). \tag{3.23}$$

By (2.9) and (3.23), we have

$$N(r, f) = N\left(r, \frac{1}{e^{\beta} - 1}\right) + S(r, f). \tag{3.24}$$

By (3.8) and (3.24) we get

$$T(r, e^{\beta}) = S(r, f). \tag{3.25}$$

From (2.9) and (3.25) we have

$$T(r, f) = T(r, e^{\alpha}) + S(r, f), \quad T(r, g) = T(r, f) + S(r, f). \tag{3.26}$$

Substituting (2.9) into (3.1) we get

$$\frac{(\alpha' e^{\beta} - \beta' e^{\beta} - \alpha')e^{2\alpha} + (\beta' e^{\beta} - ae^{2\beta} + 2ae^{\beta} - a)e^{\alpha}}{(\alpha' e^{2\beta} + \beta' e^{\beta} - \alpha' e^{\beta}) - (\beta' e^{\beta} + ae^{2\beta} - 2ae^{\beta} + a)e^{\alpha}} = e^{\delta}. \tag{3.27}$$

It is obvious that

$$\beta' e^{\beta} - ae^{2\beta} + 2ae^{\beta} - a \not\equiv 0, \quad \beta' e^{\beta} + ae^{2\beta} - 2ae^{\beta} + a \not\equiv 0. \tag{3.28}$$

If $\alpha' e^{\beta} - \beta' e^{\beta} - \alpha' \equiv 0$, then

$$\alpha' = \frac{\beta' e^{\beta}}{e^{\beta} - 1}. \tag{3.29}$$

By integration, from (3.29) we obtain

$$e^{\alpha} = C(e^{\beta} - 1), \tag{3.30}$$

where C is a nonzero constant, which is a contradiction. Thus

$$\alpha' e^{\beta} - \beta' e^{\beta} - \alpha' \not\equiv 0. \tag{3.31}$$

In the same manner as above, we have

$$\alpha' e^{2\beta} + \beta' e^{\beta} - \alpha' e^{\beta} \not\equiv 0. \tag{3.32}$$

By (3.25), (3.26), (3.27), (3.28), (3.31) and (3.32) we have

$$T(r, e^{\delta}) \geq T(r, e^{\alpha}) + S(r, f) \tag{3.33}$$

and

$$-\frac{\alpha' e^{\beta} - \beta' e^{\beta} - \alpha'}{\beta' e^{\beta} - ae^{2\beta} + 2ae^{\beta} - a} e^{\alpha} - \frac{\beta' e^{\beta} + ae^{2\beta} - 2ae^{\beta} + a}{\beta' e^{\beta} - ae^{2\beta} + 2ae^{\beta} - a} e^{\delta} + \frac{\alpha' e^{2\beta} + \beta' e^{\beta} - \alpha' e^{\beta}}{\beta' e^{\beta} - ae^{2\beta} + 2ae^{\beta} - a} e^{\delta - \alpha} \equiv 1. \tag{3.34}$$

By Lemma 6, (3.25), (3.26), (3.33) and (3.34) we obtain

$$\frac{\alpha' e^{2\beta} + \beta' e^{\beta} - \alpha' e^{\beta}}{\beta' e^{\beta} - ae^{2\beta} + 2ae^{\beta} - a} e^{\delta - \alpha} \equiv 1, \tag{3.35}$$

and

$$\frac{\alpha' e^{\beta} - \beta' e^{\beta} - \alpha'}{\beta' e^{\beta} - ae^{2\beta} + 2ae^{\beta} - a} e^{\alpha} + \frac{\beta' e^{\beta} + ae^{2\beta} - 2ae^{\beta} + a}{\beta' e^{\beta} - ae^{2\beta} + 2ae^{\beta} - a} e^{\delta} \equiv 0. \tag{3.36}$$

From (3.35) and (3.36) we get

$$\frac{(-\alpha' e^{\beta} + \beta' e^{\beta} + \alpha')(\alpha' e^{2\beta} + \beta' e^{\beta} - \alpha' e^{\beta})}{(\beta' e^{\beta} + ae^{2\beta} - 2ae^{\beta} + a)(\beta' e^{\beta} - ae^{2\beta} + 2ae^{\beta} - a)} \equiv 1. \tag{3.37}$$

From (3.37) we obtain

$$e^{\beta}\left(\alpha' - \frac{\beta'}{2}\right)^2 \equiv a^2 e^{2\beta} + \left(\frac{(\beta')^2}{4} - 2a^2\right) e^{\beta} + a^2. \tag{3.38}$$

Set

$$H = a^2 e^{2\beta} + \left(\frac{(\beta')^2}{4} - 2a^2\right) e^{\beta} + a^2, \tag{3.39}$$

then

$$H = e^{\beta}\left(\alpha' - \frac{\beta'}{2}\right)^2. \tag{3.40}$$

Applying Lemma 10 to H, from (3.39) and (3.40) we have

$$\left(\frac{(\beta')^2}{4} - 2a^2\right)^2 - 4a^4 \equiv 0. \tag{3.41}$$

From (3.41) we get

$$\beta' = 4a\omega, \tag{3.42}$$

and hence

$$e^{\beta} = Ae^{4a\omega z}, \tag{3.43}$$

where ω satisfying $\omega^2 = 1$, and A are nonzero constants. Substituting (3.42) and (3.43) into (3.38), we have

$$\alpha' = 2a\omega + a\omega_1\left(B_1 e^{2a\omega z} + \frac{1}{B_1}e^{-2a\omega z}\right),$$

where B_1 and ω_1 are constants satisfying $B_1^2 = A$ and $\omega_1^2 = 1$. Thus,

$$\alpha = 2a\omega z + \frac{\omega_1}{2\omega}\left(B_1 e^{2a\omega z} - \frac{1}{B_1}e^{-2a\omega z}\right) + C, \tag{3.44}$$

where C is a constant. Set $B = \dfrac{\omega_1 B_1}{\omega}$, then $B^2 = A$ and

$$\alpha = 2a\omega z + \frac{1}{2}\left(Be^{2a\omega z} - \frac{1}{B}e^{-2a\omega z}\right) + C. \tag{3.45}$$

From (3.45) we have

$$\alpha' = 2a\omega + Ba\omega \cdot e^{2a\omega z} + \frac{a\omega}{B}\cdot e^{-2a\omega z}. \tag{3.46}$$

Noting that $B^2 = A$, from (3.42), (3.43) and (3.46) we get

$$\alpha' e^{2\beta} + \beta' e^{\beta} - \alpha' e^{\beta} = B^5 a\omega \cdot e^{10a\omega z} + 2B^4 a\omega \cdot e^{8a\omega z} + 2B^2 a\omega \cdot e^{4a\omega z} - Ba\omega \cdot e^{2a\omega z} \tag{3.47}$$

and

$$\beta' e^{\beta} - ae^{2\beta} + 2ae^{\beta} - a = (4a\omega + 2a)B^2 \cdot e^{4a\omega z} - aB^4 \cdot e^{8a\omega z} - a. \tag{3.48}$$

Substituting (3.47) and (3.48) into (3.35), we deduce

$$\frac{e^{8a\omega z} - (4\omega + 2)/B^2 \cdot e^{4a\omega z} + 1/B^4}{e^{8a\omega z} + 2/B \cdot e^{6a\omega z} + 2/B^3 \cdot e^{2a\omega z} - 1/B^4} \equiv -B\omega e^{\delta - \alpha + 2a\omega z}. \tag{3.49}$$

Let

$$P_1(\chi) = \chi^8 - \frac{4\omega + 2}{B^2}\cdot \chi^4 + \frac{1}{B^4}, \quad P_2(\chi) = \chi^8 + \frac{2}{B}\cdot \chi^6 + \frac{2}{B^3}\cdot \chi^2 - \frac{1}{B^4}. \tag{3.50}$$

From (3.50) we can easily see that every root of $P_j(\chi) = 0$ $(j = 1, 2)$ is not equal to zero, and that there is at least one root of $P_1(\chi) = 0$ that is not any root of $P_2(\chi) = 0$. Thus, from (3.49) we can have a contradiction.

Theorem 1 is thus completely proved.

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