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PICARD CONSTANTS OF n-SHEETED ALGEBROID SURFACES

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Abstract

In this paper we construct all the surfaces defined by n-valued entire algebroid functions having at least $n+1$ exceptional values. And we investigate the number of exceptional values of entire functions on the surfaces. Furthermore we determine the Picard constants of the surfaces under certain conditions.

1. Introduction

Let $\mathfrak{M}(\boldsymbol{R})$ be the family of non-constant meromorphic functions on a Riemann surface $\boldsymbol{R}$. We call a value, which is not taken by $f \in \mathfrak{M}(\boldsymbol{R})$, an exceptional value of f. And let $p(f)$ be the cardinal number of exceptional values of $f \in \mathfrak{M}(\boldsymbol{R})$. Then we put

$$\mathscr{P}(\boldsymbol{R}) = \sup_{f \in \mathfrak{M}(\boldsymbol{R})} p(f),$$

which is called the Picard constant of $\boldsymbol{R}$. We can prove that $\mathscr{P}(\boldsymbol{R}) \geq 2$ if $\boldsymbol{R}$ is open and $\mathscr{P}(\boldsymbol{R}) = 0$ if $\boldsymbol{R}$ is compact. The Picard constant plays a very important role in the theory of analytic mappings of Riemann surfaces. Indeed Ozawa [7] proved that there exists no non-trivial analytic mapping of $\boldsymbol{R}$ into $\boldsymbol{X}$ if $\mathscr{P}(\boldsymbol{R}) < \mathscr{P}(\boldsymbol{X})$.

An n-sheeted algebroid surface is the proper existence domain of an n-valued algebroid function, which is defined by the following irreducible equation:

$$S_0(z)y^n - S_1(z)y^{n-1} + \cdots + (-1)^{n-1}S_{n-1}(z)y + (-1)^n S_n(z) = 0,$$

where $S_i(z)$ $(i = 0, 1, \ldots, n)$ are entire functions with no common zero. An algebroid function y is called transcendental if at least one of $S_i(z)/S_0(z)$ $(i = 1, 2, \ldots, n)$ is transcendental and y is called entire if all the $S_i(z)/S_0(z)$ $(i = 1, 2, \ldots, n)$ are entire. If $\boldsymbol{R}$ is an n-sheeted algebroid surface, then $\mathscr{P}(\boldsymbol{R}) \leq 2n$ by Selberg's theory of algebroid functions [14]. However it is very difficult in

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general to calculate $\mathscr{P}(\boldsymbol{R})$ of a given open Riemann surface $\boldsymbol{R}$, even an algebroid surface.

In the case of 3-sheeted surfaces we have the following

THEOREM A (Ozawa-Sawada [8]). *Let $\boldsymbol{R}$ be a 3-sheeted algebroid surface defined by*

$$y^3 - S_1(z)y^2 + S_2(z)y - S_3(z) = 0.$$

If $p(y) = 5$, then we have

$$y^3 - y_1y^2 + (y_0e^{H(z)} + y_2)y - y_3 = 0,$$

where y_0 $(\neq 0), y_1, y_2, y_3$ $(\neq 0)$ are constants and $H(z)$ is a non-constant entire function with $H(0) = 0$. And its discriminant is

$$D = 4y_0^3e^{3H} + \zeta_2y_0^2e^{2H} + \zeta_1y_0e^H + \zeta_0,$$

where ζ_0 $(\neq 0), \zeta_1, \zeta_2$ are suitable constants.

THEOREM B (Ozawa-Sawada [8], Sawada-Tohge [13]).[1] *Let $\boldsymbol{R}$ be the surface described in Theorem* A. *If $(\zeta_1, \zeta_2) \neq (0, 0)$, then $\mathscr{P}(\boldsymbol{R}) = 5$.*

Furthermore in the case of 4-sheeted surfaces we have the following

THEOREM C (Ozawa-Sawada [9]). *Let $\boldsymbol{R}$ be a 4-sheeted algebroid surface defined by*

$$y^4 - S_1(z)y^3 + S_2(z)y^2 - S_3(z)y + S_4(z) = 0.$$

If $p(y) = 7$, then we have

$$y^4 - y_1y^3 + (y_0e^{H(z)} + y_2)y^2 - (ay_0e^{H(z)} + y_3)y + y_4 = 0,$$

where y_0 $(\neq 0), y_1, y_2, y_3, y_4$ $(\neq 0)$, a $(\neq 0)$ are constants and $H(z)$ is a non-constant entire function with $H(0) = 0$. And its discriminant is

$$D = \eta_5y_0^5e^{5H} + \eta_4y_0^4e^{4H} + \eta_3y_0^3e^{3H} + \eta_2y_0^2e^{2H} + \eta_1y_0e^H + \eta_0,$$

where η_i $(i = 0, \dots, 5)$ are suitable constants with $\eta_0\eta_5 \neq 0$.

THEOREM D (Ozawa-Sawada [9], Niino-Tohge [6]).[2] *Let $\boldsymbol{R}$ be the surface described in Theorem* C. *If $(\eta_1, \eta_2, \eta_3, \eta_4) \neq (0, 0, 0, 0)$, then $\mathscr{P}(\boldsymbol{R}) = 7$.*

In this paper we extend the above results for n-sheeted algebroid surfaces and consider the following problems:

[1] Ozawa-Sawada [8] proved the above result under the condition that $\boldsymbol{R}$ is of finite order and Sawada-Tohge [13] proved that the result remains valid without the order condition.

[2] Ozawa-Sawada [9] proved the above result under the condition that $\boldsymbol{R}$ is of finite order and Niino-Tohge [6] proved that the result remains valid without the order condition.

1. How many kinds of exponential functions are there in the defining equation of an n-sheeted algebroid surface $\boldsymbol{R}$? In other words, when does there exist only one kind of exponential function in the defining equation of $\boldsymbol{R}$?

In Section 3 we construct all the surfaces defined by n-valued entire algebroid functions y with $p(y) \geq n+1$ and give an estimation for the number of exponential functions appearing in the defining equation (Theorem 1 and Corollary 1, 2 and 3).

2. Determine the discriminant of $\boldsymbol{R}$.

In Section 4 we prove that the factor of all zeros of the discriminant of $\boldsymbol{R}$, the defining equation of which has only one kind of exponential function, is representable as a polynomial with respect to the exponential function of degree $p(y) - 2$ (Theorem 2).

3. Find a representation of an entire function on $\boldsymbol{R}$.

In Section 5 we give a representation for every entire function on $\boldsymbol{R}$ by means of the defining function of $\boldsymbol{R}$ and some meromorphic functions on $\boldsymbol{C}$. Further we investigate the counting functions of poles of the meromorphic functions (Theorem 3).

4. Is $\mathscr{P}(\boldsymbol{R})$ decidable?

In Section 7 we show a relation between the number of exceptional values of an arbitrary entire function on $\boldsymbol{R}$ and a covering property of $\boldsymbol{R}$ (Theorem 4 and Corollary 4). Further we calculate $\mathscr{P}(\boldsymbol{R})$ under certain conditions (Theorem 5).

We assume that the reader is familiar with the Nevanlinna-Selberg theory of meromorphic and algebroid functions and the notations: $T(r,f)$, $m(r,f)$, $N(r,0,f)$, $N(r,\infty,f)$ and $S(r,f)$ etc. (See [3], [4] and [14]).

2. Some lemmas

In this section we introduce some lemmas used in the following sections. Let y be an n-valued algebroid function defined by the following equation:

$$F(z,y) := y^n - S_1(z)y^{n-1} + \cdots + (-1)^{n-1}S_{n-1}(z)y + (-1)^n S_n(z) = 0,$$

where S_i $(i = 1, 2, \ldots, n)$ are entire functions. Then α $(\in \boldsymbol{C})$ is not taken by y, if and only if, the following entire function:

$$F(z,\alpha) = \alpha^n - S_1(z)\alpha^{n-1} + \cdots + (-1)^{n-1}S_{n-1}(z)\alpha + (-1)^n S_n(z)$$

has no zero. In this case we call α a finite exceptional value of y. Furthermore α is called an exceptional value of the 'first kind' if $F(z,\alpha) \equiv \text{const.} \neq 0$ and α is called an exceptional value of the 'second kind' if $F(z,\alpha) \equiv \exp H(z)$, where $H(z)$ is a non-constant entire function. We have the following

Lemma 1 (Rémoundos [11]). *An n-valued transcendental entire algebroid function has at most $n-1$ exceptional values of the first kind and at most n exceptional values of the second kind.*

For our construction of n-sheeted surfaces, the following result plays an important role.

LEMMA 2 (Niino-Ozawa [5]). *Let* α_j $(j = 1, 2, \ldots, m)$ *be a set of non-zero constants and* g_j $(j = 1, 2, \ldots, m)$ *a set of entire functions satisfying*

$$\sum_{j=1}^{m} \alpha_j g_j = 1.$$

Then we have

$$\sum_{j=1}^{m} \delta(0, g_j) \le m - 1,$$

where $\delta(0, g_j)$ *denotes the Nevanlinna-deficiency.*

For our investigation of exceptional values of entire functions on n-sheeted surfaces, we need the following

LEMMA 3 (Niino-Tohge [6]). *Let* H *and* L *be non-constant entire functions with* $H(0) = L(0) = 0$, $a_m = b_n = 1$, a_μ $(\mu = 0, 1, \ldots, m-1)$ *and* b_ν $(\nu = 0, 1, \ldots, n-1)$ *meromorphic functions with* $a_0 \not\equiv 0$, $b_0 \not\equiv 0$ *and* g *a meromorphic function. Further suppose that*

$$T(r, a_\mu) = S(r, e^H) \quad \mu = 0, 1, \ldots, m-1,$$
$$T(r, b_\nu) = S(r, e^L) \quad \nu = 0, 1, \ldots, n-1,$$

and

$$N(r, 0, g) + N(r, \infty, g) = o(m(r, e^H) + m(r, e^L)) \quad r \to \infty$$

outside a set of finite measure. If $n \ge m \ge 1$, $d = (m, n)$, $m = pd$, $n = qd$ *and the identity*

$$\sum_{\nu=0}^{n} b_\nu(z) \exp(\nu L(z)) = g(z) \sum_{\mu=0}^{m} a_\mu(z) \exp(\mu H(z))$$

holds, then we have one of the following two cases:

(I) $\exp(nL(z) + mH(z)) = b_0(z)a_0(z)$, $g(z) = b_0(z)\exp(-mH(z))$,

$$b_{jq}(z) = b_0(z)a_{(d-j)p}(z)\exp\left(-\frac{j}{d}(nL(z) + mH(z))\right) \ \textit{for} \ j = 0, 1, 2, \ldots, d,$$

$a_\mu(z) \equiv 0$ *for* $\mu \ne 0, 1p, 2p, \ldots, dp = m$,

$b_\nu(z) \equiv 0$ *for* $\nu \ne 0, 1q, 2q, \ldots, dq = n$;

(II) $\exp(nL(z) - mH(z)) = b_0(z)/a_0(z)$, $g(z) = \exp(nL(z) - mH(z))$,

$$b_{jq}(z) = a_{jp}(z)\exp\left(\frac{d-j}{d}(nL(z) - mH(z))\right) \ \textit{for} \ j = 0, 1, 2, \ldots, d,$$

$a_\mu(z) \equiv 0$ *for* $\mu \neq 0,\ 1p, 2p, \dots, dp = m,$
$b_\nu(z) \equiv 0$ *for* $\nu \neq 0,\ 1q, 2q, \dots, dq = n.$

3. Construction of n-sheeted surfaces

In this section we construct n-sheeted algebroid surfaces defined by the following irreducible equation:

$$(1)\qquad F(z,y) := y^n - S_1(z)y^{n-1} + \cdots + (-1)^{n-1}S_{n-1}y + (-1)^n S_n(z) = 0,$$

where S_i $(i = 1, 2, \dots, n)$ are entire. Let us assume that the function y defined by (1) has p finite exceptional values and $p \geq n$. In this case we have $p(y) = p + 1$ $(\geq n + 1)$, since y has no pole. Let b_j $(j = 1, 2, \dots, m)$ be the set of exceptional values of the second kind of y and a_k $(k = 1, 2, \dots, p - m)$ be the set of exceptional values of the first kind of y, where a_k $(k = 1, 2, \dots, p - m)$ and b_j $(j = 1, 2, \dots, m)$ are different from each other. By Lemma 1 we have $1 \leq m \leq n$ and $0 \leq p - m \leq n - 1$. From (1) we have

$$(2)\qquad \begin{cases} F(z,b_1) = b_1^n - S_1 b_1^{n-1} + \cdots + (-1)^n S_n = \beta_1 e^{H_1(z)}, \\ \qquad\cdots \\ F(z,b_m) = b_m^n - S_1 b_m^{n-1} + \cdots + (-1)^n S_n = \beta_m e^{H_m(z)}, \\ F(z,a_1) = a_1^n - S_1 a_1^{n-1} + \cdots + (-1)^n S_n = \alpha_1, \\ \qquad\cdots \\ F(z,a_{p-m}) = a_{p-m}^n - S_1 a_{p-m}^{n-1} + \cdots + (-1)^n S_n = \alpha_{p-m}, \end{cases}$$

where β_j $(j = 1, 2, \dots, m)$ and α_k $(k = 1, 2, \dots, p - m)$ are non-zero constants and H_j $(j = 1, 2, \dots, m)$ are non-constant entire functions with $H_j(0) = 0$.

First of all let us consider the case $p = n$. In this case, from (2), each of $S_i(z)$ $(i = 1, 2, \dots, n)$ is representable as a linear combination of 1 and e^{H_j} $(j = 1, 2, \dots, m)$. Without loss of generality we may assume that

$$\begin{cases} H_1 \equiv H_2 \equiv \cdots \equiv H_{m_1} =: H_1^*, \\ H_{m_1+1} \equiv H_{m_1+2} \equiv \cdots \equiv H_{m_1+m_2} =: H_2^*, \\ \qquad\cdots \\ H_{m_1+\cdots+m_{\ell-1}+1} \equiv H_{m_1+\cdots+m_{\ell-1}+2} \equiv \cdots \equiv H_{m_1+\cdots+m_\ell} =: H_\ell^*, \end{cases}$$

where ℓ is an integer with $1 \leq \ell \leq m$, $H_i^* \not\equiv H_j^*$ $(i \neq j)$ and m_j $(j = 1, 2, \dots, \ell)$ are integers with $1 \leq m_j \leq m$ and $m_1 + m_2 + \cdots + m_\ell = m$. Hence (1) is reduced to

$$F(z,y) = P(y) + Q_1(y)e^{H_1^*(z)} + \cdots + Q_\ell(y)e^{H_\ell^*(z)} = 0,$$

where $P(y)$ is a monic polynomial of y of degree n and $Q_j(y)$ $(j = 1, 2, \dots, \ell)$ are polynomials of y with

$$\deg Q_j \leq n - 1 \quad \text{and} \quad Q_j \not\equiv 0 \quad (j = 1, 2, \dots, \ell).$$

Next let us consider the case $p \geq n+1$. From the first $n+1$ equations of (2) we have

$$\begin{bmatrix} b_1^n - \beta_1 e^{H_1} & b_1^{n-1} & b_1^{n-2} & \cdots & 1 \\ b_2^n - \beta_2 e^{H_2} & b_2^{n-1} & b_2^{n-2} & \cdots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ b_m^n - \beta_m e^{H_m} & b_m^{n-1} & b_m^{n-2} & \cdots & 1 \\ a_1^n - \alpha_1 & a_1^{n-1} & a_1^{n-2} & \cdots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n+1-m}^n - \alpha_{n+1-m} & a_{n+1-m}^{n-1} & a_{n+1-m}^{n-2} & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -S_1 \\ S_2 \\ -S_3 \\ \vdots \\ (-1)^n S_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The above equation has a non-trivial solution ${}^t[1, -S_1, S_2, \ldots, (-1)^n S_n]$. Hence we have

$$\det \begin{bmatrix} b_1^n - \beta_1 e^{H_1} & b_1^{n-1} & b_1^{n-2} & \cdots & 1 \\ b_2^n - \beta_2 e^{H_2} & b_2^{n-1} & b_2^{n-2} & \cdots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ b_m^n - \beta_m e^{H_m} & b_m^{n-1} & b_m^{n-2} & \cdots & 1 \\ a_1^n - \alpha_1 & a_1^{n-1} & a_1^{n-2} & \cdots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n+1-m}^n - \alpha_{n+1-m} & a_{n+1-m}^{n-1} & a_{n+1-m}^{n-2} & \cdots & 1 \end{bmatrix} = 0,$$

and

(3) $$\sum_{i=1}^{m} (-1)^i \beta_i A_i e^{H_i} + A_0 \equiv 0,$$

where

$$A_0 = \det \begin{bmatrix} b_1^n & b_1^{n-1} & b_1^{n-2} & \cdots & 1 \\ b_2^n & b_2^{n-1} & b_2^{n-2} & \cdots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ b_m^n & b_m^{n-1} & b_m^{n-2} & \cdots & 1 \\ a_1^n - \alpha_1 & a_1^{n-1} & a_1^{n-2} & \cdots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n+1-m}^n - \alpha_{n+1-m} & a_{n+1-m}^{n-1} & a_{n+1-m}^{n-2} & \cdots & 1 \end{bmatrix},$$

$$A_i = \det \begin{bmatrix} b_1^{n-1} & b_1^{n-2} & \cdots & \cdots & \cdots & 1 \\ \vdots & \vdots & & \ddots & & \vdots \\ b_{i-1}^{n-1} & b_{i-1}^{n-2} & \cdots & \cdots & \cdots & 1 \\ b_{i+1}^{n-1} & b_{i+1}^{n-2} & \cdots & \cdots & \cdots & 1 \\ \vdots & \vdots & & \ddots & & \vdots \\ b_m^{n-1} & b_m^{n-2} & \cdots & \cdots & \cdots & 1 \\ a_1^{n-1} & a_1^{n-2} & \cdots & \cdots & \cdots & 1 \\ \vdots & \vdots & & \ddots & & \vdots \\ a_{n+1-m}^{n-1} & a_{n+1-m}^{n-2} & \cdots & \cdots & \cdots & 1 \end{bmatrix} \neq 0 \quad (i = 1, 2, \dots, m).$$

If $A_0 \neq 0$, we have

$$\sum_{i=1}^{m} \delta(0, e^{H_i}) \leqq m - 1,$$

by (3) and Lemma 2. On the other hand we have $\delta(0, e^{H_i}) = 1$ $(i = 1, \dots, m)$. This is absurd. Therefore we have $A_0 = 0$. In this case, dividing (3) by e^{H_1}, we have

$$-\beta_1 A_1 + \sum_{i=2}^{m} (-1)^i \beta_i A_i e^{H_i - H_1} \equiv 0.$$

If $H_i \not\equiv H_1$ for any $i = 2, 3, \dots, m$, then we have $\beta_1 A_1 = 0$ by Lemma 2. This contradicts $\beta_1 A_1 \neq 0$. Therefore, without loss of generality, we may assume that

$$\exists m_1 : \text{integer} \ (2 \leqq m_1 \leqq m) \ \text{ s.t. } \begin{cases} H_i \equiv H_1 & (i = 2, 3, \dots, m_1), \\ H_i \not\equiv H_1 & (i = m_1 + 1, \dots, m). \end{cases}$$

Then we have

$$\sum_{i=1}^{m_1} (-1)^i \beta_i A_i + \sum_{i=m_1+1}^{m} (-1)^i \beta_i A_i e^{H_i - H_1} \equiv 0.$$

In this case we have $\sum_{i=1}^{m_1} (-1)^i \beta_i A_i = 0$ by the similar way of above. Furthermore, dividing (3) by $e^{H_{m_1+1} - H_1}$, we have

$$(-1)^{m_1+1} \beta_{m_1+1} A_{m_1+1} + \sum_{i=m_1+2}^{m} (-1)^i \beta_i A_i e^{H_i - H_{m_1+1}} \equiv 0.$$

By the similar way of above, we may assume that

$$\exists m_2 : \text{integer} \ (2 \leqq m_2 \leqq m - m_1) \ \text{ s.t. } \begin{cases} H_i \equiv H_{m_1+1} & (i = m_1 + 2, \dots, m_1 + m_2), \\ H_i \not\equiv H_{m_1+1} & (i = m_1 + m_2 + 1, \dots, m), \end{cases}$$

because of $\beta_{m_1+1}A_{m_1+1} \neq 0$. Therefore, repeating this process, we may put

$$\begin{cases} H_1 \equiv H_2 \equiv \cdots \equiv H_{m_1} =: H_1^*, \\ H_{m_1+1} \equiv H_{m_1+2} \equiv \cdots \equiv H_{m_1+m_2} =: H_2^*, \\ \qquad\qquad \cdots \\ H_{m_1+\cdots+m_{\ell-1}+1} \equiv H_{m_1+\cdots+m_{\ell-1}+2} \equiv \cdots \equiv H_{m_1+\cdots+m_\ell} =: H_\ell^*, \end{cases}$$

where ℓ is an integer with $1 \leq \ell \leq [m/2]$, $H_i^* \not\equiv H_j^*$ $(i \neq j)$ and m_j $(j = 1, 2, \dots, \ell)$ are integers with $2 \leq m_j \leq m$ and $m_1 + m_2 + \cdots + m_\ell = m$. In this case each of $S_i(z)$ $(i = 1, 2, \dots, n)$ is representable as a linear combination of 1 and $e^{H_j^*}$ $(j = 1, 2, \dots, \ell)$ by (2). Hence (1) is reduced to

(4) $$F(z, y) = P(y) + Q_1(y)e^{H_1^*(z)} + \cdots + Q_\ell(y)e^{H_\ell^*(z)} = 0,$$

where $P(y)$ is a monic polynomial of y of degree n and $Q_j(y)$ $(j = 1, 2, \dots, \ell)$ are polynomials of y with

$$\deg Q_j \leq n - 1 \quad \text{and} \quad Q_j \not\equiv 0 \quad (j = 1, 2, \dots, \ell).$$

In particular, in the case $\ell = 1$, we have

$$F(z, y) = P(y) + Q(y)e^{H(z)} = 0,$$

where

$$\begin{cases} P(y) = \displaystyle\prod_{i=1}^{m}(y - b_i)^{n_i}, \quad n_1 + n_2 + \cdots + n_m = n, \\ Q(y) = a\displaystyle\prod_{k=1}^{p-m}(y - a_k)^{\ell_k}, \quad \ell_1 + \ell_2 + \cdots + \ell_{p-m} \leq n - 1, \end{cases}$$

with a non-zero constant a, because that b_j $(j = 1, 2, \dots, m)$ are the exceptional values of the second kind of y and a_k $(k = 1, 2, \dots, p - m)$ are the exceptional values of the first kind of y.

Now let us consider the case $\ell > 1$ and investigate both of the sets of zeros of $P(y)$ and $Q_j(y)$, respectively. Firstly, substituting $y = a_k$ $(1 \leq k \leq p - m)$ into (4), we have

$$F(z, a_k) = P(a_k) + Q_1(a_k)e^{H_1^*} + \cdots + Q_\ell(a_k)e^{H_\ell^*} \equiv \alpha_k \ (\neq 0).$$

By Lemma 2 and $H_i^* \not\equiv H_j^*$ $(i \neq j)$, we have

(5) $$\begin{cases} Q_j(a_k) = 0 \quad \text{for } \forall j \ (1 \leqq j \leqq \ell), \\ P(a_k) = \alpha_k, \end{cases}$$

for $\forall k$ $(1 \leq k \leq p - m)$. Let us substitute $y = b_k$ $(1 \leq k \leq m)$ into (4). For every k $(1 \leq k \leq m)$ there exists only one i $(1 \leq i \leq \ell)$ such that $m_1 + \cdots + m_{i-1} + 1 \leqq k \leqq m_1 + \cdots + m_i$. And we have

$$F(z, b_k) = P(b_k) + Q_1(b_k)e^{H_1^*} + \cdots + Q_\ell(b_k)e^{H_\ell^*} \equiv \beta_k e^{H_i^*} \neq 0.$$

By Lemma 2 and $H_i^* \not\equiv H_j^*$ $(i \neq j)$, we have

$$(6) \qquad \begin{cases} P(b_k) = 0, \\ Q_j(b_k) = 0 \quad \text{for } j = 1, 2, \ldots, i-1, i+1, \ldots, \ell, \\ Q_i(b_k) = \beta_k, \end{cases}$$

for $\forall k$ $(1 \leq k \leq m)$. For the monic polynomial $P(y)$ we have

$$\begin{cases} P(b_k) = 0 & \text{for } k = 1, 2, \ldots, m, \\ P(a_k) = \alpha_k & \text{for } k = 1, 2, \ldots, p-m, \end{cases}$$

by (5) and (6). From the first n equations we can determine $P(y)$ and the remaining $p - n$ constants are decided by the following manner:

$$\alpha_k = P(a_k) \quad (k = n - m + 1, \ldots, p - m).$$

Next for $Q_j(y)$ we have

$$\begin{cases} Q_j(a_k) = 0 \quad k = 1, 2, \ldots, p-m, \\ Q_j(b_k) = 0 \quad k = 1, 2, \ldots, m_1 + \cdots + m_{j-1}, m_1 + \cdots + m_j + 1, \ldots, m, \\ Q_j(b_k) = \beta_k \quad k = m_1 + \cdots + m_{j-1} + 1, \ldots, m_1 + \cdots + m_j, \\ \deg Q_j \leq n - 1, \end{cases}$$

by (5) and (6). If the condition $\deg Q_j + 1 \leqq p - m_j$ holds, then we have $Q_j(y) \equiv 0$ by the first $p - m_j$ equations. This is absurd. Hence we may assume that $\deg Q_j + 1 > p - m_j$. In this case from the first $\deg Q_j(y) + 1$ equations we can determine $Q_j(y)$ and the remaining $p - \deg Q_j - 1$ constants are decided by the following manner:

$$\beta_k = Q_j(b_k),$$

for $m_1 + \cdots + m_{j-1} + (\deg Q_j + 1 - (p - m_j)) + 1 \leq \forall k \leq m_1 + \cdots + m_j$. Furthermore if there exists a set of ℓ polynomials from P and Q_j $(j = 1, 2, \ldots, \ell)$, which has a common zero, say c, with $c \neq a_k$ $(k = 1, 2, \ldots, p-m)$ and $c \neq b_i$ $(i = 1, 2, \ldots, m)$, then c is a finite exceptional value of y, which is different from a_k and b_i. This is absurd. Hence every set of ℓ polynomials among the $\ell + 1$ polynomials P and Q_j has no common zero, which is different from a_k and b_i.

Consequently we have the following result, which is a characterization of the n-sheeted algebroid surfaces $\boldsymbol{R}$ with $\mathscr{P}(\boldsymbol{R}) \geq n + 1$

Theorem 1. *Let y be an algebroid function defined by*

$$F(z, y) = y^n - S_1(z)y^{n-1} + \cdots + (-1)^{n-1}S_{n-1}y + (-1)^n S_n(z) = 0.$$

If the entire algebroid function y *has* p $(\geq n)$ *finite exceptional values, that is* $p(y) = p + 1 \geq n + 1$, *then* $F(z, y) = 0$ *coincides with*

$$F(z, y) = P(y) + Q_1(y)e^{H_1^*(z)} + \cdots + Q_\ell(y)e^{H_\ell^*(z)} = 0,$$

with non-constant entire functions $H_j^*(z)$ *of* $H_j^*(0) = 0$ $(j = 1, 2, \ldots, \ell)$ *and*

$$\begin{cases} P(y) = \displaystyle\prod_{i=1}^{m}(y - b_i)^{n_i}\tilde{P}(y), \\ Q_j(y) = \displaystyle\prod_{k=1}^{p-m}(y - a_k)^{n_{j,k}} \frac{\prod_{i=1}^{m}(y - b_i)^{\ell_{j,i}}}{\prod_{i=m_1+\cdots+m_{j-1}+1}^{m_1+\cdots+m_j}(y - b_i)^{\ell_{j,i}}}\tilde{Q}_j(y) \quad (j = 1, 2, \ldots, \ell), \end{cases}$$

where a_k $(k = 1, 2, \ldots, p - m)$ *and* b_i $(i = 1, 2, \ldots, m)$ *are different constants,* m *is a positive integer with* $m \leq n$, ℓ *is a positive integer such that* $\ell \leq m$ *if* $p(y) = n + 1$ *and* $\ell \leq [m/2]$ *if* $p(y) \geq n + 2$, n_i $(i = 1, \ldots, m)$, m_j $(j = 1, \ldots, \ell)$, $n_{j,k}$ $(j = 1, \ldots, \ell; k = 1, \ldots, p - m)$ *and* $\ell_{j,i}$ $(j = 1, \ldots, \ell; i = 1, \ldots, m)$ *are positive integers with* $\sum_{i=1}^{m} n_i \leq n$, $\sum_{j=1}^{\ell} m_j = m$ *and*

$$\sum_{k=1}^{p-m} n_{j,k} + \sum_{i=1}^{m} \ell_{j,i} - \sum_{i=m_1+\cdots+m_{j-1}+1}^{m_1+\cdots+m_j} \ell_{j,i} \leq n - 1 \quad (j = 1, \ldots, \ell), \tag{7}$$

$\tilde{P}(y)$ *is a monic polynomial of degree* $n - (n_1 + \cdots + n_m)$ *with* $\tilde{P}(a_k) \neq 0$ *and* $\tilde{P}(b_i) \neq 0$, $\tilde{Q}_j(y)$ $(j = 1, \ldots, \ell)$ *are polynomials of degree* $\deg \tilde{Q}_j \leq n - 1 - (\sum_{k=1}^{p-m} n_{j,k} + \sum_{i=1}^{m} \ell_{j,i} - \sum_{i=m_1+\cdots+m_{j-1}+1}^{m_1+\cdots+m_j} \ell_{j,i})$ *with* $\tilde{Q}_j(a_k) \neq 0$ *and* $\tilde{Q}_j(b_i) \neq 0$ *and every set of* ℓ *polynomials among the* $\ell + 1$ *polynomials* $\tilde{P}(y)$ *and* $\tilde{Q}_j(y)$ $(j = 1, \ldots, \ell)$ *has no common zero.*

In particular if $\ell = 1$, *then* $F(z, y) = 0$ *coincides with*

$$F(z, y) = P(y) + Q(y)e^{H(z)} = 0, \tag{E}$$

where $H(z)$ *is a non-constant entire function with* $H(0) = 0$ *and*

$$\begin{cases} P(y) = \displaystyle\prod_{i=1}^{m}(y - b_i)^{n_i}, \quad n_1 + n_2 + \cdots + n_m = n, \\ Q(y) = a\displaystyle\prod_{k=1}^{p-m}(y - a_k)^{\ell_k}, \quad \ell_1 + \ell_2 + \cdots + \ell_{p-m} \leq n - 1, \end{cases}$$

with a non-zero constant a.

In order to complete our result we prove the following

Lemma 4. *Let*

$$F(z, y) = y^n - S_1(z)y^{n-1} + \cdots + (-1)^{n-1}S_{n-1}(z)y + (-1)^n S_n(z),$$

be a polynomial of y of degree n with entire coefficients S_i $(i = 1, 2, \dots, n)$. If there exist different n constants a_j $(j = 1, 2, \dots, n)$ such that

$$F(z, a_j) \neq 0,$$

then $F(z, y)$ is irreducible.

Proof. Let us suppose that $F(z, y)$ is not irreducible. Then we may put

$$F(z, y) = \prod_{k=1}^{m} F_k(z, y),$$

where $F_k(z, y)$ $(k = 1, 2, \dots, m)$ are m irreducible polynomials of y. Furthermore we may put

$$F_k(z, y) = \prod_{j=1}^{n_k} (y - f_{k,j}) \quad (k = 1, 2, \dots, m),$$

where n_k $(k = 1, 2, \dots, m)$ are positive integers with $\sum_{k=1}^{m} n_k = n$ and $f_{k,j}$ $(j = 1, 2, \dots, n_k)$ are determinations of the algebroid function defined by $F_k(z, y) = 0$. In this case $F(z, y) = \prod_{k=1}^{m} \prod_{j=1}^{n_k} (y - f_{k,j})$ is the factorization of $F(z, y)$ over the field of algebroid functions. Hence every $f_{k,j}$ has no pole because that all coefficients of $F(z, y)$ are entire and the coefficient of y^n of $F(z, y)$ has no zero. Therefore all coefficients of $F_k(z, y)$ are entire because that every coefficient of $F_k(z, y)$ is a symmetric expression of $f_{k,j}$ $(j = 1, 2, \dots, n_k)$. Furthermore let y_k be the algebroid function defined by $F_k(z, y) = 0$ $(k = 1, 2, \dots, m)$, then y_k $(k = 1, 2, \dots, m)$ are entire, that is, every y_k has no pole. Now substituting $y = a_j$ we have

$$F(z, a_j) = \prod_{k=1}^{m} F_k(z, a_j).$$

Then we have $F_k(z, a_j) \neq 0$ $(k = 1, 2, \dots, m)$ because of $F(z, a_j) \neq 0$. Hence we have

$$n + 1 \leq \min_{1 \leq k \leq m} p(y_k) \leq 2 \min_{1 \leq k \leq m} n_k \leq n.$$

This is absurd. Q.E.D.

Every equation $F(z, y) = 0$, satisfying the conditions described in Theorem 1, is irreducible by Lemma 4.

An estimation for ℓ, which is the number of exponential functions appearing in the defining equation of the surface described in Theorem 1, is given by the following

Corollary 1. *Let $p(y)$ be the number of exceptional values of an n-valued entire algebroid function y and m be the number of exceptional values of the second* *kind of* y. *Then we have*

$$\ell \leq \frac{m}{p(y) - n}. \tag{8}$$

Proof. From (7), we have

$$\begin{aligned} n - 1 &\geq \sum_{k=1}^{p-m} n_{j,k} + \sum_{i=1}^{m} \ell_{j,i} - \sum_{i=m_1+\cdots+m_{j-1}+1}^{m_1+\cdots+m_j} \ell_{j,i} \\ &\geq p - m + m - m_j = p - m_j \quad (j = 1, 2, \ldots, \ell). \end{aligned}$$

Therefore we have

$$\ell(n-1) \geq \ell p - \sum_{j=1}^{\ell} m_j = \ell p - m,$$

and the desired result because of $p = p(y) - 1$. Q.E.D.

The following two results give us some sufficient conditions for $\ell = 1$, where ℓ is the number of exponential functions appearing in the defining equation of the surface described in Theorem 1.

Corollary 2. *Let* $p(y)$ *be the number of exceptional values of an* n*-valued entire algebroid function* y *and* m *be the number of exceptional values of the second kind of* y. *If* $m < \min(2(p(y) - n), n + 1)$, *then we have* $\ell = 1$.

Proof. By Lemma 1 we have $m < n + 1$. And by (8) and the assumption, we have $\ell \leq m/(p(y) - n) < 2$ and $\ell = 1$. Q.E.D.

Corollary 3. *Let* $p(y)$ *be the number of exceptional values of an* n*-valued entire algebroid function* y. *If* $p(y) > 3n/2$, *then we have* $\ell = 1$.

Proof. By (8), Lemma 1 and the assumption, we have

$$\ell \leq \frac{m}{p(y) - n} < \frac{n}{3n/2 - n} = 2.$$

Hence we have $\ell = 1$. Q.E.D.

In 1944 Dufresnoy [2] gave the sufficient condition for $\ell = 1$, described in Corollary 3, by the different way from that given above. The following examples show us the sharpness of these corollaries.

Example 1. Firstly let us consider the case $n = 4$. If $p(y) \geq 7$ $(>3n/2)$, then we have $\ell = 1$ by Corollary 3. In the case $p(y) = 6$, if $m \leq 3$ $(<2(6-4))$, then we have $\ell = 1$ by Corollary 2. If $m = 4$, then we have $\ell \le 4/(6-4) = 2$ and the following example:

$$F(z, y) \equiv \prod_{j=1}^{4}(y - b_j) + A_1(y - b_1)(y - b_2)(y - a)e^{H_1(z)} + A_2(y - b_3)(y - b_4)(y - a)e^{H_2(z)} = 0,$$

where a, b_j $(j = 1, 2, 3, 4)$ are different constants and A_1, A_2 are non-zero constants.

Example 2. Next let us consider the case $n = 5$. If $p(y) \ge 8$ $(>3n/2)$, then we have $\ell = 1$ by Corollary 3. In the case $p(y) = 7$, if $m \le 3$ $(<2(7-5))$, then we have $\ell = 1$ by Corollary 2. If $m = 4$, then we have $\ell \le 4/(7-5) = 2$ and the following example:

$$F(z, y) \equiv (y - b_1)^2 \prod_{j=2}^{4}(y - b_j) + A_1(y - b_1)(y - b_2)(y - a_1)(y - a_2)e^{H_1(z)} + A_2(y - b_3)(y - b_4)(y - a_1)(y - a_2)e^{H_2(z)} = 0,$$

where b_j $(j = 1, 2, 3, 4)$ and a_k $(k = 1, 2)$ are different constants and A_1 and A_2 are non-zero constants.

Example 3. Lastly we consider the case $n = 6$. If $p(y) \ge 10$ $(>3n/2)$, then we have $\ell = 1$ by Corollary 3. In the case $p(y) = 9$, if $m \le 5$ $(<2(9-6))$, then we have $\ell = 1$ by Corollary 2. If $m = 6$, then we have $\ell \le 6/(9-6) = 2$ and the following example:

$$F(z, y) \equiv \prod_{j=1}^{6}(y - b_j) + A_1 \prod_{j=1}^{3}(y - b_j)(y - a_1)(y - a_2)e^{H_1(z)} + A_2 \prod_{j=4}^{6}(y - b_j)(y - a_1)(y - a_2)e^{H_2(z)} = 0,$$

where b_j $(j = 1, \dots, 6)$, a_1 and a_2 are different constants and A_1 and A_2 are non-zero constants.

4. Discriminants of n-sheeted surfaces

In this section we confine our attention to the surface defined by the following irreducible equation:

$$\text{(E)} \qquad F(z, y) = P(y) + Q(y)e^{H(z)} = 0,$$

where $H(z)$ is a non-constant entire function with $H(0)=0$, $P(y)$ is a monic polynomial of y of degree n and $Q(y)$ is a polynomial of y of degree at most $n-1$. Let us put $Y := y$, $Z := e^{H(z)}$, then (E) is reduced to

$$P(Y)+Q(Y)Z=0. \tag{9}$$

Hence the algebroid function y defined by (E) is the composite function of the algebraic function $Y = Y(Z)$ defined by (9) and $Z = e^{H(z)}$. Therefore the discriminant of (E) is the composite function of the discriminant of (9) and $Z = e^{H(z)}$. Furthermore each of branch points of y is a pre-image of a branch point of $Y(Z)$ under $Z = e^{H(z)}$.

Now let $Z = Z_0$ $(\neq\infty)$ be a zero of the discriminant of (9). In this case the equation (9) with respect to Y has a multiple roots. Hence we have

$$\begin{cases} P(Y)+Q(Y)Z_0=0, \\ P'(Y)+Q'(Y)Z_0=0. \end{cases}$$

Therefore $Z = Z_0$ is a multiple value of the following fractional function:

$$Z=-\frac{P(Y)}{Q(Y)}. \tag{10}$$

Conversely every finite multiple value of (10) is a zero of the discriminant of (9).

Now let $Y = Y_0$ be a multiple Z_0-point of order n_0. Then the function (10) is representable as

$$Z=Z_0+Z_{n_0}(Y-Y_0)^{n_0}+\cdots \quad (Z_{n_0}\neq 0),$$

at $Y = Y_0$ and hence the function $Y = Y(Z)$ has the following form:

$$Y=Y_0+Y_1(Z-Z_0)^{1/n_0}+\cdots \quad (Y_1\neq 0),$$

at $Z = Z_0$. Therefore $Z = Z_0$ is a branch point of the algebraic function Y of multiplicity n_0. Hence the function Y takes different two values at different two points on the proper existence domain of Y, lying over a point $Z\neq\infty$. Furthermore in this case the discriminant D of (9) has the following form:

$$D=[\{(Z-Z_0)^{1/n_0}\}^{n_0(n_0-1)/2}]^2\times\tilde{D}=(Z-Z_0)^{n_0-1}\tilde{D}.$$

This expression shows us that the order of zero $Z = Z_0$ of D coincides with the sum of orders of zeros of $(d/dY)(-P/Q)$ at all the multiple Z_0-points of the function (10). Therefore the degree of D coincides with the degree of the numerator of $(d/dY)(-P/Q)$

Next we calculate $(d/dY)(P/Q)$. Let us put

$$\begin{cases} P(Y)=(Y-b_1)^{n_1}(Y-b_2)^{n_2}\cdots(Y-b_m)^{n_m}, \\ Q(Y)=a(Y-a_1)^{\ell_1}(Y-a_2)^{\ell_2}\cdots(Y-a_{p-m})^{\ell_{p-m}}, \end{cases}$$

where $n_1 + n_2 + \cdots + n_m = n$, $\ell_1 + \ell_2 + \cdots + \ell_{p-m} = \ell \leq n-1$, a is a non-zero constant and b_i $(i = 1, 2, \ldots, m)$ and a_k $(k = 1, 2, \ldots, p-m)$ are different constants. In this case we have

$$\frac{d}{dY}\frac{P(Y)}{Q(Y)} = \frac{\prod_{j=1}^{m}(Y - b_j)^{n_j-1}\{(n-\ell)aY^{p-1} + \cdots\}}{\prod_{i=1}^{p-m}(Y - a_i)^{\ell_i+1}}.$$

Consequently, because of $\sum_{j=1}^{m}(n_j - 1) = n - m$, we have

$$D = Z^{n-m}(A_{p-1}Z^{p-1} + \cdots + A_0),$$

where $A_0, \ldots, A_{p-1}$ are constants. Therefore we have the following

Theorem 2. *Let*

$$\text{(E)} \qquad F(z, y) = P(y) + Q(y)e^{H(z)} = 0,$$

be an irreducible equation with respect to y, *where* $H(z)$ *is a non-constant entire function with* $H(0) = 0$, $P(y)$ *is a monic polynomial of* y *of degree* n *and* $Q(y)$ *is a polynomial of* y *of degree at most* $n-1$. *Then the discriminant of* (E) *has the following form*:

$$D = e^{(n-m)H(z)}\{A_{p(y)-2}\exp((p(y) - 2)H(z)) + \cdots + A_0\},$$

where $p(y)$ *is the number of exceptional values of the entire algebroid function* y *defined by* (E) *and* m *is the number of exceptional values of the second kind of* y. *Further* A_i $(i = 0, 1, \ldots, p(y) - 2)$ *are polynomials with respect to the finite exceptional values of* y *with* $A_0 A_{p(y)-2} \neq 0$.

Proof. We have already shown that the factor of the discriminant D of (E), which gives all the zeros of D, is a polynomial with respect to e^H of degree at most $p-1$ $(=p(y) - 2)$, where p is the number of finite exceptional values of y. Let us assume that $A_0 A_{p(y)-2} = 0$. Firstly we have

$$\text{(11)} \qquad nT(r, y) = T(r, e^H) + O(1),$$

by (E). Secondly, by $A_0 A_{p(y)-2} = 0$, we have

$$\text{(12)} \qquad nN(r, \boldsymbol{R}) \leq N(r, 0, D) \leq (p(y) - 3 + o(1))T(r, e^H),$$

where

$$N(r, \boldsymbol{R}) = \frac{1}{n}\int_0^r \frac{n(t, \boldsymbol{R}) - n(0, \boldsymbol{R})}{t}\,dt + \frac{n(0, \boldsymbol{R})}{n}\log r,$$

with $n(r, \boldsymbol{R}) = \sum_{\boldsymbol{R}(r)}(\lambda - 1)$, where the summation $\sum$ runs through all the branch points in $\boldsymbol{R}(r)$, which is the part of $\boldsymbol{R}$ lying over $|z| < r$, and λ indicates the multiplicity of the branch point. By (11) and (12) we have

$$N(r, \boldsymbol{R}) \leq (p(y) - 3 + o(1))T(r, y),$$

and

$$\liminf_{r\to\infty} \frac{N(r, \boldsymbol{R})}{T(r, y)} = \varepsilon \leq p(y) - 3.$$

Therefore Selberg's deficiency relation [14] gives

$$\sum_{v} \delta(w_v) \leq 2 + \varepsilon \leq p(y) - 1,$$

where $\delta(w_v)$ is Nevanlinna-Selberg's deficiency at w_v of y. On the other hand we have $\sum \delta(w_v) \geq p(y)$. This is a contradiction.

In general the discriminant of the algebraic equation (E) is given as a polynomial of the coefficients of (E). On the other hand each coefficient of (E) is a polynomial of e^H and the finite exceptional values of y. Therefore A_j $(j = 0, 1, \dots, p(y) - 2)$ are polynomials of the finite exceptional values of y. Q.E.D.

5. Entire functions on $\boldsymbol{R}$

In this section we confine our attention to the family of non-constant entire functions on the n-sheeted algebroid surface defined by the following irreducible equation:

$$\text{(E)} \qquad F(z, y) = P(y) + Q(y)e^{H(z)} = 0,$$

where $H(z)$ is a non-constant entire function with $H(0) = 0$, $P(y)$ is a monic polynomial of y of degree n and $Q(y)$ is a polynomial of y of degree at most $n - 1$. We prove the following

Theorem 3. *Let $\boldsymbol{R}$ be the n-sheeted algebroid surface defined by* (E). *Let f be an entire function on $\boldsymbol{R}$. Then f is representable as*

$$f = f_0 + f_1 y + f_2 y^2 + \cdots + f_{n-1} y^{n-1}, \tag{13}$$

where f_i $(i = 1, 2, \dots, n - 1)$ are meromorphic functions on $\boldsymbol{C}$, all of which are regular at any points z satisfying $H'(z) \neq 0$.

Proof. Let z_0 be a point satisfying $H'(z_0) \neq 0$. And let us assume that at least one of the functions f_i appearing in the right hand side of (13) has a pole at $z = z_0$ of order p_i. Putting $\tilde{p} = \max_{0 \leq i \leq n-1} p_i$, we may put

$$f_i(z) = \frac{\alpha_{i,-\tilde{p}}}{(z - z_0)^{\tilde{p}}} + \cdots \quad (i = 0, 1, \dots, n - 1), \tag{14}$$

where $\alpha_{i,-\tilde{p}}$ $(i = 0, 1, \dots, n - 1)$ are constants with

$$(\alpha_{0,-\tilde{p}}, \dots, \alpha_{n-1,-\tilde{p}}) \neq (0, \dots, 0).$$

Case 1. We assume that there exists no branch point of $\boldsymbol{R}$ over z_0. In this case the algebroid function y defined by (E) has the following n determinations:

$$y_j = a_{j,0} + a_{j,1}(z - z_0) + \cdots + a_{j,k}(z - z_0)^k + \cdots \quad (j = 1, 2, \dots, n),$$

where $a_{j,k}$ are constants and $a_{i,0} \neq a_{j,0}$ $(i \neq j)$, because that the function y is the composite function of the algebraic function $Y = Y(Z)$ defined by $P(Y) + Q(Y)Z = 0$ and $Z = e^{H(z)}$, the function Y takes different two values at different two points on the proper existence domain of $Y = Y(Z)$ over each point Z $(\neq\infty)$ and $H'(z) \neq 0$ (see Section 4).

Substituting (14) into (13), we have

$$\begin{aligned} f &= f_0 + f_1 y_j + f_2 y_j^2 + \cdots + f_{n-1} y_j^{n-1} \\ &= (\alpha_{0,-\tilde{p}} + \alpha_{1,-\tilde{p}} a_{j,0} + \cdots + \alpha_{n-1,-\tilde{p}} a_{j,0}^{n-1}) \frac{1}{(z - z_0)^{\tilde{p}}} + \cdots. \end{aligned}$$

The function f has no pole. Hence we have

$$\alpha_{0,-\tilde{p}} + \alpha_{1,-\tilde{p}} a_{j,0} + \cdots + \alpha_{n-1,-\tilde{p}} a_{j,0}^{n-1} = 0 \quad (j = 1, 2, \dots, n),$$

and

$$\begin{bmatrix} 1 & a_{1,0} & a_{1,0}^2 & \cdots & a_{1,0}^{n-1} \\ 1 & a_{2,0} & a_{2,0}^2 & \cdots & a_{2,0}^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & a_{n,0} & a_{n,0}^2 & \cdots & a_{n,0}^{n-1} \end{bmatrix} \cdot \begin{bmatrix} \alpha_{0,-\tilde{p}} \\ \alpha_{1,-\tilde{p}} \\ \vdots \\ \alpha_{n-1,-\tilde{p}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The determinant of the coefficient matrix of the above equation does not vanish because of $a_{i,0} \neq a_{j,0}$ $(i \neq j)$. Therefore we have $(\alpha_{0,-\tilde{p}}, \dots, \alpha_{n-1,-\tilde{p}}) = (0, \dots, 0)$, which is absurd.

Case 2. We assume that there exists at least one branch point over z_0. In this case the function y defined by (E) has the following determinations:

$$\begin{cases} y_i^* = a_{i,0}^* + a_{i,1}^*(z - z_0)^{1/n_i} + \cdots + a_{i,n_i-1}^*(z - z_0)^{(n_i-1)/n_i} + \cdots & (i = 1, 2, \dots, \ell), \\ y_j = a_{j,0} + a_{j,1}(z - z_0) + \cdots + a_{j,k}(z - z_0)^k + \cdots & (j = 1, 2, \dots, n_{\ell+1}), \end{cases}$$

where $n_1 + \cdots + n_\ell + n_{\ell+1} = n$ and $a_{i,0}^*$ $(i = 1, 2, \dots, \ell)$ and $a_{j,0}$ $(j = 1, 2, \dots, n_{\ell+1})$ are different constants by the same reason as the above Case 1. Furthermore $a_{i,1}^* \neq 0$ $(i = 1, 2, \dots, \ell)$ because that the function y is the composite function of $Z = e^{H(z)}$ and $Y = Y(Z)$ defined by $P(Y) + Q(Y)Z = 0$, $Y = Y(Z)$ has the form: $Y = Y_0 + Y_1(Z - Z_0)^{1/n_0} + \cdots$ $(Y_1 \neq 0)$ at every branch point of Y (see Section 4) and $H'(z_0) \neq 0$ by our assumption. By the similar way of above, from (13) and (14), we have

$$
\begin{aligned}
f &= f_0 + f_1 y_i^* + f_2 y_i^{*2} + \cdots + f_{n-1} y_i^{*n-1} \\
&= \frac{\alpha_{0,-\tilde{p}}}{(z-z_0)^{\tilde{p}}} + \cdots + \left\{ \frac{\alpha_{1,-\tilde{p}}}{(z-z_0)^{\tilde{p}}} + \cdots \right\} \{ a_{i,0}^* + a_{i,1}^* (z-z_0)^{1/n_i} + \cdots \} \\
&\quad + \left\{ \frac{\alpha_{2,-\tilde{p}}}{(z-z_0)^{\tilde{p}}} + \cdots \right\} \{ a_{i,0}^* + a_{i,1}^* (z-z_0)^{1/n_i} + a_{i,2}^* (z-z_0)^{2/n_i} + \cdots \}^2 \\
&\qquad \cdots \\
&\quad + \left\{ \frac{\alpha_{n-1,-\tilde{p}}}{(z-z_0)^{\tilde{p}}} + \cdots \right\} \{ a_{i,0}^* + a_{i,1}^* (z-z_0)^{1/n_i} + a_{i,2}^* (z-z_0)^{2/n_i} + \cdots \}^{n-1} \\
&= C_0 \frac{1}{(z-z_0)^{\tilde{p}}} + a_{i,1}^* C_1 \frac{1}{(z-z_0)^{\tilde{p}-1/n_i}} + ({a_{i,1}^*}^2 C_2 + a_{i,2}^* C_1) \frac{1}{(z-z_0)^{\tilde{p}-2/n_i}} \\
&\quad + ({a_{i,1}^*}^3 C_3 + 2 a_{i,1}^* a_{i,2}^* C_2 + a_{i,3}^* C_1) \frac{1}{(z-z_0)^{\tilde{p}-3/n_i}} \\
&\qquad \cdots \\
&\quad + ({a_{i,1}^*}^{n_i-1} C_{n_i-1} + \cdots + a_{i,n_i-1}^* C_1) \frac{1}{(z-z_0)^{\tilde{p}-(n_i-1)/n_i}} + \cdots ,
\end{aligned}
$$

where

$$
\begin{cases}
C_0 = \alpha_{0,-\tilde{p}} + \cdots + \alpha_{k,-\tilde{p}} {a_{i,0}^*}^k + \cdots + \alpha_{n-1,-\tilde{p}} {a_{i,0}^*}^{n-1}, \\
C_1 = \alpha_{1,-\tilde{p}} + \cdots + \alpha_{k,-\tilde{p}} \binom{k}{1} {a_{i,0}^*}^{k-1} + \cdots + \alpha_{n-1,-\tilde{p}} \binom{n-1}{1} {a_{i,0}^*}^{n-2}, \\
C_2 = \alpha_{2,-\tilde{p}} + \cdots + \alpha_{k,-\tilde{p}} \binom{k}{2} {a_{i,0}^*}^{k-2} + \cdots + \alpha_{n-1,-\tilde{p}} \binom{n-1}{2} {a_{i,0}^*}^{n-3}, \\
\qquad \cdots \\
C_{n_i-1} = \alpha_{n_i-1,-\tilde{p}} + \cdots + \alpha_{k,-\tilde{p}} \binom{k}{n_i-1} {a_{i,0}^*}^{k-n_i+1} + \cdots + \alpha_{n-1,-\tilde{p}} \binom{n-1}{n_i-1} {a_{i,0}^*}^{n-n_i}.
\end{cases}
$$

The function f has no pole. Therefore, by $a_{i,1}^* \neq 0$, we see that all the C_k should vanish at once, that is,

$$
\begin{cases}
\alpha_{0,-\tilde{p}} + \cdots + \alpha_{k,-\tilde{p}} {a_{i,0}^*}^k + \cdots + \alpha_{n-1,-\tilde{p}} {a_{i,0}^*}^{n-1} = 0, \\
\alpha_{1,-\tilde{p}} + \cdots + \alpha_{k,-\tilde{p}} \binom{k}{1} {a_{i,0}^*}^{k-1} + \cdots + \alpha_{n-1,-\tilde{p}} \binom{n-1}{1} {a_{i,0}^*}^{n-2} = 0, \\
\alpha_{2,-\tilde{p}} + \cdots + \alpha_{k,-\tilde{p}} \binom{k}{2} {a_{i,0}^*}^{k-2} + \cdots + \alpha_{n-1,-\tilde{p}} \binom{n-1}{2} {a_{i,0}^*}^{n-3} = 0, \\
\qquad \cdots \\
\alpha_{n_i-1,-\tilde{p}} + \cdots + \alpha_{k,-\tilde{p}} \binom{k}{n_i-1} {a_{i,0}^*}^{k-n_i+1} + \cdots + \alpha_{n-1,-\tilde{p}} \binom{n-1}{n_i-1} {a_{i,0}^*}^{n-n_i} = 0,
\end{cases}
$$

and we have

$$
{}^t[\alpha_{0,-\tilde{p}} \quad \alpha_{1,-\tilde{p}} \quad \alpha_{2,-\tilde{p}} \quad \alpha_{3,-\tilde{p}} \quad \cdots \quad \alpha_{n-1,-\tilde{p}}] \cdot {}^t\boldsymbol{A} = {}^t[0 \quad 0 \quad 0 \quad 0 \quad \cdots \quad 0],
$$

where

$$A = \begin{bmatrix} 1 & a_{1,0}^* & {a_{1,0}^*}^2 & {a_{1,0}^*}^3 & \cdots & {a_{1,0}^*}^k & \cdots & {a_{1,0}^*}^{n-1} \\ 0 & 1 & \binom{2}{1}a_{1,0}^* & \binom{3}{1}{a_{1,0}^*}^2 & \cdots & \binom{k}{1}{a_{1,0}^*}^{k-1} & \cdots & \binom{n-1}{1}{a_{1,0}^*}^{n-2} \\ \vdots & \ddots & \ddots & & \cdots & & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & \cdots & \binom{k}{n_1-1}{a_{1,0}^*}^{k-n_1+1} & \cdots & \binom{n-1}{n_1-1}{a_{1,0}^*}^{n-n_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{\ell,0}^* & {a_{\ell,0}^*}^2 & {a_{\ell,0}^*}^3 & \cdots & {a_{\ell,0}^*}^k & \cdots & {a_{\ell,0}^*}^{n-1} \\ 0 & 1 & \binom{2}{1}a_{\ell,0}^* & \binom{3}{1}{a_{\ell,0}^*}^2 & \cdots & \binom{k}{1}{a_{\ell,0}^*}^{k-1} & \cdots & \binom{n-1}{1}{a_{\ell,0}^*}^{n-2} \\ \vdots & \ddots & \ddots & & \cdots & & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & \binom{k}{n_\ell-1}{a_{\ell,0}^*}^{k-n_\ell+1} & \cdots & \binom{n-1}{n_\ell-1}{a_{\ell,0}^*}^{n-n_\ell} \\ 1 & a_{1,0} & a_{1,0}^2 & a_{1,0}^3 & \cdots & a_{1,0}^k & \cdots & a_{1,0}^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n_{\ell+1},0} & a_{n_{\ell+1},0}^2 & a_{n_{\ell+1},0}^3 & \cdots & a_{n_{\ell+1},0}^k & \cdots & a_{n_{\ell+1},0}^{n-1} \end{bmatrix}.$$

It is easy to prove that $\det \boldsymbol{A} \neq 0$ because that $a_{i,0}^*$ $(i = 1, 2, \ldots, \ell)$ and $a_{j,0}$ $(j = 1, 2, \ldots, n_{\ell+1})$ are different constants. Therefore we have $(\alpha_{0,-\tilde{p}}, \ldots, \alpha_{n-1,-\tilde{p}}) = (0, \ldots, 0)$, which is absurd. Q.E.D.

Example 4. Let $\boldsymbol{R}$ be the n-sheeted algebroid surface, which is defined by the n-valued entire algebroid function y defined by

$$y^n = e^{z^n} - 1.$$

Then y has the following n branches:

$$y_j = \xi^j z(1 + c_1 z + c_2 z^2 + \cdots) \quad j = 0, 1, 2, \ldots, n-1,$$

at $z = 0$, where $\xi = e^{2\pi i/n}$ and c_k $(k = 1, 2, \ldots)$ are constants. Furthermore let us put

$$f := f_0 + f_1 y + \cdots + f_{n-1} y^{n-1}$$

and

$$f_k := \frac{F_k(z)}{z^k} \quad k = 0, 1, 2, \ldots, n-1,$$

where $F_k(z)$ $(k = 0, 1, 2, \ldots, n-1)$ are single-valued entire functions. Then we have

$$f_k y_j^k = \frac{F_k(z)}{z^k} \xi^{jk} z^k (1 + c_1 z + c_2 z^2 + \cdots)^k$$
$$= \xi^{jk} F_k(z)(1 + c_1 z + c_2 z^2 + \cdots)^k \quad j = 0, 1, \ldots, n-1.$$

Therefore f is an entire function on $\boldsymbol{R}$. Here we should notice that $z = 0$ is a zero of $(z^n)'$ and also a pole of f_j $(j = 1, 2, \ldots, n-1)$.

6. Transformation formula of discriminants

Let y be the algebroid function defined by

$$\text{(E)} \qquad F(z, y) = P(y) + Q(y)e^{H(z)} = 0,$$

where $H(z)$ is a non-constant entire function with $H(0) = 0$, $P(y)$ is a monic polynomial of y of degree n and $Q(y)$ is a polynomial of y of degree at most $n-1$. Let us assume that $p(y) \geq n+1$, where $p(y)$ is the number of exceptional values of y. In this case (E) is irreducible by Lemma 4. Let $\boldsymbol{R}$ be the algebroid surface of y. Furthermore let us assume that there exists an entire function f on $\boldsymbol{R}$ such that $p(f) \geq n+1$. Then f is representable as

$$\text{(15)} \qquad f = F_{0,1} + F_{1,1}y + F_{2,1}y^2 + \cdots + F_{n-1,1}y^{n-1},$$

where $F_{j,1}$ $(j = 0, 1, \ldots, n-1)$ are meromorphic functions on $\boldsymbol{C}$, all of which are regular at any points z satisfying $H'(z) \neq 0$ by Theorem 3. Eliminating y from (E) and (15), we have a suitable polynomial with respect to f of degree n. Hence f is at most n-valued. Furthermore the defining equation of f is irreducible by $p(f) \geq n+1$ and Lemma 4. Therefore f is just an n-valued algebroid function. So let $\boldsymbol{X}$ be the n-sheeted algebroid surface of f. Now let y_k $(k = 1, 2, \ldots, n)$ be the n determinations of y. And let us put

$$f_k = F_{0,1} + F_{1,1}y_k + F_{2,1}y_k^2 + \cdots + F_{n-1,1}y_k^{n-1} \quad (k = 1, 2, \ldots, n),$$

then f_k $(k = 1, 2, \ldots, n)$ are n determinations of f. In fact for any determination $\tilde{f}$ of f, there exists a curve C_0 such that $\tilde{f}$ is the analytic continuation of f_1 along C_0. If y_i is the analytic continuation of y_1 along C_0, then we have $\tilde{f} = F_{0,1} + F_{1,1}y_i + F_{2,1}y_i^2 + \cdots + F_{n-1,1}y_i^{n-1}$ from $f_1 = F_{0,1} + F_{1,1}y_1 + F_{2,1}y_1^2 + \cdots + F_{n-1,1}y_1^{n-1}$. This shows $\tilde{f} = f_i$.

From (E) and (15), we have

$$f^j = F_{0,j} + F_{1,j}y + F_{2,j}y^2 + \cdots + F_{n-1,j}y^{n-1} \quad (j = 1, 2, \ldots, n-1),$$

and

$$f_k^j = F_{0,j} + F_{1,j}y_k + F_{2,j}y_k^2 + \cdots + F_{n-1,j}y_k^{n-1}$$
$$(j = 1, 2, \ldots, n-1; k = 1, 2, \ldots, n),$$

where $F_{\ell,j}$ $(\ell = 0, 1, \dots, n-1; j > 1)$ are suitable polynomials with $F_{i,1}$ $(i = 0, 1, \dots, n-1)$ and e^H. Hence we have

$$(16)\qquad \begin{bmatrix} 1 & f_1 & f_1^2 & \cdots & f_1^{n-1} \\ 1 & f_2 & f_2^2 & \cdots & f_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f_n & f_n^2 & \cdots & f_n^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & y_1 & y_1^2 & \cdots & y_1^{n-1} \\ 1 & y_2 & y_2^2 & \cdots & y_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & y_n & y_n^2 & \cdots & y_n^{n-1} \end{bmatrix} \begin{bmatrix} 1 & F_{0,1} & F_{0,2} & \cdots & F_{0,n-1} \\ 0 & F_{1,1} & F_{1,2} & \cdots & F_{1,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & F_{n-1,1} & F_{n-1,2} & \cdots & F_{n-1,n-1} \end{bmatrix}.$$

The discriminants $D_{\boldsymbol{R}}$ and $D_{\boldsymbol{X}}$ of $\boldsymbol{R}$ and $\boldsymbol{X}$ are defined by

$$D_{\boldsymbol{R}} = \begin{vmatrix} 1 & y_1 & y_1^2 & \cdots & y_1^{n-1} \\ 1 & y_2 & y_2^2 & \cdots & y_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & y_n & y_n^2 & \cdots & y_n^{n-1} \end{vmatrix}^2, \quad D_{\boldsymbol{X}} = \begin{vmatrix} 1 & f_1 & f_1^2 & \cdots & f_1^{n-1} \\ 1 & f_2 & f_2^2 & \cdots & f_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f_n & f_n^2 & \cdots & f_n^{n-1} \end{vmatrix}^2,$$

respectively. Therefore from (16) we have

$$(17)\qquad D_{\boldsymbol{X}} = D_{\boldsymbol{R}} \cdot G^2,$$

where

$$G = \det \begin{bmatrix} 1 & F_{0,1} & F_{0,2} & \cdots & F_{0,n-1} \\ 0 & F_{1,1} & F_{1,2} & \cdots & F_{1,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & F_{n-1,1} & F_{n-1,2} & \cdots & F_{n-1,n-1} \end{bmatrix}.$$

This expression shows that G is meromorphic on $\boldsymbol{C}$ which is regular at any points z satisfying $H'(z) \neq 0$. Therefore we have

$$\overline{N}(r, \infty, G) \leq \overline{N}(r, 0, H').$$

Let z_0 be a pole of G of order p_0. Then, from (17), z_0 is a zero of $D_{\boldsymbol{R}}$ because that $D_{\boldsymbol{X}}$ is entire. Let m_0 be the order of zero z_0 of H'. Then from (17) we have

$$2p_0 \leq (p(y) - 2)(m_0 + 1) \leq 2(p(y) - 2)m_0 \leq 2(2n - 2)m_0,$$

by Theorem 2. Hence we have

$$(18)\qquad N(r, \infty, G) \leq (2n - 2)N(r, 0, H') = S(r, e^H).$$

Here let us assume that $\boldsymbol{X}$ is defined by

$$X : \tilde{F}(z,f) = \tilde{P}(f) + \tilde{Q}(f)e^{L(z)} = 0,$$

where $L(z)$ is a non-constant entire function with $L(0) = 0$, $\tilde{P}(f)$ is a monic polynomial of f of degree n and $\tilde{Q}(f)$ is a polynomial of f of degree $\le n-1$. By $D_{\boldsymbol{R}} \not\equiv 0$ and $D_{\boldsymbol{X}} \not\equiv 0$, (16) shows us that the function y is a function on $\boldsymbol{X}$. And by the similar way of constructing (16), we have

$$\begin{bmatrix} 1 & y_1 & y_1^2 & \cdots & y_1^{n-1} \\ 1 & y_2 & y_2^2 & \cdots & y_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & y_n & y_n^2 & \cdots & y_n^{n-1} \end{bmatrix} \tag{19}$$

$$= \begin{bmatrix} 1 & f_1 & f_1^2 & \cdots & f_1^{n-1} \\ 1 & f_2 & f_2^2 & \cdots & f_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f_n & f_n^2 & \cdots & f_n^{n-1} \end{bmatrix} \begin{bmatrix} 1 & G_{0,1} & G_{0,2} & \cdots & G_{0,n-1} \\ 0 & G_{1,1} & G_{1,2} & \cdots & G_{1,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & G_{n-1,1} & G_{n-1,2} & \cdots & G_{n-1,n-1} \end{bmatrix},$$

where $G_{i,1}$ $(i = 0, 1, \ldots, n-1)$ are meromorphic functions all of which are regular at any points z satisfying $L'(z) \neq 0$ and $G_{\ell,j}$ $(\ell = 0, 1, \ldots, n-1; j > 1)$ are suitable polynomials of $G_{i,1}$ $(i = 0, 1, \ldots, n-1)$ and e^L. From (16) and (19) we have

$$\begin{bmatrix} 1 & F_{0,1} & F_{0,2} & \cdots & F_{0,n-1} \\ 0 & F_{1,1} & F_{1,2} & \cdots & F_{1,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & F_{n-1,1} & F_{n-1,2} & \cdots & F_{n-1,n-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & G_{0,1} & G_{0,2} & \cdots & G_{0,n-1} \\ 0 & G_{1,1} & G_{1,2} & \cdots & G_{1,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & G_{n-1,1} & G_{n-1,2} & \cdots & G_{n-1,n-1} \end{bmatrix} = I_n,$$

where I_n is the unit matrix of degree n. Putting

$$\tilde{G} = \det \begin{bmatrix} 1 & G_{0,1} & G_{0,2} & \cdots & G_{0,n-1} \\ 0 & G_{1,1} & G_{1,2} & \cdots & G_{1,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & G_{n-1,1} & G_{n-1,2} & \cdots & G_{n-1,n-1} \end{bmatrix},$$

we have $G \cdot \tilde{G} = 1$. Therefore every zero of G is a pole of $\tilde{G}$. By the similar way of proving (18), we have

$$N(r,0,G) = N(r,\infty,\tilde{G}) \le (2n-2)N(r,0,L') = S(r,e^L). \tag{20}$$

7. Picard constants of $\boldsymbol{R}$

By the results of Section 4, 5 and 6 we can prove the following

THEOREM 4. *Let* $\boldsymbol{R}$ *be the n-sheeted algebroid surface defined by the following irreducible equation*:

(E) $$F(z,y) = P(y) + Q(y)e^{H(z)} = 0,$$

where $H(z)$ is a non-constant entire function with $H(0) = 0$, $P(y)$ is a monic polynomial of y of degree n and $Q(y)$ is a polynomial of y of degree at most $n-1$, Δ be the set of projections of all branch points of $\boldsymbol{R}$ and k ($<n$) be a positive integer. Assume that there exist just k different points on $\boldsymbol{R}$ over every $z \in \Delta$.

Then we have

(21) $$p(f) = m_f(n-k) + 2,$$

for an arbitrary entire function f on $\boldsymbol{R}$ with $p(f) > 3n/2$, where m_f is a suitable positive integer.

Proof. Let $\boldsymbol{X}$ be the n-sheeted algebroid surface defined by f. Then, by Lemma 4 and the assumption $p(f) > 3n/2 \geq n+1$, the defining equation of $\boldsymbol{X}$ is irreducible and contains only one exponential function, say $e^{L(z)}$, by Corollary 3 and the assumption $p(f) > 3n/2$. And, by (17), (18) and (20) in Section 6, we have

$$D_{\boldsymbol{X}} = D_{\boldsymbol{R}} \cdot G^2,$$

where G is a meromorphic function satisfying

$$N(r,\infty,G) = S(r,e^H), \quad N(r,0,G) = S(r,e^L)$$

and $D_{\boldsymbol{R}}$ and $D_{\boldsymbol{X}}$ are the discriminants of $\boldsymbol{R}$ and $\boldsymbol{X}$, respectively. Then G has no zero and no pole by Lemma 3. Therefore the factor of zeros of $D_{\boldsymbol{X}}$ coincides with that of $D_{\boldsymbol{R}}$.

Now, by Theorem 2, $D_{\boldsymbol{X}}$ is representable as

$$D_{\boldsymbol{X}} = A_{p(f)-2}e^{(n-m)L(z)}\prod_{j=1}^{m_f}(e^{L(z)} - \xi_j)^{n_j},$$

where m and m_f (≥ 1) are non-negative integers, ξ_j ($j = 0,1,\dots,m_f$) are non-zero constants and n_j ($j = 1,2,\dots,m_f$) are positive integers with $\sum_{j=1}^{m_f} n_j = p(f) - 2$. By the computations in Section 4, y is representable as

$$y(z) = w_0 + \alpha_1(z-z_0)^{1/n_0} + \cdots \quad (\alpha_1 \neq 0),$$

at every branch point z_0 satisfying $H'(z_0) \neq 0$ and y takes different two values at different two points on $\boldsymbol{R}$, lying over a point z satisfying $H'(z) \neq 0$. By the assumption that there are just k different points on $\boldsymbol{R}$ over every $z \in \Delta$, $D_{\boldsymbol{R}}$ has no zero other than an infinite number of zeros of order $n-k$. On the other hand $D_{\boldsymbol{X}}$ has an infinite number of zeros of order n_j ($j = 1,2,\dots,m_f$). Hence we have $n_j = n-k$ ($j = 1,2,\dots,m_f$). And therefore we have

$$p(f) = m_f(n-k) + 2,$$

which is the desired result. Q.E.D.

An n-sheeted algebroid surface is called regularly branched if all its branch points are of order $n-1$. As a corollary of Theorem 4 we have the following

Corollary 4. *Let* $\boldsymbol{R}$ *be the n-sheeted algebroid surface defined by the following irreducible equation*:

$$\text{(E)} \qquad F(z, y) = P(y) + Q(y)e^{H(z)} = 0,$$

where $H(z)$ *is a non-constant entire function with* $H(0) = 0$, $P(y)$ *is a monic polynomial of* y *of degree* n *and* $Q(y)$ *is a polynomial of* y *of degree at most* $n-1$. *Assume that* $\boldsymbol{R}$ *is regularly branched. Then we have* $p(f) = 2n$ *for every entire function* f *with* $p(f) > 3n/2$.

Proof. By (21) and the assumption that $\boldsymbol{R}$ is regularly branched, for every entire function f with $p(f) > 3n/2$, there is an integer m_f such that $p(f) = m_f(n-1) + 2$. Since $3n/2 < p(f) \leq 2n$ and $n \geq 2$, $m_f = 2$ must hold. Hence we have $p(f) = 2n$. Q.E.D.

In 1973 Aogai [1] proved that $\mathscr{P}(\boldsymbol{R}) = 2n$ for every n-sheeted regularly branched algebroid surface $\boldsymbol{R}$ with $\mathscr{P}(\boldsymbol{R}) > 3n/2$. Corollary 4 shows us the existence of no entire function f on $\boldsymbol{R}$ with $3n/2 < p(f) < 2n$.

At last we prove the following

Theorem 5. *Let* $\boldsymbol{R}$ *be the n-sheeted algebroid surface defined by*

$$\text{(E)} \qquad F(z, y) = P(y) + Q(y)e^{H(z)} = 0,$$

where $H(z)$ *is a non-constant entire function with* $H(0) = 0$, $P(y)$ *is a monic polynomial of* y *of degree* n *and* $Q(y)$ *is a polynomial of* y *of degree at most* $n-1$.

We assume that $p(y) > 3n/2 - 1$, *where* $p(y)$ *is the number of exceptional values of the n-valued entire algebroid function defined by* (E). *In this case, by Theorem* 2, *the discriminant of* $\boldsymbol{R}$ *is*

$$D_{\boldsymbol{R}} = e^{(n-m)H(z)}\{A_{p(y)-2}\exp((p(y)-2)H(z)) + \cdots + A_0\},$$

where m *is the number of exceptional values of the second kind of* y *and* $A_0, \ldots, A_{p(y)-2}$ *are constants with* $A_0 A_{p(y)-2} \neq 0$.

Let us put

$$J = \{d : integer \,|\, (p(y)-2, d) = d \ \ and \ \ d \leq 2n - p(y)\},$$

$$J^* = \left\{\frac{p(y)-2}{d} \,\middle|\, d \in J\right\}$$

and

$$\boldsymbol{N}J^* = \{kq \,|\, k : non\text{-}negative \ \ integer, \ \ q \in J^* \ \ and \ \ kq \leq p(y)-2\}.$$

If there exists at least one coefficient A_i *of* $D_{\boldsymbol{R}}$ *such that* $A_i \neq 0$ *and* $i \notin \boldsymbol{N}J^*$, *then we have* $\mathscr{P}(\boldsymbol{R}) = p(y)$.

Proof. First of all we have attention to the result that (E) is irreducible. In fact we have $p(y) \geq n+1$ if $p(y) > 3n/2 - 1$. Therefore (E) is irreducible by Lemma 4.

Let us assume that $\mathscr{P}(\boldsymbol{R}) > p(y)$. Then there exists a meromorphic function f on $\boldsymbol{R}$ such that $\mathscr{P}(\boldsymbol{R}) \geq p(f) > p(y)$. Without loss of generality we may assume that f is entire on $\boldsymbol{R}$. Let $\boldsymbol{X}$ be the surface defined by f. Then, by $p(f) > p(y) \geq n+1$ and Lemma 4, the defining equation of $\boldsymbol{X}$ is irreducible and has only one kind of exponential function, say $e^{L(z)}$, by Corollary 3 and the assumption: $p(f) \geq p(y) + 1 > 3n/2$. In this case we have

$$(22) \qquad e^{(n-\tilde{m})L(z)} \sum_{j=0}^{p(f)-2} B_j e^{jL(z)} = D_{\boldsymbol{X}} = G^2 D_{\boldsymbol{R}} = G^2 e^{(n-m)H(z)} \sum_{i=0}^{p(y)-2} A_i e^{iH(z)},$$

where $\tilde{m}$ is the number of exceptional values of the second kind of f, B_j $(j = 0, \ldots, p(f) - 2)$ are constants with $B_0 B_{p(f)-2} \neq 0$ and G is a meromorphic function on $\boldsymbol{C}$ satisfying

$$N(r, \infty, G) = S(r, e^{H(z)}), \quad N(r, 0, G) = S(r, e^{L(z)}),$$

by Theorem 2, (17), (18) and (20). Let us put $d := (p(y) - 2, p(f) - 2)$, then we have $d \in J$. Furthermore let q be the positive integer such that $dq = p(y) - 2$. In this case we have

$$A_i = 0 \quad (i \neq 0, q, 2q, \ldots, dq),$$

by Lemma 3 and (22). This contradicts the assumption that there exists at least one A_i such that $A_i \neq 0$ $(i \notin \boldsymbol{N}J^*)$. Q.E.D.

By Theorem 5 it is easy to verify the following result:

Let $\boldsymbol{R}$ be the n-sheeted algebroid surface defined by (E). *If* $p(y) = 2n - 1$, *then we have* $\mathscr{P}(\boldsymbol{R}) = 2n - 1$ *without* $(A_1, \ldots, A_{2n-4}) = (0, \ldots, 0)$.

This result coincides with Theorem B and D in the case $n = 3$ and $n = 4$ respectively.

Some problems

Finally we list some problems:

1. Does Theorem 4 remain valid without the discriminant condition?
In the case of 3-sheeted surfaces the author [12] proved that $\mathscr{P}(\boldsymbol{R}) = 5$ for every surface of $p(y) = 5$.

2. Let $\boldsymbol{R}$ be the surface defined by the following irreducible equation:

$$F(z, y) = P(y) + Q(y)e^{H(z)} = 0,$$

where $H(z)$ is a non-constant entire function with $H(0) = 0$, $P(y)$ is a monic polynomial of y of degree n and $Q(y)$ is a polynomial of y of degree at most $n - 1$. Is $\mathscr{P}(\boldsymbol{R})$ decidable in the case $p(y) \leq 3n/2 - 1$?

3. Let $\boldsymbol{R}$ be the surface defined by the following equation:

$$F(z,y) = P(y) + Q_1(y)e^{H_1^*(z)} + \cdots + Q_\ell(y)e^{H_\ell^*(z)} = 0,$$

with $\ell > 1$ and $p(y) \geq n+1$, where $H_j^*(z)$ $(j = 1, \ldots, \ell)$ are non-constant entire of $H_j^*(0) = 0$, $P(y)$ is a monic polynomial of y of degree n and $Q_j(y)$ $(j = 1, \ldots, \ell)$ are polynomial of y of degree at most $n-1$. In this case, is $\mathscr{P}(\boldsymbol{R})$ decidable?

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References

[1] H. Aogai, Picard constant of a finitely sheeted covering surface, Kōdai Math. Sem. Rep., **25** (1973), 219–224.

[2] J. Dufresnoy, Théorie nouvelle des familles complexes normales. Applications à l'étude des fonctions algébroïdes, Ann. Sci. École Norm. Sup. (3), **61** (1944), 1–44.

[3] W. K. Hayman, Meromorphic Functions, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[4] R. Nevanlinna, Analytic Functions, Springer-Verlag, New York-Berlin, 1970.

[5] K. Niino and M. Ozawa, Deficiencies of an entire algebroid function, I, Kōdai Math. Sem. Rep., **22** (1970), 98–113.

[6] K. Niino and K. Tohge, Some functional equations and Picard constants of algebroid surfaces, J. Math. Soc. Japan, **48** (1996), 649–665.

[7] M. Ozawa, On complex analytic mappings, Kōdai Math. Sem. Rep., **17** (1965), 93–102.

[8] M. Ozawa and K. Sawada, Three-sheeted algebroid surfaces whose Picard constants are five, Kodai Math. J., **17** (1994), 101–124.

[9] M. Ozawa and K. Sawada, Picard constants of four-sheeted algebroid surfaces, I, Kodai Math. J., **18** (1995), 99–141.

[10] M. Ozawa and K. Sawada, Picard constants of four-sheeted algebroid surfaces, II, Kodai Math. J., **18** (1995), 199–233.

[11] G. Rémoundos, Extension aux fonctions algébroïdes multiformes du théorème de M. Picard et de ses généralisations, Mém. Sci. Math. 23, Gauthier-Villars, Paris, 1927.

[12] K. Sawada, Picard constants of three-sheeted algebroid surfaces with $p(y) = 5$, Kodai Math. J., **23** (2000), 242–254.

[13] K. Sawada and K. Tohge, A remark on three-sheeted algebroid surfaces whose Picard constants are five, Kodai Math. J., **18** (1995), 142–155.

[14] H. L. Selberg, Algebroide Funktionen und Umkehrfunktionen Abelscher Integrale, Avh. Norske Vid. Akad. Oslo, **8** (1934), 1–72.

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