

HOLOMORPHIC MOTIONS IN THE PARAMETER SPACE FOR THE RELAXED NEWTON'S METHOD

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Abstract

It is a well known fact, that for certain polynomials f the relaxed Newton's method $N_{f,h}(z) = z - h(f(z)/f'(z))$ associated with f has some extraneous attracting cycles. In the case of cubic polynomials the set of these bad conditioned polynomials has been intensively studied and described by means of quasi-holomorphic surgery and holomorphic motions, cf. [12]. In the present paper we will generalize this description to polynomials of higher degree.

1. Introduction

It is a well known fact, that for certain polynomials f the relaxed Newton's method $N_{f,h}(z) = z - h(f(z)/f'(z))$ associated with f has some extraneous attracting cycles. In order to illustrate the seriousness of the problem we look at the family of cubic polynomials

$$(1) \quad f_\lambda(z) = z^3 + (\lambda - 1)z - \lambda,$$

where $\lambda \in \mathbb{C}$. Figure 1 shows the set of parameters λ such that the Newton's method associated with f_λ fails with positive probability.

Barna seems to be the first who established the existence of the extraneous attractors, cf. [1]. Since then, the dynamics of the Newton's method has received much attention [6, 8, 12, 14, 22, 27]. Patterns of non-convergent the Newton's method have been shown in [5, 23].

The occurrence of extraneous attractors gives rise to the following questions

- *What is the probability that the Newton's method will converge for a randomly chosen initial value?*
- *What can be said about the set of polynomials such that the Newton's method has extraneous attractors?*
- *How to improve the convergence of the Newton's method?*

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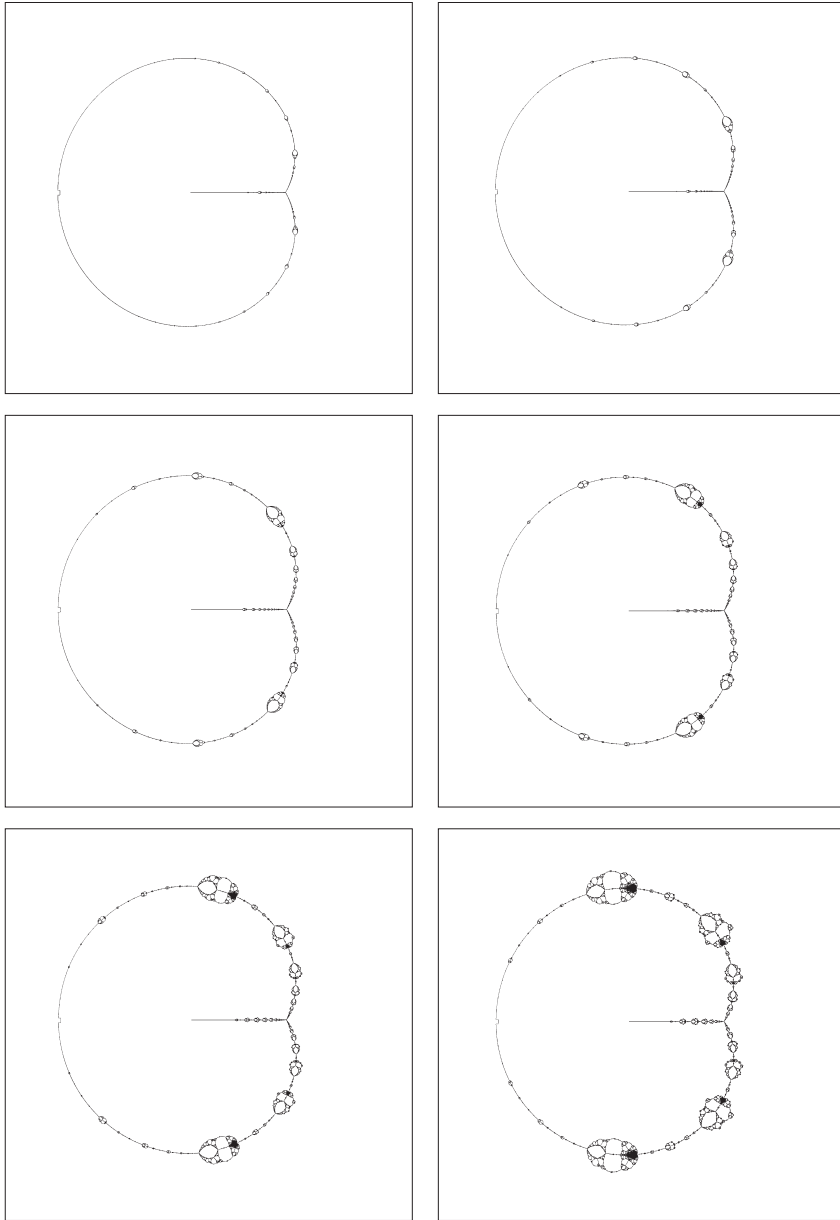


FIGURE 1. The pictures show the parameters λ satisfying $|\operatorname{Re}(\lambda)|, |\operatorname{Im}(z)| < 2.5$ such that the relaxed Newton's method $N_{f_\lambda, h}$ associated with $f_\lambda(z) = z^3 + (\lambda - 1)z - \lambda$ fails to converge with positive probability, that is, have some extraneous attracting cycle, where $h = 0.2$ (upper left), $h = 0.4$ (upper right), $h = 0.6$ (middle left), $h = 0.8$ (middle right), $h = 1.0$ (lower left), $h = 1.2$ (lower right).

This paper is devoted to the study of the set of bad conditioned polynomials, that is the set of polynomials such that the relaxed Newton's method has at least one extraneous attracting cycle.

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2. The relaxed Newton's method

Instead of the standard (unrelaxed) Newton's method, the relaxed Newton's method is frequently applied for finding approximations of the roots of a given polynomial f .

DEFINITION 1. Let $f : \mathbf{C} \rightarrow \mathbf{C}$ be a polynomial and $h \in \mathbf{C}$. The *relaxed Newton's method* for finding the roots of f consists of iterating the function

$$(2) \quad N_{f,h} : \bar{\mathbf{C}} \rightarrow \bar{\mathbf{C}}; \quad z \mapsto z - h \frac{f(z)}{f'(z)}.$$

Closely related to the relaxed Newton's method is the so-called Newton flow

$$(3) \quad w'(t) = - \frac{f(w(t))}{f'(w(t))}.$$

The defining equations (2) and (3) elucidate the relation between them: The relaxed Newton's method is nothing else but the Euler method (with step size $h \in]0, 1[$) of the Newton flow. We refer to [3, 15] for a detailed discussion of the Newton flow.

Note that the roots of f are sinks of the Newton flow and attracting fixed points of the rational function $N_{f,h}$. Let $\mathcal{G}(f,h)$ denote the set of initial values z_0 such that the sequence of iterates with respect to $N_{f,h}$ converges to a root of f . In other words, $\mathcal{G}(f,h)$ is the set of 'good' initial values for the Newton's method $N_{f,h}$. The complement $\mathcal{B}(f,h) := \bar{\mathbf{C}} \setminus \mathcal{G}(f,h)$ is the set of 'bad' initial values causing the Newton's method to fail. As commonly is known from the general theory of differential equations, these sets $\mathcal{G}(f,h)$ converge to the union of the basins of the roots of f for the Newton flow. The complement of the latter consists of all points whose trajectories with respect to the Newton flow land in some singularity of the flow, that is to say, roots of f' . In particular, the set of points whose trajectories with respect to the Newton's flow do *not* land in a root of f equals a finite union of analytic Jordan arcs. This is the underlying idea in the proof of the following theorem which has independently been established in [19] and [9], see also [20] and [11].

THEOREM 2. For every polynomial f the (spherical) Lebesgue measure of $\bar{\mathbf{C}} \setminus \mathcal{G}(f,h)$ tends to zero as $h \searrow 0$.

In other words, the probability, that the relaxed Newton's method converges, tends to 1 as $h \searrow 0$. Does this mean that the extraneous attractors disappear as h tends to zero? It is known that, for generic polynomial f , there exists some $h^* \in]0, 1]$ such that for $0 < h < h^*$ the set of bad initial values $\mathcal{B}(f, h) := \overline{\mathcal{C}} \setminus \mathcal{G}(f, h)$ has Lebesgue measure zero, cf. [7, 11, 20]. Note that the number h^* heavily depends of f . On the other hand, using qc-surgery for fixed $h \in D_1(1)$, one can establish a correspondence between the set of all f such that $N_{f,1}$ has some extraneous attractors and the set of all polynomials g such that $N_{g,h}$ has some extraneous attracting cycle. In particular, for each $h \in D_1(1)$ there is an open set of polynomials f , such that $N_{f,h}$ has some extraneous attracting cycle. The purpose of the present paper is to provide a detailed discussion of this phenomenon.

In order to illustrate the problem we again look at the family of cubic polynomials (1). Figure 1 shows the set of parameters λ such that the relaxed Newton's method associated with f_λ fails with positive probability.

As a consequence, we have to modify the question we will address in this paper: *What can be said about the set of polynomials such that the relaxed Newton's method has some extraneous attractors?*

Theorem 4, the main result of this paper, is a description of this set in terms of hyperbolic motions.

3. Description of the parameter space

In [10] quasiconformal surgery has been used to establish the following result.

THEOREM 3 (von Haeseler-Kriete, 1993). *Let f be an arbitrary polynomial of degree d having $m \geq 2$ roots. For each pair of parameters $h_1, h_2 \in D_1(1)$ there exists a polynomial g , again of degree d and having m roots, such that the Julia sets $\mathcal{J}(N_{f,h_1})$ and $\mathcal{J}(N_{g,h_2})$ are homeomorphically equivalent. Let $\mathcal{B}(f, h_1)$ and $\mathcal{B}(g, h_2)$ denote the set of initial values where N_{f,h_1} respectively N_{g,h_2} do not converge to a root of f . In addition, $N_{f,h_1}|_{\mathcal{B}(f,h_1)}$ and $N_{g,h_2}|_{\mathcal{B}(g,h_2)}$ are quasiconformally conjugated.*

Remark. Actually, it has been proved that the conjugacy is a biholomorphic conjugacy between $N_{f,h_1}|_{\text{Int}(\mathcal{B}(f,h_1))}$ and $N_{g,h_2}|_{\text{Int}(\mathcal{B}(g,h_2))}$. In particular, if N_{f,h_1} has an extraneous cycle then N_{g,h_2} also has one (with the same period and the same multiplier). Furthermore, the conjugacy can quasiconformally be extended to some open neighbourhoods of $\mathcal{B}(f, h_1)$ respectively $\mathcal{B}(g, h_2)$.

Since the space of polynomials of degree d admits a holomorphic parameterization one might expect the conjugacy to depend holomorphically on the polynomial f . Then it should be possible to describe the parameter space in terms of holomorphic motions. Unfortunately, this is not possible, cf. Lemma 9 and Lemma 13.

We fix an integer $d \geq 3$. Throughout this paper we shall deal with centered and normalized polynomials, that are polynomials of the form

$$f(z) = z^d + \lambda_{d-2}z^{d-2} + \cdots + \lambda_2z^2 + (\lambda_1 - 1)z - \sum_{\nu=1}^{d-2} \lambda_\nu$$

with some $\lambda \in \mathbf{C}^{d-2}$. Each root ζ of such a polynomial f is an attracting fixed point of the Newton's method $N_{f,h}$, where $h \in D_1(1)$, and its basin of attraction is defined as

$$\mathcal{A}_{f,h}(\zeta) := \left\{ z_0 \in \overline{\mathbf{C}} \mid \lim_{n \rightarrow \infty} N_{f,h}^n(z_0) = \zeta \right\}.$$

Note that the immediate basin of attraction $\mathcal{A}_{f,h}^*(\zeta)$ of ζ (with respect to $N_{f,h}$), that is the component of $\mathcal{A}_{f,h}(\zeta)$ containing ζ , contains at least one critical point of $N_{f,h}$.

Let $M_h^d \subset \mathbf{C}^{d-2}$ be the set of those polynomials f of degree d such that every root of f is simple and that the immediate basin of attraction $\mathcal{A}_{f,h}^*(\zeta)$ of each root ζ of f contains exactly one simple critical point of $N_{f,h}$ but no further (multiple) critical points. This can be regarded as the worst case, because this case covers the possibility that the maximal possible number of free critical points do not converge to a root of the polynomial under iteration of the relaxed Newton's method. The main result is:

THEOREM 4 (Main Theorem). *There exists a holomorphic motion $L : M_1^d \times D_1(1) \rightarrow \mathbf{C}^{d-2}$ satisfying:*

1. $L(\cdot, 1) = \text{id}$,
2. $L(f, \cdot)$ is holomorphic on $D_1(1)$ for every $f \in M_1^d$,
3. $L(\cdot, h) : M_1^d \rightarrow M_h^d$ is a homeomorphism,
4. $N_{f,1}|_{\mathcal{B}(f,1)}$ and $N_{L(f,h),h}|_{\mathcal{B}(L(f,h),h)}$ are homeomorphic.

Remark. The case $d = 3$ has been settled in the paper [10]. The construction is the base of the proof of the preceding theorem in the case of arbitrary degree $d \geq 3$. However, extra arguments have had to be inserted because the λ -lemma cannot be used in this context. In addition, further arguments have been added because of the need of clarification of certain parts of the original proof.

4. Holomorphic motions and quasiconformal surgery

In this paper we shall make intensive use of the quasiconformal surgery. The reader interested in further details of this technique is referred to [16]. One of the main ingredients is the theory of quasiconformal mappings; for an introduction to this theory we refer to [2] and [17]. The quasiconformal surgery for polynomials and, more generally, rational functions was developed by Douady-Hubbard [6] and Shishikura [24]. The main tool is

LEMMA 5 (qc-lemma, Shishikura). *Let $F : \bar{\mathcal{C}} \rightarrow \bar{\mathcal{C}}$ be a proper and orientation preserving mapping such that the complex dilatation of F is bounded almost everywhere by a constant $K < 1$. Assume that for $\mu \in I = \{1, \dots, m\}$ there are pairwise disjoint domains G_μ and quasiconformal mappings $\Phi_\mu : G_\mu \rightarrow G'_\mu$ such that:*

1. G_1, \dots, G_m are invariant with respect to F ,
2. $\Phi_\mu \circ F \circ \Phi_\mu^{-1} : G'_\mu \rightarrow G'_\mu$ is holomorphic on G'_μ for every $\mu \in I$,
3. $(\partial/\partial\bar{z})F \equiv 0$ almost everywhere (with respect to the spherical Lebesgue measure) on $\bar{\mathcal{C}} \setminus F^{-1}(G_1 \cup \dots \cup G_m)$.

Then there exists a quasiconformal mapping $\phi : \bar{\mathcal{C}} \rightarrow \bar{\mathcal{C}}$ such that $R := \phi \circ F \circ \phi^{-1}$ is holomorphic on $\bar{\mathcal{C}}$, $(\partial/\partial\bar{z})\phi \equiv 0$ almost everywhere (with respect to the spherical Lebesgue measure) on $\bar{\mathcal{C}} \setminus (G_1 \cup \dots \cup G_m)$ and $\phi \circ \Phi_\mu^{-1} : G'_\mu \rightarrow \phi(G_j)$ is biholomorphic. ϕ can be normalized to have the fixed points $0, 1$ and ∞ .

We refer to Shishikura [24] for a proof. In the case of cubic polynomials, the so-called λ -lemma, which may be found in [18, 25], was used as the second main ingredient.

LEMMA 6 (λ -lemma, Sullivan-Thurston). *Let $E \subset \bar{\mathcal{C}}$ be an arbitrary set and $\Phi : E \times D_1(1) \rightarrow \bar{\mathcal{C}}$ a mapping satisfying:*

1. $\Phi(\cdot, h_0) = \text{id}$ for some $h_0 \in D_1(1)$,
2. $\Phi(\cdot, h)$ is injective on E ,
3. $\Phi(\lambda, \cdot)$ is holomorphic on $D_1(1)$.

Then Φ extends to a continuous mapping $\Phi : \text{cl}(E) \times D_1(1) \rightarrow \bar{\mathcal{C}}$ such that $\Phi(\cdot, h)$ is a quasiconformal mapping for every $h \in D_1(1)$.

In this lemma the set E should be regarded as a subset of the parameter space. In our setting, the parameter space is not \mathcal{C} but \mathcal{C}^m (with some $m \geq 1$). Unfortunately, in this more general setting the λ -lemma does not hold! A counterexample will be given at the end of this section. Another one can be found in [25, p. 224].

The proof of Theorem 4 bases on the proof in the case of cubic polynomials as given in [10]. But the following changes have to be made. Firstly, we have to redefine the notion of *holomorphic motion*, taking into account the fact that we have to work with a multidimensional parameter space. Secondly, whenever the λ -lemma is used, an additional argument has to be inserted. At this place we recall the multidimensional version of the definition of holomorphic motions.

DEFINITION 7 (Holomorphic motion). *Let $E \subset \mathcal{C}^m$ for some $m \in \mathbb{N} \setminus \{0\}$ and $G \subset \mathcal{C}$. A mapping $\Phi : E \times G \rightarrow \mathcal{C}^m$ is called *holomorphic motion* if the following conditions are satisfied:*

1. Φ is continuous,
2. $\Phi(\cdot, h_0) = \text{id}$ for some $h_0 \in G$,
3. $\Phi(\lambda, \cdot)$ is holomorphic on G for every $\lambda \in E$, and
4. $\Phi(\cdot, h)$ is injective on E for every $h \in G$.

Instead of the λ -lemma we shall use the following ‘theorem’.

THEOREM 8. *Let $\Phi : E \times G \rightarrow \mathbf{C}^m$ be a holomorphic motion for some set $E \subset \mathbf{C}^m$ and some $m \in \mathbf{N} \setminus \{0\}$. If Φ is equicontinuous on $E \times K$ for every relatively compact set $K \subset\subset G$, then Φ extends to a mapping $\hat{\Phi} : \bar{E} \times G \rightarrow \mathbf{C}^m$ having the following properties:*

1. $\hat{\Phi}$ is continuous,
2. $\hat{\Phi}(\cdot, h_0) = \text{id}$ for some $h_0 \in G$, and
3. $\hat{\Phi}(\lambda, \cdot)$ is holomorphic on G for every $\lambda \in \bar{E}$.

WARNING. *The extension $\hat{\Phi}$ is not necessarily a holomorphic motion. In fact, it is not obvious that the extension is injective again.*

Proof. The proof is quite elementary. The equicontinuity of Φ yields that Φ can be extended to a continuous mapping $\hat{\Phi} : \bar{E} \times G \rightarrow \mathbf{C}^m$. Since the limit of holomorphic functions is holomorphic again, $\hat{\Phi}(\lambda, \cdot)$ is holomorphic for every $\lambda \in \bar{E}$. Finally, $\Phi(\cdot, h_0)|_E = \text{id}$ carries over to $\hat{\Phi}(\cdot, h_0)|_{\bar{E}} = \text{id}$. ■

Counterexample. Let

$$E := \left\{ \left(\frac{1}{k}, 0 \right) \in \mathbf{C}^2 \mid k \in \mathbf{N} \setminus \{0\} \right\} \subset \mathbf{C}^2$$

and

$$\Phi : E \times D_1(1) \rightarrow \mathbf{C}^2; \quad \Phi\left(\frac{1}{k}, 0, h\right) := \left(\frac{1}{k}, (1-h) \cdot (-1)^k\right).$$

Then Φ has the following properties:

1. Φ is continuous,
2. $\Phi(\cdot, 1) = \text{id}$,
3. $\Phi(\lambda, \cdot)$ is holomorphic on $D_1(1)$ for every $\lambda \in E$, and
4. $\Phi(\cdot, h)$ is injective on E for every $h \in D_1(1)$.

Hence, Φ satisfies the hypothesis of the λ -lemma. But for every $h \in D_1(1) \setminus \{1\}$ we obtain

$$\Phi\left(\left(\frac{1}{k}, 0\right), h\right) = \left(\frac{1}{k}, (1-h) \cdot (-1)^k\right).$$

Since the sequence $\{(1-h) \cdot (-1)^k\}_{k \in \mathbf{N}}$ is not converging but oscillating, Φ does not have a continuous extension to $\bar{E} = \{0\} \cup E$.

5. Surgery for the Relaxed Newton's Method

The purpose of this section is to prepare the reader for the proof of the Main Theorem of this paper. Here we shall introduce and describe the techniques which will be used in the next section for proving the Main Theorem. In this first paragraph we briefly *sketch* the surgery procedure which is the core of this paper. The details will be given in the proof of Theorem 10. We choose some

polynomial f of degree d with d simple zeroes ζ_v , where $v = 1, \dots, d$. Note that each ζ_v is a simple root and hence it is a superattracting fixed point of the (unrelaxed) Newton's method $N_{f,1}$. Furthermore, $N_{f,1}$ is biholomorphically conjugate to $z \mapsto z^{m_v}$ on a neighbourhood U of ζ_v for some $m_v \geq 2$, cf. [4, Theorem II.4.1]. Using quasiconformal surgery on U we shall replace $N_{f,1}$ by a function which is biholomorphically conjugate to the restriction of the Blaschke product

$$(4) \quad B_h(z) = z \cdot \left(\frac{z + \frac{m_v-1}{m_v} \sqrt{1-h}}{1 + \frac{m_v-1}{m_v} \sqrt{1-\bar{h}z}} \right)^{m_v-1}, \quad h \in D_1(1) \setminus \{1\},$$

to a neighbourhood V of 0, such that $B_h : V \rightarrow B_h(V)$ is proper of degree m_v . Note that $B'_h(0) = 1 - h$. Applying the qc-lemma 5 gives a rational function R of degree d with the following properties: $R(\infty) = \infty$, R has exactly d finite fixed points ξ_v , and $R'(\xi_v) = 1 - h$. Thus R is the relaxed Newton's method $N_{g,h}$ for $g(z) = \prod_{v=1}^d (z - \xi_v)$. Note that the original function $N_{f,1}$ has been changed on the immediately basin of attraction (with respect to $N_{f,1}$) of the roots of f , only. These basins are mapped by the conjugacy ϕ given by the qc-lemma onto the immediate basins of attraction (with respect to $N_{g,h}$) of the roots ξ_v of g . Thus, the conjugacy ϕ in fact is a conjugacy between $N_{f,1}$ and $N_{g,h}$ restricted to their respective Julia sets. In particular, the Julia sets for $N_{f,1}$ and $N_{g,h}$ are homeomorphic.

Clearly, the properties of the mapping $(f, h) \mapsto g$ depend on the concrete realization of the surgery. We shall discuss the details later. For the surgery we need a 'good' parameterization of the polynomials. In the sequel we will consider polynomials of degree $d > 2$, only. The case $d = 1$ is trivial, and the case $d = 2$ is handled in [26].

A short calculation yields $T^{-1} \circ N_{f,h} \circ T = N_{g,h}$ with $g = f \circ T$ for every polynomial f and every affine transformation $T(z) = az + b$. Thus we may assume that the point $z = 1$ is a root of f and that all roots of f sum up to 0. The definition of the relaxed Newton's method immediately implies $N_{f,h} = N_{g,h}$ with $g = af$ for every $a \in \mathbf{C}^* := \mathbf{C} \setminus \{0\}$. As a canonical parameterization of the polynomials of degree d we obtain

$$(5) \quad f_\lambda(z) = z^d + \lambda_{d-2}z^{d-2} + \dots + \lambda_2z^2 + (\lambda_1 - 1)z - \sum_{v=1}^{d-2} \lambda_v,$$

where $\lambda = (\lambda_1, \dots, \lambda_{d-2}) \in \mathbf{C}^{d-2}$. We shall also denote the associated relaxed Newton's method by $N_{\lambda,h}$, and we shall simultaneously use the notions $N_{f_\lambda,h}$ and $N_{\lambda,h}$.

Remark. Let α be a zero of f_λ and $g(z) := \alpha^d f_\lambda(z/\alpha)$. Then $N_{g,h}$ and $N_{f_\lambda,h}$ are conjugated via $T_z = \alpha z$. However, the parameterization (5) is unique in the following sense: $f_\lambda^{-1}(0) = f_\mu^{-1}(0) \Leftrightarrow \lambda = \mu$.

A first result on the set M_h^d is the following lemma.

LEMMA 9. *Let $d \geq 3$. There exists some set $\mathfrak{C} \subset \mathbf{C}^{d-2}$ with $\#\mathfrak{C} \geq 2$ such that $\mathfrak{C} \cap M_h^d = \emptyset$.*

Remark. Actually, one can show that $\mathbf{C}^{d-2} \setminus M_h^d$ contains an open subset, for example the set of all polynomials f such that all the critical points of $N_{f,h}$ are absorbed by the roots of f .

Proof. Let

$$f_1(z) = (z-1)^{d-1}(z-(1-d))$$

and

$$f_2(z) = \left(z - \left(\frac{-1}{d-1} \right) \right)^{d-1} (z-1)$$

and λ_1 and λ_2 the respective parameter values in \mathbf{C}^{d-2} . Note that in both cases 1 is a root, that the leading coefficient is one and that the origin is the center of the roots, hence these polynomials in fact belong to the family considered. In addition, both polynomials have a multiple root, hence $\{\lambda_1, \lambda_2\} \cap M_h^d = \emptyset$. ■

Remark. Note that a short calculation shows that the components of the two parameter values are distinct: $0 \neq \lambda_{1,j} \neq \lambda_{2,j} \neq 0$ for $j = 1, \dots, d-2$.

Next, we give further details about the surgery procedure which has been sketched at the beginning of this section. Since we shall need similar constructions in the sequel, we include the proof, although it is already contained in [10].

THEOREM 10 (von Haeseler-Kriete, 1993). *Let $d \geq 3$. For $\lambda \in \mathbf{C}^{d-2}$ let f_λ be a polynomial with d' zeroes ζ_v (counted without multiplicity). Then for each $h \in D_1(1)^* := D_1(1) \setminus \{1\}$ there exists a polynomial g of degree d' and a quasi-conformal mapping $\phi: \bar{\mathbf{C}} \rightarrow \bar{\mathbf{C}}$ satisfying $\phi \circ N_{\lambda,1} = N_{g,h} \circ \phi$ on $\bar{\mathbf{C}} \setminus \bigcup_{v=1}^{d'} \mathcal{A}_{\lambda,1}(\zeta_v)$.*

Proof. Throughout this paper for a Jordan curve $\Gamma \subset \mathbf{C}$ let $\text{Int}(\Gamma)$ denote the bounded component of $\mathbf{C} \setminus \Gamma$.

We fix $h \in D_1(1)^*$. Using quasiconformal surgery we want to transform $N_{\lambda,1}$ into a relaxed Newton's method. To this end we shall construct a selfmapping \tilde{N} of $\bar{\mathbf{C}}$ which has a repelling fixed point at ∞ and attracting fixed points with prescribed multiplier at the roots of f_λ . Using Shishikura's qc-lemma 5 we shall conjugate \tilde{N} to a rational function R which will turn out to be a relaxed Newton's method for some polynomial g .

We fix a zero ζ of the polynomial f_λ . Clearly, ζ is an attracting fixed point for $N_{\lambda,1}$. We have to distinguish two cases:

1. ζ is a root of f_λ of multiplicity $m \geq 2$, that is, ζ is an attracting (but not superattracting) fixed point for $N_{\lambda,1}$ with $N'_{\lambda,1}(\zeta) = (m-1)/m =: \tilde{\mu}$,
2. ζ is a simple root of f_λ , that is, ζ is a superattracting fixed point for $N_{\lambda,1}$ of, say, order m .

CASE 1. Recall that in this case ζ is an attracting but not superattracting fixed point of $N_{\lambda,1}$. By K oenig's Theorem, cf. [4, Theorem II.2.1], there exist an open neighbourhood G of ζ and a biholomorphic mapping $\psi : G \rightarrow D_r(0)$ for some $r > 0$ satisfying $(\psi \circ N_{\lambda,1})(z) = \tilde{\mu} \cdot \psi(z)$ on G and $\psi'(\zeta) = 1$. Note that the latter normalization makes the mapping ψ to be unique. We write $\Gamma := \psi(\partial G) = \partial D_r(0)$ and $\gamma := \partial((\psi \circ N_{\lambda,1})(G))$, that is $\gamma = \tilde{\mu} \partial D_r(0)$. We may and will assume r to be maximal, that is to say, r is the radius of convergence of the power series of ψ^{-1} . On G we want to replace $N_{\lambda,1}$ by a function which is on $N_{\lambda,1}(G)$ conjugate to the mapping $z \mapsto (1-h)z$.

We start with defining $\hat{\Gamma} := \Gamma$ and $\hat{\gamma} := |1-h|\hat{\Gamma}$. Next we construct a self mapping $\hat{\psi}$ of $\text{cl}(\text{Int } \Gamma)$. We define $\hat{\psi} = \text{id}$ on Γ and $\hat{\psi}(z) = \tilde{\mu}^{-1} \cdot (1-h) \cdot z$ on $\text{cl}(\text{Int}(\gamma))$. Then $\hat{\psi}$ satisfies

$$(6) \quad z \in \Gamma \Rightarrow \hat{\psi}(\tilde{\mu}z) = (1-h) \cdot \hat{\psi}(z).$$

Now we want extend $\hat{\psi}$ to $\text{cl}(\text{Int}(\Gamma))$. The existence of the extension, which will also be denoted by $\hat{\psi}$, is given by the following interpolation lemma. The proof of this lemma will be postponed and can be found on page 100.

LEMMA 11 (Interpolation Lemma). *Let $r_1, r_2, R_1, R_2 \in \mathbf{R}$ satisfy $0 < r_1 < R_1 < \infty$ and $0 < r_2 < R_2 < \infty$. Let C_r denote the circle $\{z \in \mathbf{C} \mid |z| = r\}$. For $j = 1, 2$ let A_j denote the open annulus $D_{R_j} \setminus \bar{D}_{r_j}$. Let $\alpha : C_{r_1} \rightarrow C_{r_2}$ and $\beta : C_{R_1} \rightarrow C_{R_2}$ be real analytic diffeomorphisms. Then there exists a mapping $A : \bar{A}_1 \rightarrow \bar{A}_2$ satisfying*

1. $A|_{A_1} : A_1 \rightarrow A_2$ is a diffeomorphism,
2. $A|_{C_{r_1}} \equiv \alpha$,
3. $A|_{C_{R_1}} \equiv \beta$, and
4. the complex dilatation of A is bounded by some constant $K < 1$.

In addition, A depends continuously on the data r_1, r_2, R_1, R_2 . If the mappings α and β depend holomorphically on some parameter η running through some complex space E as parameter space, then the joint extension A depends holomorphically on $\eta \in E$, too.

This interpolation lemma Lemma 11 assures the extension of $\hat{\psi}$ to an orientation preserving C^1 -diffeomorphism $\hat{\psi} : \text{cl}(\text{Int}(\Gamma)) \rightarrow \text{cl}(\text{Int}(\hat{\Gamma}))$ satisfying (6) on Γ . We define $\Phi := \hat{\psi} \circ \psi$ and $\tilde{N} := \Phi^{-1}((1-h) \cdot \Phi(z))$ on $\text{cl}(G)$. This new function \tilde{N} agrees on ∂G with $N_{\lambda,1}$, and on $N_{\lambda,1}(G)$ it is conjugate via Φ to the holomorphic function $z \mapsto (1-h)z$.

CASE 2. Since ζ is a superattracting fixed point of $N_{\lambda,1}$ of order m , by B ottcher's Theorem, cf. [4, Theorem II.4.1], near ζ the mapping $N_{\lambda,1}$ is conjugated to $z \rightarrow z^m$. In particular, there exists a biholomorphic mapping $\psi : A_{\lambda,1}^*(\zeta) \rightarrow D_r(0)$ satisfying $(\psi \circ N_{\lambda,1})(z) = (\psi(z))^m$ on $A_{\lambda,1}^*(\zeta)$ for some number $r \in]0, 1[$ and $\psi'(\zeta) = 1$. We define $G := \psi^{-1}(D_r(0))$. We write $\Gamma := \psi(\partial G)$ and $\gamma := \partial((\psi \circ N_{\lambda,1})(G))$. We want to replace $N_{\lambda,1}$ on G by a function which is conjugated to a Blaschke product B_h on $N_{\lambda,1}(G)$. To this end we fix a Blaschke

product B_h of degree m satisfying $B_h(0) = 0$ and $B'_h(0) = (1 - h)$, for example the Blaschke product (4). Then there exists an analytic Jordan curve $\tilde{\Gamma} \subset D_1(1)$ such that B_h maps $\text{Int}(\tilde{\Gamma})$ proper and of degree m onto $\text{Int}(\hat{\gamma})$, where $\hat{\gamma} := B_h(\tilde{\Gamma}) \subset \text{Int}(\tilde{\Gamma})$.

We fix some biholomorphic mapping $\Psi : \text{Int}(\gamma) \rightarrow \text{Int}(\hat{\gamma})$. By construction, the boundaries of the domains $\text{Int}(\gamma)$ and $\text{Int}(\hat{\gamma})$ are real analytic curves, hence Ψ has an extension to a real analytic mapping $\Psi : \text{cl}(\text{Int}(\gamma)) \rightarrow \text{cl}(\text{Int}(\hat{\gamma}))$. We define $\hat{\psi} := \Psi$ on $\text{cl}(\text{Int}(\gamma))$, and $\hat{\psi} = \text{id}$ on Γ . Note that both domains, $\text{Int}(\Gamma) \setminus \text{cl}(\text{Int}(\gamma))$ and $\text{Int}(\tilde{\Gamma}) \setminus \text{cl}(\text{Int}(\hat{\gamma}))$, are biholomorphically equivalent to concentric annuli and that this equivalence extends up to the boundaries. Thus we can again apply Lemma 11 to extend $\hat{\psi}$ to a diffeomorphism $\hat{\psi} : \text{cl}(\text{Int}(\Gamma)) \rightarrow \text{cl}(\text{Int}(\tilde{\Gamma}))$ satisfying

$$(7) \quad \hat{\psi}(z^m) = (B_h \circ \phi)(z) \quad \text{on } \Gamma.$$

We write $\phi := \hat{\psi} \circ \psi$ and define $\tilde{N} := \phi^{-1} \circ B_h \circ \phi$ on $\text{cl}(G)$. This new function \tilde{N} agrees on ∂G with $N_{\lambda,1}$, and on $N_{\lambda,1}(G)$ it is conjugate via ϕ to a holomorphic function having an attracting fixed point with derivative $(1 - h)$.

Having defined \tilde{N} on the neighbourhoods G_v of the zeroes ζ_v of p_λ we define $\tilde{N} := N_{\lambda,1}$ on the complement of the G_v 's. By construction, \tilde{N} is a proper and orientation preserving self mapping of \bar{C} . Furthermore, it is holomorphic on $\bar{C} \setminus \bigcup G_v$. The degree of \tilde{N} equals the degree of $N_{\lambda,1}$. Applying the qc-lemma 5 gives a rational function R . R is conjugate to \tilde{N} via a quasiconformal mapping ϕ (which fixes 0, 1 and ∞). Therefore R has the same number of finite fixed point as \tilde{N} has. All these fixed points of R have derivative $1 - h$, in particular, they are attracting fixed point. Furthermore, since ϕ fixes 0 and 1, these attracting fixed points sum up to 0 and 1 is one of these fixed points. Note that ∞ is a fixed point of both, $N_{\lambda,1}$ and R . But ϕ is *not* a conjugacy between $N_{\lambda,1}$ and R . However, the fact that each rational function has at least one weakly repelling fixed point assures ∞ to be a weakly repelling fixed point. Thus, R is a relaxed Newton's method associated with some polynomial $g := f_\mu$ of degree d' and some $\mu \in \mathbf{C}^{d-2}$. Note that this in turn implies ∞ to be a repelling fixed point. ■

All important in what follows is the continuity of the construction.

PROPOSITION 12. *Let $\lambda_0 \in \mathbf{C}^{d-2}$ and $\{\lambda_n\}_{n \in \mathbf{N}^*} \subset \mathbf{C}^{d-2}$ a sequence converging to λ_0 having the following properties:*

1. *If f_{λ_0} has a multiple root ζ_0 of multiplicity m , then for all but finitely many λ_n the polynomial f_{λ_n} have a root ζ_n of multiplicity m such that $\lim_{n \rightarrow \infty} \zeta_n = \zeta_0$ holds,*
2. *If f_{λ_0} has a simple root ζ_0 such that ζ_0 is a superattracting fixed point of order m of $N_{\lambda_0,1}$, then for all but finitely many λ_n the polynomial f_{λ_n} has a simple root ζ_n such that ζ_n is a superattracting fixed point of $N_{\lambda_n,1}$ of order m , and satisfying $\lim_{n \rightarrow \infty} \zeta_n = \zeta_0$.*

Let $g := f_{\mu_0}$ respectively $g_n := f_{\mu_n}$ be polynomials given by Theorem 10 with $\mu_n, \mu_0 \in \mathbf{C}^{d-2}$. Then $\lim_{n \rightarrow \infty} \mu_n = \mu_0$.

Proof. We fix f_{λ_0} and choose a sequence $\{f_{\lambda_n}\}_{n \in \mathbf{N}}$ converging to f_{λ_0} . Clearly, for each root ζ_v of f_{λ_0} there exists a sequence $\{\zeta_{v,n}\}_{n \in \mathbf{N}}$ converging to ζ_v . Let $\mathcal{A}_{\lambda_0,1}^*(\zeta_v)$ and $\mathcal{A}_{\lambda_n,1}^*(\zeta_{v,n})$ denote the corresponding immediate basin of attraction of the associated Newton's methods. Note, that each $\mathcal{A}_{\lambda_0,1}^*(\zeta_v)$ is kernel of the sequence $\{\mathcal{A}_{\lambda_n,1}^*(\zeta_{v,n})\}_{n \in \mathbf{N}^*}$. This yields, that the Riemann mappings $\psi_{v,n}$ of the domains $\mathcal{A}_{\lambda_n,1}^*(\zeta_{v,n})$ converge to the Riemann mapping ψ_v of the domain $\mathcal{A}_{\lambda_0,1}^*(\zeta_v)$ uniformly on compact subsets. For the same reason, the radius of convergence of the mappings $\psi_{v,n}$ converges to the radius of convergence of ψ_v . Thus, at the point λ_0 the data of the surgery, in particular $\partial\Phi_v/\partial\bar{z}$, depend continuously on the parameter λ . Since the solution of the Beltrami equation depend continuously on the prescribed complex dilatation, we conclude the convergence of the quasiconformal conjugacies ϕ_n to the conjugacy ϕ_0 determine the polynomials $g := f_{\mu_0}$ respectively $g_n := f_{\mu_n}$. The latter in turn implies the convergence of the corresponding parameters μ_n to μ_0 . ■

Remarks.

1. This proposition (and its proof) is the key to settle the problems arising from the λ -lemma.
2. Note that ϕ has been chosen such that the resulting polynomial has the desired form:

$$g = f_\mu \quad \text{for some } \mu \in \mathbf{C}^{d'-2}.$$

3. Note that N and \tilde{N} coincide on $\bar{\mathbf{C}} \setminus (\mathcal{F}(N) \cup \bigcup G_v)$ and that none of the G_v contains a repelling periodic point. In particular, both mappings have the same repelling periodic points. Since ϕ is a homeomorphism, $\phi(\mathcal{F}(N))$ is the closure of the set of all the repelling periodic points of \tilde{N} . This in turn implies $\phi(\mathcal{F}(N_{\lambda,1})) = \mathcal{F}(N_{\mu,h})$. Furthermore, $\tilde{N}|_{\mathcal{F}(N)} \equiv N|_{\mathcal{F}(N)}$, hence ϕ conjugates $N_{\lambda,1}|_{\mathcal{F}(N_{\lambda,1})}$ and $N_{\mu,h}|_{\mathcal{F}(N_{\mu,h})}$.
4. Finally, if $N_{\lambda,1}$ has an attracting cycle different from the ζ_v , then $N_{\mu,h}$ has an attracting cycle of same order and with the same multiplier.

Finally, we add the proof of the Interpolation Lemma.

Proof of Lemma 11. For $j = 1, 2$ let $S_j := \{z \in \mathbf{C} \mid \ln(r_j) \leq \operatorname{Re}(z) \leq \ln(R_j)\}$. Note that these parallel strips are the universal coverings of the annuli $\operatorname{cl}(D_{R_j} \setminus D_{r_j})$ (via the exponential function). Let $\hat{\alpha}$ and $\hat{\beta}$ denote the lifts of α respectively β . Note that these lifts are periodic with period $2\pi i$ and that they define a diffeomorphism \hat{A} of ∂S_1 onto ∂S_2 by $\hat{A}|_{\{\operatorname{Re}(z)=\ln(r_1)\}} = \hat{\alpha}$ and $\hat{A}|_{\{\operatorname{Re}(z)=\ln(R_1)\}} = \hat{\beta}$. By linear interpolation this can be extended to all of S_1 : For some point $z = (\ln(r_1) + t(\ln(R_1) - \ln(r_1)) + iy) \in S_1$, where $t \in [0, 1]$ and $y \in \mathbf{R}$, we define

$$\hat{A}(z) := \hat{A}(\ln(r_1) + iy) + t(\hat{A}(\ln(R_1) + iy) - \hat{A}(\ln(r_1) + iy)).$$

Clearly, \hat{A} is periodic with period $2\pi i$, too. Thus the push-down A of \hat{A} with respect to the exponential function is well defined. It is a diffeomorphism by construction. If the data r_1, r_2, R_1, R_2 and the mappings α, β depend continuously respectively holomorphically on some parameter, then so do S_1, S_2 respectively $\hat{\alpha}, \hat{\beta}$. This carries over to the extension \hat{A} and thereby to A . Note that \hat{A} extends to a real analytic mapping defined on some open neighbourhood of \bar{S}_1 . Thus the complex dilatation of \hat{A} is bounded by some constant $K < 1$ and by the holomorphy of the exponential function this carries over to A . This completes the proof. ■

6. Parameter space and holomorphic motions

In this section we consider the relaxed Newton method for polynomials of the form (5). Recall that Theorem 10 assigns to each pair (f_λ, h) a new polynomial f_μ such that $N_{f_\lambda, 1}$ and $N_{f_\mu, h}$ are conjugated. In particular, both Newton's methods have the same degree. In particular, if f_λ has simple roots only, then so must f_μ . We want to study the function $(\lambda, h) \mapsto \mu$. A first result is the following lemma, cf. [10, Lemma 4.7].

LEMMA 13 (von Haeseler-Kriete, 1993). *The function $(\lambda, h) \mapsto \mu$ can not be chosen to be both, injective in λ and holomorphic in h .*

Remark. There are two reasons for this trouble.

1. The first is the fact, that for $\lambda \notin M_h^d$, e.g., λ sufficiently close to 0, there is a zero ζ of the polynomial f_λ such that $\mathcal{A}_{\lambda, 1}^*(\zeta)$ contains at least two critical points. We assume for a moment, that $\mathcal{A}_{\lambda, 1}^*(\zeta)$ contains two critical points different from ζ . Then one has to restrict the quasiconformal surgery to some neighbourhood of ζ containing the 'nearest' critical point but not the other one. If λ varies then different critical points will be the closest to ζ . Hence, there does not exist any continuous or holomorphic parameterization of the nearest critical point.
2. The second reason, which has been used in the proof of the above lemma, is the following. Assume for a moment that $d = 3$. Then $\lambda = -2, 1/4$ are the parameter values where the polynomial in question has a multiple root. Consequently, $N_{-2, h}$ and $N_{1/4, h}$ are rational functions of degree 2 and therefore have to be fixed points of the function $(\lambda, h) \mapsto \mu$. Another crucial role is playing the parameter value $\lambda = 0$. This is the only parameter value where 0 is a simple root of the polynomial in question and a double critical point of the associated Newton's method $N_{0, 1}$. For $0 < h < 1$ however, there are three different values $\mu_1(h), \mu_2(h), \mu_3(h)$ satisfying $\lim_{h \rightarrow 0} \mu_j(h) = 0$, where $j = 1, 2, 3$, such that $N_{\mu_j(h), h}$ has a double critical point. Note that having a double critical point or having degree two are invariants of the surgery procedure. But the function $(\lambda, h) \mapsto \mu$ cannot map $\mathcal{C} \setminus \{-2, 0, 1/4\}$ injectively and holomorphically into $\mathcal{C} \setminus \{-2, 0, 1/4, \mu_1(h), \mu_2(h), \mu_3(h)\}$ for generic h .

In order to avoid these problems, in the sequel we will restrict the quasiconformal surgery to the parameter set M_1^d . We want to establish a construction for the surgery which guarantees, that the mapping $L : M_1^d \times D_1(1) \rightarrow \mathbf{C}$; $(\lambda, h) \mapsto \mu$ is homeomorphic in λ and holomorphic in h . In other words, we want to describe M_h^d in terms of holomorphic motions. Note that the construction described in the proof of Theorem 10 does not work if h is chosen out of some open neighbourhood of 1. In fact, one might say, that ‘the method has a singularity at $h = 1$ ’. Hence, we have to add another preliminary step. This will be a canonical construction which works for h sufficiently close to 1. Then we shall introduce a special family of quadratic polynomials and describe a method how these polynomials can be inserted into the relaxed Newton’s method $N_{\lambda,1}$. These parts are taken from [10] but not without modifications. Finally, we shall show that this construction can be used to obtain the holomorphic motion L whose existence has been announced in Theorem 4. Since we restrict the qc-surgery to parameters $\lambda \in M_h^d$, the assumptions of Proposition 12 are always satisfied. Hence, this proposition (or arguments similar to those given in the proof) assures the continuity of the construction, and, therefore, we may and will (implicitly) apply Theorem 8.

We fix $\lambda \in M_1^d$ and a zero ζ of f_λ . Recall that $\mathcal{A}_{\lambda,1}^*(\zeta)$ is simply connected and that there exists a biholomorphic mapping $\psi : \mathcal{A}_{\lambda,1}^*(\zeta) \rightarrow \mathbf{D}$ satisfying

$$\begin{aligned} \psi(\zeta) &= 0, \\ \psi'(\zeta) &> 0 \quad \text{and} \\ (\psi \circ N_{\lambda,1})(z) &= (\psi(z))^2 \quad \text{on } \mathcal{A}_{\lambda,h}^*(\zeta). \end{aligned}$$

Note that ψ is unique by this normalization. We consider the holomorphic family of polynomials $g_h(z) = (1-h)z + z^2$. Clearly, for h near 1 the polynomial g_h is a small perturbation of $z \mapsto z^2$. We fix a number $r \in]0, 1[$ and define curves:

$$\Gamma := \{|z| = r\} \quad \text{and} \quad \gamma := \{|z| = r^2\}.$$

If ε sufficiently small and $h \in D_\varepsilon(1)$, for the Jordan curve $\hat{\Gamma}_h := g_h^{-1}(\gamma)$ we have that $g_h|_{\hat{\Gamma}_h} : \hat{\Gamma}_h \rightarrow \hat{\gamma} := \gamma$ is a covering of order 2. The curves $\hat{\Gamma}_h$ have a parameterization $\hat{\Gamma}$ which is holomorphic in h , i.e.,

$$\hat{\Gamma} : S^1 \times D_\varepsilon(1) \rightarrow \mathbf{C} \quad \text{such that} \quad \hat{\Gamma}(S^1, h) = \hat{\Gamma}_h.$$

Now we construct quasiconformal mappings $\phi(\cdot, h) : \text{Int}(\Gamma) \rightarrow \text{Int}(\hat{\Gamma}_h)$ depending holomorphically on h . We start by setting $\phi(\cdot, h) = \text{id}$ on $\{|z| \leq r^2\}$ and $\phi(\cdot, h)(z) = g_h^{-1}(z^2)$ on Γ , where the inverse branches are chosen such that

$$(8) \quad (g_h \circ \phi(\cdot, h))(z) = \phi(z^2, h) \quad \text{on } \Gamma.$$

As described in the proof of Theorem 10 we extend ϕ to a mapping $\phi : \text{Int}(\Gamma) \times D_\varepsilon(1) \rightarrow \mathbf{C}$ such that

1. $\phi(\cdot, h) = \phi$ on $\text{Int}(\Gamma)$, $\phi(\cdot, 1) = \text{id}$,
2. $\phi(\cdot, h)$ is a quasiconformal mapping, and
3. $\phi(z, \cdot)$ is holomorphic on $D_\varepsilon(1)$.

We define $\Phi := \phi(\cdot, h) \circ \psi$, and $\tilde{N} := \Phi^{-1} \circ g_h \circ \Phi$ on $G_\zeta := \psi^{-1}(\text{Int}(\Gamma))$. Outside the domains G_ζ we set $\tilde{N} := N_{\lambda,1}$. According to Theorem 10 and the qc-lemma \tilde{N} is conjugate via a quasiconformal mapping $F_{\lambda,h}$ to a relaxed Newton's method $N_{\mu,h}$. This gives a mapping $\hat{L} : M_1^d \times D_\varepsilon(1) \rightarrow \mathbf{C}$.

- LEMMA 14. 1. $\hat{L}(\cdot, 1) = \text{id}$,
 2. $\hat{L}(\lambda, \cdot)$ is holomorphic on $D_\varepsilon(1)$, and
 3. $\hat{L}(\cdot, h) : M_1^d \rightarrow M_h^d$ is a homeomorphism.

Proof. 1. This is true by construction.

2. ϕ is a holomorphic family, thus for fixed z the complex dilatations of $\phi(\cdot, h)$ and $\Phi(\cdot, h)$ are holomorphic in h (cf. [21, Theorem 2]). Therefore the conjugacies $F_{\lambda,h}$ are holomorphic functions in h . In particular, for every zero ζ of f_λ the values $F_{\lambda,h}(\zeta)$ are holomorphic in h . This implies $\hat{L}(\lambda, \cdot)$ to be holomorphic on $D_\varepsilon(1)$.

3. We show the injectivity of $\hat{L}(\cdot, h)$. Then $\hat{L}(\cdot, h)$ is an embedding. Furthermore one obtains the inverse $\hat{L}^{-1}(\cdot, h)$ by the reverse construction, hence the image of M_1^d under $\hat{L}(\cdot, h)$ is M_h^d . Combining these two statements yield that $\hat{L}(\cdot, h)$ is a homeomorphism.

We assume $\hat{L}(\lambda_1, h) = \hat{L}(\lambda_2, h)$ for some $\lambda_1, \lambda_2 \in M_1^d$ and $h \in D_\varepsilon(1)$. The quasiconformal mapping $F := F_{\lambda_2, h}^{-1} \circ F_{\lambda_1, h}$ maps $\bar{\mathbf{C}} \setminus \bigcup \mathcal{A}_{\lambda_1, 1}(\zeta_\nu)$ biholomorphically onto $\bar{\mathbf{C}} \setminus \bigcup \mathcal{A}_{\lambda_2, 1}(\zeta_\nu)$. Furthermore we have $F(G_{\lambda_1, \zeta_\nu}) = G_{\lambda_2, \zeta_\nu}$. By construction the mappings

$$F_{\lambda_i, h} \circ \psi_{\lambda_i, \zeta_\nu}^{-1} : \text{Int}(\Gamma_h) \rightarrow F_{\lambda_i, h}(G_{\lambda_i, \zeta_\nu})$$

are biholomorphic. Thus $F : G_{\lambda_1, \zeta_\nu} \rightarrow G_{\lambda_2, \zeta_\nu}$ is biholomorphic. On $A_{\lambda_1, 1}(\zeta_\nu) \setminus G_{\lambda_1, \zeta_\nu}$ we obtain the complex dilatation of $F_{\lambda_i, h}$ by pulling back the complex dilatation of $F_{\lambda_i, h}|_{G_{\lambda_i, \alpha_\nu}}$ via $N_{\lambda_i, h}$. This proves the complex dilatation of F to vanish on every $A_{\lambda_1, h}(\zeta_\nu)$. Hence $F : \bar{\mathbf{C}} \rightarrow \bar{\mathbf{C}}$ is a biholomorphic mapping. By construction we have $F(\infty) = \infty$ and $F(1) = 1$. This yields F to be a linear transformation. We have normalized the polynomials such that their zeroes sum up to 0. F maps the zeroes of f_{λ_1} onto the zeroes of f_{λ_2} , therefore 0 is a fixed point of F . The identity is the only transformation having three fixed points, thus we obtain $f_{\lambda_1} = f_{\lambda_2}$ and therefore $\lambda_1 = \lambda_2$. ■

Remark. Note that in the proof we have fixed some $r \in]0, 1[$ and that we have required that $g_h^{-1}(\gamma)$, where $\gamma = \{|z| = r\}$ and $g_h(z) = (1-h)z + z^2$, is a Jordan curve which is mapped properly (of degree 2) onto γ . This is true for certain combinations for r and h , only, for example, for h sufficiently near 1. For arbitrary $h \in D_1(1)$ we will modify the construction.

In order to extend \hat{L} to the whole of $M_1^d \times D_1(1)$ we consider the polynomials $\phi_h(z) = (1-h)(z + z^2)$ for $h \in D_1(1)^*$. For $h \in D_1(1)^*$ let $\psi_h(z) = z + \dots$ be the formal conjugacy between ϕ_h and $z \mapsto (1-h)z$:

$$(\phi_h \circ \psi_h)(z) = \psi_h((1-h)z).$$

The coefficients of the power series of ψ_h are holomorphic functions in h . Let $R(h)$ denote its radius of convergence. Then $\psi_h(\partial D_{R(h)}(0))$ and $\phi_h(\psi_h(\partial D_{R(h)}(0))) = \partial\psi_h(|1-h|D_{R(h)}(0))$ are piecewise analytic Jordan curves. The following result is classical, see [13].

LEMMA 15. *For every $h \in D_1(1)^*$ the limit $u(h) := \lim_{n \rightarrow \infty} (\phi_h^n(c)/(1-h)^n)$, where $c = -1/2$, exists. The function u is holomorphic on $D_1(1)^*$ with a simple pole in $h = 0$. Furthermore, the equality $|u(h)| = R(h)$ holds on $D_1(1)^*$.*

We now apply this lemma in order to establish

COROLLARY 16. *The basin of attraction \mathcal{A}_h of ϕ_h have a holomorphic parameterization.*

Proof. Since the coefficients of the power series of ψ_h holomorphically depend on h , the mapping $h \rightarrow \psi_h(z)$ is holomorphic. Hence due to Lemma 15 the function $\Gamma : D_1(1)^* \times S^1 \rightarrow \mathbf{C}$; $(h, z) \mapsto \psi_h(z \cdot u(h))$ is holomorphic in h , too. The mapping $\Gamma(h, \cdot)$ is a homeomorphism of S^1 onto a piecewise analytic Jordan curve Γ_h . ϕ_h injectively maps $\text{Int}(\Gamma_h)$ into itself: $\gamma_h := \partial\phi_h(\text{Int}(\Gamma_h)) = \phi_h(\Gamma_h) \subset \text{Int}(\Gamma_h)$. The annular domain $\text{Int}(\Gamma_h) \setminus \overline{\text{Int}(\gamma_h)}$ is a fundamental domain for ϕ_h in the sense of [18]. If $K_h := \text{cl}(\text{Int}(\Gamma_h) \setminus \overline{\text{Int}(\gamma_h)})$, then $\hat{K}_h := \psi_h^{-1}(K_h)$ is equal to the closed annulus $\{|1-h| \cdot R(h) \leq |z| \leq R(h)\}$. For a given $h_0 \in D_1(1)^*$ one easily finds a holomorphic family $\hat{\phi}_{h_0} : D_1(1)^* \times \hat{K}_{h_0} \rightarrow \mathbf{C}$ of quasiconformal mappings $\hat{\phi}_{h_0}(h, \cdot) : \hat{K}_{h_0} \rightarrow \hat{K}_h$. Applying ψ_h we obtain a holomorphic family $\varphi_{h_0} : D_1(1)^* \times K_{h_0} \rightarrow \mathbf{C}$ of quasiconformal mappings $\varphi_{h_0}(h, \cdot) : K_{h_0} \rightarrow K_h$. φ_{h_0} extends via forward and backward iteration of ϕ_h to a holomorphic family of quasiconformal mappings $\varphi_{h_0}(h, \cdot) : \mathcal{A}_{h_0} \rightarrow \mathcal{A}_h$. ■

Remark. The mapping $\varphi_{h_0}(h, \cdot)$ constructed above conjugate f_{h_0} and ϕ_h :

$$\phi_h \circ \varphi_{h_0}(h, \cdot) = \varphi_{h_0}(h, f_{h_0}(\cdot)) \quad \text{on } \mathcal{A}_{h_0}.$$

Another immediate consequence is

COROLLARY 17. *There exists an analytic family $\hat{\Gamma}$ of Jordan curves, such that $\hat{\Gamma}_h \subset \mathcal{A}_h$ and $\phi_h(\hat{\Gamma}_h) = \hat{\gamma}_h$ is of degree 2.*

$$\hat{\Gamma} : D_1(1)^* \times \hat{\Gamma}_{h_0} \rightarrow \mathbf{C}; \quad (h, z) \mapsto \varphi_{h_0}(h, z) \quad \text{and}$$

$$\hat{\gamma} : D_1(1)^* \times \hat{\gamma}_{h_0} \rightarrow \mathbf{C}; \quad (h, z) \mapsto \varphi_{h_0}(h, z).$$

At this point we summarize what we have gotten hold of so far. We have obtained domains $\text{Int}(\hat{\Gamma}_h)$ which are parameterized holomorphically on $D_1(1)^*$ and which are mapped by ϕ_h proper of degree 2 onto the domains $\text{Int}(\hat{\gamma}_h)$. For every $h \in D_1(1)^*$ the equality

$$(9) \quad \phi_h \circ \hat{\Gamma}_h = \hat{\gamma}_h \circ f_{h_0}$$

holds on Γ_{h_0} . Furthermore we have $\hat{\Gamma}_{h_0} = \text{id}$ and $\hat{\gamma}_{h_0} = \text{id}$.

In the next part we shall insert the restrictions $\phi_h|_{\text{Int}(\tilde{\Gamma}_h)}$ in the relaxed Newton's method N_{λ, h_0} . The result of this surgery will be a relaxed Newton's method $N_{\mu, h}$. The mapping $L_{h_0} : (\lambda, h) \mapsto \mu$ turns out to be a holomorphic motion. If h_0 tends to 1 then L_{h_0} will tend to a holomorphic motion L with the desired properties.

For every zero ζ of f_λ , where $\lambda \in M_{h_0}^d$, there exists a biholomorphic conjugacy $g_{\lambda, \zeta} : \mathcal{A}_{\lambda, h_0}^*(\zeta) \rightarrow \mathcal{A}_{h_0}$ such that: $g_{\lambda, \zeta} \circ N_{\lambda, h_0} = f_{h_0} \circ g_{\lambda, \zeta}$ on $\mathcal{A}_{\lambda, h_0}^*(\zeta)$. We write $G_\zeta := g_{\lambda, \zeta}^{-1}(\text{Int}(\Gamma_{h_0}))$. For $h \in D_1(1)^*$ we define a new function by

$$\tilde{N}_{\lambda, h} := g_{\lambda, \zeta}^{-1} \circ \phi_{h_0}(h, \cdot)^{-1} \circ \phi_h \circ \phi_{h_0}(h, \cdot) \circ g_{\lambda, \zeta} \quad \text{on } G_{\lambda, \zeta}$$

and $\tilde{N}_{\lambda, h} := N_{\lambda, h_0}$ otherwise. According to Theorem 10 $\tilde{N}_{\lambda, h}$ is conjugate via some quasiconformal mapping to a relaxed Newton's method $N_{\mu, h}$. We define $L_{h_0}(\lambda, h) := \mu$ and summarize some properties of L_{h_0} .

PROPOSITION 18. *The mapping $L_{h_0} : M_{h_0}^d \times D_1(1)^* \rightarrow \mathbf{C}^{d-2}$ is a holomorphic motion satisfying $L_{h_0}(M_{h_0}^d, h) \subset M_h^d$ for all $h \in D_1(1)^*$.*

LEMMA 19. *For every $\lambda \in M_{h_0}^d$ the functions $L_{h_0}(\lambda, \cdot) = (L_{1, h_0}, \dots, L_{d-2, h_0})$ extend to meromorphic functions on $D_1(1)$ with an unessential singularity at $h = 1$.*

Proof. By Proposition 18, each component L_{j, h_0} of the mapping $L_{h_0}(\lambda, \cdot)$ is well defined and holomorphic on the punctured disk $D_1(1)^*$ for each $\lambda \in M_{h_0}^d$. Therefore, in order to prove the lemma, it is sufficient to discuss the behavior of the function $L_{h_0}(\lambda, \cdot)$ near the origin. Recall that by Lemma 9 and the remark following its proof, each L_{j, h_0} omits two distinct values in \mathbf{C} . By construction, it takes values in \mathbf{C} , only, thus it misses ∞ . Applying Picard's theorem shows that the singularity at 0 is not essential, that is, each L_{j, h_0} and therefore L_{h_0} is meromorphic on the disc $D_1(1)$. ■

As in Lemma 14 we prove $L_{h_0}(M_{h_0}^d, h) = M_h^d$ for every $h \in D_1(1)^*$. Then Lemma 9 yields

LEMMA 20. *For every $h \in D_1(1)^*$ there exist a positive number ε_h , depending on h , such that $\mathcal{E} \cap M_\eta^d = \emptyset$ for every $\eta \in D_1(1) \cap D_{\varepsilon_h}(h)$.*

For $\eta \in D_1(1)^*$ we define $L_\eta : M_\eta^d \times D_1(1) \rightarrow \mathbf{C}^{d-2}$; $(\lambda, h) \mapsto (L_{h_0}(\cdot, h) \circ L_{h_0}(\cdot, \eta)^{-1})(\lambda)$. By construction we have

1. $L_\eta(\cdot, \eta) = \text{id}$,
2. $L_\eta(\lambda, \cdot)$ is meromorphic on $D_1(1)$ for every $\lambda \in M_\eta^d$ and
3. $L_{\eta_1}(\lambda, \cdot) = L_{\eta_2}(L_{\eta_1}(\lambda, \eta_2), \cdot)$ on $M_{\eta_1}^d$ for every $\eta_1, \eta_2 \in D_1(1)^*$.

Now let $\{\eta_n\}_{n \in \mathbf{N}} \subset D_1(1) \setminus \{1\}$ be a sequence such that $\lim_{n \rightarrow \infty} \eta_n = 1$ and M' a countable dense subset of M_1^d . We write $M' = \{\lambda_0, \lambda_1, \dots\}$. For λ_0 there exists a sequence $\{\lambda_{0, n}\}_{n \in \mathbf{N}}$ satisfying $\lambda_{0, n} \in M_{\eta_n}^d$ and due to Lemma 14 we have that the sequence $\{M_{\eta_n}^d\}_{n \in \mathbf{N}}$ converges towards M_1^d . This implies $\lim_{n \rightarrow \infty} \lambda_{0, n} = \lambda_0$. As in the proof of Lemma 19, Lemma 9 and Proposition 18 yield that the sequence

$\{L_{\eta_n}(\lambda_{0,n}, \cdot)\}_{n \in \mathbb{N}}$ is a normal family, thus for a suitable subsequence $\{\eta_{0,n}\}_{n \in \mathbb{N}}$ of $\{\eta_n\}_{n \in \mathbb{N}}$ we may assume the existence of $\lim_{n \rightarrow \infty} L_{\eta_{0,n}}(\lambda_{0,n}, \cdot)$ on $D_1(1)$ (with $\lambda_{0,n} \in M_{\eta_{0,n}}^d$). Now we proceed by induction. For $j \in \mathbb{N} \setminus \{0\}$ there exists a sequence $\{\lambda_{j,n}\}_{n \in \mathbb{N}}$ satisfying $\lambda_{j,n} \in M_{\eta_{j-1,n}}^d$ and $\lim_{n \rightarrow \infty} \lambda_{j,n} = \lambda_j$. As above for a subsequence $\{\eta_{j,n}\}_{n \in \mathbb{N}}$ of $\{\eta_{j-1,n}\}_{n \in \mathbb{N}}$ we may assume the existence of $\lim_{n \rightarrow \infty} L_{\eta_{j,n}}(\lambda_{j,n}, \cdot)$ on $D_1(1)$. Now,

$$L(\lambda_j, \cdot) := \lim_{n \rightarrow \infty} L_{\eta_{j,n}}(\lambda_{j,n}, \cdot)$$

is well defined. Moreover $L_{\eta_{n,n}}(\lambda, \eta_{n,n}) = \lambda$ converges to $\lambda = L(\lambda, 1)$ for all $\lambda \in M'$. Therefore $L : M' \times D_1(1) \rightarrow \mathcal{C}^{d-2}$ is well defined and has the following properties

1. $L(\cdot, 1) = \text{id}$,
2. $L(\lambda, \cdot)$ is holomorphic on $D_1(1)$,
3. $L(\lambda, h) = L_{\eta}(L(\lambda, \eta), h)$ on M' for every $h, \eta \in D_1(1)^*$, and
4. $L(\cdot, h)$ is injective.

Only 4 needs a proof. For this purpose we choose $\lambda_1, \lambda_2 \in M'$ such that $\lambda_1 \neq \lambda_2$. By 1 this yields $L(\lambda_1, \eta_n) \neq L(\lambda_2, \eta_n)$ for n sufficiently large. $L_{\eta}(\cdot, \eta_n)$ is an injective mapping, thus 3 completes the proof. Now by applying Theorem 8 we extend L to a holomorphic motion.

Thus we have proved the existence of an holomorphic motion $L : M_1^d \times D_1(1) \rightarrow \mathcal{C}^{d-2}$ satisfying:

1. $L(\cdot, 1) = \text{id}$,
2. $L(\lambda, \cdot)$ is holomorphic on $D_1(1)$ for every $\lambda \in M_1^d$,
3. $L(\lambda, h) = L_{\eta}(L(\lambda, \eta), h)$ on M_1 for every $\eta, h \in D_1(1)^*$, and
4. $L(\cdot, h) : M_1^d \rightarrow M_h^d$ is a homeomorphism.

This completes the proof of the Main Theorem. ■

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