J. LU
KODAI MATH. J.
26 (2003), 221–229

THEOREMS OF PICARD TYPE FOR ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

Jin Lu*

Abstract

In this paper, some theorems of Picard type relating to the total derivative for entire functions of several complex variables are proved.

1. Introduction

In 1940, H. Milloux showed that for a meromorphic function f on the complex plane, the following inequality

$$T_f(r) \leq N_f(r,0) + N_f(r,\infty) + N_{f^{(k)}}(r,1) - N_{f^{(k+1)}}(r,0) + S(r,f)$$

holds, where $T_f(r)$ is the characteristic function of f and $S(r,f) = O(\log rT_f(r))$ holds for all large r outside a set with finite measure ([2], [3] and [6]). The important characteristic of the above inequality is that the right side of it contains a counting function of $f^{(k)}$, and hence we can derive theorems of Picard type relating to derivatives. For example, we can directly derive from the above inequality the following: Let f be an entire function on the complex plane, and let a, b $(b \neq 0)$ be two distinct complex numbers. If $f \neq a$ and $f^{(k)} \neq b$, then f is constant ([2]). It is natural to ask the following question: Whether such kinds of theorems hold for entire functions of several complex variables? In this paper we discuss this question.

For $z \in \boldsymbol{C}^n$, we write $z = (z_1, z_2, \dots, z_n)$. First we give the definition of *total derivative*.

Definition 1.1. Let f be an entire function on $\boldsymbol{C}^n$, the total derivative Df of f is defined by

$$Df(z) = \sum_{j=1}^{n} z_j f_{z_j}(z),$$

2000 *Mathematics Subject Classification*: Primary 32H30, 32A22; Secondary 32H04.
Keywords and phrases: Theorem of Picard type, entire function, total derivative.
*This work was supported by National Natural Science Foundation of China (No. 10271029).
Received September 30, 2002; revised February 3, 2003.

where f_{z_j} is the partial derivative of f with respect to z_j $(j = 1, 2, \ldots, n)$. The k-th order total derivative $D^k f$ of f is defined by

$$D^k f = D(D^{k-1} f),$$

inductively.

The merit of the total derivative is the following: If f is a transcendental entire function on $\boldsymbol{C}^n$, then for any positive integer k, $D^k f$ is also a transcendental entire function on $\boldsymbol{C}^n$ (see Lemma 2.2 bellow). However the partial derivative may not have this property. The main theorems in this paper are the following:

Theorem 1.1. *Let f be an entire function on $\boldsymbol{C}^n$, and let a and b $(b \neq 0)$ be two distinct complex numbers and k be a positive integer. If $f \neq a$ and $D^k f \neq b$, then f is constant.*

Theorem 1.2. *Let f be an entire function on $\boldsymbol{C}^n$, and let $b \neq 0$ be a complex number and $k \geq 2$ a positive integer. If $f^k \cdot Df \neq b$, then f is constant.*

This theorem is also a generalization of a result of [6] on entire function of one complex variable. In the Section 4 of this paper, we will give an example to indicate that these two theorems are not valid if the total derivative is replaced by the partial derivative.

2. Notations and lemmas

For $z = (z_1, \ldots, z_n) \in \boldsymbol{C}^n$, define $|z| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$. Let

$$S_n(r) = \{z \in \boldsymbol{C}^n; |z| = r\}, \quad \bar{B}_n(r) = \{z \in \boldsymbol{C}^n; |z| \leq r\}.$$

Set $d = \partial + \bar{\partial}$ and $d^c = (\partial - \bar{\partial})/4\pi i$. Define

$$\omega_n(z) = dd^c \log|z|^2, \quad \sigma_n(z) = d^c \log|z|^2 \wedge \omega_n^{n-1}(z), \quad v_n(z) = dd^c|z|^2.$$

Then $\sigma_n(z)$ is a positive measure on $S_n(r)$ with the total measure one. Let $a \in \boldsymbol{P}^1$. If $f^{-1}(a) \neq \boldsymbol{C}^n$, we denote by Z_a^f the a-divisor of f, write $Z_a^f(r) = \bar{B}_n(r) \cap Z_a^f$ and define

$$n_f(r, a) = r^{2-2n} \int_{Z_a^f(r)} v_n^{n-1}(z).$$

Then the counting function $N_f(r, a)$ is defined by

$$N_f(r, a) = \int_0^r [n_f(t, a) - n_f(0, a)] \frac{dt}{t} + n_f(0, a) \log r,$$

where $n_f(0, a)$ is the Lelong number of Z_a^f at the origin. Then Jensen's formula gives that

$$N_f(r,0) - N_f(r,\infty) = \int_{S_n(r)} \log|f(z)|\sigma_n(z) + O(1).$$

We define the proximity function $m_f(r,a)$ by

$$\begin{aligned} m_f(r,a) &= \int_{S_n(r)} \log^+ \frac{1}{|f(z)-a|}\sigma_n(z) \quad \text{if } a \neq \infty; \\ &= \int_{S_n(r)} \log^+|f(z)|\sigma_n(z) \qquad \text{if } a = \infty. \end{aligned}$$

We also define the characteristic function $T_f(r)$ by

$$T_f(r) = m_f(r,\infty) + N_f(r,\infty).$$

The first main theorem states that ([4], Chapter 4, A5.1)

$$T_f(r) = m_f(r,a) + N_f(r,a) + O(1).$$

Let $I = (\alpha_1, \ldots, \alpha_n)$ be a multi-index, where α_j $(j = 1, 2, \ldots, n)$ are nonnegative integers. We denote by $|I|$ the length of I, that is, $|I| = \sum_{j=1}^n \alpha_j$. Define

$$\partial^I f = \frac{\partial^{|I|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}.$$

Lemma 2.1 ([7], Theorem 1). *Let f be a non-constant meromorphic function on $\boldsymbol{C}^n$, and let $I = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a multi-index. Then*

$$m_{\partial^I f/f}(r,\infty) = \int_{S_n(r)} \log^+\left|\frac{\partial^I f}{f}(z)\right|\sigma_n(z) = O(\log rT_f(r))$$

holds for all large r outside a set with finite Lebesgue measure.

We say f to be transcendental if

$$\lim_{r\to\infty} \frac{T_f(r)}{\log r} = \infty.$$

Lemma 2.2. *Let f be a transcendental entire function on $\boldsymbol{C}^n$. Then for any positive integer k, $D^k f$ is also a transcendental entire function on $\boldsymbol{C}^n$, and*

$$m_{D^k f/f}(r,\infty) = O(\log rT_f(r))$$

holds for all large r outside a set with finite Lebesgue measure.

Proof. Since f is an entire function on $\boldsymbol{C}^n$, then we have a convergent series on $\boldsymbol{C}^n$ as follows:

$$f(z) = \sum_{m=0}^{\infty} P^m(z),$$

where $P^m(z)$ is either identically zero or a homogeneous polynomial of degree m in z $(m = 0, 1, 2, \dots)$. By the homogeneity of $P^m(z)$ we have

$$\sum_{j=1}^{n} z_j P^m_{z_j}(z) = mP^m(z) \quad (m = 1, 2, \dots).$$

Hence we see

$$Df(z) = \sum_{j=1}^{n} z_j f_{z_j}(z) = \sum_{m=1}^{\infty} mP^m(z).$$

By induction, we have

$$D^k f(z) = \sum_{m=1}^{\infty} m^k P^m(z) \quad (k = 1, 2, \dots).$$

Since f is transcendental, there are infinitely many terms of $\{P^m(z)\}$ which are not identically zero. Hence there are infinitely many terms of $\{m^k P^m(z)\}$ which are not identically zero. Thus $D^k f$ is a transcendental entire function on $\boldsymbol{C}^n$ for all positive integers k.

It is clear that, for any positive integer k, there are multi-indices $I_1, \dots, I_p$ such that

$$D^k f(z) = \sum_{j=1}^{p} Q_{I_j}(z) \partial^{I_j} f(z),$$

where $Q_{I_j}(z)$ $(j = 1, 2, \dots, p)$ are polynomials in z. Note that, for any rational function $R(z)$, we have $m_R(r, \infty) = O(\log r)$. Hence

$$\begin{aligned}
m_{D^k f/f}(r, \infty) &= \int_{S_n(r)} \log^+ \left| \sum_{j=1}^{p} Q_{I_j}(z) \frac{\partial^{I_j} f}{f}(z) \right| \sigma_n(z) \\
&\le \sum_{j=1}^{p} \int_{S_n(r)} \log^+ \left| \frac{\partial^{I_j} f}{f}(z) \right| \sigma_n(z) + \sum_{j=1}^{p} \int_{S_n(r)} \log^+ |Q_{I_j}(z)| \sigma_n(z) + O(1) \\
&= \sum_{j=1}^{p} m_{\partial^{I_j} f/f}(r, \infty) + \sum_{j=1}^{p} m_{Q_{I_j}}(r, \infty) + O(1) \\
&= \sum_{j=1}^{p} m_{\partial^{I_j} f/f}(r, \infty) + O(\log r).
\end{aligned}$$

Thus by Lemma 2.1, we have completed the proof. □

LEMMA 2.3. *Let f be a transcendental entire function on $\boldsymbol{C}^n$, and let a be a complex number. Then for any positive integer k,*

$$m_{D^{k+1} f/(D^k f - a)}(r, \infty) = O(\log r T_f(r))$$

holds for all large r outside a set with finite Lebesgue measure.

Proof. It is easy to see that $D(D^k f - a) = D^{k+1} f$. By Lemma 2.2, we see that $D^k f - a$ is a transcendental entire function, and

$$m_{D^{k+1}f/(D^k f - a)}(r, \infty) = m_{D(D^k f - a)/(D^k f - a)}(r, \infty) \\ = O(\log rT_{D^k f - a}(r)) = O(\log rT_{D^k f}(r)) \tag{2.1}$$

holds for all large r outside a set with finite Lebesgue measure. Note that

$$T_{D^k f}(r) = m_{D^k f}(r, \infty) \le m_{D^k f/f}(r, \infty) + m_f(r, \infty) = m_{D^k f/f}(r, \infty) + T_f(r). \tag{2.2}$$

By Lemma 2.2, (2.1) and (2.2), we get the desired conclusion. □

Lemma 2.4. *Let f be a polynomial of degree p. If Df is constant, then f is constant and $Df \equiv 0$.*

Proof. We write f as

$$f(z) = \sum_{m=0}^{p} P^m(z),$$

where $P^m(z)$ is either identically zero or a homogeneous polynomial of degree m $(m = 0, 1, 2, \dots, p)$. As in the proof of Lemma 2.2, we have

$$Df(z) = \sum_{m=1}^{p} mP^m(z),$$

If Df is constant, every $mP^m(z)$ must be identically zero, so is $P^m(z)$ $(m = 1, 2, \dots, p)$. Thus f is constant and $Df \equiv 0$. □

3. Main inequalities

In order to prove our theorems we first give some estimates for the characteristic function relating to the total derivative. As usual, the notation "$\|$ P" means that the assertion P holds for all large $r \in [0, +\infty)$ outside a set with finite Lebesgue measure.

Theorem 3.1. *Let f be a transcendental entire function on $\boldsymbol{C}^n$. Then for any positive integer k,*

$$\| \quad T_f(r) \le N_f(r, 0) + N_{D^k f}(r, 1) - N_{D^{k+1} f}(r, 0) + O(\log rT_f(r)).$$

Proof. By the equality

$$\frac{1}{f} = \frac{D^k f}{f} - \frac{D^k f - 1}{D^{k+1} f} \cdot \frac{D^{k+1} f}{f}$$

and the definition of the proximity function, we see

$$m_f(r, 0) \le m_{D^k f/f}(r, \infty) + m_{(D^k f - 1)/D^{k+1} f}(r, \infty) + m_{D^{k+1} f/f}(r, \infty) + O(1). \tag{3.1}$$

By the first main theorem, we have

$$(3.2)\qquad m_{(D^kf-1)/D^{k+1}f}(r,\infty) = m_{D^{k+1}f/(D^kf-1)}(r,0)$$
$$= m_{D^{k+1}f/(D^kf-1)}(r,\infty) + N_{D^{k+1}f/(D^kf-1)}(r,\infty)$$
$$- N_{D^{k+1}f/(D^kf-1)}(r,0) + O(1).$$

By Lemma 2.2, we know that D^kf and $D^{k+1}f$ are transcendental entire functions on $\boldsymbol{C}^n$, and hence $N_{D^kf}(r,\infty) = N_{D^{k+1}f}(r,\infty) = 0$. Then by Jensen's formula, we see

$$(3.3)\qquad N_{D^{k+1}f/(D^kf-1)}(r,0) - N_{D^{k+1}f/(D^kf-1)}(r,\infty)$$
$$= \int_{S_n(r)} \log\left|\frac{D^{k+1}f}{D^kf-1}(z)\right|\sigma_n(z) + O(1)$$
$$= \int_{S_n(r)} \log|D^{k+1}f(z)|\sigma_n(z) + \int_{S_n(r)} \log\left|\frac{1}{D^kf-1}(z)\right|\sigma_n(z) + O(1)$$
$$= N_{D^{k+1}f}(r,0) - N_{D^{k+1}f}(r,\infty) - N_{D^kf-1}(r,0) + N_{D^kf-1}(r,\infty) + O(1)$$
$$= N_{D^{k+1}f}(r,0) - N_{D^{k+1}f}(r,\infty) - N_{D^kf}(r,1) + N_{D^kf}(r,\infty) + O(1)$$
$$= N_{D^{k+1}f}(r,0) - N_{D^kf}(r,1) + O(1).$$

By (3.1), (3.2) and (3.3), we have

$$T_f(r) = m_f(r,0) + N_f(r,0) + O(1)$$
$$\le N_f(r,0) + N_{D^kf}(r,1) - N_{D^{k+1}f}(r,0)$$
$$+ m_{D^kf/f}(r,\infty) + m_{D^{k+1}f/f}(r,\infty) + m_{D^{k+1}f/(D^kf-1)}(r,\infty) + O(1).$$

Therefore, by Lemmas 2.2 and 2.3, we obtain the conclusion of the theorem 3.1. □

As usual, we use the notation $\overline{N}_f(r,a)$ for the counting function of the a-divisor of f which does not count multiplicities.

THEOREM 3.2. *Let f be a transcendental entire function on $\boldsymbol{C}^n$. Then*

$$\| \quad T_f(r) \le 2\overline{N}_f(r,0) + N_{Df}(r,1) + O(\log rT_f(r)).$$

Proof. If the zero multiplicity r of f at $z^0 = (z_1^0, z_2^0, \ldots, z_n^0)$ is at least three (see [1] for the definition of multiplicity of zero), then in a neighborhood of z^0, we can expand f as a convergent series of homogeneous polynomials in $z - z^0$:

$$f(z) = \sum_{m=r}^{\infty} P^m(z - z^0),$$

where r is a positive integer with $r \geq 3$. By the homogeneity of $P^m(z - z^0)$, we have

$$\sum_{j=1}^{n} (z_j - z_j^0) P_{z_j}^m (z - z^0) = m P^m (z - z^0), \quad m = r, r+1, \ldots.$$

Hence we see

$$\begin{aligned} Df(z) &= \sum_{j=1}^{n} z_j f_{z_j}(z) = \sum_{j=1}^{n} (z_j - z_j^0) f_{z_j}(z) + \sum_{j=1}^{n} z_j^0 f_{z_j}(z) \\ &= \sum_{m=r}^{\infty} m P^m (z - z^0) + \sum_{m=r}^{\infty} G^{m-1}(z - z^0) \\ &= \sum_{m=r-1}^{\infty} \tilde{P}^m (z - z^0), \end{aligned}$$

where $G^m(z - z^0)$ and $\tilde{P}^m(z - z^0)$ are either identically zero or a homogeneous polynomials in $z - z^0$ of degree m, respectively. By the same way we have

$$D^2 f(z) = D(Df)(z) = \sum_{m=r-2}^{\infty} \tilde{\tilde{P}}^m (z - z^0),$$

where $\tilde{\tilde{P}}^m(z - z^0)$ is either identically zero or a homogeneous polynomial in $z - z^0$ of degree m $(m = r-2, r-1, r, \ldots)$. Therefore, the zero multiplicity of $D^2 f$ at z^0 is at least $r - 2$.

Hence by the definition of the counting function, we have

$$N_f(r, 0) - N_{D^2 f}(r, 0) \leq 2\overline{N}_f(r, 0) + O(\log r).$$

Thus, by Theorem 3.1, we have

$$\begin{aligned} \| \quad T_f(r) &\leq N_f(r, 0) + N_{Df}(r, 1) - N_{D^2 f}(r, 0) + O(\log r T_f(r)) \\ &\leq 2\overline{N}_f(r, 0) + N_{Df}(r, 1) + O(\log r T_f(r)). \end{aligned}$$

This completes the proof. □

4. Proofs of Theorems

Proof of Theorem 1.1. First we prove that f is a polynomial. Assume the contrary. Then f is a transcendental entire function ([1] or [5]), and hence

$$F(z) = \frac{f(z) - a}{b}$$

is a transcendental entire function. By Theorem 3.1, we have

$$\| \quad T_F(r) \leq N_F(r, 0) + N_{D^k F}(r, 1) + O(\log r T_F(r)).$$

Since $D^kF = D^kf/b$, $T_f(r) = T_F(r) + O(1)$ and the assumptions, we deduce from above inequality that

$$\text{(4.1)} \qquad \| \quad T_f(r) \le N_f(r,a) + N_{D^kf}(r,b) + O(\log rT_f(r)) = O(\log rT_f(r)).$$

Now f is transcendental, we can get a contradiction by (4.1).

Therefore f is a polynomial ([1] or [5]). Since $f \neq a$, f must be constant. □

Proof of Theorem 1.2. First we prove that f is a polynomial. Assume the contrary. Then f is a transcendental entire function, and hence

$$F(z) = \frac{f^{k+1}(z)}{(k+1)b}$$

is also a transcendental entire function. Obviously, $DF(z) = f^k(z) \cdot Df(z)/b$, and the zero multiplicity at each point of 0-divisor of F is at least $k+1 \ge 3$. Hence

$$\overline{N}_F(r,0) \le \frac{1}{3}N_F(r,0) + O(\log r).$$

By the assumption we deduce that $DF(z) \neq 1$, and from Theorem 3.2 we have

$$\begin{aligned} \| \quad T_F(r) &\le 2\overline{N}_F(r,0) + N_{DF}(r,1) + O(\log rT_F(r)) \\ &\le \frac{2}{3}N_F(r,0) + O(\log rT_F(r)) \le \frac{2}{3}T_F(r) + O(\log rT_F(r)). \end{aligned}$$

Hence we see

$$\text{(4.2)} \qquad \| \quad \frac{1}{3}T_F(r) \le O(\log rT_F(r)).$$

Now F is transcendental, (4.2) gives a contradiction. Therefore f is a polynomial, so is $f^k \cdot Df$. Since $f^k \cdot Df \neq b$, $f^k \cdot Df$ must be constant. Since the degree of $f^k \cdot Df$ is not less than the degree of Df, then Df is constant. By Lemma 2.4, we conclude that f is constant. □

The following example shows that Theorems 1.1 and 1.2 are not valid if the total derivative is replaced by the partial derivative.

Example 4.1. Let $f(z_1, z_2) = e^{z_2}$. It is clear that $f \neq 0$. Since $f_{z_1}(z_1, z_2) \equiv 0$, then $f_{z_1} \neq 1$ and for any positive integer k, $f^k \cdot f_{z_1} \neq 1$. However f is not constant.

References

[1] H. Fujimoto, On families of meromorphic maps into the complex projective space, Nagoya Math. J., **54** (1974), 21–51.

[2] W. K. Hayman, Meromorphic Functions, Oxford Univ. Press, Oxford, 1964.
[3] H. Milloux, Extension d'un théorème de M. R. Nevanlinna et applications, Act. Scient. et Ind., 888, Hermann et Cie., Paris, 1940.
[4] M. Ru, Nevanlinna Theory and Its Relation to Diophantine Approximation, World Scientific Publishing Co., Singapore, 2001.
[5] W. Stoll, Die beiden Hauptsätze der Wertverteilungstheorie bei Funktionen mehrerer komplexer Veränderlichen (I), Acta Math., **90** (1953), 1–115; (II), Acta Math., **92** (1954), 55–169.
[6] L. Yang, Value Distribution Theory, Springer-Verlag, Berlin-Heidelberg, 1993.
[7] Z. Ye, On Nevanlinna's second main theorem in projective space, Invent. Math., **122** (1995), 475–507.

Department of Mathematics
Fudan University
Shanghai 200433
China
e-mail: jinluk@online.sh.cn