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ON THE EXISTENCE OF SPHERICALLY BENT SUBMANIFOLDS, AN ANALOGUE OF A THEOREM OF E. CARTAN

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Abstract

In this article an analogue of E. Cartan's theorem about the existence of (local) totally geodesic submanifolds with a prescribed tangent plane is proved, namely for the existence of spherically bent submanifolds (=extrinsic spheres); also a global version is deduced.

Introduction

In a riemannian manifold M the simplest submanifolds are the totally geodesic ones. An inhabitant of M will consider them as uncurved. The one-dimensional examples are the geodesics. However the examples of proper totally geodesic submanifolds of dimensions ≥ 2 may be very rare. Already E. Cartan has investigated this situation: Starting from a point $p \in M$, a linear subspace $U \subsetneqq T_pM$ of dimension ≥ 2 and some $\varepsilon > 0$ such that $U_\varepsilon(p)$ is a normal neighbourhood he obtained the following

Theorem. *If $S(U)$ denotes the unit sphere in U and $c_u :]-\varepsilon, \varepsilon[\to M$ the geodesic $t \mapsto \exp_p(tu)$ for every $u \in S(U)$, then the "geodesic ε-umbrella"*

$$N_\varepsilon(p, U) := \bigcup_{u \in S(U)} c_u(]-\varepsilon, \varepsilon[) \tag{1}$$

is a totally geodesic submanifold of M if and only if for every $u \in S(U)$ and every $t \in]-\varepsilon, \varepsilon[$ the parallel translate

$$U_u(t) := (\overset{t}{\underset{0}{\|}} c_u)(U) \subset T_{c_u(t)}M \ \text{of } U \text{ along } c_u, \quad \text{see (5)},$$

is curvature invariant, that means

$$\forall v, v', v'' \in U_u(t): \quad R(v, v')v'' \in U_u(t).$$

The most famous special case of this theorem is the well known relation between Lie triple systems and totally geodesic submanifolds in the theory of symmetric spaces.

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In this article we solve the analogous problem for *spherical submanifolds*, known also as *extrinsic spheres* from the article [NY] of Nomizu and Yano. In the euclidean space the open parts of linear subspaces and of ordinary spheres (of any dimension) are the spherical submanifolds. In an arbitrary riemannian space M the spherical submanifolds are those which are bent uniformally, that means, they are totally umbilical and have a parallel mean curvature normal (see Definition 1). If N is a spherical submanifold, $p \in N$ a fixed point and z the mean curvature normal of N at p, then every normal ε-neighbourhood of N about p is uniquely determined by the data $(p, U := T_pN, z)$, because the unit speed geodesics c of N through p are *circular arcs* of M, that means, they satisfy the third order differential equation

$$\nabla_\partial \nabla_\partial \dot{c} + \langle \nabla_\partial \dot{c}, \nabla_\partial \dot{c} \rangle \cdot \dot{c} = 0$$

(∂ denotes the canonical unit vector field of $\boldsymbol{R}$), and the solutions of this equation are uniquely determined by their initial values $c(0), \dot{c}(0)$ and $(\nabla_\partial \dot{c})(0)$ (see the Propositions 1(f), 2 and 3).

This observation has led us to construct "circular ε-umbrellas" $N_\varepsilon(p, U, z)$ for prescribed data (p, U, z) analogously to (1) (see Proposition 5); they are locally (the only possible) candidates for spherical submanifolds of M associated to (p, U, z). In fact, we will prove in section 6:

Theorem 1 (Main result). *Let be given a point $p \in M$, a non-trivial linear subspace $U \subsetneqq T_pM$ and a vector $z \in U^\perp \setminus \{0\} \subset T_pM$; furthermore, let $S(U)$ denote the unit sphere in U and $c_u : J_u \to M$ the maximal circular arc with the initial data $c_u(0) = p$, $\dot{c}_u(0) = u$ and $(\nabla_\partial \dot{c}_u)(0) = z$ for every $u \in S(U)$, choose an $\varepsilon > 0$ such that the circular ε-umbrella*

$$N_\varepsilon(p, U, z) := \bigcup_{u \in S(U)} c_u(]-\varepsilon, \varepsilon[) \tag{2}$$

is defined and put $V := U \oplus \boldsymbol{R}z$. For every $u \in S(U)$ and $t \in]-\varepsilon, \varepsilon[$ we define

$$\begin{aligned} &V_u(t) := (\overset{t}{\underset{0}{\|}} c_u)(V), \quad z_u(t) := (\nabla_\partial \dot{c}_u)(t) \in V_u(t) \quad \textit{and} \\ &U_u(t) := \{v \in V_u(t) \mid v \perp z_u(t)\}, \end{aligned} \tag{3}$$

and suppose for all $v, v', v'' \in U_u(t)$

$$R(v, v')v'' \in U_u(t) \quad \textit{and} \quad R(v, v')z_u(t) = 0. \tag{4}$$

Then $N_\varepsilon(p, U, z)$ is a spherical submanifold of M.

The proof of this theorem is inspired by a discovery of K. Tsukada (see [T]), namely that n-dimensional totally geodesic submanifolds are related to the integral manifolds of a canonical (horizontal) distribution defined on the Grassmann bundle $G_n(TM)$ over the riemannian manifold M. Already in [PR] we used this idea for a short proof of E. Cartan's theorem. Here in this article we present an analogue to TSUKADA's result by constructing such a horizontal distribution $\mathscr{D}$ on a submanifold $\hat{E}$ of the fibre product $G_{n+1}(TM) \times_M TM$ that the n-dimensional spherical (not totally geodesic) submanifolds are related to the integral manifolds of $\mathscr{D}$ (Theorem 2). Furthermore, we prove that $\mathscr{D}$ is involutive in $(U \oplus \boldsymbol{R}z, z) \in \hat{E}$ if and only if (U, z) satisfies condition (4) appropriately modified (see Proposition 9(e)); here the *curvature form of a horizontal structure* on $G_{n+1}(TM) \times_M TM$ is the essential tool. After that the proof of Theorem 1 is similar to our short proof of CARTAN's theorem: Over the circular ε-umbrella $N_\varepsilon(p, U, z)$ we construct a submanifold $S_\varepsilon(p, U, z) \subset \hat{E}$ built up by horizontal lifts of the circular arcs c_u (used in Theorem 1). As these lifts are tangential to $\mathscr{D}$, we call $S_\varepsilon(p, U, z)$ a *$\mathscr{D}$-umbrella* (see Definition 3). Since condition (4) implies the involutivity of $\mathscr{D}$ at all points of $S_\varepsilon(p, U, z)$, this $\mathscr{D}$-umbrella is an integral manifold of $\mathscr{D}$ according to Theorem 3 (a theorem usefull also in different situations), and therefore $N_\varepsilon(p, U, z)$ is a spherical submanifold of M.

Of course, maximal connected integral manifolds of $\mathscr{D}$ give rise to maximally extended spherical manifolds of M, possibly with selfintersections; they are described correctly by maximally extended spherical immersions $f : N \to M$. Theorem 4 shows that they are *geodesically closed* under suitable hypotheses (see Definition 4); in particular, if in this situation M is complete, then N is complete, too.

1. Notations and general basic facts

At first we remark that all manifolds, maps etc. are assumed to be C^∞ differentiable if not otherwise stated.

In this article M always denotes a connected riemannian manifold of dimension m; $\pi_M : TM \to M$, $\langle \cdot , \cdot \rangle$, ∇ and R denote its tangent bundle, riemannian metric, Levi-Civita connection and curvature tensor, respectively. For all other (riemannian) manifolds the analogous geometric objects will be marked by an appropriate index. Furthermore, for any curve $\alpha : J \to M$ and any $t_1, t_2 \in J$ let

$$(5) \qquad \overset{t_2}{\underset{t_1}{\Vert}} \alpha : T_{\alpha(t_1)}M \to T_{\alpha(t_2)}M$$

denote the parallel displacement in M along α. In this article an essential geometric object will be the connection map $K : TTM \to TM$ of M (see [D], [P], [L] p. 284). It is a vector bundle morphism along the projection π_M; in particular, the following diagram

$$\begin{array}{ccc} TTM & \xrightarrow{K} & TM \\ \Big\downarrow{\scriptstyle \pi_{TM}} & & \Big\downarrow{\scriptstyle \pi_M} \\ TM & \xrightarrow[\pi_M]{} & M \end{array}$$

commutes. K is defined by[1]

$$Kw := \nabla_w \mathrm{id}_{TM}, \tag{6}$$

where id_{TM} is considered as a vector field of M along the projection π_M. Consequently, for every curve $\xi : J \to TM$ we have

$$K \circ \dot{\xi} = \nabla_\partial \xi; \tag{7}$$

for ∂ and $T_v\pi_M$ see the end of this section. If for $p \in M$ and $v \in T_pM$ the vertical subspace $\ker(T_v\pi_M) = T_v(T_pM)$ is denoted by $\mathscr{V}_v(\pi_M)$, we obtain that the restriction

(8) $K \mid \mathscr{V}_v(\pi_M)$ coincides with the canonical isomorphism $T_v(T_pM) \to T_pM$.

Furthermore, the restriction

(9) $(T\pi_M, K) \mid T_vTM : T_vTM \to T_pM \oplus T_pM$ is an isomorphism.

Therefore one obtains: If $g : L \to TM$ is some differentiable map and $Y_1, Y_2 : L \to TM$ are two vector fields along $\pi_M \circ g$, then there exists one and only one vector field $X : L \to TTM$ (of TM) along g such that

$$(T\pi_M) \circ X = Y_1 \quad \text{and} \quad K \circ X = Y_2. \tag{10}$$

For any differentiable map $f : N \to M$ its *differential* is denoted by f_* or Tf and its restriction to T_pN sometimes by T_pf. A submanifold N of M is said to be *regular*, if its topology is induced by the topology of M. At last, by ∂ we denote the canonical unit vector field of $\boldsymbol{R}$; thus, $\nabla_\partial \xi$ denotes the covariant derivative of a vector field $\xi : J \to TM$ with respect to $t \in J$.

2. Basic properties of spherical maps and submanifolds

DEFINITION 1.

(a) An isometric immersion $f : N \to M$ is said to be *spherical* if it is totally umbilical, i.e., there exists a normal vector field H along f such that the second fundamental form h of f is given by

$$h(X, Y) = \langle X, Y \rangle \cdot H \quad \text{for all } X, Y \in \mathfrak{X}(N), \tag{11}$$

and if this field H is parallel in the normal bundle of f, i.e.,

$$\nabla_v H \in f_* T_pN \text{ for every } v \in T_pN \quad (p \in N). \tag{12}$$

H is the so called *mean curvature normal* of f. We put

$$\varkappa(f) := \langle H, H \rangle.$$

(b) A 1-dimensional isometric immersion, i.e., a unit speed curve $c : J \to M$ from an open interval, is called a *circular arc* if it is spherical.

[1] For the covariant differentiation of vector fields along maps see [P] p. 36.

(c) A submanifold N of M is said to be spherical, if its inclusion map $N \hookrightarrow M$ is spherical.

Remark 1. (a) Spherical submanifolds (with non-vanishing mean curvature normal) were introduced by Nomizu and Yano in [NY] under the name *extrinsic spheres*. In fact, they were treated already before, e.g., in [LN] and [No].

(b) Every totally geodesic map (resp. submanifold) is also a spherical map (resp. submanifold), namely with $H \equiv 0$.

(c) If M is a space of constant curvature, then condition (4) of Theorem 1 is satisfied automatically. Thus, every initial data (p, U, z) are induced by a spherical submanifold of M. In the euclidean space the linear subspaces and ordinary spheres of any dimension are the complete spherical submanifolds. In the euclidean sphere S^m the complete spherical submanifolds are the intersections of S^m with affine subspaces of $\boldsymbol{R}^{m+1}$. If we describe the hyperbolic space $\boldsymbol{H}^m$ by the hyperboloid model in the Lorentzian vector space $\boldsymbol{R}_1^{m+1}$ (see [O'N] p. 111), then the complete spherical submanifolds are obtained as the analogous intersections.

(d) If M is a symmetric space and $\dim U \geq 2$, then the circular ε-umbrella $N_\varepsilon(p, U, z)$ of Theorem 1 is a spherical submanifold if and only if there exists a $c \in \boldsymbol{R}$ such that for all vectors $v, v', v'' \in V := U \oplus \boldsymbol{R}z$ we have

$$R(v, v')v'' = c \cdot (\langle v', v'' \rangle \cdot v - \langle v, v'' \rangle \cdot v'). \tag{13}$$

Proof. If condition (13) is satisfied, then V and every parallel translate of V is curvature invariant (because of $\nabla R = 0$). According to the theorem of E. Cartan (mentioned in the introduction) there exists a totally geodesic submanifold $\tilde{N}$ of M with $p \in \tilde{N}$ and $T_p\tilde{N} = V$; moreover, $\tilde{N}$ is a space of constant curvature c. Applying the last remark (c) we see that there exists a spherical submanifold N of $\tilde{N}$ adapted to the initial data (p, U, z) and which therefore contains $N_\varepsilon(p, U, z)$ as open part. — Conversely, according to a result of B. Y. Chen every spherical, not totally geodesic submanifold N of M with $\dim N \geq 2$ is a hypersurface of a totally geodesic submanifold $\tilde{N}$ of M, which has constant curvature (see [Ch]). Therefore condition (13) is satisfied. □

Proposition 1. *Every spherical immersion $f : N \to M$ with mean curvature normal H and shape operator A has the following properties for all $X, Y, Z \in \mathfrak{X}(N)$:*

(a) $A_H X = \varkappa(f) \cdot X$

(b) $\nabla_X H = -\varkappa(f) \cdot f_* X$

(c) $R(f_*X, f_*Y)f_*Z = f_*(R^N(X, Y)Z - \varkappa(f) \cdot (\langle Y, Z \rangle X - \langle X, Z \rangle Y))$

(d) $R(f_*X, f_*Y)H = 0$

(e) *$f_*TN \oplus \boldsymbol{R}H$ is a parallel subbundle of TM along f, that means: if $\alpha : J \to N$ is any curve, then we have*

$$\Big(\overset{t_2}{\underset{t_1}{\|}} f \circ \alpha\Big)(f_* T_{\alpha(t_1)} N \oplus \boldsymbol{R}H_{\alpha(t_1)}) = (f_* T_{\alpha(t_2)} N \oplus \boldsymbol{R}H_{\alpha(t_2)}).$$

(f) *The image* $c := f \circ \alpha$ *of any unit speed geodesic* $\alpha : J \to N$ *in* N *is a circular arc in* M *with mean curvature normal* $\nabla_\partial \dot{c} = H \circ \alpha$; *hence* $\varkappa(c) = \varkappa(f)$ (*see Definition* 1(b)).

Proof. (a) is deduced from (11) by use of the relation between h and A. From (a) and (12) one obtains (b) immediately with the Weingarten identity. (c) and (d) are obtained by repeating the proof for the curvature equations of Gauss and Codazzi. For (e) and (f) the formulas (11) and (12) are used again. □

3. Circular arcs and circular ε-umbrellas

At first we quote a result from Leung and Nomizu, see §1 in [LN].

Proposition 2. *A unit speed curve* $c : J \to M$ *is a circular arc if and only if it satisfies the differential equation*

$$\nabla_\partial \nabla_\partial \dot{c} + \langle \nabla_\partial \dot{c}, \nabla_\partial \dot{c} \rangle \cdot \dot{c} = 0.$$

The term $\nabla_\partial \dot{c}$ *is the mean curvature normal of* c, *and hence we have* $\langle \nabla_\partial \dot{c}, \nabla_\partial \dot{c} \rangle \equiv \varkappa(c)$ *for circular arcs.*

In order to make full use of Proposition 2 we consider differential equations of this type more generally.

Proposition 3. *Let a vector field* $Y : TM \times_M TM \to TM$ *along the canonical projection* $TM \times_M TM \to M$ *be given. Then the following holds:*[2]

(a) *For every* $(v, a) \in TM \times_M TM$ *there exists a solution* $\alpha = \alpha_{(v,a)} : J_{(v,a)} \to M$ *of the differential equation*

$$\nabla_\partial \nabla_\partial \dot{\alpha} = Y(\dot{\alpha}, \nabla_\partial \dot{\alpha}) \tag{14}$$

defined on an open interval $J_{(v,a)}$, *satisfying the initial conditions* $0 \in J_{(v,a)}$, $\dot{\alpha}_{(v,a)}(0) = v$ *and* $(\nabla_\partial \dot{\alpha}_{(v,a)})(0) = a$, *and which is unique and maximal in the following sense: If* $\tilde{\alpha} : \tilde{J} \to M$ *is another solution of* (14) *with the same initial data, then* $\tilde{J} \subset J_{(v,a)}$ *and* $\tilde{\alpha} = \alpha_{(v,a)} | \tilde{J}$.

(b) *If* $\alpha : J \to M$ *is a solution of* (14), *then also* $\beta : s + J \to M$, $t \mapsto \alpha(t - s)$ *is a solution of* (14).

(c) *If in the situation* (a) $\delta := \sup J_{(v,a)} < \infty$ (*resp.* $\delta := \inf J_{(v,a)} > -\infty$), *then for every compact subset* $C \subset TM \times_M TM$ *there exists a parameter* $t_0 \in J_{(v,a)}$ *such that*

$$(\dot{\alpha}_{(v,a)}, \nabla_\partial \dot{\alpha}_{(v,a)})(]t_0, \delta[) \cap C = \emptyset \quad (\textit{resp.} \quad (\dot{\alpha}_{(v,a)}, \nabla_\partial \dot{\alpha}_{(v,a)})(]\delta, t_0[) \cap C = \emptyset).$$

[2] The results keep valid, if M is only equipped with a linear connection ∇, and also (with obvious modifications) if Y is only defined on an open subset $G \subset TM \times_M TM$.

(d) *The subset*

$$D^Y := \{(t, v, a) \in \boldsymbol{R} \times (TM \times_M TM) \mid t \in J_{(v,a)}\}$$

is an open neighbourhood of the set $\{0\} \times (TM \times_M TM)$ *in* $\boldsymbol{R} \times (TM \times_M TM)$, *and the map*

$$\Phi^Y : D^Y \to M, \quad (t, v, a) \mapsto \alpha_{(v,a)}(t)$$

is differentiable.

Proof. Let $P_i : TM \times_M TM \to TM$ denote the canonical projection $(v_1, v_2) \mapsto v_i$ for $i = 1, 2$. Then there exists one and only one vector field $X \in \mathfrak{X}(TM \times_M TM)$ such that its components $X_i := P_{i*}X : TM \times_M TM \to TTM$ satisfy

$$\pi_{M*}X_1 = \pi_{M*}X_2 = P_1, \quad KX_1 = P_2 \quad \text{and} \quad KX_2 = Y;$$

see (10). Using these formulas one easily checks:

- If α is a solution of the differential equation (14), then $(\dot{\alpha}, \nabla_\partial \dot{\alpha})$ is an integral curve of X.
- If ξ is an integral curve of X, then $\alpha := \pi_M \circ P_1 \circ \xi = \pi_M \circ P_2 \circ \xi$ is a solution of the differential equation (14).

Therefore, the theory of integral curves of vector fields implies all assertions (e.g., see [W]). □

Let us apply Proposition 3 to get some information on circular arcs. For that we fix a value $\varkappa \in \boldsymbol{R}$ and introduce the vector field[3]

$$Y_\varkappa : TM \times_M TM \to TM, \quad (v, a) \mapsto -\varkappa \cdot \langle v, v \rangle \cdot v,$$

which via (14) is associated to the differential equation

$$\nabla_\partial \nabla_\partial \dot{\alpha} + \varkappa \cdot \langle \dot{\alpha}, \dot{\alpha} \rangle \cdot \dot{\alpha} = 0. \tag{15}$$

Proposition 4.

(a) *Let* $c : J \to M$ *be a curve with the initial data* $0 \in J$, $u := \dot{c}(0)$, $a := (\nabla_\partial \dot{c})(0)$, $\|u\| = 1$, $u \perp a$ *and* $\langle a, a \rangle = \varkappa$. *Then c is a circular arc if and only if it is a solution of the differential equation* (15).

(b) *If* $c : J \to M$ *is a maximal circular arc* (*see the following Remark*) *and* $\delta := \sup J < \infty$ (*resp.* $\delta := \inf J > -\infty$), *then* $c(t)$ *tends to the boundary of M for* $t \to \delta$, *that means*: *for every compact subset* $C \subset M$ *there exists a parameter* $t_0 \in J$ *such that*

$$c(]t_0, \delta[) \cap C = \emptyset \quad (\textit{resp.} \quad c(]\delta, t_0[) \cap C = \emptyset).$$

Therefore, every maximal circular arc is defined on the entire real line $\boldsymbol{R}$, *if M is complete.*

[3] The vector field $Y : (v, a) \mapsto -\langle a, a \rangle \cdot v$ would represent the differential equation of circular arcs (see Proposition 2) exactly; but with respect to the proof for Proposition 5 the choice of $Y_\varkappa$ is more convenient.

Remark 2. The assertion (a) and Proposition 3(a) show that for any initial data $(u,a) \in TM \times_M TM$ with $\|u\| = 1$ and $u \perp a$ there exists a circular arc $c : J \to M$ satisfying $0 \in J$, $\dot{c}(0) = u$ and $(\nabla_\partial \dot{c})(0) = a$, which is unique and maximal in the sense of Proposition 3(a). So we know what is meant by *maximal circular arcs*.

Proof. The "only if part" of (a) is clear. Because of Proposition 2, for the "if part" we have to prove only that $c := \alpha_{(u,a)}$ is a unit speed curve with $\mu := \langle \nabla_\partial \dot{c}, \nabla_\partial \dot{c} \rangle = \varkappa$ under the prescribed initial assumptions. For that we put $\lambda := \langle \dot{c}, \dot{c} \rangle$ in addition and calculate that λ and μ satisfy the differential equations

$$\lambda'' = 2(\mu - \varkappa\lambda^2) \quad \text{and} \quad \mu' = -\varkappa\lambda\lambda' \tag{16}$$

and the initial conditions $\lambda(0) = 1$, $\lambda'(0) = 0$ and $\mu(0) = \varkappa$; since $(\lambda \equiv 1,\ \mu \equiv \varkappa)$ also solves (16), we get $\langle \dot{c}, \dot{c} \rangle \equiv 1$ and $\langle \nabla_\partial \dot{c}, \nabla_\partial \dot{c} \rangle = \varkappa$.

For (b): Let be given a maximal circular arc $c : J \to M$ and a compact subset C of M and assume $\delta := \sup J < \infty$. We may assume $0 \in J$. We put $u := \dot{c}(0)$, $a := (\nabla_\partial \dot{c})(0)$ and $\varkappa := \langle a, a \rangle$. From (a) we obtain that c is the maximal solution $\alpha_{(u,a)}$ of (15). As the curve $\xi := (\dot{c}, \nabla_\partial \dot{c})$ runs in the subset $S := \{(v,b) \in TM \times_M TM \mid \|v\| = 1 \text{ and } \|b\|^2 = \varkappa\}$ and $\tilde{C} := \{(v,b) \in S \mid \pi_M(v) \in C\}$ is a compact subset of $TM \times_M TM$, according to Proposition 3(c) there exists a parameter $t_0 \in J$ such that $\xi(]t_0, \delta[) \cap \tilde{C} = \emptyset$. But this situation can only occur, if $c(]t_0, \delta[) \cap C = \emptyset$. — If in the foregoing situation M would be complete, then $C := \{p \in M \mid d(p, c(0)) \leq \delta\}$ would be compact (according to the theorem of Hopf-Rinow) and $c([0, \delta[)$ could not leave C in contradiction to the last statement; hence $\delta < \infty$ is impossible for complete M. — The case $\delta := \inf J > -\infty$ is proved analogously. □

Proposition 5. *Let be given a point $p \in M$, a non-trivial linear subspace $U \subsetneqq T_pM$ and a vector $z \in U^\perp \subset T_pM$; furthermore, for every unit vector $u \in U$ let $c_u : J_u \to M$ denote the maximal circular arc with the initial data $c_u(0) = p$, $\dot{c}_u(0) = u$ and $(\nabla_\partial \dot{c}_u)(0) = z$. Then there exists an $\varepsilon > 0$ such that each of these circular arcs is defined at least on the interval $]-\varepsilon, \varepsilon[$ and*

$$N_\varepsilon(p, U, z) := \bigcup_{u \in U,\, \|u\| = 1} c_u(]-\varepsilon, \varepsilon[)$$

is a regular submanifold of M (see section 1*), which we will call the* circular ε-umbrella associated to the data (p, U, z).

Proof. We will mimic the construction of the exponential map of a riemannian manifold at a point, where the geodesics are replaced by circular arcs. For that we apply Proposition 3 to the differential equation (15) with $\varkappa = \langle z, z \rangle$, it means to the vector field $Y := Y_\varkappa$ and the associated differentiable map $\Phi^Y : D^Y \to M$. Since for every unit vector $u \in U$ we have $\varkappa(c_u) = \langle z, z \rangle = \varkappa$, Proposition 4(a) implies (with the notations of Proposition 3)

$$J_u = J_{(u,z)} \quad \text{and} \quad c_u = \alpha_{(u,z)}.$$

Therefore, we may abbreviate $J_u := J_{(u,\langle u,u\rangle\cdot z)}$ and $c_u := \alpha_{(u,\langle u,u\rangle\cdot z)}$ for arbitrary vectors $u \in U$. Obviously we have $J_0 = \boldsymbol{R}$ and $c_0 \equiv p$. Furthermore, for every $u \in U$ and every $s \in \boldsymbol{R}\backslash\{0\}$ the curve $t \mapsto c_u(st)$ $\left(t \in \frac{1}{s}\cdot J_u\right)$ is seen to be a solution of the differential equation (15), too; hence we obtain

$$J_{su} = \frac{1}{s}\cdot J_u \quad \text{and} \quad c_{su} : t \mapsto c_u(st). \tag{17}$$

Therefore,

$$D := \{u \in U \mid (1, u, \langle u,u\rangle \cdot z) \in D^Y\}$$

is a star shaped open neighbourhood of 0 in U,

$$\Phi : D \to M, \quad u \mapsto \Phi^Y(1, u, \langle u,u\rangle \cdot z) = c_u(1)$$

is a differentiable map with the special value $\Phi(0) = p$, and

$$\Phi(tu) = c_u(t) \quad \text{for every vector } u \in U \text{ and every } t \in J_u. \tag{18}$$

Hence the differential $T_0\Phi : T_0U \to T_pM$ is the inclusion map $U \hookrightarrow T_pM$, if T_0U is canonically identified with U. Therefore, there exists an $\varepsilon > 0$ such that the ε-neighbourhood $U_\varepsilon(0)$ of the euclidean space $(U, \langle\cdot,\cdot\rangle_p)$ is imbedded into M by Φ. Because of (17) and (18) we have $]-\varepsilon,\varepsilon[\subset J_u$ for every unit vector $u \in U$ and $\Phi(U_\varepsilon(0)) = N_\varepsilon(p, U, z)$. Thus the proof is complete. □

Remark 3. (a) In case $z = 0$ the map Φ is the restriction of the exponential map $\exp_p$ to $D \subset U$.

(b) Although we have $\Phi(u) = c_{u/\|u\|}(\|u\|)$ for $u \in U_\varepsilon(0)\backslash\{0\}$, where $c_{u/\|u\|}$ is a simple circle arc, we can not use this formula for a definition of Φ; it is the purpose of the above construction to remove the "singularity" at $u = 0$. Quite a similar situation will occur in section 6 when we prove Theorem 1.

(c) In [NY] the theory of circular arcs is based on the development of curves. But this method would not simplify the proof of Proposition 5.

4. Horizontal structures

Let $\pi : E \to M$ be an arbitrary fibre bundle[4] and $\mathscr{V} := \ker(T\pi)$ its vertical subbundle. If E_p denotes the fibre of π over $p \in M$, then $\mathscr{V}_e = T_eE_p$ for every $e \in E_p$.

Definition 2. By a *horizontal structure* of π we understand a subbundle $\mathscr{H} \subset TE$, which is complementary to $\mathscr{V}$, i.e., $TE = \mathscr{V} \oplus \mathscr{H}$, and by its *curvature form* the tensor field Ω of type $(1,2)$ on E characterized by the equation

[4] The theory of horizontal structures can be developed for arbitrary surjective submersions as well.

$$\forall X, Y \in \mathfrak{X}(E): \quad \Omega(X, Y) = -[X_{\mathscr{H}}, Y_{\mathscr{H}}]_{\mathscr{V}};$$

the indizes $\mathscr{H}$ and $\mathscr{V}$ mean that one has to regard the horizontal resp. vertical part of the respective vector field. Furthermore, a C^∞ map $g : L \to E$ is said to be *horizontal* if it satisfies

$$g_* T_q L \subset \mathscr{H}_{g(q)} \quad \text{for every } q \in L.$$

Remark 4. The restriction $(T_e\pi) \mid \mathscr{H}_e : \mathscr{H}_e \to T_pM$ is an isomorphism for every $e \in E_p$, and for every $X, Y \in \Gamma(\mathscr{H})$ we have

$$[X, Y](e) \in \mathscr{H}_e \Leftrightarrow \Omega(X(e), Y(e)) = 0. \tag{19}$$

Differential geometers are familiar with special horizontal structures: In the article [E] C. Ehresmann introduced *connections*, which are horizontal structures with a further property (see Remark 5); the best known examples are the connections on principal fibre bundles and the induced connections on bundles which are associated to them (see [KN]). Other examples appear in the theory of riemannian submersions (see [O'N]). With the methods of the latter book one can prove:

Proposition 6. *If $\mathscr{H}$ is a horizontal structure of π, $g : L \to E$ a C^∞ map and $X : L \to TM$ a vector field along $\pi \circ g$, then there exists one and only one vector field $\tilde{X} : L \to TE$ along g such that*

$$\pi_* \tilde{X} = X \quad \textit{and} \quad \tilde{X}_q \in \mathscr{H}_{g(q)} \quad \textit{for every } q \in L.$$

$\tilde{X}$ is called the horizontal lift *of X.*

For $X \in \mathfrak{X}(M)$ the horizontal lift of $X \circ \pi$ is simply called the horizontal lift of X.

Proposition 7. *For any horizontal structure $\mathscr{H}$ of π the following is true:*

(a) *For every curve $\alpha : J \to M$ defined on an open interval J and every initial data $(s, e) \in J \times E$ with $\pi(e) = \alpha(s)$ there exists a horizontal curve $\tilde{\alpha} : \tilde{J} \to E$ defined on an open interval $\tilde{J}$ satisfying*

$$s \in \tilde{J} \subset J, \quad \tilde{\alpha}(s) = e \quad \textit{and} \quad \pi \circ \tilde{\alpha} = \alpha|\tilde{J},$$

which is unique and maximal in the sense of Proposition 3(a). *The curve $\tilde{\alpha}$ will be called the* maximal horizontal lift *of α with the initial data (s, e).*

(b) *Let L be a further manifold, B an open neighbourhood of $\{0\} \times L$ in $\boldsymbol{R} \times L$ and $F : B \to M$ resp. $g : L \to E$ C^∞ maps satisfying $\pi \circ g(p) = F(0, p)$ for all $p \in L$. We assume that $J_p := \{t \in \boldsymbol{R} \mid (t, p) \in B\}$ is an interval for every $p \in L$. If $\tilde{\alpha}_p : \tilde{J}_p \to E$ denotes the maximal horizontal lift of the curve $\alpha_p : J_p \to M$, $t \to F(t, p)$ satisfying $\tilde{\alpha}_p(0) = g(p)$, then $\tilde{B} := \{(t, p) \in B \mid t \in \tilde{J}_p\}$ is an open neighbourhood of $\{0\} \times L$ in B and the map $\tilde{F} : \tilde{B} \to E$, $(t, p) \mapsto \tilde{\alpha}_p(t)$ is C^∞ differentiable.*

Remark 5. A horizontal structure $\mathscr{H}$ is a connection in the sense of EHRESMANN, if in the situation of Proposition 7(a) we always have $\tilde{J} = J$.

Proof. Accepting assertion (a) we first prove (b). For that we pull back the fibre bundle π by means of the map F and get the fibre bundle $\tilde{E} \to B$ shown in the diagram

$$\begin{array}{ccc} \tilde{E} := B \times_M E & \xrightarrow{\mathrm{pr}_E} & E \\ {\scriptstyle \mathrm{pr}_B}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{F} & M. \end{array}$$

Analogously to the pullback of connections (see [P] p. 57) we can define the pullback $\tilde{\mathscr{H}}$ of the horizontal structure $\mathscr{H}$; it is characterized by

$$\tilde{\mathscr{H}}_{(q,e)} = \{w \in T_{(q,e)}\tilde{E} \mid \mathrm{pr}_{E*} w \in \mathscr{H}_e\} \quad \text{for all } (q,e) \in \tilde{E}.$$

Now let $X \in \mathfrak{X}(\tilde{E})$ be the horizontal lift of the canonical vector field $\partial/\partial t \in \mathfrak{X}(B)$ (notice $B \subset \boldsymbol{R} \times L$) and $\Phi : D \to \tilde{E}$ its maximal flow. If for $p \in L$ we put $\tilde{p} := ((0,p), g(p)) \in \tilde{E}$ and denote the maximal integral curve of X starting at $\tilde{p}$ by $\xi_{\tilde{p}}$, then $\mathrm{pr}_E \circ \xi_{\tilde{p}}$ is the maximal horizontal lift $\tilde{\alpha}_p$ of the assertion (b). Therefore, we get $\tilde{B} = \{(t,p) \in B \mid (t,\tilde{p}) \in D\}$ and $\tilde{F} : (t,p) \mapsto \mathrm{pr}_E \circ \Phi(t,\tilde{p})$. Hence $\tilde{B}$ and $\tilde{F}$ have the stated properties.

Assertion (a) is proved with the same methods; work with the diagram

$$\begin{array}{ccc} \tilde{E} := J \times_M E & \xrightarrow{\mathrm{pr}_E} & E \\ {\scriptstyle \mathrm{pr}_J}\downarrow & & \downarrow{\scriptstyle \pi} \\ J & \xrightarrow{\alpha} & M \end{array}$$

and use the horizontal lift X of $\partial \in \mathfrak{X}(J)$ (compare [P] p. 59). □

PROPOSITION 8. *Let be given a second fibre bundle $\tilde{\pi} : \tilde{E} \to M$ with a horizontal structure $\tilde{\mathscr{H}}$ and a fibre bundle morphism $F : E \to \tilde{E}$ and assume $F_* \mathscr{H}_e \subset \tilde{\mathscr{H}}_{F(e)}$ for every $e \in E$. Then the curvature forms Ω and $\tilde{\Omega}$ of $\mathscr{H}$ resp. $\tilde{\mathscr{H}}$ are related to each other by*

$$F_* \Omega(X,Y) = \tilde{\Omega}(F_* X, F_* Y) \quad \textit{for all } X, Y \in \mathfrak{X}(E).$$

For the proof one uses similar arguments as for the proof of the structure equation for the curvature tensor (e.g., see [P] p. 83).

We give now three examples of horizontal structures, which will become important for our investigation.

Example 1. **A horizontal structure of the tangent bundle related to circular arcs.** We modify the construction of the canonical connection of the tangent bundle over the riemannian manifold M (see [P] p. 77): If K denotes the connection map of M (see (6)–(10)), then the kernels

$$\mathscr{H}_z^c(\pi_M) := \ker(T_zTM \to TM, w \mapsto Kw + \langle z, z\rangle \cdot \pi_{M*}w) \quad \text{for } z \in TM$$

define a horizontal structure $\mathscr{H}^c(\pi_M)$ of the tangent bundle π_M, as is easily seen. If $\alpha : J \to M$ is a curve with $0 \in J$ and $z \in T_{\alpha(0)}M$, then

(20) z_α denotes the maximal $\mathscr{H}^c(\pi_M)$-horizontal lift of α with $z_\alpha(0) = z$;

according to (7) it is the maximal solution ζ of the differential equation

$$\nabla_\partial \zeta + \langle \zeta, \zeta\rangle \cdot \dot\alpha = 0 \quad \text{with } \zeta(0) = z. \tag{21}$$

Therefore, Proposition 2 shows that for a circular arc $c : J \to M$ with $0 \in J$ and $z := (\nabla_\partial \dot c)(0)$ its mean curvature normal $\nabla_\partial \dot c$ is the maximal $\mathscr{H}^c(\pi_M)$-horizontal lift z_c; hence in this case z_c is defined over the entire interval J. But nevertheless, $\mathscr{H}^c(\pi_M)$ is no connection in the sense of Ehresmann (see Remark 5); in particular, it can not be induced by a covariant derivative of M. Now we show that the curvature form Ω^c_{TM} of $\mathscr{H}^c(\pi_M)$ is given by

$$\begin{aligned} &K\Omega^c_{TM}(w, w') \\ &\quad = R(\pi_{M*}w, \pi_{M*}w')z + 2\langle z, z\rangle \cdot (\langle \pi_{M*}w', z\rangle \pi_{M*}w - \langle \pi_{M*}w, z\rangle \pi_{M*}w') \end{aligned} \tag{22}$$

for all $w, w' \in T_z(TM)$ $(z \in TM)$; notice (9) and that $\pi_{M*}\Omega^c_{TM} \equiv 0$ is satisfied automatically. For the proof of (22) let two vector fields $X, Y \in \mathfrak{X}(M)$ be given, denote their horizontal lifts by $\tilde X$ and $\tilde Y$ and abbreviate $\iota := \mathrm{id}_{TM}$, $\mathscr{H} := \mathscr{H}^c(\pi_M)$ and $\pi := \pi_M$. Then $[\tilde X, \tilde Y]_{\mathscr{H}}$ is the horizontal lift of $[X, Y]$ because of $\pi_*[\tilde X, \tilde Y] = [X, Y] \circ \pi$. Now we calculate

$$\nabla_{\tilde X}\iota = K\tilde X = -\langle \iota, \iota\rangle \cdot X \circ \pi$$

and

$$\nabla_{[\tilde X, \tilde Y]}\iota = K([\tilde X, \tilde Y]_{\mathscr{V}} + [\tilde X, \tilde Y]_{\mathscr{H}}) = -K\Omega^c_{TM}(\tilde X, \tilde Y) - \langle \iota, \iota\rangle \cdot [X, Y] \circ \pi.$$

With this expressions we continue using the structure equation for the curvature tensor (e.g., see [P] p. 83):

$$\begin{aligned} R(\pi_*\tilde X, \pi_*\tilde Y)\iota &= \nabla_{\tilde X}\nabla_{\tilde Y}\iota - \nabla_{\tilde Y}\nabla_{\tilde X}\iota - \nabla_{[\tilde X, \tilde Y]}\iota \\ &= 2\langle \iota, \iota\rangle \cdot (\langle X \circ \pi, \iota\rangle \cdot Y \circ \pi - \langle Y \circ \pi, \iota\rangle \cdot X \circ \pi) + K\Omega^c_{TM}(\tilde X, \tilde Y). \end{aligned}$$

From this formula (22) can easily be derived.

Example 2. **The canonical connection of a Grassmann bundle over a riemannian manifold.** For each point $p \in M$ let L_pM be the set of frames of T_pM, which we describe by isomorphisms $u : \boldsymbol{R}^m \to T_pM$ as in [KN] p. 56 (remember $m = \dim M$); by $\pi : \mathrm{L}M \to M$ we denote the entire frame bundle.

For some fixed number $r \in \{1, \dots, m\}$ let $\gamma_r : \mathrm{G}_r(TM) \to M$ denote the Grassmann bundle; its fibre over p is the Grassmann manifold $\mathrm{G}_r(T_pM)$ of the

r-dimensional subspaces $V \subset T_pM$. This bundle is associated to the frame bundle via the map

$$\varrho : \mathrm{L}M \times \mathrm{G}_r(\boldsymbol{R}^m) \to \mathrm{G}_r(TM), \quad (u, V) \mapsto u(V);$$

see [B] sect. 6.5.1. For each $u \in \mathrm{L}_pM$ the map $\varrho_u : \mathrm{G}_r(\boldsymbol{R}^m) \to \mathrm{G}_r(T_pM)$, $V \mapsto u(V)$ is a diffeomorphism; and if $\boldsymbol{V} \in \mathrm{G}_r(\boldsymbol{R}^m)$ denotes the subspace which is spanned by the first canonical unit vectors $e_1, \ldots, e_r \in \boldsymbol{R}^m$, then the fibre bundle morphism

$$\varrho^{\boldsymbol{V}} : \mathrm{L}M \to \mathrm{G}_r(TM), \quad u \mapsto u(\boldsymbol{V}) = \mathrm{span}\{u(e_1), \ldots, u(e_r)\} \tag{23}$$

is a surjective submersion (even a principal fibre bundle). The linear connection $\mathscr{H}(\mathrm{L}M)$ of $\mathrm{L}M$ corresponding to ∇ induces a connection $\mathscr{H}(\gamma_r)$ (in the sense of Ehresmann, see Remark 5) on the Grassmann bundle (see [KN] p. 87); it is given by

$$\mathscr{H}_{\varrho(u,\boldsymbol{V})}(\gamma_r) = \varrho_*^{\boldsymbol{V}} \mathscr{H}_u(\mathrm{L}M) \subset T_{\varrho(u,\boldsymbol{V})}(\mathrm{G}_r(TM)) \quad (u \in \mathrm{L}M). \tag{24}$$

If $\alpha : J \to M$ is a (broken) curve with $0 \in J$ and $V \in \mathrm{G}_r(T_{\alpha(0)}M)$, then

(25) V_α denotes the maximal $\mathscr{H}(\gamma_r)$-horizontal lift of α with $V_\alpha(0) = V$;

it is exactly the parallel displacement of V in the riemannian manifold M along the curve α:

$$V_\alpha(t) = (\overset{t}{\underset{0}{\|}}\alpha)(V). \tag{26}$$

In [PR] section 5 we have calculated the curvature form $\Omega_{\mathrm{G}_r(TM)}$ at an arbitrary point $V \in \mathrm{G}_r(TM)$ for $w, w' \in T_V\mathrm{G}_r(TM)$

$$\Omega_{\mathrm{G}_r(TM)}(w, w') = \frac{\dot{\mathrm{d}}}{\mathrm{d}t}\bigg|_{t=0} (\exp(t \cdot R(v, v'))(V)) \quad \text{with } v := \gamma_{r*}w \text{ and } v' := \gamma_{r*}w';$$

here $R(v, v')$ is considered as an endomorphism of T_pM; hence, $t \mapsto \exp(t \cdot R(v, v'))$ is a 1-parameter subgroup of $\mathrm{GL}(T_pM)$ and $t \mapsto \exp(t \cdot R(v, v'))(V)$ a curve in $\mathrm{G}_r(T_pM)$. From this result we have derived for arbitrary $w, w' \in T_V\mathrm{G}_r(TM)$

$$\Omega_{\mathrm{G}_r(TM)}(w, w') = 0 \Leftrightarrow R(\gamma_{r*}w, \gamma_{r*}w')(V) \subset V. \tag{27}$$

We bring this example to an end by mentioning a canonical C^∞ distribution $\mathscr{T} = \mathscr{T}_r$ on $\mathrm{G}_r(TM)$ introduced by K. Tsukada in [T] p. 400 in order to investigate totally geodesic submanifolds (in his paper it is denoted by $\mathscr{D}$). For every point $V \in \mathrm{G}_r(TM)$ the linear subspace $\mathscr{T}_V \subset T_V\mathrm{G}_r(TM)$ is characterized by

$$\mathscr{T}_V \subset \mathscr{H}_V(\gamma_r) \quad \text{and} \quad \gamma_{r*}\mathscr{T}_V = V.$$

Example 3. **The essential horizontal structure of the article.** Now we glue together the foregoing examples. For some fixed number $n \in \{1, \ldots, m-1\}$ we take the fibre product of $\gamma_{n+1} : \mathrm{G}_{n+1}(TM) \to M$ and $\pi_M : TM \to M$

$$\tau : E := \mathrm{G}_{n+1}(TM) \times_M TM \to M;$$

it is also associated to the frame bundle of M, namely via the map

$$\tilde{\varrho} : \mathrm{L}M \times (\mathrm{G}_{n+1}(\boldsymbol{R}^m) \times \boldsymbol{R}^m) \to E, \quad (u, V, v) \mapsto (u(V), u(v)).$$

Furthermore we introduce the canonical projections $P_1 : E \to \mathrm{G}_{n+1}(TM)$ and $P_2 : E \to TM$ and define a horizontal structure $\mathscr{H}(\tau)$ of τ in canonical way, namely by

$$\begin{aligned}\mathscr{H}_{(V,z)}(\tau) &:= \{w \in T_{(V,z)}E \mid P_{1*}w \in \mathscr{H}_V(\gamma_{n+1}) \text{ and } P_{2*}w \in \mathscr{H}_z^c(\pi_M)\} \\ &\cong \{(w_1, w_2) \in \mathscr{H}_V(\gamma_{n+1}) \times \mathscr{H}_z^c(\pi_M) \mid \gamma_{n+1\,*}w_1 = \pi_{M*}w_2\}\end{aligned}$$

for all $(V, z) \in E$. From this construction the following is clear: If $\alpha : J \to M$ is a curve with $0 \in J$ and $e = (V, z) \in E_{\alpha(0)} := \tau^{-1}(\alpha(0))$, and if $V_\alpha : J \to \mathrm{G}_{n+1}(TM)$ and $z_\alpha : \tilde{J} \to TM$ denote the maximal horizontal lifts described in (26) and the remark behind (21), then

(28)
$e_\alpha := (V_\alpha, z_\alpha) : \tilde{J} \to E$ is the maximal $\mathscr{H}(\tau)$-horizontal lift of α with $e_\alpha(0) = e$.

Furthermore, using Proposition 8 we can calculate the curvature form Ω_E of $\mathscr{H}(\tau)$, namely for $e \in E$ and $w, w' \in T_eE$ we obtain $P_{1*}\Omega_E(w, w') = \Omega_{\mathrm{G}_{n+1}(TM)}(P_{1*}w, P_{1*}w')$ and $P_{2*}\Omega_E(w, w') = \Omega^c_{TM}(P_{2*}w, P_{2*}w')$, or more loosely speaking

$$\Omega_E(w, w') \cong (\Omega_{\mathrm{G}_{n+1}(TM)}(w_1, w_1'), \Omega^c_{TM}(w_2, w_2')), \qquad \text{if } w \cong (w_1, w_2), \quad w' \cong (w_1', w_2'). \tag{29}$$

5. A characterization of spherical isometric immersions

By $\tau : E \to M$ and $\mathscr{H}(\tau)$ we denote the fibre bundle and horizontal structure of Example 3.

PROPOSITION 9.
(a) *The set*

$$\hat{E} := \{(V, z) \in E \mid z \in V \backslash \{0\}\}$$

is a regular submanifold of E *(see section* 1*), and the restriction* $\tau|\hat{E} : \hat{E} \to M$ *is a subbundle of* $\tau : E \to M$ *with typical fibre*

$$\hat{\boldsymbol{F}} := \{(V, v) \in \mathrm{G}_{n+1}(\boldsymbol{R}^m) \times \boldsymbol{R}^m \mid v \in V \backslash \{0\}\}$$

associated to the frame bundle of M *via the restriction* $\tilde{\varrho}\,|\,(\mathrm{L}M \times \hat{\boldsymbol{F}})$ *of the map* $\tilde{\varrho}$ *described in Example* 3. *In the following we will shortly write* τ *instead of* $\tau|\hat{E}$. *In addition we introduce the vector field*

$$\hat{H} : \hat{E} \to TM, \quad (V, z) \mapsto z$$

in M along τ and the fibre bundle morphism

$$G : \hat{E} \to \mathrm{G}_n(TM), \quad (V, z) \mapsto \{v \in V \mid v \perp z\}.$$

(b) *There exists a subbundle $\mathscr{D}$ of the tangent bundle $T\hat{E}$ of rank n, which is characterized by*

(30) $$\mathscr{D}_e \subset \mathscr{H}_e(\tau) \quad \textit{and} \quad \tau_* \mathscr{D}_e = G(e) \quad \textit{for all } e \in \hat{E}.$$

(c) *If $c : J \to M$ is a circular arc with $0 \in J$, $z := (\nabla_\partial \dot{c})(0) \neq 0$ and $V \in \mathrm{G}_{n+1}(T_{c(0)}M)$ a subspace containing $\dot{c}(0)$ and z, then $\xi := (V_c, z_c := \nabla_\partial \dot{c})$ is a $\mathscr{D}$-integral curve in $\hat{E}$ (see the following Remark).*

(d) *For all sections $X, Y \in \Gamma(\mathscr{D})$ and all points $e = (V, z) \in \hat{E}$ the following formulas hold:*

(31) $$\nabla_{X(e)} \tau_* Y \in V, \quad \langle \nabla_{X(e)} \tau_* Y, z \rangle = \langle z, z \rangle \cdot \langle \tau_* X(e), \tau_* Y(e) \rangle \quad \textit{and}$$

(32) $$\nabla_X \hat{H} = -\langle \hat{H}, \hat{H} \rangle \cdot \tau_* X.$$

(e) *The distribution $\mathscr{D}$ is involutive at $e \in \hat{E}$, that means*

$$[X, Y](e) \in \mathscr{D}_e \quad \textit{for all sections } X, Y \in \Gamma(\mathscr{D}),$$

if and only if for all $v, v', v'' \in G(e)$ the following is true

(33) $$R(v, v')v'' \in G(e) \quad \textit{and} \quad R(v, v')\hat{H}(e) = 0.$$

Remark 6. (a) It is intended that the notations $\hat{H}$ and G remind the reader to "mean curvature normal" (see Definition 1(a)) and "Gauss map".

(b) A differentiable map $g : L \to \hat{E}$ is said to be $\mathscr{D}$-integral, if $g_* T_p L \subset \mathscr{D}_{g(p)}$ for every $p \in L$.

Proof. *For* (a): We put $\boldsymbol{F} := \{(V, v) \in \mathrm{G}_{n+1}(\boldsymbol{R}^m) \times \boldsymbol{R}^m \mid v \in V\}$ and consider the canonical vector bundle $\boldsymbol{F} \to \mathrm{G}_{n+1}(\boldsymbol{R}^m)$, $(V, v) \mapsto V$ as subbundle of the trivial vector bundle $\mathrm{G}_{n+1}(\boldsymbol{R}^m) \times \boldsymbol{R}^m \to \mathrm{G}_{n+1}(\boldsymbol{R}^m)$. It is well known that $\boldsymbol{F}$ is a regular submanifold of $\mathrm{G}_{n+1}(\boldsymbol{R}^m) \times \boldsymbol{R}^m$. This result is carried over to $\hat{\boldsymbol{F}}$, because this set is open in $\boldsymbol{F}$. And so also $\hat{E}$ is seen to be a regular submanifold of E via local trivializations of the bundle $\tau : E \to M$ constructed by means of the map $\tilde{\varrho}$; notice that $\hat{\boldsymbol{F}}$ is invariant under the action $\mathrm{GL}(m) \times (\mathrm{G}_{n+1}(\boldsymbol{R}^m) \times \boldsymbol{R}^m) \to \mathrm{G}_{n+1}(\boldsymbol{R}^m) \times \boldsymbol{R}^m$, $(A, V, v) \mapsto (A(V), A(v))$; obviously $\tau | \hat{E}$ is a subbundle of τ.

For (b): Let $\tilde{\mathscr{T}}$ denote the inverse image of the Tsukada distribution $\mathscr{T}_{n+1}$ by the vector bundle morphism $TP_1 \mid \mathscr{H}(\tau) : \mathscr{H}(\tau) \to \mathscr{H}(\gamma_{n+1})$, where TP_1 is the differential of the canonical projection $P_1 : E \to \mathrm{G}_{n+1}(TM)$ (see the Examples 2 and 3). It is a differentiable subbundle of the horizontal structure $\mathscr{H}(\tau)$, namely

$$\tilde{\mathscr{T}}_{(V,z)} = \{w \in \mathscr{H}_{(V,z)}(\tau) \mid \tau_* w \in V\} \quad \text{for all } (V, z) \in E.$$

Now we prove

(34) $$\tilde{\mathscr{T}}_e \subset T_e \hat{E} \quad \text{for all } e \in \hat{E},$$

that means, $\tilde{\mathscr{T}}|_{\hat{E}}$ is a distribution on $\hat{E}$ of rank $n+1$. For the proof of (34) let a point $e = (V, z) \in \hat{E}$ and a vector $w \in \tilde{\mathscr{T}}_e$ be given. Then there exists a $\tilde{\mathscr{T}}$-integral curve $\xi : J \to E$ with $\dot{\xi}(0) = w$. If we put $\alpha := \tau \circ \xi$, then we get $\xi = (V_\alpha, z_\alpha)$ from Example 3, because ξ is horizontal. Moreover, $\dot{\xi}(t) \in \tilde{\mathscr{T}}_{\xi(t)}$ implies $\dot{\alpha}(t) = \tau_* \dot{\xi}(t) \in V_\alpha(t)$. Since $\zeta = z_\alpha$ satisfies the differential equation (21) we find $(\nabla_\partial z_\alpha)(t) \in V_\alpha(t)$; as V_α is a parallel subbundle of TM along α according to Example 2 and the initial value $z_\alpha(0) = z$ lies in $V \backslash \{0\} = V_\alpha(0) \backslash \{0\}$, one can easily derive $z_\alpha(t) \in V_\alpha(t) \backslash \{0\}$ for every t in some interval $]-\varepsilon, \varepsilon[$. Therefore, $\xi\,|\,]-\varepsilon, \varepsilon[$ is a curve in $\hat{E}$ and therefore $w \in T_e \hat{E}$. So (34) is proved. Now the proof of assertion (b) is quickly completed, namely $\mathscr{D}$ is the kernel of the linear vector bundle map $\tilde{\mathscr{T}} \to \boldsymbol{R}$, $w \mapsto \langle \hat{H}(e), \tau_* w \rangle$ if $w \in \tilde{\mathscr{T}}_e$.

For (c): From the remark behind (21) and the formulas (25) and (28) we know that ξ is horizontal. Furthermore, $t \mapsto \eta(t) := \boldsymbol{R}\dot{c}(t) \oplus \boldsymbol{R}(\nabla_\partial \dot{c})(t)$ is a parallel subbundle of TM along c with $\eta(0) \subset V$; therefore we get $z_c(t) \in \eta(t) \subset V_c(t)$, $\tau_* \dot{\xi}(t) = \dot{c}(t) \in \eta(t) \subset V_c(t)$ and $\tau_* \dot{\xi}(t) \perp z_c(t)$ for every t. Thus we see that ξ is a curve in $\hat{E}$, which is $\mathscr{D}$-integral.

For (d): If $X, Y \in \Gamma(\mathscr{D})$ and $e = (V, z) \in \hat{E}$ are given, let $\xi : J \to \hat{E}$ be the integral curve of X with $\xi(0) = e$ and put $\alpha := \tau \circ \xi$. Then we have $\xi = (V_\alpha, z_\alpha)$ and $\dot{\alpha} = \tau_* \dot{\xi}$. Since in particular ξ is horizontal, we get:

(i) the parallelity of V_α (see (26)) and (ii) $\nabla_\partial z_\alpha + \langle z_\alpha, z_\alpha \rangle \cdot \dot{\alpha} = 0$ (see (21)).

On the other hand, since $Y \circ \xi$ is tangential to $\mathscr{D}$, formula (30) shows in particular $\tau_* Y \circ \xi(t) \in V_\alpha(t)$ for every t. Because of (i) we obtain therefore also $(\nabla_\partial \tau_* Y \circ \xi)(t) \in V_\alpha(t)$, particularly: $\nabla_{X(e)} \tau_* Y = (\nabla_\partial \tau_* Y \circ \xi)(0) \in V_\alpha(0) = V$; this is the first part of (31). Next we prove (32) by means of (ii):

$$\nabla_{X(e)} \hat{H} = (\nabla_\partial \hat{H} \circ \xi)(0) = (\nabla_\partial z_\alpha)(0) = -\langle z, z \rangle \cdot \dot{\alpha}(0) = -(\langle \hat{H}, \hat{H} \rangle \cdot \tau_* X)|_e.$$

After this result we continue using the Ricci identity and obtain the second identity of (31):

$$\begin{aligned} \langle \nabla_{X(e)} \tau_* Y, \hat{H}(e) \rangle &= X(e) \underbrace{\langle \tau_* Y, \hat{H} \rangle}_{=0} - \langle \tau_* Y(e), \nabla_{X(e)} \hat{H} \rangle \\ &\underset{(32)}{=} \langle \hat{H}(e), \hat{H}(e) \rangle \cdot \langle \tau_* X(e), \tau_* Y(e) \rangle. \end{aligned}$$

For (e): We continue with the data given in the proof of (d). According to the structure equation for the torsion (see [P] p. 101) and using (31) we obtain

$$\tau_*[X, Y](e) \in G(e) \quad \text{for all } e \in \hat{E}.$$

Therefore, $\mathscr{D}$ is involutive in $e = (V, z)$ if and only if $[X, Y](e) \in \mathscr{H}_e(\tau)$ for all $X, Y \in \Gamma(\mathscr{D})$. Because of (19) and (29) this property is equivalent to

$$\Omega_{G_{n+1}(TM)}(w_1, w_1') = 0 \quad \text{and} \quad \Omega^c_{TM}(w_2, w_2') = 0$$

$$\text{for all } w \cong (w_1, w_2), \quad w' \cong (w_1', w_2') \in \mathscr{D}_e. \tag{35}$$

Now we use (27) and (22), take notice of $\gamma_{n+1*}w_1 = \pi_{M*}w_2 = \tau_* w \in G(e)$ and get that (35) is equivalent to

$$R(v, v')V \subset V \quad \text{and} \quad R(v, v')z = 0 \quad \text{for all } v, v' \in G(e).$$

With $\langle R(v, v')v'', z\rangle = -\langle R(v, v')z, v''\rangle$ we finaly obtain the statement (e). □

Theorem 2.

(a) *If $S \subset \hat{E}$ is an integral manifold of the distribution $\mathscr{D}$, then there exists a riemannian metric $\langle \cdot, \cdot \rangle_S$ on S such that $\tau|S : S \to M$ is an isometric immersion; in fact, it is spherical and $\hat{H}|S$ is its mean curvature normal.*

(b) *If $f : N \to M$ is a spherical isometric immersion from a connected n-dimensional riemannian manifold with mean curvature normal $H \neq 0$, then*

$$\hat{f} : p \mapsto (f_* T_p N \oplus \boldsymbol{R} H_p, H_p)$$

is a $\mathscr{D}$-integral map into $\hat{E}$ (see Remark 6(b)) with the following properties

$$(36) \qquad \tau \circ \hat{f} = f, \quad \hat{H} \circ \hat{f} = H \quad \text{and} \quad G \circ \hat{f} \text{ is the Gauss map of } f.$$

Furthermore, $S := \hat{f}(N)$ is an integral manifold of $\mathscr{D}$ and $\hat{f}$ is a local isometry onto S, if S is equipped with the riemannian metric $\langle \cdot, \cdot \rangle_S$ described in (a).

Proof. For (a): Because of Remark 4 $\tau|S$ is an immersion into M. Thus $\langle \cdot, \cdot \rangle_S$ can be defined in the appropriate way. In order to show that $\tau|S$ is spherical, let vector fields $X, Y \in \mathfrak{X}(S) = \Gamma(\mathscr{D}|_S)$ and $e = (V, z) \in S$ be given. Then we calculate the second fundamental form h of $\tau|S$ using the Gauss equation and Proposition 9(d): Because we have

$$\tau_* \nabla^S_{X(e)} Y + h(X(e), Y(e)) = \nabla_{X(e)} \tau_* Y \in V \quad \text{with } \tau_* \nabla^S_{X(e)} Y \in \tau_* T_e S = \tau_* \mathscr{D}_e = G(e),$$

where $h(X(e), Y(e))$ is perpendicular to $G(e)$, the second identity of (31) implies $h(X(e), Y(e)) = \langle \tau_* X(e), \tau_* Y(e)\rangle \cdot z = \langle X(e), Y(e)\rangle_S \cdot \hat{H}(e)$. Eventually the parallelity of $\hat{H}|S$ follows from (32).

For (b): Obviously $\hat{f}$ is a differentiable map into $\hat{E}$ with the properties stated in (36). For the $\mathscr{D}$-integrability of $\hat{f}$ it suffice to prove: For every point $p \in N$ and every unit vector $v \in T_p N$ the image $\hat{f}_* v$ lies in $\mathscr{D}_e$ with $e := (V, z) := \hat{f}(p)$. For that let $\alpha : J \to N$ denote the maximal geodesic with $\dot{\alpha}(0) = v$ and put $c := f \circ \alpha$. According to Proposition 1(f) c is a circular arc in M with $(\nabla_\partial \dot{c})(0) = H_p = z$. Obviously V contains the vectors $\dot{c}(0)$ and z. Then Proposition 9(c) implies that $e_c = (V_c, z_c)$ is a $\mathscr{D}$-integral curve; and from Proposition 1(e), (f) we derive $e_c = \hat{f} \circ \alpha$. Therefore $\hat{f}_* v = \dot{e}_c(0) \in \mathscr{D}_e$. The further assertions of (b) are obvious. □

6. Proof of Theorem 1

Since in Proposition 5 the existence of the circular umbrella $N_\varepsilon(p, U, z)$ already was proved, it remains to show that it is a spherical submanifold under the hypothesis (4). For that we put $n := \dim U$ and continue with the notations used in the proof of Proposition 5. If $V, S(U), V_u, z_u$ and U_u have the meaning of Theorem 1, then (V_u, z_u) is the horizontal lift $e_u : J_u \to \hat{E}$ of c_u with $e_u(0) = e := (V, z)$ for every vector $u \in S(U)$; according to Proposition 9(c) e_u is $\mathscr{D}$-integral.

Furthermore, for every $s \in \boldsymbol{R}$ the curve $e_{su} : J_{su} \to \hat{E}$ defined by $e_{su}(t) = e_u(st)$ is also $\mathscr{D}$-integral; in addition it is the horizontal lift of c_{su} with $e_{su}(0) = e$. If we put $B := \{(t,u) \in \boldsymbol{R} \times U \mid t \in J_u\}$, then $F : B \to M$, $(t,u) \mapsto c_u(t) = \Phi^Y(t, u, \langle u, u\rangle \cdot z)$ is a differentiable map. Applying Proposition 7(b) (with $g \equiv e$) we get that $\tilde{F} : B \to \hat{E}$, $(t,u) \mapsto e_u(t)$ is differentiable, too. Consequently,

$$\tilde{\Phi} : U_\varepsilon(0) \to \hat{E},\ u \mapsto e_u(1) \text{ is a differentiable map with } \tau \circ \tilde{\Phi} = \Phi.$$

As $\Phi \mid U_\varepsilon(0)$ is an embedding, the same is true for $\tilde{\Phi}$. In particular,

$$\tilde{\Phi}(U_\varepsilon(0)) = \bigcup_{u \in S(U)} e_u(]-\varepsilon, \varepsilon[),$$

is a regular submanifold of $\hat{E}$. Because for every $u \in S(U)$ the curve $t \mapsto \tilde{\Phi}(tu) = e_u(t)$ was proved to be $\mathscr{D}$-integral, $\tilde{\Phi}$ is a $\mathscr{D}$-umbrella in the sense of the following Definition 3. Moreover, since $G(e_u(t)) = U_u(t)$, according to Proposition 9(e) the assumption (4) says that the distribution $\mathscr{D}$ is involutive at all points of $S := \tilde{\Phi}(U_\varepsilon(0))$. Therefore, the following Theorem 3 shows that $\tilde{\Phi}$ is $\mathscr{D}$-integral; that means, S is an integral manifold of $\mathscr{D}$, and according to Theorem 2(a) $\tau|S$ is a spherical immersion with the mean curvature normal $\hat{H}|S$. Consequently the image $\tau(S) = N_\varepsilon(p, U, z)$ is a spherical submanifold with tangent space $G(e) = U$ and mean curvature normal $\hat{H}(e) = z$ at the point p. Thus the proof is complete. □

For the end of this section let $\mathscr{D}$ be a C^∞ distribution on a manifold E.

Definition 3. If U is a linear space, $B \subset U$ a star shaped neighbourhood of 0 in U, $J_u := \{t \in \boldsymbol{R} \mid tu \in B\}$ for every $u \in U$ and $\varphi : B \to E$ a C^∞ imbedding such that all curves $J_u \to E$, $t \mapsto \varphi(tu)$ are $\mathscr{D}$-integral, then we call φ a *$\mathscr{D}$-umbrella*.

Theorem 3. *If $\varphi : B \to E$ is a $\mathscr{D}$-umbrella and $\mathscr{D}$ is involutive at all points of $\varphi(B)$, then the entire map φ is $\mathscr{D}$-integral.*

This theorem was proved in the last section of [PR] by using a result of Blumenthal and Hebda (see [BH] p. 165).

7. A global theorem

If in the foregoing proof S is replaced by the maximal connected integral manifold $\tilde{S}$ containing S, then $\tau|\tilde{S}$ describes a maximally extended spherical submanifold (with selfintersections, if $\tau|\tilde{S}$ is not injective). It is the purpose of the following theorem to show that $\tau|\tilde{S}$ has some completeness property under suitable hypotheses. In order to formulate them we define:

Definition 4. A totally geodesic resp. spherical immersion $f : N \to M$ is said to be *geodesically closed*, if the image $f \circ \tilde{\alpha}$ of any maximal unit speed geodesic $\tilde{\alpha}$ is

(a) a maximal geodesic in case that f is totally geodesic, and
(b) a maximal circular arc in case that f is spherical (see Remark 2).

Remark 7. If M is complete, then f is geodesically closed if and only if N, too, is complete (see Proposition 4(b)).

Let again be given a point $p \in M$, a linear subspace $U \subsetneqq T_pM$ and a vector $z \in U^\perp \setminus \{0\} \subset T_pM$ and put $V := U \oplus \boldsymbol{R}z$ and $e := (V, z) \in \hat{E}$. Under these assumptions we introduce a special class of *broken circular curves*: A broken unit speed curve $c : [0, b_c] \to M$ with the "break points" $0 < t_1 < \cdots < t_n < b_c$ is called a *broken circular curve*, if every "section" $c_i := c \,|\, [t_i, t_{i+1}]$ (with $t_0 := 0$, $t_{n+1} := b_c$) is a circular arc and if the mean curvature normals $\nabla_\partial \dot{c}_i$ add to a *continuous* vector field along c, which we will denote by $\nabla_\partial \dot{c}$. Now, by $C(p, U, z)$ we denote the set of all such broken circular curves with $c(0) = p$ and $(\nabla_\partial \dot{c})(0) = z$ such that $\dot{c}_i(t_i) \in V_c(t_i)$ holds for every i, where $V_c : [0, b_c] \to \mathrm{G}_{n+1}(TM)$ denotes the $\mathscr{H}(\gamma_{n+1})$-horizontal lift of c (see (25) and (26)); then automatically one has $\dot{c}_i(t)$, $(\nabla_\partial \dot{c})(t) \in V_c(t)$ for every $t \in [t_i, t_{i+1}]$, $\nabla_\partial \dot{c}$ coincides with the $\mathscr{H}^c(\pi_M)$-horizontal lift z_c of c (see Example 1) and (V_c, z_c) is the $\mathscr{H}(\tau)$-horizontal lift e_c of c (see (28)); it is a (continuous) broken curve in $\hat{E}$ such that every section $e_c \,|\, [t_i, t_{i+1}]$ is $\mathscr{D}$-integral. Of course every simple circular arc with the prescribed initial data belongs to $C(p, U, z)$.

Remark 8. If $f : N \to M$ is a spherical immersion with mean curvature normal H and $q \in N$ a point such that $f(q) = p$, $f_* T_q N = U$ and $H(q) = z$ and if $\alpha : [0, b] \to N$ is a broken unit speed geodesic of N with $\alpha(0) = p$, then the image $c := f \circ \alpha$ is an element of $C(p, U, z)$.

Theorem 4. *If in the above situation for every broken circular curve* $(c : [0, b_c] \to M) \in C(p, U, z)$ *and every* $t \in [0, b_c]$ *we have*

$$\forall v, v', v'' \in G(e_c(t)) : (R(v, v')v'' \in G(e_c(t)) \text{ and } R(v, v')z_c(t) = 0), \tag{37}$$

then there exists one (*and up to an isometry exactly one*) *geodesically closed, spherical immersion* $F : N \to M$ *with mean curvature normal* H *from a simply connected riemannian manifold* N *and a point* $q \in N$ *such that* $F(q) = p$, $F_* T_q N = U$ *and* $H(q) = z$.

In the special case of a complete riemannian manifold M Theorem 4 is an analogue of a result of R. Hermann [H] on the existence of *complete totally geodesic* submanifolds.

Proof. At first we see from (37) that the assumptions (4) of Theorem 1 are satisfied for some circular ε-umbrella $N_\varepsilon(p,U,z)$; hence we find that $N_\varepsilon(p,U,z)$ is a spherical submanifold of M. Applying Theorem 2(b) with the inclusion map $f := (N_\varepsilon(p,U,z) \hookrightarrow M)$ we get an integral manifold $\hat{f}(N_\varepsilon(p,U,z))$ of $\mathscr{D}$ containing $e = (V,z)$. Let $\tilde{S}$ be the maximal connected integral manifold of $\mathscr{D}$ which contains $\hat{f}(N_\varepsilon(p,U,z))$ (see [Nu, Theorem 4] or [BH, Theorem 1.3 and 1.4], a proof of the paracompactness of $\tilde{S}$ can be found in [LR] p. 94). Because of Theorem 2(a) $\tau|\tilde{S}$ is a spherical immersion with respect to the induced riemannian metric $\langle\cdot,\cdot\rangle_{\tilde{S}}$. Of course, we have $e \in \tilde{S}$, $\tau(e) = p$, $\tau_* T_e\tilde{S} = \tau_*\mathscr{D}_e = G(e) = U$ and z is its mean curvature normal at e.

Now the crucial point is to prove that $\tau|\tilde{S}$ is geodesically closed. For that, let a maximal unit speed geodesic $\tilde{\alpha} : \tilde{J} \to \tilde{S}$ be given. As $\tau \circ \tilde{\alpha}$ is a circular arc in M, it can be extended to a maximal circular arc $\alpha : J \to M$. Let us assume $\delta := \sup \tilde{J} < \sup J$. Then we choose some broken unit speed geodesic $\tilde{\beta} : [0,d] \to \tilde{S}$ starting from $\tilde{\beta}(0) = e$ with $\tilde{\beta}(d) \in \tilde{\alpha}(\tilde{J})$. We may assume $\tilde{\beta}(d) = \tilde{\alpha}(d)$. As

$$\tilde{c} : [0,\delta[\to \tilde{S}, \quad t \mapsto \begin{cases} \tilde{\beta}(t) & \text{for } t \in [0,d] \\ \tilde{\alpha}(t) & \text{for } t \in]d,\delta[\end{cases}$$

is a broken unit speed geodesic, the curve

$$c : [0,\delta] \to M, \quad t \mapsto \begin{cases} \tau \circ \tilde{\beta}(t) & \text{for } t \in [0,d] \\ \alpha(t) & \text{for } t \in]d,\delta] \end{cases}$$

is an element of $C(p,U,z)$ according to Remark 8. Its horizontal lift $e_c : [0,\delta] \to \hat{E}$ is $\mathscr{D}$-integral and it satisfies $e_c \,|\, [0,\delta[= \tilde{c}$. Thus $e' := e_c(\delta)$ is a good candidate in order to continue $\tilde{\alpha}$.

For realizing this idea we put $p' := c(\delta)$, $U' := G(e')$ and $z' := \hat{H}(e')$. Then there exists a suitable $\varepsilon' > 0$ such that the circular ε'-umbrella $N' := N_{\varepsilon'}(p',U',z')$ exists. In particular, for every unit vector $u \in U'$ the circular arc $c_u :]-\varepsilon',\varepsilon'[\to N'$ with initial values $\dot{c}_u(0) = u$ and $(\nabla_\partial \dot{c}_u)(0) = z'$ exists, and the curve

$$[0,\delta+\varepsilon'] \to M, \quad t \mapsto \begin{cases} c(t) & \text{for } t \leq \delta \\ c_u(t-\delta) & \text{for } t > \delta \end{cases}$$

is an element of $C(p,U,z)$. Because of (37) Theorem 1 can be applied in order to obtain that N' is a spherical submanifold of M, too.

Now we show that the curves c_u (with $u \in U'$) are geodesics of N': Applying the Gauss equation (combined with Definition 1(a)) and Proposition 1(b) we get

$$-\varkappa \cdot \dot{c}_u = \nabla_\partial \nabla_\partial \dot{c}_u = \nabla_\partial^{N'} \nabla_\partial^{N'} \dot{c}_u - \varkappa \cdot \dot{c}_u \quad \text{with } \varkappa := \varkappa(c_u) = \langle z,z\rangle.$$

Hence, $\nabla_\partial^{N'} \dot{c}_u$ is a parallel vector field in N' along c_u; it vanishes identically, because $(\nabla_\partial \dot{c}_u)(0) = z' \perp T_{p'}N'$. So we have seen that c_u is a geodesic of N'.

Of course, we may assume $\varepsilon' < \delta - d$. Because $u' := \dot{\alpha}(\delta) = \dot{c}(\delta) \in U'$ and $(\nabla_\partial \dot{\alpha})(\delta) = (\nabla_\partial \dot{c})(\delta) = z'$, we get $\alpha(t) = c_{u'}(t-\delta)$ for every $t \in]\delta - \varepsilon', \delta]$. Hence we even obtain $]\delta - \varepsilon', \delta + \varepsilon'[\subset J$ and $\alpha(t) = c_{u'}(t-\delta)$ for every $t \in]\delta - \varepsilon', \delta + \varepsilon'[$. Consequently, $\alpha \,|\,]\delta - \varepsilon', \delta + \varepsilon'[$ is a geodesic arc of N'.

With the inclusion map $f' := (N' \hookrightarrow M)$ of the spherical submanifold N' we get a further integral manifold $S' := \widehat{f'}(N')$; it contains $\widehat{f'}(p') = e' = e_c(\delta)$. Moreover, $\widehat{f'} \circ \alpha \,|\,]\delta - \varepsilon', \delta]$ and $e_c \,|\,]\delta - \varepsilon', \delta]$ are $\mathscr{H}(\tau)$-horizontal lifts of $\alpha \,|\,]\delta - \varepsilon', \delta]$ with $\widehat{f'} \circ \alpha(\delta) = e_c(\delta)$. Therefore we get

$$\widehat{f'} \circ \alpha(]\delta - \varepsilon', \delta[) = e_c(]\delta - \varepsilon', \delta[) = \tilde{c}(]\delta - \varepsilon', \delta[) = \tilde{\alpha}(]\delta - \varepsilon', \delta[) \subset S' \cap \tilde{S}.$$

Consequently S' is a subset of $\tilde{S}, \widehat{f'}$ an isometry into $\tilde{S}$ and $\widehat{f'} \circ \alpha \,|\,]\delta - \varepsilon', \delta + \varepsilon'[$ a geodesic in $\tilde{S}$ continuing $\tilde{\alpha}$ beyond δ in contradiction to the maximality of $\tilde{\alpha}$. Thus we have proved $\sup \tilde{J} = \sup J$. In the same way we get $\inf \tilde{J} = \inf J$, hence $\tilde{J} = J$.

In order to define the spherical immersion $F : N \to M$ of Theorem 4 we use the universal covering $\varphi : N \to \tilde{S}$ of $\tilde{S}$, put $F := (\tau|\tilde{S}) \circ \varphi$ and choose some point $q \in \varphi^{-1}(\{e\})$. Let us now prove the uniqueness of F. For that let $f : \tilde{N} \to M$ be another spherical immersion from a simply connected riemannian manifold $\tilde{N}$ and $\tilde{q} \in \tilde{N}$ a point which have the same properties as F and q. According to Theorem 2(b) the induced map $\hat{f} : \tilde{N} \to \hat{E}$ satisfies (36); in particular we have $\hat{f}(\tilde{q}) = e = \varphi(q)$. Hence, $\hat{f}$ is a local isometry into $\tilde{S}$. Since $\tilde{N}$ is simply connected, there exists a local isometry $\Psi : \tilde{N} \to N$ such that $\varphi \circ \Psi = \hat{f}$ and $\Psi(\tilde{q}) = q$. From the construction we get $F \circ \Psi = f$. As f is geodesically closed, Ψ is geodesically closed, too. Therefore, according to the following Lemma Ψ is a covering map, in fact even an isometry because of the simple connectedness of N. □

Lemma. *If N and $\tilde{N}$ are connected riemannian manifolds of the same dimension, then each geodesically closed local isometry $f : \tilde{N} \to N$ is a covering map.*

This lemma is a generalization of Theorem 4.6(1) in [KN] p. 176, in which $\tilde{N}$ is assumed to be complete. One can follow the proof of [KN]; where they use the completeness of $\tilde{N}$ the argumentation keeps valid if instead of that we use that f is geodesically closed.

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