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FANO THREEFOLDS WITH PICARD NUMBER 2 IN POSITIVE CHARACTERISTIC

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Abstract

The smooth Fano threefolds with Picard number two in positive characteristic are classified. The each class has the same description as in characteristic zero. We give also a new example of Fano threefolds with Picard number three having a wild conic bundle structure.

0. Introduction

Let X be a smooth projective variety over an algebraically closed field k. We say that X is a *Fano variety* if its anticanonical divisor $-K_X$ is ample.

Fano threefolds are one of the fundamental classes of varieties in three dimensional birational geometry. In characteristic zero, the classification of smooth Fano threefolds was completed by many authors:

(1) Iskovskih studied Fano threefolds in the 1970s and classified them when Picard number is one (see [Isk77, Isk78]).
(2) Shokurov proved that there exist lines on Fano threefolds. This is a crucial result for the classification, though his proof was complicated (see [Sho79]).
(3) Using Mori's description of extremal rays, the classification of Fano threefolds with bigger Picard numbers was established by Mori and Mukai (see [MM81, MM83, MM86]). This depends on the existence of lines on Fano threefolds with Picard number one.
(4) Fujita developed the theory of Δ-genera to classify Fano varieties of coindex two in any dimension (see [Fuj90]).
(5) Takeuchi reproduced the classification of Fano threefolds with Picard number one by simple numerical calculations based on the theory of extremal rays (see [Tak89]). In particular, his approach gives automatically the existence of lines on them, thus simplifying substantially the method by Iskovskih and Shokurov.

There are several difficulties for conducting the classification of Fano three-

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folds in positive characteristic, including the failures of Kodaira vanishing or Bertini's theorems. Shepherd-Barron [SB97] has overcome these obstacles and given a birational classification of Fano threefolds of Picard number one. He also proved the existence of lines on them, an essential fact for the analysis when Picard number is at least two. On the other hand, Megyesi [Meg98] has classified Fano threefolds in positive characteristic of Fano index at least two, using the method of Δ-genera by Fujita.

In view of these works, it is natural to investigate Fano threefolds with bigger Picard numbers. Here we have to cope with the difficulties that conic bundle structures may be wild and that del Pezzo fibrations may have singular general fiber. Instead of the original Mori-Mukai's argument, we depend on numerical arguments due to Takeuchi.

Our main theorem is the following:

Theorem 0.1. *Let X be a Fano threefold with $\rho(X) = 2$. Then*
(i) *X belongs to one of the* 36 *classes which have the same description as the classification by Mori and Mukai.*
(ii) *If X has a wild conic bundle structure, then X belongs to the class* No. 24 *in the Mori-Mukai's table, that is, X is a divisor in $\boldsymbol{P}^2 \times \boldsymbol{P}^2$ of bidegree* $(1,2)$.
(iii) *X does not have any non-normal del Pezzo fibrations.*
(iv) *If X has a quadric cone fibration structure, then X belongs to the class* No. 29 *in the Mori-Mukai's table, that is, X is the blow-up of a quadric threefold $Q \subset \boldsymbol{P}^4$ along a conic.*

As a consequence, we obtain a new example of Fano threefolds with Picard number three having a wild conic bundle structure.

Throughout this paper, the ground field k is algebraically closed of characteristic $p > 0$.

Remark. Recently, the author heard that Mori had already obtained the classification of Fano threefolds with Picard number at least 2 in any characteristic.

1. Preliminaries

1.1. For studying higher dimensional algebraic varieties, the classification of extremal rays by Mori is one of the key results. In [Kol91], Kollár proved that extremal rays on smooth threefolds in positive characteristic have almost the same description as in characteristic zero:

Theorem 1.1. *Let X be a smooth projective threefold over k. Let R be an extremal ray of the cone of curves. Then*
(i) *There is a normal projective variety Y and a surjective morphism $f : X \to Y$ such that an irreducible curve $C \subset X$ is mapped to a point by f if and only if $[C] \in R$.*

(ii) *The possibilities for* f *and* Y *are as follows*:
The case (E). *There is a unique irreducible divisor* E *on* X *with* $E.C < 0$ *such that* f *contracts* E. *There are five different types*:
(E1): f *is the blow-up of a smooth curve in* Y. E *is a smooth ruled surface and* Y *is also smooth.*
(E2): f *is the blow-up of a point in* Y. $E \cong \boldsymbol{P}^2$ *and* Y *is smooth.*
(E3): f *is the blow-up of a singular point* P *in* Y. E *is a smooth quadric surface in* $\boldsymbol{P}^3$. $\hat{\mathcal{O}}_{P,Y} \cong k[[x,y,z,t]]/(xy - zt)$.
(E4): f *is the blow-up of a singular point* P *in* Y. E *is a quadric cone in* $\boldsymbol{P}^3$. $\hat{\mathcal{O}}_{P,Y} \cong k[[x,y,z,t]]/(xy - z^2 - t^3)$.
(E5): f *is the blow-up of a singular point* P *in* Y. $E \cong \boldsymbol{P}^2$. $\hat{\mathcal{O}}_{P,Y} \cong k[[x^2, y^2, z^2, xy, yz, zx]]$.
The case (C). Y *is a smooth surface*, *and* f *is a flat conic bundle. If* $\operatorname{char} k \neq 2$, *the general fiber is smooth.*
The case (D). Y *is a smooth curve and every fiber of* f *is irreducible. Any fiber with reduced scheme structure is a* (*possibly non-normal*) *del Pezzo surface.*
The case (F). X *is a Fano threefold with* $\rho(X) = 1$.

Put $\mu(R) := \min\{(-K_X).C \mid C \in R\}$. For the cases (C) and (D), we can divide f into subcases:
(C1): f has a singular fiber, that is, a line pair or a double line. We have $\mu(R) = 1$.
(C2): f is a smooth morphism. We have $\mu(R) = 2$.
(D1): The general fiber of f is a del Pezzo surface of degree d, $1 \leq d \leq 6$. It may be singular. If it is normal, then it has only rational double points or it is the cone over an elliptic curve by [HW81]. We have $\mu(R) = 1$.
(D2): The general fiber of f is isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ or a quadric cone in $\boldsymbol{P}^3$. We have $\mu(R) = 2$.
(D3): f is a $\boldsymbol{P}^2$-bundle. We have $\mu(R) = 3$.
We recall the following significant result on Fano threefolds in positive characteristic.

Theorem 1.2 (Shepherd-Barron [SB97]). *Let* X *be a Fano threefold with* $\rho(X) = 1$. *Then* X *contains a line.*

Theorems 1.1 and 1.2 enable us to apply the following result in [MM86, Theorem 5.2] to the positive characteristic case:

Theorem 1.3. *Let* X *be a Fano threefold with* $\rho(X) = 2$, *and* $f : X \to Y$ *be a contraction of an extremal ray of type* (E1). *Then* Y *is a Fano threefold of index* $r \geq 2$.

Proof. Thanks to Theorem 1.2, the proof by Mori-Mukai works. Let Y be a Fano threefold with $\rho(Y) = 1$ and of index 1, and $f : X \to Y$ the blow-up of Y along a smooth curve C. When $-K_Y$ is not very ample, we have $(-K_Y)^3 \leq 4$, hence $(-K_X)^3 = 2$. Then by the classification of hyperelliptic Fano threefolds ([Meg98]), $-K_X$ must be very ample. However, we see that $X \cong \boldsymbol{P}^3$ since $h^0(X, -K_X) = 4$, which is a contradiction.

Assume that $-K_Y$ is very ample and $(-K_Y).C > 1$. Then by Theorem 1.2, there exists a curve l such that $(-K_Y).l = 1$. Since

$$\mathcal{N}_{l/Y} \cong \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(-1) \quad \text{or} \quad \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-2),$$

by Iskovskih ([Isk78]), we have $h^0(\mathcal{N}_{l/Y}) - h^1(\mathcal{N}_{l/Y}) > 0$, hence l moves in Y. So there exists a surface $S \subset Y$ swept out by lines. Therefore we can take a line l' on Y intersecting C. However, we have $-K_X.f^{-1}(l') \leq 0$, which shows that X is not a Fano threefold. When C is a line, we can also prove in the same way as in [MM86, Theorem 5.2]. □

1.2. We recall some known results about the structure of double covers in characteristic 2 (see [CD89] or [Meg98]). Let $h : X \to Y$ be a finite morphism of degree 2 from Cohen-Macaulay scheme X to a smooth variety Y. Then the natural inclusion $\mathcal{O}_Y \to h_*\mathcal{O}_X$ defines a line bundle $\mathcal{L} = (h_*\mathcal{O}_X/\mathcal{O}_Y)^\vee$. X is locally a hypersurface in $Y_i \times \boldsymbol{A}^1$ given by an equation $x^2 + a_i x + b_i = 0$, where $\{Y_i\}$ is an open affine covering of Y which trivializes $\mathcal{L}$, and $a_i, b_i \in \mathcal{O}_Y(Y_i)$. The local data $(b_i, a_i, 1)$ glue together to give a section s of a vector bundle $\mathcal{E}$ of rank 3, which is isomorphic to an extension

$$0 \to \mathcal{L}^{\otimes 2} \to \mathcal{E} \to \mathcal{L} \oplus \mathcal{O} \to 0.$$

We see that the triple $(\mathcal{L}, \mathcal{E}, s)$ determines a double cover $h : X \to Y$. $\mathcal{E}$ is said to be *splittable* if it splits into the sum of line bundles $\mathcal{L}^{\otimes 2} \oplus \mathcal{L} \oplus \mathcal{O}$. If $\mathrm{H}^1(Y, \mathcal{L}) = 0$, $\mathcal{E}$ is splittable. Obviously, $\{b_i\}$ form a global section $b \in \Gamma(Y, \mathcal{L}^{\otimes 2})$ if $\mathcal{E}$ is splittable.

If $a = \{a_i\} \neq 0$, we see that h is separable and branched along $\operatorname{Supp}\operatorname{div}(a)$. Then the singularity of X is the inverse image of the zeros of $x\,\mathrm{d}a_i + \mathrm{d}b_i$ lying on $\operatorname{Supp}\operatorname{div}(a)$. On the other hand, if $a = 0$, then h is inseparable and the inverse image of the zeros of $\mathrm{d}b_i$ is the singularity of X. In both cases, we have

$$\omega_X \cong h^*(\omega_Y \otimes \mathcal{L}).$$

$\{x\,\mathrm{d}a_i + \mathrm{d}b_i\}$ is glued together to form a section of the sheaf $\mathcal{L}^{\otimes 2} \otimes \Omega^1_Y$. Thus by checking $c_3(\mathcal{L}^{\otimes 2} \otimes \Omega^1_Y)$ is not equal to zero, we can exclude the possibility that the double cover given is purely inseparable.

1.3. We recall also the classification table of normalized del Pezzo surfaces by Reid ([Rei94, Theorem 1.1]).

THEOREM 1.4. *Let S be a non-normal del Pezzo surface, $\tilde{S}$ its normalization, and C the curve on $\tilde{S}$ defined by the conductor ideal. Then pairs $C \subset \tilde{S}$ are listed as follows*:

Case	$(\tilde{S}, \mathcal{O}_{\tilde{S}}(1))$	deg $\tilde{S}$	Class of C	Nature of C
(a)	$(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(1))$	1	$\mathcal{O}_{\boldsymbol{P}^2}(2)$	(a1) smooth conic (a2) line pair (a3) double line
(b)	$(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(2))$	4	$\mathcal{O}_{\boldsymbol{P}^2}(1)$	smooth conic
(c)	$(\boldsymbol{F}_{a;0}, aA)$ for $a \geq 2$	a	$2A$	(c1) line pair (c2) double line (c0) smooth conic (only if $a = 2$)
(d)	$(\boldsymbol{F}_{a;1}, (a+1)A + B)$ for $a \geq 0$	$a+2$	$A+B$	(d1) line pair (d0) smooth conic (only if $a \leq 1$)
(e)	$(\boldsymbol{F}_{a;2}, (a+2)A + B)$	$a+4$	B	smooth conic

Here A is a fiber of $\boldsymbol{F}_a$, and B a negative section.

In the following, we investigate Fano threefolds for each pair of extremal contractions. Unless otherwise stated, we assume that X is a Fano threefold with $\rho(X) = 2$.

2. Conic bundles

We treat here Fano threefolds having an extremal contraction of type (C), that is, a conic bundle structure. In particular, in characteristic 2, we must take account of the possibility that the conic bundle may be wild.

DEFINITION. Let $f : X \to S$ be a surjective morphism from a smooth projective threefold X to a smooth projective surface S. We say that f is a *wild conic bundle* if every fiber of f is a double line. Clearly, wild conic bundle occurs only in characteristic 2. We call an *ordinary conic bundle* if it is not wild.

PROPOSITION 2.1 (Shepherd-Barron [SB97]). *If X is a Fano threefold and $f : X \to S$ is a wild conic bundle, then $S \cong \boldsymbol{P}^2$ or $\boldsymbol{P}^1 \times \boldsymbol{P}^1$.*

Put $W := f^{-1}(\ell)$ where ℓ is a smooth rational curve on S. Let m be a fiber of f with reduced structure.

LEMMA 2.2. *W is reduced.*

Proof. If W is non-reduced, then it follows that $W = 2U$ where U is a $\boldsymbol{P}^1$-bundle over ℓ with fiber m. We have $\mathcal{N}_{U/X}.m = 0$ since m moves in X, so the adjunction formula gives $-K_U.m = 1$. Obviously, this is impossible. □

Let $\mu : \tilde{W} \to W$ be the normalization. Then there exists a factorization $\tilde{W} \to \ell^{(-1)} \xrightarrow{F} \ell$ of $f \circ \mu$, where F is the geometric Frobenius map.

Lemma 2.3. *$\tilde{W} \to \ell^{(-1)}$ is a smooth $\boldsymbol{P}^1$-bundle.*

Proof. Since $\mu^*(-K_X|_W).m = 1$, every fiber m is generically reduced. m is a Cartier divisor on the normal surface $\tilde{W}$, so m has no embedded point. This implies that $\tilde{W}$ is smooth. $\square$

Let $C \subset \tilde{W}$ be the curve defined by the conductor ideal. Since

$$\begin{aligned}\omega_{\tilde{W}} &\cong \mu^*\omega_W \otimes \mathcal{O}_{\tilde{W}}(-C)\\ &\cong \mu^*\mathcal{O}_W(K_X|_W) \otimes \mu^*\mathcal{N}_{W/X} \otimes \mathcal{O}_{\tilde{W}}(-C),\end{aligned}$$

we have $C.m = 1$, so we can set $C \equiv C_0 + rm$, where C_0 is the negative section of $\tilde{W}$ and r an integer. In particular, the singularities of W are unibranched.

We recall also the fact about ordinary conic bundles.

Definition. Let $f : X \to S$ be a surjective morphism from a smooth projective threefold X to a smooth projective surface S such that a general fiber is a smooth conic. Then the set Δ defined by

$$\{P \in S \mid f^{-1}(P) \text{ is not smooth}\}$$

is said to be the *discriminant locus* of a conic bundle.

Lemma 2.4. *Let $f : X \to S$ be an ordinary conic bundle over a smooth projective surface S, and Δ its discriminant locus. Then*:

(i) *Δ is smooth at every point over which the fiber is a reduced line pair. If the characteristic of k is not equal to* 2, *Δ is a curve with at worst ordinary double points.*

(ii) *We have the formula*

$$\Delta \equiv -f_*(-K_X)^2 - 4K_S.$$

Proof. (i) If the characteristic $\neq 2$, the proof by Beauville ([Bea77]) works. So assume that char $k = 2$. Let $P \in \Delta$ be a point in Δ such that the fiber over P is a line pair. In a neighborhood of P in S, we may take local coordinates u, v such that X is locally given as

$$q(x, y) = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6 = 0,$$

where a_i $(i = 1, \dots, 6)$ are functions of (u, v). We may assume that the fiber over P has a singularity at $(x, y) = (0, 0)$. Then by $q_x = a_2y + a_4$ and $q_y = a_2x + a_5$, we see that a_2 has a constant term. The equation of discriminant locus is locally given as

$$P(u, v) = a_1a_5^2 + a_2a_4a_5 + a_3a_4^2 + a_2^2a_6 = 0.$$

On the other hand, since X is smooth at $(x, y; u, v) = (0, 0; 0, 0)$, a_6 must have a linear term for (u, v). Therefore we have

$$P_u(0,0) = a_2^2 \frac{\partial a_6}{\partial u}(0,0) \neq 0, \quad P_v(0,0) = a_2^2 \frac{\partial a_6}{\partial v}(0,0) \neq 0,$$

which shows that Δ is smooth at P.

(ii) This is proved just as in characteristic zero (see [MM83, Proposition 6.2 (4)], [MM86, Corollary 4.6]). □

Now we consider Fano threefolds X with $\rho(X) = 2$ having a conic bundle structure $f : X \to S$ including the case where it is wild. Clearly, S is isomorphic to $\boldsymbol{P}^2$, and $f : X \to \boldsymbol{P}^2$ is one of the two extremal contractions. We denote the other extremal contraction by $g : X \to Y$:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \boldsymbol{P}^2 & & Y. \end{tikzcd}$$

We set $H := f^*\mathcal{O}_{\boldsymbol{P}^2}(1)$. By definition, we obtain

$$H^3 = 0, \quad (-K_X).H^2 = 2.$$

Moreover, we set $c := (-K_X)^2.H$. If $f : X \to \boldsymbol{P}^2$ is ordinary, then $c = 12 - \deg \Delta$ by Lemma 2.4, where Δ is the discriminant locus of $f : X \to \boldsymbol{P}^2$.

If f is of type (C2), namely a $\boldsymbol{P}^1$-bundle, then the argument in [MM83] are valid in positive characteristic. Thus we assume that $f : X \to \boldsymbol{P}^2$ is of type (C1). Let m be an irreducible component of a reducible fiber, or a reduced part of a general fiber when the conic bundle is wild. We analyze X for each type of g.

2.1. The case (E). First assume that g is of type (E). We use the method inspired by [Tak89]. Let E be the exceptional divisor of $g : X \to Y$. Since $\rho(X) = 2$, we can write $E \equiv z(-K_X) - uH$, where $z, u \in \boldsymbol{Q}$. Then from the equalities

$$E.m = z(-K_X).m - uH.m = z \quad \text{and}$$

$$E.(-K_X).H = z(-K_X)^2.H - u(-K_X).H^2 = zc - 2u,$$

we have $z \in \boldsymbol{Z}_{>0}$, $u \in \frac{\boldsymbol{Z}_{>0}}{2}$. Indeed, if $u \leq 0$, then we get $\kappa(z(-K_X) - uH) = 3$, which is absurd.

Consider the case where g is of type (E2), (E3), (E4) or (E5). Then we have the following equalities:

$$\begin{aligned} E^3 &= z^3(-K_X)^3 - 3z^2u(-K_X)^2.H + 3zu^2(-K_X).H^2 - u^3H^3 \\ &= z(z^2(-K_X)^3 - 3czu + 6u^2), \end{aligned} \tag{2.1}$$

$$(2.2)\qquad (-K_X).E^2 = z^2(-K_X)^3 - 2zu(-K_X)^2.H + u^2(-K_X).H^2$$
$$= z^2(-K_X)^3 - 2czu + 2u^2,$$

$$(2.3)\qquad (-K_X)^2.E = z(-K_X)^3 - u(-K_X)^2.H$$
$$= z(-K_X)^3 - cu.$$

In the (E2) case, we have

$$E^3 = 1, \quad (-K_X).E^2 = -2, \quad (-K_X)^2.E = 4.$$

Hence it follows that

$$(z, z^2(-K_X)^3 - 3czu + 6u^2) = (1,1) \text{ or } \left(2, \frac{1}{2}\right)$$

from (2.1). However, by (2.2) and (2.3), it is easily checked that $(-K_X)^3 \notin 2\boldsymbol{Z}$. Thus this case is impossible.

We can also rule out the possibility that g is of type (E5) in a similar way.

In the (E3), (E4) cases, since we have

$$E^3 = 2, \quad (-K_X).E^2 = -2, \quad (-K_X)^2.E = 2,$$

we obtain

$$(-K_X)^3 = 14, \quad c = 6, \quad (z, u) = (1, 2)$$

by the similar argument as the case before. As in characteristic zero, it follows that X is a double cover of $V_7 = \boldsymbol{P}_{\boldsymbol{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$ ([MM83, p. 115]). Note that in characteristic 2, if the double cover is separable then the branch locus B is non-reduced. This belongs to the class No. 8 in the Mori-Mukai's table ([MM81]).

Assume that the conic bundle structure of X is wild. Then since $E.m = 1$ where m is a reduced general fiber of $f : X \to \boldsymbol{P}^2$, we see that $g|_E : E \to \boldsymbol{P}^2$ is a purely inseparable morphism of degree 2. However, by [GR82, Theorem 2.1], there exists no inseparable cover of $\boldsymbol{P}^2$. Thus we can exclude the possibility that X has a wild conic bundle structure in this case.

Now suppose that g is of type (E1). By Theorem 1.3, Y is a Fano threefold of index at least 2, which is classified as in characteristic zero by [Meg98]. Put $g(E) = C$. We denote a fiber of $g|_E : E \to C$ by l. Since $E.l = -1$, we can write $z + 1 = uk$ where k is an integer.

Claim 2.5. *If $u \in \boldsymbol{Z}_{>0}$, then $u = F(Y)$, where $F(Y)$ is the Fano index of Y.*

Proof. Since $E \equiv z(-K_X) - uH$, we have

$$\operatorname{Pic} X/\boldsymbol{Z}E \oplus \boldsymbol{Z}(-K_X) \cong \operatorname{Pic} X/\boldsymbol{Z}uH \oplus \boldsymbol{Z}(-K_X).$$

Take any divisor $D \in \operatorname{Pic} X$, and set $\alpha = D.m$. By $(D - \alpha(-K_X)).m = 0$, we have $D - \alpha(-K_X) \in f^*\operatorname{Pic} \boldsymbol{P}^2$, so we can write $D - \alpha(-K_X) \equiv \beta H$ where $\beta \in \boldsymbol{Z}$. Hence we have

$$\boldsymbol{Z}/u\boldsymbol{Z} \cong \operatorname{Pic} X/\boldsymbol{Z}uH \oplus \boldsymbol{Z}(-K_X).$$

Moreover, by a similar argument, we also have $\operatorname{Pic} X \cong \boldsymbol{Z}g^*\operatorname{Pic} Y \oplus \boldsymbol{Z}E$, hence

$$\operatorname{Pic} X/\boldsymbol{Z}E \oplus \boldsymbol{Z}g^*(-K_Y) \cong \operatorname{Pic} Y/\boldsymbol{Z}(-K_Y).$$

Therefore we have $\boldsymbol{Z}/u\boldsymbol{Z} \cong \operatorname{Pic} Y/\boldsymbol{Z}(-K_Y)$. □

Claim 2.6. *If $u \in \frac{\boldsymbol{Z}_{>0}}{2}$ and $u \notin \boldsymbol{Z}_{>0}$, then $2u \mid F(Y)$.*

Proof. Let M be a fundamental divisor of Y. Since $2E \equiv 2z(-K_X) - 2uH$, we have $2z(-K_Y) \equiv 2ug(H)$, hence

$$2ug(H) \equiv 2zF(Y)M.$$

Since E is reduced, $(2z, 2u) = 1$ by assumption, so we are done. □

We have the following equalities:

$$\begin{aligned}
(2.4)\qquad (-K_Y)^3 &= (-K_X + E)^2.(-K_X)\\
&= ((z+1)(-K_X) - uH)^2.(-K_X)\\
&= (z+1)^2(-K_X)^3 - 2c(z+1)u + 2u^2,\\
(2.5)\qquad 0 &= (-K_X + E)^2.E\\
&= ((z+1)(-K_X) - uH)^2.(z(-K_X) - uH)\\
&= z(z+1)^2(-K_X)^3 - c(z+1)(3z+1)u + 2(3z+2)u^2,\\
(2.6)\qquad (-K_Y).C &= (-K_X + E).E.(-K_X)\\
&= ((z+1)(-K_X) - uH).(z(-K_X) - uH).(-K_X)\\
&= z(z+1)(-K_X)^3 - c(2z+1)u + 2u^2,\\
(2.7)\qquad (-K_E)^2 &= (-K_X - E)^2.E\\
&= 4(z^2(-K_X)^3 - 2zu(-K_X)^2.H + u^2(-K_X).H^2)\\
&= -4(z^2(-K_X)^3 - 2czu + 2u^2).
\end{aligned}$$

The case $F(Y) = 4$. Clearly, $Y \cong \boldsymbol{P}^3$ and $(-K_Y)^3 = 64$. By Claims 2.5 and 2.6, we have $u = 4$ or $\frac{1}{2}$. Assume that $u = 4$. Then by (2.4), $k = \frac{z+1}{u} = 1$ or 2. If $k = 1$, then by (2.5)–(2.7), we obtain

$$z = 3, \quad (-K_X)^3 = 16, \quad \deg C = 7 \quad \text{and} \quad p_a(C) = 5.$$

This belongs to the class No. 9. We also have $c = 7$, which means $\deg \Delta = 5$ when $f : X \to \boldsymbol{P}^2$ is ordinary. If $k = 2$, then we see that $(-K_X)^3 \notin \boldsymbol{Z}$, which is absurd.

On the other hand, if $u = \frac{1}{2}$, by (2.4) and (2.5), we get $(-K_X)^3 < 0$, which is impossible.

The case $F(Y) = 3$. Clearly, $Y \cong Q$, where Q is a quadric threefold in $\boldsymbol{P}^4$, and $(-K_Y)^3 = 54$. By Claims 2.5 and 2.6, we have $u = 3, \frac{1}{2}$ or $\frac{3}{2}$. If $u = 3$, then by (2.4), $k = 1, 2$ or 4. When $k = 1$, we obtain

$$z = 2, \quad (-K_X)^3 = 20, \quad \deg C = 6 \quad \text{and} \quad p_a(C) = 2$$

by (2.5)–(2.7). If $f : X \to \boldsymbol{P}^2$ is ordinary, we have $\deg \Delta = 4$. This belongs to the class No. 13.

We have no solutions when $k = 2, 4$. An argument similar to that when $F(Y) = 4$ also gives a contradiction if $u = \frac{1}{2}$ or $\frac{3}{2}$.

The case $F(Y) = 2$. In this case, Y is a del Pezzo threefold of degree d, and $(-K_Y)^3 = 8d$. We obtain $u = 2$ and

$$z = 1, \quad (-K_X)^3 = 2(2d + 3), \quad \deg C = d - 2 \quad (d = 3, 4 \text{ or } 5)$$

by the same argument as the two cases above. These belong to the classes Nos. 11, 16 and 20, respectively.

Proposition 2.7. *If X is a Fano threefold with $\rho(X) = 2$ such that the two extremal contractions are* (C1) *and* (E1) *types, then the conic bundle structures of X is not wild.*

Proof. Assume that $f : X \to \boldsymbol{P}^2$ is wild. Then every fiber of f is a double line $2m$. If $F(Y) = 2$, then we can prove in the same way as the case where g is of type (E3) or (E4); indeed, we have $E.m = 1$, which means the restriction morphism $f|_E : E \to \boldsymbol{P}^2$ is purely inseparable. However, there exists no purely inseparable cover of $\boldsymbol{P}^2$ by [GR82, Theorem 2.1], hence we have a contradiction.

Suppose that $F(Y) = 3$. In this case, X is the blow-up of a quadric threefold Q along a curve of degree 6, and $g(H)$ is a quadric section in Q, namely, a non-normal del Pezzo surface of degree 4. We denote the curve $E|_H$ by C'. C' is a Cartier divisor on E. Moreover, C' is smooth. Indeed, otherwise C' is the union of a section and fibers on E, but it contradicts that H has unibranched singularities. Hence H is smooth along C' and isomorphic to $g(H)$. Since $\mathrm{Sing}(H)$ are unibranched and the normalization $\tilde{H}$ of H is a smooth $\boldsymbol{P}^1$-bundle by Lemma 2.3, $\tilde{H}$ belongs to the case (e) in the classification table in Theorem 1.4. Since C' is of degree 6, its inverse image on $\tilde{H}$ is equivalent to $4A + B$ or $2A + 2B$. However, in both cases, it must intersect the curve defined by the conductor ideal. Thus we have a contradiction.

We can show the claim in the same way when $F(Y) = 4$. □

2.2. The case (D). Next, we assume that g is of type (D), namely a del Pezzo fibration. Clearly, $Y \cong \boldsymbol{P}^1$. We can write $H \equiv x(-K_X) - yS$ with $x, y \in \boldsymbol{Q}$, where S is a fiber of $g : X \to \boldsymbol{P}^1$.

Consider first the case where g is of type (D1). Let d be the degree of a general fiber. Since $S^3 = (-K_X).S^2 = 0$, $(-K_X)^2.S = d$, we have the following relations:

$$H^3 = x^3(-K_X)^3 - 3x^2yd = 0, \tag{2.8}$$

$$(-K_X).H^2 = x^2(-K_X)^3 - 2xyd = 2, \tag{2.9}$$

$$(-K_X)^2.H = x(-K_X)^3 - yd = c. \tag{2.10}$$

Since a general fiber S has a line l, we have

$$x = x(-K_X).l - yS.l = H.l \in \boldsymbol{Z}.$$

We also see that $x/y = S.m \in \boldsymbol{Z}$. Moreover, since $\kappa(H) = 2$, we have $x > 0$, $y > 0$. Under these conditions, the equations (2.8)–(2.10) have the following solutions:

$$(-K_X)^3 = 6, \quad c = 4, \quad (x, y, d) = (1, 1, 2) \text{ or } (1, 1/2, 4).$$

However, since S is reduced, the latter case is excluded. Hence S is a del Pezzo surface of degree 2. Take $h := (f, g) : X \to \boldsymbol{P}^2 \times \boldsymbol{P}^1$. h is finite and surjective. We have

$$\deg h = H^2.S = H^2.((-K_X) - H) = 2,$$

therefore X is a double cover of $\boldsymbol{P}^2 \times \boldsymbol{P}^1$. This belongs to the class No. 2.

Now assume that the conic bundle $f : X \to \boldsymbol{P}^2$ is wild. Then h is set-theoretically one-to-one, so it is a purely inseparable cover of degree 2. By comparing the canonical divisors, we see that the invertible sheaf defined by h is $\mathscr{L} \cong \mathcal{O}_{\boldsymbol{P}^2}(2) \boxtimes \mathcal{O}_{\boldsymbol{P}^1}(1)$ (see Section 1.2). Hence $c_3(\mathscr{L}^{\otimes 2} \otimes \Omega^1) = 40$ which implies the sheaf $\mathscr{L}^{\otimes 2} \otimes \Omega^1$ cannot have any nowhere vanishing sections. Therefore we conclude that there exists no wild conic bundle structure in this case.

PROPOSITION 2.8. *Under the notation above, a general fiber X_η of del Pezzo fibration $g : X \to \boldsymbol{P}^1$ is normal.*

Proof. Since $f : X \to \boldsymbol{P}^2$ is not wild and $\mathrm{H}^1(\boldsymbol{P}^2 \times \boldsymbol{P}^1, \mathscr{L}) = 0$, X is a separable double cover of $\boldsymbol{P}^2 \times \boldsymbol{P}^1$ and given by an equation $y^2 + ay + b = 0$, where $a \in \mathrm{H}^0(\boldsymbol{P}^2 \times \boldsymbol{P}^1, \mathscr{L})$ and $b \in \mathrm{H}^0(\boldsymbol{P}^2 \times \boldsymbol{P}^1, \mathscr{L}^{\otimes 2})$. Let $[s, t]$ be the homogeneous coordinates of $\boldsymbol{P}^1$. Then we can write

$$a = sp(x_0, x_1, x_2) + tq(x_0, x_1, x_2),$$
$$b = s^2 r_1(x_0, x_1, x_2) + str_2(x_0, x_1, x_2) + t^2 r_2(x_0, x_1, x_2),$$

where $p(x_0, x_1, x_2)$ and $q(x_0, x_1, x_2)$ are quadrics, and $r_i(x_0, x_1, x_2)$ $(i = 1, 2, 3)$ are quartic homogeneous polynomials. Assume that X_η is non-normal. Then the singularity on X_η is one-dimensional locus, and contained in the locus defined by $a = 0$. On the other hand, since the total space X is smooth, the equations

$$p(x_0, x_1, x_2)y + tr_2(x_0, x_1, x_2) = 0, \tag{2.11}$$

$$q(x_0, x_1, x_2)y + sr_2(x_0, x_1, x_2) = 0 \tag{2.12}$$

have no common zeros. Since X_η is of degree 2, its normalization $\tilde{X}_\eta$ is of type (c) or (d) in Theorem 1.4. Then in both cases the inverse image of the non-normal locus is ample, so we can take the point P which is the intersection of the locus defined by (2.11) and the non-normal locus on X_η. However, P also satisfies (2.12) since $a(P) = 0$. Hence X must have a singularity at P. □

Next suppose that g is of type (D2). Since $S^3 = (-K_X).S^2 = 0$ and $(-K_X)^2.S = 8$, by computing in the same way as the (D1) case, we have the following:

$$(-K_X)^3 = 24, \quad c = 8, \quad (x, y) = (1/2, 1/2).$$

This belongs to the class No. 18. As in the case before, we see that $h := (f, g) : X \to \boldsymbol{P}^2 \times \boldsymbol{P}^1$ is a double cover of $\boldsymbol{P}^2 \times \boldsymbol{P}^1$.

If $f : X \to \boldsymbol{P}^2$ is a wild conic bundle, then X is a purely inseparable cover of $\boldsymbol{P}^2 \times \boldsymbol{P}^1$. Therefore we get a contradiction in the same way as in the case before.

For the possibility that the general fiber is not smooth in this case, we discuss in Section 5.

2.3. The case (C). Now we assume that g is of type (C). Clearly, $Y \cong \boldsymbol{P}^2$. First we consider the (C1) case. Let L be the inverse image of a line in $\boldsymbol{P}^2$ and set $H \equiv x(-K_X) - yL$. We have the following relations:

$$L^3 = 0,$$

$$(-K_X).L^2 = 2,$$

$$(-K_X)^2.L = 12 - \deg \Delta,$$

where Δ is the discriminant locus on $\boldsymbol{P}^2$ of the conic bundle $g : X \to \boldsymbol{P}^2$. The last equality follows from Lemma 2.4. Hence we obtain

$$H^3 = x^3(-K_X)^3 - 3x^2y(12 - \deg \Delta) + 6xy^2 = 0, \tag{2.13}$$

$$(-K_X).H^2 = x^2(-K_X)^3 - 2xy(12 - \deg \Delta) + 2y^2 = 2, \tag{2.14}$$

$$(-K_X)^2.H = x(-K_X)^3 - y(12 - \deg \Delta) = c. \tag{2.15}$$

If $x < 0$, then we have $y < 0$ since $L.m = x/y \in \boldsymbol{Z}_{>0}$. Thus we see that $\kappa(-yL) = \kappa(-x(-K_X) + H) = 3$, which is absurd. Hence $x > 0$, $y > 0$. Moreover, x is an integer since $H.l = x(-K_X).l - yL.l = x$, where l is a component of degenerate fibers on the discriminant locus Δ. On the other hand, the equality $-K_X.L.H = x(-K_X)^2.L - y(-K_X).L^2$ gives

$$y = \frac{-K_X.L.H - x(12 - \deg \Delta)}{2} \in \frac{\boldsymbol{Z}_{>0}}{2}.$$

We have also that $1/y \in \frac{\boldsymbol{Z}_{>0}}{2}$ in a similar way. Hence $y = 1/2, 1$ or 2. Under these conditions, we solve (2.13)–(2.15) and obtain

$$(-K_X)^3 = 12, \quad c = 6, \quad (x, y, 12 - \deg \Delta) = (1, 1, 6).$$

Take the morphism $h = (f, g) : X \to \boldsymbol{P}^2 \times \boldsymbol{P}^2$. By an argument similar to that in [MM83, pp. 114–115], we see that h is either an embedding and $h(X)$ is a divisor of bidegree $(2, 2)$ in $\boldsymbol{P}^2 \times \boldsymbol{P}^2$, or a double cover and $h(X)$ is a divisor W of bidegree $(1, 1)$. This belongs to the class No. 6.

Now consider the case where the conic bundle $f : X \to \boldsymbol{P}^2$ is wild. Consider the former case. Let P be a singular point of a degenerate fiber of the ordinary conic bundle $g : X \to \boldsymbol{P}^2$. Clearly, $g(P)$ is on the discriminant locus Δ in $\boldsymbol{P}^2$. We can take an open set $U \subset \boldsymbol{P}^2$ with local coordinates (y_1, y_2) containing $g(P)$ such that the local equation of X over U is expressed as

$$f_1(x)y_1^2 + f_2(x)y_2^2 + \alpha f_3(x) = 0,$$

where $f_1(x), f_2(x), f_3(x)$ are homogeneous polynomials of degree 2 and α is a constant. Then it is easy to see that the total space X also has a singularity at P. Thus we exclude this possibility. In the latter case, the sheaf defining the double cover $\mathscr{L}$ is isomorphic to $\mathcal{O}_{\boldsymbol{P}^2}(1) \boxtimes \mathcal{O}_{\boldsymbol{P}^2}(1)$. Thus we obtain $c_3(\mathscr{L}^{\otimes 2} \otimes \Omega_W^1) = 18$, which implies that X has singularities. Hence this case is also impossible.

Next suppose that g is of type (C2). By some numerical calculations similar to ones above, we obtain

$$(-K_X)^3 = 30, \quad c = 9, \quad (x, y) = (1/2, 1/2).$$

This belongs to the class No. 24. Consider the morphism $h = (f, g) : X \to \boldsymbol{P}^2 \times \boldsymbol{P}^2$. Then X is a divisor of bidegree $(2, 1)$ in $\boldsymbol{P}^2 \times \boldsymbol{P}^2$.

The example below by Kollár shows that there exists a Fano threefold having a wild conic bundle structure in this class.

Example 2.9 ([Kol91]). Let X be the smooth hypersurface in $\boldsymbol{P}^2 \times \boldsymbol{P}^2$ defined by

$$x_0y_0^2 + x_1y_1^2 + x_2y_2^2 = 0,$$

where $[x_0, x_1, x_2]$ and $[y_0, y_1, y_2]$ are homogeneous coordinates. Then projection to the first factor makes it into a wild conic bundle. X is a Fano threefold, and has two extremal contractions of types (C1) and (C2). X belongs to the class No. 24.

3. Del Pezzo fibrations

We treat here Fano threefolds having a structure of del Pezzo fibrations. Note that in positive characteristic, a general fiber of del Pezzo fibrations may have singularities.

Let X be a Fano threefold with $\rho(X) = 2$. Suppose that X has a del Pezzo fibration, say $f : X \to \boldsymbol{P}^1$. Let $g : X \to Y$ be the other extremal contraction:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \boldsymbol{P}^1 & & Y. \end{tikzcd}$$

Let S be a general fiber of f, and d its degree. By the argument in [MM83, p. 124], g is either of type (E1) or (C). The case where g is of type (C1) was treated in the previous section. On the other hand, $f : X \to \boldsymbol{P}^1$ is just a $\boldsymbol{P}^2$-bundle when it is of type (D3), and we can classify them as in characteristic zero. So we consider the case where f is of type (D1) or (D2), and g is of type (E1).

Suppose first that f is of type (D1). Then we have the following intersection numbers:

$$S^3 = (-K_X).S^2 = 0, \quad (-K_X)^2.S = d.$$

Suppose that g is of type (E1). Let E be the exceptional divisor of g, and C the center of the blow-up. Since $\rho(X) = 2$, we can put $E \equiv z(-K_X) - uS$, where $z, u \in \boldsymbol{Q}$. In the same way as in the case where X has a conic bundle structure, we have the following equalities:

$$(z+1)^2(-K_X)^3 - 2(z+1)ud = (-K_Y)^3, \tag{3.1}$$

$$z(-K_X)^3 - (3z+1)ud = 0, \tag{3.2}$$

$$z(z+1)(-K_X)^3 - (2z+1)ud = (-K_Y).C, \tag{3.3}$$

$$-4(z^2(-K_X)^3 - 2zud) = (-K_E)^2. \tag{3.4}$$

Since S has a line, we have $z \in \boldsymbol{Z}_{>0}$. Moreover, for a fiber l of $g|_E : E \to C$, we have

$$z(-K_X).l - uS.l = E.l = -1,$$

hence we have $\frac{z+1}{u} = S.l \in \boldsymbol{Z}_{>0}$. Since $E.(-K_X)^2 = z(-K_X)^3 - ud$, we have $u \in \frac{\boldsymbol{Z}_{>0}}{d}$. Note that if S is not smooth, the fibration $f : X \to \boldsymbol{P}^1$ does not necessarily have a section.

Claim 3.1. *If $u \in \boldsymbol{Z}_{>0}$, then $u = F(Y)$, where $F(Y)$ is the Fano index of Y.*

Proof. Just as Claim 2.5. □

Claim 3.2. *If $u \notin \boldsymbol{Z}_{>0}$ and $u \in \frac{\boldsymbol{Z}_{>0}}{d}$, one of the following holds*:
(1) $ud \mid F(Y)$.
(2) $d = 4$, $u \in \frac{\boldsymbol{Z}_{>0}}{2}$.

Proof. Let M be a fundamental divisor of Y. Since $dE \equiv dz(-K_X) - duS$, we have $dz(-K_Y) \equiv dug(S)$, hence $dug(S) \equiv dzF(Y)M$. Since E is reduced, $(dz, du) = 1$ if $d \neq 4$ by assumption. If $d = 4$, (dz, du) may be 2, which is the case $u \in \frac{\boldsymbol{Z}_{>0}}{2}$. □

By Theorem 1.3, we have $F(Y) = 2, 3$ or 4.

The case $F(Y) = 4$. Clearly, $Y \cong \boldsymbol{P}^3$ and $(-K_Y)^3 = 64$. Then for the equations (3.1)–(3.4), there exists the only one solution under the claims above and the other numerical conditions, which is as follows:

$$(z, u) = (3, 4), \quad d = 3, \quad (-K_X)^3 = 10, \quad \deg C = 9.$$

Hence X is the blow-up of $\boldsymbol{P}^3$ with center an intersection of two cubic hypersurfaces. This belongs to the class No. 4.

The case $F(Y) = 3$. Clearly, $Y \cong Q$ and $(-K_Y)^3 = 54$. We obtain the solution for (3.1)–(3.4) in the same way as in the case $F(Y) = 4$:

$$(z, u) = (2, 3), \quad d = 4, \quad (-K_X)^3 = 14, \quad \deg C = 4.$$

Hence X is the blow-up of Q with center an intersection of two members in $|\mathcal{O}_Q(2)|$. This belongs to the class No. 7.

The case $F(Y) = 2$. In this case, Y is a del Pezzo threefold of degree d, and $(-K_Y)^3 = 8d$. By the classification of del Pezzo threefolds, we have $d \leq 5$. We obtain

$$(z, u) = (1, 2), \quad (-K_X)^3 = 4d, \quad \deg C = d \quad (d = 1, \dots, 5)$$

by the same argument as the two cases above. These belong to the classes Nos. 1, 3, 5, 10 and 14, respectively.

We also see that S is normal as in the previous section. Assume that S is non-normal, and denote its normalization by $\mu : \tilde{S} \to S$. Let Γ be the non-normal locus on S. The inverse image of Γ on $\tilde{S}$ is described in the list of Theorem 1.4. Consider the case where $F(Y) = 2$. Then as in the previous section, we see that S is smooth along the curve $E|_S$. However, $E|_S$ is of degree d, hence its inverse image must intersect $\mu^{-1}(\Gamma)$ which is a conic. Thus we have a contradiction. We can show the normality of S in the same way when $F(Y) \geq 3$.

Suppose next that f is of type (D2). We may assume that g is of type (E1). Let E be the exceptional divisor of $g : X \to Y$, and C the center of the blow-up. We set $E \equiv z(-K_X) - uH$. Then the numerical argument as the case before gives the two classes:

$$(z, u) = (1, 2), \quad (-K_X)^3 = 32, \quad \deg C = 4 \quad \text{or}$$

$$(z, u) = (1/2, 3/2), \quad (-K_X)^3 = 40, \quad \deg C = 2.$$

These correspond to the classes Nos. 25 and 29 in the table.

4. Divisorial contractions

Let X be a Fano threefold with $\rho(X) = 2$. When the both extremal contractions are divisorial, Mori-Mukai's method works in any characteristic. We give here the classification using the numerical argument only for the case where the both contractions are of (E1) type. We can classify them having the contractions of the other types in the similar way.

We denote the two contractions by f, g:

$$\begin{array}{ccc} & X & \\ {}^{f}\swarrow & & \searrow^{g} \\ Y_1 & & Y_2. \end{array}$$

We set $L := f^*M$, where M is the fundamental divisor in Y_1. Let E be the exceptional divisor of $g : X \to Y_2$ and C its center. By Theorem 1.3, Y_1 and Y_2 are both Fano threefolds of index at least 2. Let $F(Y_i)$ be the Fano index of Y_i. We may assume that $F(Y_1) \geq F(Y_2)$. Since $\rho(X) = 2$, we can write $E \equiv z(-K_X) - uH$, where $z, u \in \boldsymbol{Q}$. We set $c := (-K_X)^2.H$. Then as the case before, we have the following equalities:

$$\begin{aligned}
(4.1)\qquad (-K_{Y_2})^3 &= (-K_X + E)^2.(-K_X) \\
&= ((z+1)(-K_X) - uH)^2.(-K_X) \\
&= (z+1)^2(-K_X)^3 - 2c(z+1)u + u^2(-K_X).H^2, \\
(4.2)\qquad 0 &= (-K_X + E)^2.E \\
&= ((z+1)(-K_X) - uH)^2.(z(-K_X) - uH) \\
&= z(z+1)^2(-K_X)^3 - c(z+1)(3z+1)u \\
&\quad + (3z+2)u^2(-K_X).H^2 - u^3H^3, \\
(4.3)\qquad (-K_{Y_2}).C &= (-K_X + E).E.(-K_X) \\
&= ((z+1)(-K_X) - uH).(z(-K_X) - uH).(-K_X) \\
&= z(z+1)(-K_X)^3 - c(2z+1)u + u^2(-K_X).H^2, \\
(4.4)\qquad (-K_E)^2 &= (-K_X - E)^2.E \\
&= 4(z^2(-K_X)^3 - 2zu(-K_X)^2.H + u^2(-K_X).H^2) \\
&= -4(z^2(-K_X)^3 - 2czu + u^2(-K_X).H^2).
\end{aligned}$$

As in the previous sections, we see that $z \in \boldsymbol{Z}_{>0}$ and $\frac{z+1}{u} \in \boldsymbol{Z}_{>0}$. The following claim is proved just as Claim 2.5:

CLAIM 4.1. *If $u \in \boldsymbol{Z}_{>0}$, then $u = F(Y_2)$.*

Suppose first $Y_1 \cong \boldsymbol{P}^3$. Then we have $H^3 = 1$ and $(-K_X).H^2 = F(Y_1).H^2 = 4$. Moreover, $E.H^2 = z(-K_X).H^2 - uH^3$ implies $u \in \boldsymbol{Z}_{>0}$, hence $u = F(Y_2)$. Hence from the equalities (4.1)–(4.4), we obtain the classes of Fano threefolds having the following invariants:

$$(z,u) = (3,4), \quad (-K_X)^3 = 20, \quad \deg C = 6 \quad \text{if} \quad Y_2 \cong \boldsymbol{P}^3,$$

$$(z,u) = (2,3), \quad (-K_X)^3 = 24, \quad \deg C = 5 \quad \text{if} \quad Y_2 \cong Q,$$

$$(z,u) = (1,2), \quad (-K_X)^3 = 26, \quad \deg C = 1 \quad \text{if} \quad Y_2 \cong V_4,$$

$$(z,u) = (1,2), \quad (-K_X)^3 = 30, \quad \deg C = 2 \quad \text{if} \quad Y_2 \cong V_5,$$

where Q is a quadric threefold in $\boldsymbol{P}^4$, and V_d is a del Pezzo threefold of degree d. These correspond to the classes Nos. 12, 17, 19 and 22, respectively.

When $Y_1 \cong Q$, we have $H^3 = 2$ and $(-K_X).H^2 = 6$. Moreover, we have $u \in \frac{\boldsymbol{Z}_{>0}}{2}$. If $u \in \boldsymbol{Z}_{>0}$, then since $u = F(Y_2)$, we solve the equalities (4.1)–(4.4) to obtain the following:

$$(z,u) = (2,3), \quad (-K_X)^3 = 28, \quad \deg C = 4 \quad \text{if} \quad Y_2 \cong Q,$$

$$(z,u) = (1,2), \quad (-K_X)^3 = 34, \quad \deg C = 1 \quad \text{if} \quad Y_2 \cong V_5.$$

These correspond to the classes Nos. 21 and 26, respectively.

If $u \notin \boldsymbol{Z}_{>0}$ and $u \in \frac{\boldsymbol{Z}_{>0}}{2}$, it is easy to see that $2u \mid 3$. Hence we have $u = \frac{1}{2}$ or $\frac{3}{2}$. However, we obtain $(-K_X)^3 \notin \boldsymbol{Z}$ in both cases, which is impossible.

Assume that $Y_1 \cong V_{d_1}$. We have $H^3 = d_1$ and $(-K_X).H^2 = 2d_1$. Since $E.H^2 = 2d_1 z - ud_1$, we have $u \in \frac{\boldsymbol{Z}_{>0}}{d_1}$. If $u \in \boldsymbol{Z}_{>0}$, then $u = F(Y_2) = 2$. Let d_2 be the degree of Y_2. Then from (4.1) and (4.2), we have $(-K_X)^3 = 4(d_1 + d_2)$. However, by (4.3) we obtain $(-K_{Y_2}).C = -20d_1 - 4d_2$, which is a contradiction. When $u \notin \boldsymbol{Z}_{>0}$, the assertion just as Claim 3.2 holds. For each d_1 and d_2, we solve the equations (4.1)–(4.4) to see that $(-K_X)^3$ is negative or not integer. Clearly, this is impossible.

Thus we have done for (i)–(iii) of Theorem 0.1.

5. Quadric cone fibrations

Definition. Let $f : X \to \boldsymbol{P}^1$ be a surjective morphism from a smooth projective threefold X to $\boldsymbol{P}^1$. We say that f is a *quadric cone fibration* if every fiber of f is a quadric cone in $\boldsymbol{P}^3$. As an extremal contraction, f is of type (D2).

By the classification, Fano threefolds having the extremal contraction of type (D2) are in the following three classes:

No. 18: a double cover of $\boldsymbol{P}^2 \times \boldsymbol{P}^1$ whose branch locus is a divisor of bidegree $(2,2)$.

No. 25: the blow-up of $\boldsymbol{P}^3$ along an elliptic curve which is an intersection of two quadrics.
No. 29: the blow-up of $Q \subset \boldsymbol{P}^4$ along a conic.

Proposition 5.1. *Fano threefolds of the classes No.* 18 *and No.* 25 *cannot have a quadric cone fibration structure.*

Proof. Consider first the case No. 18. X has two extremal contractions of types (C1) and (D2). We denote the corresponding morphisms by $f : X \to \boldsymbol{P}^2$ and $g : X \to \boldsymbol{P}^1$. Now assume that $g : X \to \boldsymbol{P}^1$ is a quadric cone fibration. We have $-K_X \equiv 2H + S$, where $H = f^*\mathcal{O}_{\boldsymbol{P}^2}(1)$ and $S = g^*\mathcal{O}_{\boldsymbol{P}^1}(1)$. Let C be the curve defined by the locus of vertices of quadric cones. Since $S.C = 2$, $g|_C : C \to \boldsymbol{P}^1$ is a purely inseparable morphism of degree 2. In particular, C is a conic. Hence we have $H.C = 0$ which means C is a fiber of $f : X \to \boldsymbol{P}^2$.

Let $\varphi : Z \to X$ be the blow-up of X along C, and E its exceptional divisor. Then Z has two conic bundle structures: one of them is $f' : Z \to \boldsymbol{F}_1$, which derives from the conic bundle structure $f : X \to \boldsymbol{P}^2$. The base surface $\boldsymbol{F}_1$ is obtained as the blow-up of $\boldsymbol{P}^2$ at the point $f(C)$. The other one is $g' : Z \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Its general fiber is the inverse image of two rulings of a quadric cone. Hence $g' : Z \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ is a wild conic bundle over $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. Since $E.m = 1$ where m is a reduced fiber of g', $g'|_E : E \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ is a purely inseparable morphism. Take $H_0 := f^{-1}(\ell)$ where ℓ is a line on $\boldsymbol{P}^2$ through $f(C)$, and set $L := \varphi^*(H_0)$. Then L is the inverse image of a ruling on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ and covered by lines. Hence $H_0.S$ is double rulings of S. However, this means each fiber of f over every point on ℓ is a double line. This is a contradiction.

Next we consider Fano threefold X in the class No. 25. Suppose that $f : X \to \boldsymbol{P}^1$ is a quadric cone fibration. Set $S := f^*\mathcal{O}_{\boldsymbol{P}^1}(1)$. X has the divisorial contraction of type (E1). We denote the corresponding morphism by $g : X \to \boldsymbol{P}^3$ and its exceptional divisor by E_1. Note that $g(E_1)$ is an elliptic curve which is an intersection of quadric cones. We have $-K_X \equiv E_1 + 2S$ by the classification. Let C_2 be the curve defined by the locus of vertices of quadric cones. Since $S.C_2 = 2$, we see that $g|_{C_2} : C_2 \to \boldsymbol{P}^1$ is purely inseparable, and $g(C_2)$ is a line in $\boldsymbol{P}^3$. We consider the blow-up $\varphi : Z \to X$ along C_2, and denote its exceptional divisor by E_2. Then Z has a wild conic bundle structure over $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ such that each fiber is the inverse image of two rulings of a quadric cone. We denote it by $h : Z \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Now we take the blow-down $\psi : Z \to Y$ of Z contracting E_1 to obtain a new Fano threefold Y. This is the blow-up of $\boldsymbol{P}^3$ along a line, hence belongs to the class No. 33:

$$\begin{tikzcd}
\rho = 3 & & Z \arrow[dl, "\varphi"'] \arrow[d, "h"] \arrow[dr, "\psi"] & \\
\rho = 2 & X \text{ (No. 25)} \arrow[dr, "g"'] & \boldsymbol{P}^1 \times \boldsymbol{P}^1 & Y \text{ (No. 33)} \arrow[dl] \\
\rho = 1 & & \boldsymbol{P}^3. &
\end{tikzcd}$$

We denote the inverse image of a ruling of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ by H, and set $C_1 := \psi(E_1)$. We have $-K_Z \equiv 2H + E_1 - E_2$. $\psi(H)$ is a quadric in Y, hence a non-normal del Pezzo surface of degree 2 containing C_1. We see that H is smooth along $E_1|_H$ as in the previous section. On the other hand, since the normalization $\tilde{H}$ of H is a smooth $\boldsymbol{P}^1$-bundle by Lemma 2.3, it is of type (d) in Theorem 1.4. It is easy to see that an elliptic curve of degree 4 must intersect the inverse image of the non-normal locus. Hence we obtain a contradiction. □

We have an example of Fano threefolds of the class No. 29 such that its general fiber is a quadric cone.

Example 5.2. Let Y be a quadric threefold in $\boldsymbol{P}^4$ defined by

$$x_0^2 + x_1x_2 + x_3x_4 = 0.$$

Then every hyperplane section derived from a hyperplane through the point $[1,0,0,0,0]$ is a quadric cone. In particular, the smooth conic defined by

$$C\colon\ x_0^2 + x_1x_2 = 0, \quad x_3 = x_4 = 0$$

is the intersection of one-dimensional family of quadric cones. Let $f : X \to Y$ be the blow-up along C. Then X is a Fano threefold and has a del Pezzo fibration with a singular general fiber, namely, a quadric cone.

Example 5.3. We have also a new example of Fano threefolds having a wild conic bundle structure. Let Y be a quadric threefold defined in Example 5.2, and C_1 and C_2 smooth conics defined by

$$C_1\colon\ x_0^2 + x_1x_2 = 0, \quad x_3 = x_4 = 0 \quad \text{and}$$
$$C_2\colon\ x_0^2 + x_3x_4 = 0, \quad x_1 = x_2 = 0,$$

respectively. Note that C_1, C_2 are disjoint. Let $f : X \to Y$ be the blow-up along C_1 and C_2. Then X has a wild conic bundle structure over $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. It is easy to see that X is a Fano threefold with $\rho(X) = 3$, and $(-K_X)^3 = 10$.

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References

[Bea77] A. Beauville, Variétés de Prym et jacobiennes intermédiaires, Ann. Sci. École Norm. Sup. (4), **10** (1977), 309–391.

[CD89] F. R. Cossec and I. V. Dolgachev, Enriques surfaces. I, Progr. Math. 76, Birkhäuser Boston, Boston, 1989.

[Fuj90] T. Fujita, Classification Theories of Polarized Varieties, London Math. Soc. Lecture Note Ser. 155, Cambridge University Press, Cambridge, 1990.

[GR82] R. Ganong and P. Russell, Derivations with only divisorial singularities on rational and ruled surfaces, J. Pure Appl. Algebra, **26** (1982), 165–182.

[HW81] F. Hidaka and K. Watanabe, Normal Gorenstein surfaces with ample anticanonical divisor, Tokyo J. Math., **4** (1981), 319–330.

[Isk77] V. A. Iskovskih, Fano threefolds. I, Izv. Akad. Nauk SSSR Ser. Mat., **41** (1977), 516–562.

[Isk78] V. A. Iskovskih, Fano threefolds. II, Izv. Akad. Nauk SSSR Ser. Mat., **42** (1978), 506–549.

[Kol91] J. Kollár, Extremal rays on smooth threefolds, Ann. Sci. École Norm. Sup. (4), **24** (1991), 339–361.

[Meg98] G. Megyesi, Fano threefolds in positive characteristic, J. Algebraic Geom., **7** (1998), 207–218.

[MM81] S. Mori and S. Mukai, Classification of Fano 3-folds with $B_2 \geq 2$, Manuscripta Math., **36** (1981), 147–162.

[MM83] S. Mori and S. Mukai, On Fano 3-folds with $B_2 \geq 2$, Algebraic Varieties and Analytic Varieties (Tokyo, 1981), Adv. Stud. Pure Math. 1, North-Holland, Amsterdam, 1983, 101–129.

[MM86] S. Mori and S. Mukai, Classification of Fano 3-folds with $B_2 \geq 2$. I, Algebraic and Topological Theories (Kinosaki, 1984), Kinokuniya, Tokyo, 1986, 496–545.

[Rei94] M. Reid, Nonnormal del Pezzo surfaces, Publ. Res. Inst. Math. Sci., **30** (1994), 695–727.

[SB97] N. I. Shepherd-Barron, Fano threefolds in positive characteristic, Compositio Math., **105** (1997), 237–265.

[Sho79] V. V. Shokurov, The existence of a line on Fano varieties, Izv. Akad. Nauk SSSR Ser. Mat., **43** (1979), 922–964.

[Tak89] K. Takeuchi, Some birational maps of Fano 3-folds, Campositio Math., **71** (1989), 265–283.

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