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STABILITY AND QUANTUM PHENOMENEN AND LIOUVILLE THEOREMS OF p-HARMONIC MAPS WITH POTENTIAL[1]

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Abstract

In this paper, we discuss the stability and the pointwise gap phenomenen of p-harmonic maps with potential. Stability theorems of p-H-harmonic maps from or into general submanifolds of the shpere and the Euclidean space are established, and Sealey's quantum theorem is extended. We also discuss the conservation law and the Liouville theorems of p-H-harmonic maps. As a consequence of our stability theorem, we not only generalize Leung's stability theorem to rather general case, but also improve it by replacing the sectional curvature bound by a Ricci curvature bound. In order to discuss the gap property of p-harmonic maps, we establish a Bochner-typed formula which is used by some authors in a uncorrect form.

1. Introduction

Let M^m and N^n be Riemannian manifolds, $u : M \to N$ a smooth map, H a smooth function on N. We call u a p-harmonic map with potential H or a p-H-harmonic map if it is a critical point of the p-H-energy:

$$E_{p,H}(u) = \frac{1}{p}\int_M |du|^p - \int_M H \circ u. \tag{1}$$

If H is constant, a p-H-harmonic map is called p-harmonic. A 2-harmonic map is called harmonic. Hence p-H-harmonic maps are a generalization of the usual ones.

In this paper, we always assume that all initial manifolds are compact and that $p \geq 2$.

Y. L. Xin in [13] proved that any stable harmonic map from S^m $(m > 2)$ is constant and P. F. Leung in [8] proved that any stable harmonic map from M^m $(m > 2)$ to a hypersurface of Euclidean space is constant. Q. Chen in [2] generalized them to harmonic maps with potential. Ohnita in [9] verified that stable harmonic maps from or into minimal submanifolds of the sphere is constant if the Ricci curvatures of the submanifolds are bigger than half the

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dimensions. In Section 3, we investigate the stability of p-H-harmonic maps from or into general submanifolds of the sphere and the Euclidean space. Even back to original harmonic maps, our results are new and optimal.

H. C. J. Sealey in [10] demonstrated a quantum theorem on harmonic maps. A. M. Matei in [6] discuss the quantum properties of p-harmonic maps. In Section 4, we investigave quantum phenomena of p-H-harmonic maps. Our method is different from A. M. Matei's. We use a different Bochner formula which is verified in Section 2.

Y. L. Xin in [14] used conservation law to obtain a Liouville theorem for harmonic maps and Q. Chen in [2] generalized Karcher and Wood's theorem to harmonic maps with potential taking use of the same technique. In Section 5, we introduce a stress p-energy tensor and discuss the corresponding conservation law, and obtain two Liouville theorems of p-H-harmonic maps.

2. Preliminaries

In the following, we denote the exterior differential operator on bundle-valued r-forms by d, its adjoint by d^*. We use ∇ and $\langle\, ,\rangle$ stand for connections and inner products, respectively, of various vector bundles which are evident according to the contexts, $\sum$ for summation of repeated indices.

2.1. The first variation

Let $u : M^m \to N^n$ be a smooth map, u_t its variation, $\psi = (du_t/dt)|_{t=0}$ the variational field. Then

$$\left.\frac{\mathrm{d}E_{p,H}(u_t)}{dt}\right|_{t=0} = -\int_M \langle \tau_{p,H}(u), \psi\rangle \tag{2}$$

where $\tau_{p,H}(u) = \tau_p(u) + \operatorname{grad} H \circ u$, $\tau_p(u) = -\mathrm{d}^*(|du|^{p-2}du)$. Therefore, the Euler-Lagrange equation of $E_{p,H}$ is

$$\tau_{p,H}(u) = 0. \tag{3}$$

Theorem 1. *Let M be a Riemannian manifold, u be a p-H-harmonic map from M. If* $\operatorname{Hess} H \circ u \le 0$, *then* $due_i \in \operatorname{Ker} \operatorname{Hess} H \circ u$. *Especially, if* $\operatorname{Hess} H \circ u < 0$, *then u is constant.*

Proof. By the Euler-Lagrange equation we have

$$\begin{aligned}
\langle \mathrm{dd}^*(|du|^{p-2}du), |du|^{p-2}du\rangle &= \langle \mathrm{d}(\operatorname{grad} H \circ u), |du|^{p-2}du\rangle \\
&= \langle \nabla^{u^{-1}TN} \operatorname{grad} H \circ u, |du|^{p-2}du\rangle \\
&= \sum |du|^{p-2}\langle \nabla^{u^{-1}TN}_{e_i} \operatorname{grad} H \circ u, due_i\rangle \\
&= \sum |du|^{p-2}\langle \nabla^{TN}_{due_i} \operatorname{grad} H, due_i\rangle \\
&= \sum |du|^{p-2} \operatorname{Hess} H(due_i, due_i).
\end{aligned} \tag{4}$$

On the other hand,

$$\int_M \langle \mathrm{dd}^*(|du|^{p-2}du), |du|^{p-2}du\rangle = \int_M |\mathrm{d}^*(|du|^{p-2}du)|^2. \tag{5}$$

So, we have

$$\int_M \sum |du|^{p-2} \operatorname{Hess} H(due_i, due_i) = \int_M |\mathrm{d}^*(|du|^{p-2}du)|^2 \tag{6}$$

from which we can draw the conclusion of the theorem. Q.E.D.

2.2. The second variation

Let u_{st} be a variation with double parameters s and t, $v = (\partial u_{st}/\partial s)|_{s=0,\, t=0}$ and $w = (\partial u_{st}/\partial t)|_{s=0,\, t=0}$ the variational fields. Let $e_p(u) = (1/p)|du|^p$ be the p-energy density and $E_p(u) = \int_M e_p(u)$ the p-energy. Set $\Phi(x, s, t) = u_{st}(x)$. Then by a standard calculation, we have

$$\begin{aligned}\left.\frac{\partial^2 E_p(u_{st})}{\partial s \partial t}\right|_{s=0,\, t=0} &= (p-2)\int_M |du|^{p-4} \sum \langle \nabla_{e_i} w, due_i\rangle\langle \nabla_{e_j} v, due_j\rangle \\ &\quad + \int_M \langle \nabla_w v, \mathrm{d}^*(|du|^{p-2}du)\rangle \\ &\quad + \int_M |du|^{p-2} \sum \langle R^N(due_i, w)v, due_i\rangle \\ &\quad + \int_M |du|^{p-2}\langle \nabla v, \nabla w\rangle.\end{aligned} \tag{7}$$

On the other hand,

$$\begin{aligned}\left.\frac{\partial^2 H \circ u_{st}}{\partial s \partial t}\right|_{s=0,\, t=0} &= \langle \nabla_w \operatorname{grad} H, v\rangle + \langle \operatorname{grad} H, \nabla_w v\rangle \\ &= \operatorname{Hess} H(w, v) + \langle \mathrm{d}^*(|du|^{p-2}du), \nabla_w v\rangle\end{aligned} \tag{8}$$

where we used the Euler-Lagrange equation. Therefore we obtain

$$\begin{aligned}\left.\frac{\partial^2 E_{p,H}(u_{st})}{\partial s \partial t}\right|_{s=0,\, t=0} &= (p-2)\int_M |du|^{p-4} \sum \langle \nabla_{e_i} w, due_i\rangle\langle \nabla_{e_j} v, due_j\rangle \\ &\quad - \int_M \operatorname{Hess} H(v, w) \\ &\quad - \int_M |du|^{p-2} \sum \langle R^N(due_i, w)due_i, v\rangle \\ &\quad + \int_M |du|^{p-2}\langle \nabla v, \nabla w\rangle.\end{aligned} \tag{9}$$

This implies

$$\left.\frac{\partial^2 E_{p,H}(u_t)}{\partial t^2}\right|_{t=0} \le (p-1)\int_M |du|^{p-2}|\nabla w|^2 - \int_M \text{Hess } H(w,w) - \int_M |du|^{p-2}\sum \langle R^N(due_i, w)due_i, w\rangle. \tag{10}$$

In the following, we will denote $(\partial^2 E_{p,H}(u_{st})/\partial s\partial t)|_{s=0,t=0}$ by $I_{p,H}(v,w)$.

2.3. Bochner formulas

Bochner type formulas are important in harmonic maps theories. There are several versions of these formulas for p-harmonic maps. Here, we introduce two of them which will be used in this paper.

Let V be a vector bundle over a Riemannian manifold M^m, $\{e_i; i = 1, \cdot, m\}$ a local field of orthonormal tangent frame on M^m such that $\nabla_{e_i} e_j = 0$ at a fixed point under consideration. For any V-valued 1-form σ, we have (see [3])

$$\frac{1}{2}\Delta|\sigma|^2 = \langle \Delta\sigma, \sigma\rangle + |\nabla\sigma|^2 - \sum \langle R^V(e_i, e_j)\sigma(e_i), \sigma(e_j)\rangle + \sum \langle \sigma(\text{Ric}^M\, e_i), \sigma(e_i)\rangle. \tag{11}$$

Here $\Delta\sigma = -(\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)\sigma$. Let $\sigma = |du|^{p-2}du$ which is a $u^{-1}TN$-valued 1-form. Substituting it into the above formula, we have

$$\frac{1}{2}\Delta|du|^{2p-2} = \langle \Delta(|du|^{p-2}du), |du|^{p-2}du\rangle + |\nabla(|du|^{p-2}du)|^2 - \sum |du|^{2p-4}\langle R^N(due_i, due_j)due_i, due_j\rangle + \sum |du|^{2p-4}\langle du\, \text{Ric}^M\, e_i, due_i\rangle. \tag{12}$$

If $p = 2$ and u is harmonic, then the first term of the right hand side above vanishes. But for general p, it does not. Generally,

$$\begin{aligned}\int_M \langle \Delta(|du|^{p-2}du), |du|^{p-2}du\rangle &= -\int_M |\mathrm{d}^*(|du|^{p-2}du)|^2 - \int_M |\mathrm{d}(|du|^{p-2}du)|^2 \\ &= -\int_M |\tau_p|^2 - \int_M |\mathrm{d}(|du|^{p-2}du)|^2.\end{aligned} \tag{13}$$

If u is a p-H-harmonic map, then the above equality becomes as

$$(14)\qquad \int_M \langle \Delta(|du|^{p-2}du), |du|^{p-2}du\rangle = -\int_M |du|^{p-2}\sum \mathrm{Hess}\, H(due_i, due_i) - \int_M |\mathrm{d}(|du|^{p-2}du)|^2$$

where we have used (6). Integrating (12) and then taking (14) into account, we have

$$(15)\qquad 0 = -\int_M |du|^{p-2}\sum \mathrm{Hess}\, H(due_i, due_i) + \int_M |S\nabla(|du|^{p-2}du)|^2 - \int_M |du|^{2p-4}\langle R^N(due_i, due_j)due_i, due_j\rangle + \int_M |du|^{2p-4}\langle du\,\mathrm{Ric}^M\, e_i, due_i\rangle$$

where $S\nabla(|du|^{p-2}du)$ denotes the symmetric part of $\nabla(|du|^{p-2}du)$.

Remark. In general, $\nabla(|du|^{p-2}du)$ is not symmetric. But it can be decomposite into a summation of symmetric and anti-symmetric parts. In fact, we have

$$(16)\qquad \begin{aligned}\nabla(|du|^{p-2}du)(X, Y) &= \frac{1}{2}((\nabla(|du|^{p-2}du))(X, Y) + (\nabla(|du|^{p-2}du))(Y, X)) \\ &\quad + \frac{1}{2}((\nabla(|du|^{p-2}du))(X, Y) - (\nabla(|du|^{p-2}du))(Y, X)) \\ &= S\nabla(|du|^{p-2}du)(X, Y) + \mathrm{d}(|du|^{p-2}du)(X, Y)\end{aligned}$$

and

$$(17)\qquad |S\nabla(|du|^{p-2}du)|^2 + |\mathrm{d}(|du|^{p-2}du)|^2 = |\nabla(|du|^{p-2}du)|^2.$$

The following formula of Bochner type is also needed in this paper:

$$(18)\qquad \begin{aligned}\frac{1}{p}\Delta|du|^p &= \nabla_{e_i}\langle\tau(u), |du|^{p-2}due_i\rangle - \langle\tau(u), \tau_p(u)\rangle + |du|^{p-2}|\nabla du|^2 \\ &\quad - |du|^{p-2}\sum\langle R^N(due_i, due_j)due_i, due_j\rangle \\ &\quad + |du|^{p-2}\sum\langle du\,\mathrm{Ric}^M\, e_i, due_i\rangle + (p-2)|du|^{p-2}|\nabla|du|\,|^2\end{aligned}$$

whenever $|du| \neq 0$. Here $\tau(u) := \tau_2(u)$. This formula deduce easily from the sequel three equalities; it can also be found in [12].

$$\frac{1}{p}\Delta|du|^p = \frac{1}{p}\Delta(|du|^2)^{p/2} = \frac{1}{2}|du|^{p-2}\Delta|du|^2 + (p-2)|du|^{p-2}|\nabla|du|\,|^2, \tag{19}$$

$$\begin{aligned}\frac{1}{2}\Delta|du|^2 = \langle \Delta du, du\rangle + |\nabla du|^2 - \sum \langle R^N(due_i, due_j)due_i, due_j\rangle \\ + \sum \langle du\,\mathrm{Ric}^M\, e_i, due_i\rangle,\end{aligned} \tag{20}$$

$$|du|^{p-2}\langle \Delta du, du\rangle = \sum \nabla_{e_i}\langle \tau(u), |du|^{p-2}due_i\rangle - \langle \tau(u), \tau_p(u)\rangle. \tag{21}$$

3. Stability

A p-H-harmonic map is called stable, if $(\partial^2 E_{p,H}(u_t)/\partial t^2)|_{t=0} \geq 0$ for all variations u_t of u.

Now, we suppose that M^m be a submanifold of the Euclidean space $\boldsymbol{R}^{m+k_0}$. In order to examine the stability of p-H-harmonic maps from M^m or into M^m, we need to establish the second varational estimates.

Let $\{X_A, A = 1, \ldots, m+k_0\}$ be an orthonormal base of $\boldsymbol{R}^{m+k_0}$, each member of which is a constant vector, and let $\{e_i, i = 1, \ldots, m; e_\mu, \mu = m+1, \ldots, m+k_0\}$ be a local orthonormal field of frame of $\boldsymbol{R}^{m+k_0}$ around a point x of M^m, of which, restricting to M^m, the first m members are tangent to M^m, and the others are normal to M^m. We can let such that $\nabla_{e_i} e_j = 0$ at a fixed point under consideration. Denote the tangent part and the normal part of X_A by X_A^T and X_A^N respectively. Then

$$\begin{aligned} X_A^T &= \sum \langle X_A, e_i\rangle e_i =: \sum v_A^i e_i, \\ X_A^N &= \sum \langle X_A, e_\mu\rangle e_\mu =: \sum v_A^\mu e_\mu. \end{aligned} \tag{22}$$

It is not difficult to check

$$\begin{aligned} &\sum v_A^B v_A^C = \delta_{BC} \quad A, B, C \in \{1, \ldots, m+k_0\} \\ &\nabla_{e_i}^M X_A^T = \sum v_A^\mu h_{ij}^\mu e_j \end{aligned} \tag{23}$$

where h_{ij}^μ is the second fundamental tensor of M^m in $\boldsymbol{R}^{m+k_0}$.

In fact,

$$\begin{aligned} \delta_{BC} &= \langle e_B, e_C\rangle \\ &= \sum \langle X_A, e_B\rangle X_A \cdot \sum \langle X_A, e_C\rangle X_A \\ &= \sum v_A^B v_A^C, \end{aligned} \tag{24}$$

$$\nabla^M_{e_i} X^T_A = (\nabla^{\boldsymbol{R}^{m+k_0}}_{e_i} X^T_A)^T = (\nabla^{\boldsymbol{R}^{m+k_0}}_{e_i} (X_A - X^N_A))^T = -(\nabla^{\boldsymbol{R}^{m+k_0}}_{e_i} X^N_A)^T = A_{e_i} X^N_A = h^\mu_{ij} v^\mu_A e_j. \tag{25}$$

Let N^n be any Riemannian manifold. On it, we always take a local field of frame $\{\varepsilon_\alpha, \alpha = 1, \dots, n\}$ near a point under consideration. We have

LEMMA 1. *Let u be a p-H-harmonic map from M^m to N^n. then*

$$\sum I_{p,H}(duX^T_A, duX^T_A) \le (p-2)\int_M \sum \text{Hess } H(due_i, due_i) + (p-1)\int_M |du|^{p-2} \sum h^\mu_{jk} h^\mu_{ii} \langle due_j, due_k\rangle - 2(p-1)\int_M |du|^{p-2} \sum R^M_{ij} \langle due_i, due_j\rangle + (p-2)\int_M |du|^{p-2} \sum \langle R^N(due_i, due_j) due_i, due_j\rangle \tag{26}$$

where R^M_{ij} is the Ricci curvature tensor, and $R^N(\cdot,\cdot)$ is the Riemannian curvature operator of N^n.

Proof. From (23), we have

$$\nabla_{e_i}(duX^T_A) = \sum v^j_A (\nabla_{e_i} du) e_j + \sum v^\mu_A h^\mu_{ij} due_j.$$

Taking use of this together with Gaussian equation of M^m, we get

$$\sum |\nabla(duX^T_A)|^2 = |\nabla du|^2 + \sum h^\mu_{ij} h^\mu_{ik} \langle due_j, due_k\rangle = |\nabla du|^2 + \sum h^\mu_{jk} h^\mu_{ii} \langle due_j, due_k\rangle - R^M_{jk}\langle due_j, due_k\rangle. \tag{27}$$

On the other hand, by (23) again, one has

$$\sum \langle R^N(due_i, duX^T_A) due_i, duX^T_A\rangle = \sum \langle R^N(due_i, due_j) due_i, due_j\rangle. \tag{28}$$

Substituting (27) and (28) into (10), we reach

$$
\begin{aligned}
(29) \qquad \sum I_{p,H}(duX_A^T, duX_A^T) \le (p-1)\int_M |du|^{p-2}|\nabla du|^2 \\
+ (p-1)\int_M |du|^{p-2} \sum h_{jk}^{\mu} h_{ii}^{\mu} \langle due_j, due_k\rangle \\
- (p-1)\int_M |du|^{p-2} \sum R_{jk}^M \langle due_j, due_k\rangle \\
- \int_M |du|^{p-2} \sum \langle R^N(due_i, due_j)due_i, due_j\rangle \\
- \int_M \sum \text{Hess } H(due_i, due_i).
\end{aligned}
$$

Integrating both sides of (18), we know

$$
\begin{aligned}
(30) \qquad \int_M |du|^{p-2}|\nabla du|^2 &\le \int_M \langle \tau(u), \tau_p(u)\rangle \\
&+ \int_M |du|^{p-2} \sum \langle R^N(due_i, due_j)due_i, due_j\rangle \\
&- \int_M |du|^{p-2} \sum R_{ij}^M \langle due_j, due_i\rangle \\
&= \int_M \sum \text{Hess } H(due_i, due_i) \\
&+ \int_M |du|^{p-2} \sum \langle R^N(due_i, due_j)due_i, due_j\rangle \\
&- \int_M |du|^{p-2} \sum R_{ij}^M \langle due_j, due_i\rangle
\end{aligned}
$$

where we have used $\int_M \langle \tau(u), \tau_p(u)\rangle = \int_M \sum \text{Hess } H(due_i, due_i)$. Insert (30) in (29). Then the lemma follows.

LEMMA 2. *Let u be a p-H-harmonic map from N^n to M^m. Then*

$$
\begin{aligned}
(31) \qquad I_{p,H}(X_A^T, X_A^T) \le (p-1)\int_N |du|^{p-2} \sum u_\alpha^i u_\alpha^k h_{ik}^{\mu} h_{jj}^{\mu} \\
- p\int_N |du|^{p-2} \sum R_{ik}^M u_\alpha^i u_\alpha^k - \int_N \Delta_M H \circ u
\end{aligned}
$$

where we have denoted $du\varepsilon_\alpha = \sum u_\alpha^i e_i$.

Proof. By (23), we have $\nabla_{\varepsilon_\alpha} X_A^T = \nabla_{du\varepsilon_\alpha}^M X_A^T = \sum u_\alpha^i \nabla_{e_i}^M X_A^T = \sum u_\alpha^i v_A^{\mu} h_{ij}^{\mu} e_j$. So by (23) again together with Gaussian equation, we have

$$\sum |\nabla_{\varepsilon_\alpha} X_A^T|^2 = \sum u_\alpha^i u_\alpha^k h_{ij}^\mu h_{jk}^\mu$$
$$= \sum u_\alpha^i u_\alpha^k h_{ik}^\mu h_{jj}^\mu - \sum u_\alpha^i u_\alpha^k R_{ik}^M. \tag{32}$$

We easily check

$$\sum \langle R^M(du\varepsilon_\alpha, X_A^T)du\varepsilon_\alpha, X_A^T\rangle = \sum R_{ij}^M u_\alpha^i u_\alpha^j, \tag{33}$$

and

$$\sum \text{Hess } H(X_A^T, X_A^T) = \sum \text{Hess } H(e_i, e_i) = \Delta_M H. \tag{34}$$

Inserting (32), (33) and (34) in (10), we obtain the estimate of Lemma 2.

Now we are in a position to discuss the stability.

THEOREM 2. *Let M^m be a submanifold of $\boldsymbol{R}^{m+k_0}$ with the second fundamental tensor h_{ij}^μ. Set $A^\mu = (h_{ij}^\mu)$, $\lambda =$ the maximal eigenvalue of $\sum A^\mu A^\mu$, and $\eta =$ the mean curvature vector of M^m in $\boldsymbol{R}^{m+k_0}$. Let N^n be a Riemannian manifold with κ as the upper bound of the sectional curvatures.*

(i) *Assume that u is a stable p-H-harmonic map from M^m to N^n. If* $\text{Ric}^M > (1/(2(p-1)))((p-1)m|\eta|\sqrt{\lambda} + (p-2)\kappa|du|^2)$ *and* $(p-2)\,\text{Hess } H \circ u \le 0$, *then u is constant.*

(ii) *Assume that u is a stable p-H-harmonic map from N^n to M^m. If* $\text{Ric}^M > ((p-1)m/p)|\eta|\sqrt{\lambda}$ *and* $\Delta_M H \circ u \ge 0$, *then u is constant.*

Proof. By Lemma 1 and the assumptions, we have

$$\sum I_{p,H}(duX_A^T, duX_A^T)$$
$$\le (p-1)\int_M |du|^{p-2} \sum h_{jk}^\mu h_{ii}^\mu \langle due_j, due_k\rangle$$
$$- 2(p-1)\int_M |du|^{p-2} \sum R_{ij}^M \langle due_i, due_j\rangle$$
$$+ (p-2)\int_M \kappa |du|^{p+2}. \tag{35}$$

It is easy to check

$$\sum h_{jk}^\mu h_{ii}^\mu \langle due_j, due_k\rangle$$
$$\le \sum_{j,k} \sqrt{\sum_\mu (h_{jk}^\mu \langle due_j, due_k\rangle)^2} \cdot \sqrt{\sum_\mu \left(\sum_i h_{ii}^\mu\right)^2}$$
$$\le m|\eta|\sqrt{\lambda}|du|^2. \tag{36}$$

Therefore

$$\sum I_{p,H}(duX_A^T, duX_A^T) \leq \int_M [(p-1)m|\eta|\sqrt{\lambda} + (p-2)\kappa|du|^2 - 2(p-1)\rho]|du|^p \tag{37}$$

where ρ is a Ricci lower bound of M^m. From this estimate, the first part of the theorem is proven.

The proof of the second part follows from Lemma 2 and an estimate similar to (36). Q.E.D.

Theorem 3. *Let M^m be a submanifold of S^{m+k_0} with the second fundamental tensor h_{ij}^μ. Set $A^\mu = (h_{ij}^\mu)$, $\lambda =$ the maximal eigenvalue of $\sum A^\mu A^\mu$, and $\eta =$ the mean curvature vector of M^m in S^{m+k_0}. Let N^n be a Riemannian manifold with κ as the upper bound of the sectional curvatures.*

(i) *Assume that u is a stable p-H-harmonic map from M^m to N^n. If* $\mathrm{Ric}^M > (1/(2(p-1)))((p-1)m + (p-1)m|\eta|\sqrt{\lambda} + (p-2)\kappa|du|^2)$ *and* $(p-2)\,\mathrm{Hess}\, H \circ u \leq 0$, *then u is constant.*

(ii) *Assume that u is a stable p-H-harmonic map from N^n to M^m. If* $\mathrm{Ric}^M > ((p-1)m/p)(|\eta|\sqrt{\lambda} + 1)$ *and $\Delta_M H \circ u \geq 0$, then u is constant.*

Proof. Regard M^m as a submanifold of $\boldsymbol{R}^{m+k_0+1}$. Then Lemmas 1 and 2 can be used. Denote the second fundamental tensor of M^m in $\boldsymbol{R}^{m+k_0+1}$ by $\bar{h}_{ij}^\mu$, where $\mu = 1, \ldots, m+k_0+1$. Note that we take e_{m+k_0+1} to be the unit outward normal vector of R^{m+k_0}. Then $\bar{h}_{ij}^{m+k_0+1} = -\delta_{ij}$. So similar to the proof of Theorem 2 we can verify Theorem 3. Q.E.D.

Remark. Theorem 2(ii) can be regarded as a generalization and an improvement of Leung's stability ([8]); Theorem 3 extends the Ohnita's theorem ([9]). Theorems 2 and 3 origins from Y.-L. Xin's and Leung's stability theorems (see [13] and [8]).

4. Pointwise quantum theorem

For harmonic maps, Sealey obtained a pointwise quantum theorem (see [10]). Here we generalize his result to p-H-harmonic maps.

Theorem 4. *Let $u : M^m \to N^n$ be a p-H-harmonic map such that* $\mathrm{Hess}\, H \circ u \leq 0$. *Suppose that* $\mathrm{Ric}^M \geq B > 0$, *and that the sectional curvatures K^N of N are not more than another positive number A, and that the rank of u is not great than q. If $|du|^2 \leq (q/(q-1))(B/A)$, then we have $|du| = 0$ or $|du|^2 = (q/(q-1))(B/A)$. And the latter implies that u is totally geodesic.*

Remark. Matei in [6] proved that $E_p(u) \leq ((q-1)/q)(A/B)E_{p+2}(u)$ for p-

harmonic maps and the equality implies that u is constant or geodesic. On the other hand if $|du|^2 \le (q/(q-1))(B/A)$, then Matei's inequality becomes an equality. Hence we have that u is constant or geodesic. In the following we give a different proof of this theorem for p-H-harmonic maps.

Proof. By (15) we have

$$0 \ge -A\int_M |du|^{2p-4}|due_i \wedge due_j|^2 + B\int_M |du|^{2p-2}. \tag{38}$$

By the assumption on the rank, without loss of generality, we can suppose that $due_{q+1} = \cdots = due_m = 0$. We have

$$\begin{aligned}\sum |due_i \wedge due_j|^2 &= \sum \langle due_i, due_i\rangle\langle due_j, due_j\rangle - \sum \langle due_i, due_j\rangle^2 \\ &\le |du|^4 - \sum_i^q \langle due_i, due_i\rangle^2 \\ &\le |du|^4 - \frac{1}{q}\left(\sum_i^q |due_i|^2\right)^2 \\ &= \frac{q-1}{q}|du|^4.\end{aligned} \tag{39}$$

Hence

$$0 \ge \int_M |du|^{2p-2}\left(B - \frac{q-1}{q}A|du|^2\right) \tag{40}$$

from which we have $|du| = 0$ or $|du|^2 = (q/(q-1))(B/A)$. When $|du|^2 = (q/(q-1))(B/A)$, all inequalities above become equalities. Hence by (15) and (38), we get $S\nabla(|du|^{p-2}du) = 0$, and hence $\nabla du = 0$ since $|du|^{p-2} = \text{const}$.

Q.E.D.

For a p-H-harmonic map u with Hess $H \circ u \le \lambda$, then by Bochner formula we have

$$0 \ge \int_M |du|^p\left(|du|^{p-2}B - \lambda - \frac{q-1}{q}A|du|^p\right). \tag{41}$$

When $p = 2$, we obtain the following

Theorem 5. *Let $u: M^m \to N^n$ be a harmonic map with potential H, where M^m and N^n are as in Theorem* 4, *and* Hess $H \circ u \le \lambda$, $0 \le \lambda < B$. *If $|du|^2 \le (q/(q-1))((B-\lambda)/A)$, then $|du| = 0$ or $\lambda = 0$ and $|du|^2 = (q/(q-1))(B/A)$. The latter implies u is totally geodesic.*

Proof. By (41), if $du \neq 0$, then $|du|^2 = (q/(q-1))((B-\lambda)/A)$. Thus (41) becomes as an equality which implies that u is totally geodesic. Q.E.D.

Remark. In Theorems 4 and 5, when $|du|^2 = (q/(q-1))((B-\lambda)/A)$, then M is a Riemannian product of two totally geodesic submanifolds, one of which is a q dimensional space form. Here $q = \operatorname{rank} u$.

5. Conservation law and Liouville theorems

5.1. Conservation law

Set $u : (M, g) \to (N, h)$ be a smooth map whose tangent map is du and cotangent map is u^*. Define stress p-energy tensor by

(42) $$S_{p,u} := e_p(u)g - |du|^{p-2}u^*h.$$

Then, for any $X \in TM$

(43) $$\begin{aligned}(\operatorname{div} S_{p,u})(X) &:= (\nabla_{e_i}S_{p,u})(e_i, X)\\ &= \nabla_{e_i}S_{p,u}(e_i, X) - S_{p,u}(e_i, \nabla_{e_i}X) - S_{p,u}(\nabla_{e_i}e_i, X)\\ &= \nabla_{e_i}\left(\frac{1}{p}\langle due_j, due_j\rangle^{p/2}\langle e_i, X\rangle - \langle |du|^{p-2}due_i, duX\rangle\right)\\ &\quad - e_p(u)\langle e_i, \nabla_{e_i}X\rangle + |du|^{p-2}\langle due_i, du\nabla_{e_i}X\rangle\\ &= (|du|^{p-2}\langle\nabla_{e_i}due_j, due_j\rangle\langle e_i, X\rangle + e_p(u)\langle e_i, \nabla_{e_i}X\rangle\\ &\quad - \langle\nabla_{e_i}(|du|^{p-2}due_i), duX\rangle - |du|^{p-2}\langle due_i, \nabla_{e_i}duX\rangle)\\ &\quad - e_p(u)\langle e_i, \nabla_{e_i}X\rangle + |du|^{p-2}\langle due_i, du\nabla_{e_i}X\rangle\\ &= |du|^{p-2}\langle(\nabla_{e_i}du)e_j, due_j\rangle\langle e_i, X\rangle\\ &\quad - \langle(\nabla_{e_i}|du|^{p-2}du)e_i, duX\rangle - \langle |du|^{p-2}due_i, (\nabla_{e_i}du)X\rangle\\ &\quad - \langle |du|^{p-2}due_i, du\nabla_{e_i}X\rangle + |du|^{p-2}\langle due_i, du\nabla_{e_i}X\rangle\\ &= |du|^{p-2}\langle(\nabla_X du)e_j, due_j\rangle - \langle(\nabla_{e_i}|du|^{p-2}du)e_i, duX\rangle\\ &\quad - |du|^{p-2}\langle due_i, (\nabla_{e_i}du)X\rangle\\ &= -\langle\tau_p(u), duX\rangle\end{aligned}$$

where $\tau_p(u) = -\mathrm{d}^*(|du|^{p-2}du) = (\nabla_{e_i}(|du|^{p-2}du))(e_i)$.

(44) $$\begin{aligned}\operatorname{div}(e_p(u)X) &= \langle\nabla_{e_i}(e_p(u)X), e_i\rangle\\ &= \langle(\nabla_{e_i}e_p(u))X, e_i\rangle + e_p(u)\langle\nabla_{e_i}X, e_i\rangle\\ &= \nabla_X e_p(u) + e_p(u)\langle\nabla_{e_i}X, e_i\rangle,\end{aligned}$$

$$
\begin{aligned}
\nabla_X e_p(u) &= \frac{1}{p}\nabla_X\langle due_i, due_i\rangle^{p/2} \\
&= |du|^{p-2}\langle(\nabla_X du)e_i, due_i\rangle \\
&= |du|^{p-2}\langle(\nabla_{e_i} du)X, due_i\rangle \\
&= \langle\nabla_{e_i}(duX), |du|^{p-2}due_i\rangle - |du|^{p-2}\langle du\nabla_{e_i}X, due_i\rangle \\
&= \nabla_{e_i}\langle duX, |du|^{p-2}due_i\rangle - \langle duX, (\nabla_{e_i}(|du|^{p-2}du))e_i\rangle \\
&\quad - |du|^{p-2}\langle du\nabla_{e_i}X, due_i\rangle \\
&= \operatorname{div}(|du|^{p-2}\langle duX, due_i\rangle e_i) - \langle duX, \tau_p(u)\rangle \\
&\quad - |du|^{p-2}\langle\nabla X, u^*h\rangle,
\end{aligned} \tag{45}
$$

where $\nabla X(V, W) := \langle\nabla_V X, W\rangle$. Hence we have

$$
\begin{aligned}
\operatorname{div}(e_p(u)X) &= \operatorname{div}(|du|^{p-2}\langle duX, due_i\rangle e_i) - \langle duX, \tau_p(u)\rangle \\
&\quad - |du|^{p-2}\langle\nabla X, u^*h\rangle + e_p(u)\langle\nabla_{e_i}X, e_i\rangle \\
&= \operatorname{div}(|du|^{p-2}\langle duX, due_i\rangle e_i) - \langle duX, \tau_p(u)\rangle \\
&\quad + \langle S_{p,u}, \nabla X\rangle.
\end{aligned} \tag{46}
$$

If Supp X is compact, by Green formula, we have

$$
\int_M (\operatorname{div} S_{p,u})(X) + \int_M \langle S_{p,u}, \nabla X\rangle = 0. \tag{47}
$$

Take $D \subset M$, we have

$$
\begin{aligned}
\int_{\partial D} e_p(u)\langle X, \boldsymbol{n}\rangle &= \int_{\partial D} |du|^{p-2}\langle duX, du\boldsymbol{n}\rangle + \int_D (\operatorname{div} S_{p,u})(X) \\
&\quad + \int_D \langle S_{p,u}, \nabla X\rangle,
\end{aligned} \tag{48}
$$

where $\boldsymbol{n}$ is the outward normal vector field of ∂D.

5.2. Liouville theorems

Let $u : M^m \to N^n$ be a map. For any fixed $x_0 \in M^m$, $r(x)$ denotes the distance function from x_0 to x, $B_R(x_0)$ stands for the geodesic ball with radius R and center x_0. We say that the energy of u is divergent slowly if there exists a positive function $\psi(t)$ with $\int_{R_0}^{\infty}(dt/t\psi(t)) = \infty$ $(R_0 > 0)$, such that

$$
\lim_{R\to\infty}\int_{B_R(x_0)} \frac{e(u)(x)}{\psi(r(x))} < \infty. \tag{49}
$$

In this section, we prove two Liouville theorems. One is for p-harmonic maps and another for p-H-harmonic maps.

Theorem 6. *Let M^m be a complete, simply connected Riemannian manifold with non-positive sectional curvature K. Assuming that K satisfies*

(1) $-a^2 \le K \le -b^2$, *where* $a > 0$, $b > 0$ *and* $(m-1)b - pa \ge 0$; *or*

(2) $-A/(1+r^2) \le K \le 0$, *where* $0 < A < (1/4)(2m/p - 1)^2 - 1/4$.

If u is a p-harmonic map $(m > p)$ from M^m whose energy is divergent slowly, then, u is constant.

Proof. By the definitions, we have

$$\langle S_{p,u}, \nabla X\rangle = (e_p(u)\langle e_\alpha, e_\beta\rangle - |du|^{p-2}\langle due_\alpha, due_\beta\rangle)\langle \nabla_{e_\alpha} X, e_\beta\rangle \\ = e_p(u)\langle \nabla_{e_\alpha} X, e_\alpha\rangle - |du|^{p-2}\langle due_\alpha, due_\beta\rangle\langle \nabla_{e_\alpha} X, e_\beta\rangle. \tag{50}$$

Let $X = r(\partial/\partial r)$, then

$$\nabla_{\partial/\partial r} X = \frac{\partial}{\partial r}; \tag{51}$$

$$\nabla_{e_s} X = r\nabla_{e_s}\frac{\partial}{\partial r} = r\,\mathrm{Hess}(r)(e_s, e_t)e_t; \tag{52}$$

$$\langle \nabla_{e_\alpha} X, e_\alpha\rangle = 1 + r\,\mathrm{Hess}(r)(e_s, e_s) \tag{53}$$

where $\{e_s, \partial/\partial r\}$ is the orthonormal frame field of $B_R(x_0)$. Substituting (51), (52) and (53) into (50), we have

$$\begin{aligned} \langle S_{p,u}, \nabla X\rangle &= e_p(u)(1 + r\,\mathrm{Hess}(r)(e_s, e_s)) \\ &\quad - |du|^{p-2}\langle due_s, due_t\rangle\langle \nabla_{e_s} X, e_t\rangle \\ &\quad - |du|^{p-2}\left\langle du\frac{\partial}{\partial r}, du\frac{\partial}{\partial r}\right\rangle\left\langle \nabla_{\partial/\partial r} X, \frac{\partial}{\partial r}\right\rangle \\ &\quad - |du|^{p-2}\left\langle du\frac{\partial}{\partial r}, due_t\right\rangle\langle \nabla_{\partial/\partial r} X, e_t\rangle \\ &\quad - |du|^{p-2}\left\langle due_s, du\frac{\partial}{\partial r}\right\rangle\left\langle \nabla_{e_s} X, \frac{\partial}{\partial r}\right\rangle \\ &= e_p(u)(1 + r\,\mathrm{Hess}(r)(e_s, e_s)) \\ &\quad - |du|^{p-2}\langle due_s, due_t\rangle r\,\mathrm{Hess}(r)(e_s, e_t) \\ &\quad - |du|^{p-2}\left\langle du\frac{\partial}{\partial r}, du\frac{\partial}{\partial r}\right\rangle. \end{aligned} \tag{54}$$

Under the assumption in Theorem 6, by Hessian comparison theorem

$$b\coth(br)(g - dr\otimes dr) \le \operatorname{Hess}(r) \le a\coth(ar)(g - dr\otimes dr) \tag{55}$$

we have

$$\begin{aligned}
\langle S_{p,u}, \nabla X\rangle &\ge e_p(u)(1+(m-1)(br)\coth(br)) \\
&\quad - |du|^{p-2}(ar)\coth(ar)\langle du e_s, du e_s\rangle \\
&\quad - |du|^{p-2}\left\langle du\frac{\partial}{\partial r}, du\frac{\partial}{\partial r}\right\rangle \\
&= |du|^{p-2}\left(\frac{m-1}{p}(br)\coth(br) + \frac{1-p}{p}\right)\left\langle du\frac{\partial}{\partial r}, du\frac{\partial}{\partial r}\right\rangle \\
&\quad + |du|^{p-2}\left(\frac{1}{p} + \frac{m-1}{p}(br)\coth(br) - (ar)\coth(ar)\right)\langle du e_s, du e_s\rangle \\
&\ge \frac{m-p}{p}|du|^{p-2}\left\langle du\frac{\partial}{\partial r}, du\frac{\partial}{\partial r}\right\rangle \\
&\quad + |du|^{p-2}\left(\frac{1}{p} + r\coth(br)\left(\frac{m-1}{p}b - a\right)\right)\langle du e_s, du e_s\rangle \\
&\ge C e_p(u)
\end{aligned} \tag{56}$$

where C is a positive constant.

Under the assumption (2), by Hessian comparison theorem, we also have

$$\langle S_{p,u}, \nabla X\rangle \ge C e_p(u). \tag{57}$$

In fact, in this case, the Hessian comparison theorem is

$$\frac{1}{r}(g - dr\otimes dr) \le \operatorname{Hess}(r) \le \frac{\beta}{r}(g - dr\otimes dr) \tag{58}$$

where $\beta = 1/2 + (1/2)(1+4A)^{1/2}$. Applying it to (54), we have

$$\begin{aligned}
\langle S_{p,u}, \nabla X\rangle &\ge m e_p(u) - |du|^{p-2}\beta\langle du e_s, du e_s\rangle \\
&\quad - |du|^{p-2}\left\langle du\frac{\partial}{\partial r}, du\frac{\partial}{\partial r}\right\rangle \\
&= \frac{m-p}{p}|du|^{p-2}\left\langle du\frac{\partial}{\partial r}, du\frac{\partial}{\partial r}\right\rangle \\
&\quad + \frac{m-p\beta}{p}|du|^{p-2}\langle du e_s, du e_s\rangle \\
&\ge C e_p(u)
\end{aligned} \tag{59}$$

as desired.

For any fixed $x_0 \in M^m$, take $D = B_R(x_0)$. Then on ∂D, $\boldsymbol{n} = \partial/\partial r$. Hence we have

$$\int_{\partial D} e_p(u)\langle X, \boldsymbol{n}\rangle - \int_{\partial D} |du|^{p-2}\langle duX, du\boldsymbol{n}\rangle \tag{60}$$
$$= \int_{\partial D} Re_p(u) - \int_{\partial D} R|du|^{p-2}\left\langle du\frac{\partial}{\partial r}, du\frac{\partial}{\partial r}\right\rangle$$
$$\leq R\int_{\partial D} e_p(u).$$

More precisely, we have

$$\int_{\partial D} e_p(u)\langle X, \boldsymbol{n}\rangle - \int_{\partial D} |du|^{p-2}\langle duX, du\boldsymbol{n}\rangle \tag{61}$$
$$\leq \frac{(p-1)R}{p}\int_{\partial D} |du|^{p-2} e_{p-1}(u|_{\partial D}).$$

From (43), (48), (56), (57) and (60) or (61), we have

$$R\int_{\partial B_R(x_0)} e_p(u) \geq C\int_{B_R(x_0)} e_p(u) \tag{62}$$

or

$$\frac{(p-1)R}{p}\int_{\partial B_R(x_0)} |du|^{p-2} e_{p-1}(u|_{\partial D}) \geq C\int_{B_R(x_0)} e_p(u). \tag{63}$$

If u is not constant, then

$$\int_{B_\varepsilon(x_0)} e_p(u) =: E(\varepsilon) > 0. \tag{64}$$

So when $R \geq \varepsilon$, we have from (62)

$$\int_{\partial B_R(x_0)} e_p(u) \geq \frac{CE(\varepsilon)}{R}. \tag{65}$$

Therefore

$$\lim_{R\to\infty}\int_{B_R(x_0)} \frac{e_p(u)(x)}{\psi(r(x))} = \int_0^\infty \frac{dR}{\psi(R)}\int_{\partial B_R(x_0)} e_p(u) \tag{66}$$
$$\geq CE(\varepsilon)\int_0^\infty \frac{dR}{R\psi(R)}$$
$$\geq CE(\varepsilon)\int_{R_0}^\infty \frac{dR}{R\psi(R)} = \infty$$

which is a contradiction. Q.E.D.

Remark. From (63), if $u|_{\partial B_R(x_0)} = P$, then u is constant. This is a generalization of Karcher and Wood's result. For p-H-harmonic maps, we have

THEOREM 7. *Let M^m be as in Theorem 6, $B_R(x_0)$ be a geodesic ball of M^m with radius R and center x_0. Assume that u is a p-H-harmonic map from $B_R(x_0)$ to N^n with $u|_{\partial B_R(x_0)} = P$, where $P \in N^n$ satisfies $H(P) = \max_{y \in N^n} H(y)$. Then, u is constant.*

For $p = 2$, this theorem is proven by Qun Chen in [2]; for general p, the proof is similar. For completeness, we prove it as follows.

Proof. From (43), (48), (56), (57) and (61), we have

$$
\begin{aligned}
(67)\qquad \frac{(p-1)R}{p} \int_{\partial B_R(x_0)} |du|^{p-2} e_{p-1}(u|_{\partial B_R(x_0)}) \\
&\geq \int_{B_R(x_0)} \operatorname{div} S_{p,u}(X) + \int_{B_R(x_0)} \langle S_{p,u}, \nabla X \rangle \\
&\geq \int_{B_R(x_0)} r \frac{\partial H \circ u}{\partial r} + C \int_{B_R(x_0)} e_p(u).
\end{aligned}
$$

Let $J(\theta, r)\, d\theta dr$ be the volume element of $B_R(x_0)$ in polar coordinates around x_0. Because $(\partial/\partial r)(rJ(\theta, r)) > 0$ (see [2]), we have

$$
\begin{aligned}
(68)\qquad \int_0^R r \frac{\partial H \circ u}{\partial r} J(\theta, r)\, dr &= RJ(\theta, R) H(P) - \int_0^R H \circ u(\theta, r) \frac{\partial}{\partial r}(rJ(\theta, r))\, dr \\
&\geq RJ(\theta, R) H(P) - H(P) \int_0^R \frac{\partial}{\partial r}(rJ(\theta, r))\, dr \\
&= 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(69)\qquad \int_{B_R(x_0)} r \frac{\partial H \circ u}{\partial r} &= \int_{\partial B_R(x_0)} \left(\int_0^R r \frac{\partial H \circ u}{\partial r} J(\theta, r)\, dr \right) d\theta \\
&\geq 0.
\end{aligned}
$$

By (67) and (69) and $e_{p-1}(u|_{\partial B_R(x_0)}) = 0$, we have $\int_{B_R(x_0)} e_p(u) \leq 0$. Q.E.D.

REFERENCES

[1] M. ARA, Geometry of F-harmonic maps, Kodai Math. J., **22** (1999), 243–263.

[2] Q. CHEN, Stability and constant boundary-value problems of harmonic maps with potential, J. Austral. Math. Soc. Ser. A, **68** (2000), 145–154.

[3] J. EELLS AND L. LEMAIRE, Selected Topics in Harmonic Maps, CBMS Regional Conference Series in Mathematics 50, American Mathematical Society, Providence, 1983.

[4] R. E. GREENE AND H. WU, Function Theory on Manifolds Which Possess a Pole, Lecture Notes in Math. 699, Springer-Verlag, Berlin, 1979.

[5] H. S. Hu, A nonexistence theorem for harmonic maps with slowly divergent energy, Chinese Ann. Math. Ser. B, **5** (1984), 737–740.
[6] A.-M. Matei, Gap phenomena for *p*-harmonic maps, Ann. Global Anal. Geom., **18** (2000), 541–554.
[7] N. Nakauchi and S. Takakuwa, A remark on *p*-harmonic maps, Nonlinear Anal., **25** (1995), 169–185.
[8] P. F. Leung, On the stability of harmonic maps, Harmonic Maps (R. J. Knill, M. Kalka and H. C. J. Sealey eds.), Lecture Notes in Math. 949, Springer-Verlag, Berlin, 1982, 122–129.
[9] Y. Ohnita, Stability of harmonic maps and standard minimal immersions, Tohoku Math. J. (2), **38** (1986), 259–267.
[10] H. C. J. Sealey, Harmonic maps of small energy, Bull. London Math. Soc., **13** (1981), 405–408.
[11] H. Takeuchi, Stability and Liouville theorems of *p*-harmonic maps, Japan. J. Math. (N.S.), **17** (1991), 317–332.
[12] S. W. Wei, Representing homotopy groups and spaces of maps by *p*-harmonic maps, Indiana Univ. Math. J., **47** (1998), 625–670.
[13] Y. L. Xin, Some results on stable harmonic maps, Duke Math. J., **43** (1980), 609–613.
[14] Y. L. Xin, Liouville type theorems and regularity of harmonic maps, Differential Geometry and Differential Equations (Gu Chaohao, M. Berger and R. L. Bryant eds.), Lecture Notes in Math. 1255, Springer-Verlag, Berlin, 1987, 198–208.
[15] Y. L. Xin, Harmonic Maps, Shanghai Scientific & Technical Publishers, Shanghai, 1995.

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