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RICCI TENSOR OF SLANT SUBMANIFOLDS IN COMPLEX SPACE FORMS

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Abstract

B.-Y. Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for a submanifold in a Riemannian space form with arbitrary codimension. The Lagrangian version of this inequality was proved by the same author.

In this article, we obtain a sharp estimate of the Ricci tensor of a slant submanifold M in a complex space form $\tilde{M}(4c)$, in terms of the main extrinsic invariant, namely the squared mean curvature. If, in particular, M is a Kaehlerian slant submanifold which satisfies the equality case identically, then it is minimal.

1. Preliminaries

Let M be a real n-dimensional submanifold of a complex m-dimensional complex space form $\tilde{M}(4c)$ of constant holomorphic sectional curvature $4c$. We denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}(4c)$, respectively. Let J be the complex structure on $\tilde{M}(4c)$. Also, we denote by h the second fundamental form and R the Riemann curvature tensor of M.

Then the Gauss equation is given by

$$(1.1)\qquad \tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W))$$

for any vectors X, Y, Z, W tangent to M, where

$$(1.2)\qquad \tilde{R}(X,Y,Z,W) = c\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z) - g(JX,W)g(JY,Z) + g(JX,Z)g(JY,W) + 2g(X,JY)g(Z,JW)\}.$$

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Let $p \in M$ and $\{e_1, \dots, e_{2m}\}$ an orthonormal basis at p, such that $e_1, \dots, e_n$ are tangent to M and $e_{n+1}, \dots, e_{2m}$ are normal to M.

We denote by H the mean curvature vector, i.e.,

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i). \tag{1.3}$$

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j \in \{1, \dots, n\}, \ r \in \{n+1, \dots, 2m\} \tag{1.4}$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{1.5}$$

For any $p \in M$ and $X \in T_pM$, we put $JX = PX + FX$, where PX and FX are the tangential and normal components of JX, respectively.

We denote by

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j). \tag{1.6}$$

We recall that for a submanifold M in a Riemannian manifold, the relative null space of M at a point $p \in M$ is defined by

$$\mathcal{N}_p = \{X \in T_pM \mid h(X, Y) = 0, \text{ for all } Y \in T_pM\}.$$

2. Ricci tensor and squared mean curvature

B.-Y. Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in real space forms (see [3]). Afterwards, he obtained the Lagrangian version of this relationship (see [4]).

First, we prove a similar inequality for an n-dimensional slant submanifold M of an m-dimensional complex space form $\tilde{M}(4c)$ of constant holomorphic sectional curvature $4c$.

A submanifold M of a complex space form $\tilde{M}(4c)$ is said to be a *slant submanifold* [1] if for any $p \in M$ and any nonzero vector $X \in T_pM$, the angle between JX and the tangent space T_pM is constant $(= \theta)$.

It is obvious that both complex submanifolds and totally real submanifolds are slant submanifolds, corresponding to $\theta = 0$ and $\theta = \pi/2$, respectively.

Theorem 2.1. *Let M be an n-dimensional θ-slant submanifold in an m-dimensional complex space form $\tilde{M}(4c)$ of constant holomorphic sectional curvature $4c$. Then*:

i) *For each unit vector* $X \in T_pM$, *we have*

$$\mathrm{Ric}(X) \le \frac{n^2}{4}\|H\|^2 + (n-1)c + 3c\cos^2\theta. \tag{2.1}$$

ii) *If* $H(p) = 0$, *then a unit tangent vector* X *at* p *satisfies the equality case of* (2.1) *if and only if* $X \in \mathcal{N}_p$.

iii) *The equality case of* (2.1) *holds identically for all unit tangent vectors at* p *if and only if either* p *is a totally geodesic point or* $n = 2$ *and* p *is a totally umbilical point.*

In the proof of this theorem, we will use the following result of B.-Y. Chen.

Lemma [2]. *Let* $n \ge 2$ *and* $a_1, \dots, a_n, b$ *real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right). \tag{2.2}$$

Then $2a_1a_2 \ge b$, *with equality holding if and only if*

$$a_1 + a_2 = a_3 = \cdots = a_n.$$

We will give a very short proof, different from the original one in [2].

Proof. By the Cauchy-Schwartz inequality, we have

$$[(a_1 + a_2) + a_3 + \cdots + a_n]^2 \le (n-1)[(a_1 + a_2)^2 + a_3^2 + \cdots + a_n^2].$$

The equation (2.2) implies

$$\sum_{i=1}^{n} a_i^2 + b \le (a_1 + a_2)^2 + a_3^2 + \cdots + a_n^2$$

or equivalently, $2a_1a_2 \ge b$.

The equality holds if and only if

$$a_1 + a_2 = a_3 = \cdots = a_n. \qquad \square$$

Proof of Theorem 2.1. i) Let $X \in T_pM$ be a unit tangent vector X at p. We choose an orthonormal basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m}\}$ such that $e_1, \dots, e_n$ are tangent to M at p, with $e_n = X$ and e_{n+1} is parallel to the mean curvature vector $H(p)$.

Then, from the Gauss equation, we have

$$n^2\|H\|^2 = 2\tau + \|h\|^2 - [n(n-1) + 3n\cos^2\theta]c, \tag{2.3}$$

where τ denotes the scalar curvature at p, that is,

$$\tau = \sum_{1 \le i < j \le n} K(e_i \wedge e_j) = \sum_{1 \le i < j \le n} R(e_i, e_j, e_i, e_j).$$

We put

$$\delta = 2\tau - \frac{n^2}{2}\|H\|^2 - [n(n-1) + 3n\cos^2\theta]c.$$

Then, from (2.3), we get

$$n^2\|H\|^2 = 2(\delta + \|h\|^2). \tag{2.4}$$

With respect to the above orthonormal basis, (2.4) takes the following form:

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left\{\delta + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i \ne j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\right\}.$$

If we put $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n-1} h_{ii}^{n+1}$ and $a_3 = h_{nn}^{n+1}$, the above equation becomes

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left\{\delta + \sum_{i=1}^{3}(a_i)^2 + \sum_{i \ne j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 - \sum_{2 \le \alpha \ne \beta \le n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1}\right\}.$$

Thus a_1, a_2, a_3 satisfy the Lemma of Chen (for $n = 3$), i.e.,

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left(b + \sum_{i=1}^{3}(a_i)^2\right).$$

Then $2a_1a_2 \ge b$, with equality holding if and only if $a_1 + a_2 = a_3$.
In the case under consideration, this means

$$\sum_{1 \le \alpha \ne \beta \le n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} \ge \delta + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2$$

or equivalently,

$$\begin{aligned} &\frac{n^2}{2}\|H\|^2 + [n(n-1) + 3n\cos^2\theta]c \\ &\quad \ge 2\tau - \sum_{1 \le \alpha \ne \beta \le n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2. \end{aligned} \tag{2.5}$$

Using again the Gauss equation, we have

$$
\begin{aligned}
(2.6)\qquad 2\tau - \sum_{1\le\alpha\ne\beta\le n-1} h^{n+1}_{\alpha\alpha}h^{n+1}_{\beta\beta} + 2\sum_{i<j}(h^{n+1}_{ij})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h^r_{ij})^2 \\
= 2S(e_n,e_n) + [(n-1)(n-2) + 3(n-2)\cos^2\theta]c + 2\sum_{i=1}^{n-1}(h^{n+1}_{in})^2 \\
+ \sum_{r=n+2}^{2m}\left\{(h^r_{nn})^2 + 2\sum_{i=1}^{n-1}(h^r_{in})^2 + \left(\sum_{\alpha=1}^{n-1}h^r_{\alpha\alpha}\right)^2\right\},
\end{aligned}
$$

where S is the Ricci tensor of M.

Combining (2.5) and (2.6), we obtain

$$
\begin{aligned}
&\frac{n^2}{2}\|H\|^2 + [2(n-1) + 6\cos^2\theta]c \\
&\qquad \ge 2S(e_n,e_n) + 2\sum_{i=1}^{n-1}(h^{n+1}_{in})^2 + \sum_{r=n+2}^{2m}\left\{\sum_{i=1}^{n}(h^r_{in})^2 + \left(\sum_{\alpha=1}^{n-1}h^r_{\alpha\alpha}\right)^2\right\}
\end{aligned}
$$

which implies (2.1).

ii) Assume $H(p) = 0$. Equality holds in (2.1) if and only if

$$
(2.7)\qquad \begin{cases} h^r_{1n} = \cdots = h^r_{n-1,n} = 0 \\ h^r_{nn} = \sum_{i=1}^{n-1} h^r_{ii} \end{cases}, \quad r \in \{n+1,\dots,2m\}.
$$

Then $h^r_{in} = 0$, $\forall i \in \{1,\dots,n\}$, $r \in \{n+1,\dots,2m\}$, i.e., $X \in \mathcal{N}_p$.

iii) The equality case of (2.1) holds for all unit tangent vectors at p if and only if

$$
(2.8)\qquad \begin{cases} h^r_{ij} = 0, \quad i \ne j,\ r \in \{n+1,\dots,2m\}, \\ h^r_{11} + \cdots + h^r_{nn} - 2h^r_{ii} = 0, \quad i \in \{1,\dots,n\},\ r \in \{n+1,\dots,2m\}. \end{cases}
$$

We distinguish two cases:

a) $n \ne 2$, then p is a totally geodesic point;

b) $n = 2$, it follows that p is a totally umbilical point.

The converse is trivial. □

Corollary 2.2. *Let M be an n-dimensional totally real submanifold in an m-dimensional complex space form $\tilde{M}(4c)$ of constant holomorphic sectional curvature $4c$. Then:*

i) *For each unit vector $X \in T_pM$, we have*

$$
(2.9)\qquad \mathrm{Ric}(X) \le \frac{n^2}{4}\|H\|^2 + (n-1)c.
$$

ii) *If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case of (2.9) if and only if $X \in \mathcal{N}_p$.*

iii) *The equality case of* (2.9) *holds identically for all unit tangent vectors at* p *if and only if either* p *is a totally geodesic point or* $n = 2$ *and* p *is a totally umbilical point.*

It is known that every complex submanifold of a Kaehlerian manifold is minimal.

COROLLARY 2.3. *Let* M *be an* n*-dimensional complex submanifold in an* m*-dimensional complex space form* $\tilde{M}(4c)$ *of constant holomorphic sectional curvature* $4c$. *Then*:

i) *For each unit vector* $X \in T_pM$, *we have*

$$\mathrm{Ric}(X) \leq 2(n+1)c. \tag{2.10}$$

ii) *A unit tangent vector* X *at* p *satisfies the equality case of* (2.10) *if and only if* $X \in \mathcal{N}_p$.

iii) *The equality case of* (2.10) *holds identically for all unit tangent vectors at* p *if and only if* p *is a totally geodesic point.*

By polarization, from Theorem 2.1, we derive:

THEOREM 2.4. *Let* M *be an* n*-dimensional* θ*-slant submanifold in an* m*-dimensional complex space form* $\tilde{M}(4c)$ *of constant holomorphic sectional curvature* $4c$. *Then the Ricci tensor* S *satisfies*

$$S \leq \left(\frac{n^2}{4}\|H\|^2 + (n-1)c + 3c\cos^2\theta\right)g. \tag{2.11}$$

The equality case of (2.11) *holds identically if and only if either* M *is a totally geodesic submanifold or* $n = 2$ *and* M *is a totally umbilical submanifold.*

In particular, for totally real and complex submanifolds, respectively, we state:

COROLLARY 2.5 [4]. *Let* M *be an* n*-dimensional totally real submanifold in an* m*-dimensional complex space form* $\tilde{M}(4c)$ *of constant holomorphic sectional curvature* $4c$. *Then the Ricci tensor* S *satisfies*

$$S \leq \left(\frac{n^2}{4}\|H\|^2 + (n-1)c\right)g. \tag{2.12}$$

The equality case of (2.12) *holds identically if and only if either* M *is a totally geodesic submanifold or* $n = 2$ *and* M *is a totally umbilical submanifold.*

For a classification of totally umbilical submanifolds in nonflat complex space forms we refer to [6].

COROLLARY 2.6. *Let M be an n-dimensional complex submanifold in an m-dimensional complex space form $\tilde{M}(4c)$ of constant holomorphic sectional curvature $4c$. Then the Ricci tensor S satisfies*

$$S \le 2(n+1)cg. \tag{2.13}$$

The equality case of (2.13) *holds identically if and only if M is a totally geodesic submanifold.*

3. Minimality of Kaehlerian slant submanifolds

Let $\tilde{M}(4c)$ be an n-dimensional complex space form of constant holomorphic sectional curvature $4c$ and M an n-dimensional θ-slant submanifold of $\tilde{M}(4c)$.

By reference to [1], M is said to be a *Kaehlerian slant submanifold* if it is proper (i.e., $\theta \notin \{0, \pi/2\}$) and the endomorphism P of the tangent bundle TM is parallel with respect to the Riemannian connection ∇ of M (i.e. $\nabla P = 0$). A Kaehlerian slant submanifold is a Kaehler manifold with respect to the induced metric and the almost complex structure $\tilde{J} = (1/\cos\theta)P$.

It is known that every proper slant surface in a Kaehler manifold is Kaehlerian slant (see [1]). An example of a 4-dimensional Kaehlerian slant submanifold in $\boldsymbol{C}^4$ is given by the following immersion.

$$x(u, v, w, z) = (u, v, k \sin w, k \sin z, kw, kz, k \cos w, k \cos z),$$

where $k > 0$ is a constant. In this case, $\theta = \pi/4$ (see [1]).

We denote by $\mathscr{R}$ the maximum Ricci curvature function on M (see [4]), defined by

$$\mathscr{R}(p) = \max\{S(u,u) \mid u \in T_p^1 M\}, \quad p \in M,$$

where $T_p^1 M = \{u \in T_p M \mid g(u,u) = 1\}$.

If $n = 3$, $\mathscr{R}$ is the Chen first invariant δ_M defined in [2]. For $n > 3$, $\mathscr{R}$ is the Chen invariant $\delta(n-1)$ (see [5]).

In this section, we derive an inequality for the Chen invariant $\mathscr{R}$ and prove that any Kaehlerian slant submanifold which satisfies the equality case is minimal. This is a generalization of a result of B.-Y. Chen [4] for Lagrangian submanifolds in complex space forms.

THEOREM 3.1. *Let M be an n-dimensional Kaehlerian slant submanifold in an n-dimensional complex space form $\tilde{M}(4c)$ of constant holomorphic sectional curvature $4c$. Then*

$$\mathscr{R} \le \frac{n^2}{4}\|H\|^2 + (n-1)c + 3c\cos^2\theta. \tag{3.1}$$

If M satisfies the equality case of (3.1) *identically, then M is a minimal submanifold.*

Proof. The inequality (3.1) is an immediate consequence of the inequality (2.11).

We assume that M is a Kaehlerian slant submanifold of $\tilde{M}(4c)$, which satisfies the equality case of (3.1) at a point $p \in M$. We may choose an orthonormal basis $\{\bar{e}_1, \dots, \bar{e}_n\}$ of T_pM such that $\mathscr{R}(p) = S(\bar{e}_n, \bar{e}_n)$. We set $\bar{e}_{n+j} = (1/\sin\theta)F\bar{e}_j$, $j \in \{1, \dots, n\}$. By the proof of Theorem 2.1, it follows that the equations (2.7) hold, where h_{ij}^r are the coefficients of the second fundamental form with respect to the orthonormal basis $\{\bar{e}_1, \dots, \bar{e}_n, \bar{e}_{n+1}, \dots, \bar{e}_{2n}\}$.

Let A denote the shape operator of M in $\tilde{M}(4c)$. It is known (see [1]) that P is parallel if and only if

$$A_{FX}Y = A_{FY}X, \tag{3.2}$$

for all vector fields X, Y tangent to M.

We distinguish two cases:

i) If $g(h(u,v), Fw) = 0$, $\forall u, v, w \in T_pM$, then obviously $H(p) = 0$.

ii) We assume that case i) does not hold. Then we define

$$f_p : T_p^1M \to \boldsymbol{R}, \quad f_p(v) = g(h(v,v), Fv).$$

Since T_p^1M is compact, there exists a vector $v \in T_p^1M$ such that f_p attains an absolute maximum at v. Let denote $e_1 = v$ and $f_p(v) = \lambda_1 > 0$. It follows that $A_{Fe_1}e_1 = \lambda_1 e_1$.

We can choose an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM such that e_i is an eigenvector of A_{Fe_1} with corresponding eigenvalue λ_i, for all $i \in \{1, \dots, n\}$.

We consider the function $f_i(t) = f_p((\cos t)e_1 + (\sin t)e_i)$, $i \in \{2, \dots, n\}$.

It is easily seen that f_i has a relative maximum at $t = 0$. Thus, $f_i'(0) = 0$ and $f_i''(0) \leq 0$. By a straightforward computation, one finds

$$0 \geq f_i''(0) = -3\lambda_1 + 6\lambda_i,$$

i.e., $\lambda_1 \geq 2\lambda_i$, $\forall i \geq 2$. Since $\lambda_1 > 0$, one gets $\lambda_1 \neq \lambda_i$, $\forall i \geq 2$. Thus, the multiplicity of the eigenvalue λ_1 is 1.

We have $e_1 \neq \pm\bar{e}_n$. Otherwise

$$A_{Fe_i}\bar{e}_n = \pm A_{Fe_i}e_1 = \pm A_{Fe_1}e_i = \pm\lambda_i e_i \perp \bar{e}_n, \quad i \in \{2, \dots, n\},$$

implies $\lambda_2 = \cdots = \lambda_n = 0$, and hence, using (2.7), $\lambda_1 = 0$, which is a contradiction.

On the other hand, by (2.7) it is easily seen that $\bar{e}_n$ is an eigenvector of A_{Fe_1}. Thus, we can choose $e_n = \bar{e}_n$, and, consequently, we may assume $e_j = \bar{e}_j$, $\forall j \in \{1, \dots, n\}$.

By (3.2) and (2.7), we have

$$A_{Fe_n}e_1 = A_{Fe_1}e_n = \lambda_n e_n = 0.$$

Thus, (2.7) implies $\lambda_1 + \cdots + \lambda_{n-1} = \lambda_n = 0$. Therefore $\operatorname{tr} A_{Fe_1} = 0$.

For $i \in \{2, \dots, n-1\}$, one has

$$\operatorname{tr} A_{Fe_i} = \sum_{j=1}^{n} g(A_{Fe_i} e_j, e_j) = \sum_{j=1}^{n} g(h(e_j, e_j), Fe_i) = 2g(h(e_n, e_n), Fe_i)$$
$$= 2g(h(e_i, e_n), Fe_n) = 0.$$

Similarly

$$\operatorname{tr} A_{Fe_n} = \sum_{j=1}^{n} g(h(e_j, e_j), Fe_n) = 2\sum_{j=1}^{n-1} g(h(e_j, e_j), Fe_n) = 2\sum_{j=1}^{n-1} g(h(e_j, e_n), Fe_j) = 0.$$

Thus, $\operatorname{tr} A_{Fe_i} = 0$, $\forall i \in \{1, \dots, n\}$.
Consequently, $H(p) = 0$. □

Corollary 3.2. *Let M be an n-dimensional Kaeherian slant submanifold of an n-dimensional complex space form $\tilde{M}(4c)$. If* $\dim \mathcal{N}_p$ *is positive constant, then M satisfies the equality case of* (3.1) *identically and is foliated by totally geodesic submanifolds.*

Proof. By the above proof, it follows that M satisfies the equality case of (3.1) at a point $p \in M$ if and only if $\dim \mathcal{N}_p \geq 1$.
Assume that $\dim \mathcal{N}_p$ is positive constant.
It is known that $\mathcal{N}$ is involutive and its leaves are totally geodesic (see, for instance, [4], [10]). This achieves the proof. □

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