J. CHOI, J. LEE AND H. M. SRIVASTAVA
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A GENERALIZATION OF WILF'S FORMULA

Junesang Choi, Jungseob Lee and H. M. Srivastava

Abstract

A remarkable infinite product formula was recently posed as a problem by Wilf [6]. Subsequently, Choi and Seo [2] proved Wilf's formula as well as three other analogous product formulas. The main object of this sequel to these earlier works is to present several general infinite product formulas which include, as their special cases, the aforementioned product formulas of Wilf [6] and Choi and Seo [2]. Some other related results, involving the Riemann Zeta function, are also considered briefly.

1. Introduction

In 1997, Wilf [6] posed as a problem the following elegant infinite product formula which contains some of the most important mathematical constants such as π, e and the Euler-Mascheroni constant γ defined by Equation (6) below:

$$\prod_{j=1}^{\infty}\left\{e^{-1/j}\left(1+\frac{1}{j}+\frac{1}{2j^2}\right)\right\}=\frac{e^{\pi/2}+e^{-\pi/2}}{\pi e^{\gamma}}. \tag{1}$$

Subsequently, Choi and Seo [2] proved (1) as well as three other similar product formulas by making use of well-known infinite product formulas for circular and hyperbolic functions and the familiar Stirling formula:

$$n! \sim \left(\frac{n}{e}\right)^n\sqrt{2\pi n} \quad (n\to\infty). \tag{2}$$

Here, in this sequel to the works of Wilf [6] and Choi and Seo [2], we present the following two general infinite product formulas which include Wilf's formula (1) and other similar formulas in [2] as their special cases:

$$\prod_{j=1}^{\infty}\left\{e^{-1/j}\left(1+\frac{1}{j}+\frac{\alpha^2+1/4}{j^2}\right)\right\}=\frac{2(e^{\alpha\pi}+e^{-\alpha\pi})}{(4\alpha^2+1)\pi e^{\gamma}} \quad \left(\alpha\in\boldsymbol{C};\quad \alpha\neq\pm\frac{1}{2}i;\quad i=\sqrt{-1}\right) \tag{3}$$

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and

$$\prod_{j=1}^{\infty}\left\{e^{-2/j}\left(1+\frac{2}{j}+\frac{\beta^2+1}{j^2}\right)\right\}=\frac{e^{\beta\pi}-e^{-\beta\pi}}{2\beta(\beta^2+1)\pi e^{2\gamma}} \quad (\beta\in \boldsymbol{C}\backslash\{0\};\quad \beta\neq\pm i). \tag{4}$$

2. Derivation of the product formulas (3) and (4)

We begin by letting

$$\mathscr{A}(p,q):=\prod_{j=1}^{\infty}\left\{e^{-p/j}\left(1+\frac{p}{j}+\frac{q}{j^2}\right)\right\} \quad (p,q\in \boldsymbol{C};\quad \Re(p)>0) \tag{5}$$

and take logarithms on both sides of (5). Then, using the Euler-Mascheroni constant γ defined by

$$\gamma:=\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n}-\log N\right)\cong 0.577215664901532860606512\cdots, \tag{6}$$

we obtain

$$\log\mathscr{A}(p,q)=-p\gamma+\lim_{n\to\infty}\left[-p\log n+\sum_{j=1}^{n}\log(j^2+pj+q)-2\sum_{j=1}^{n}\log j\right],$$

which, upon applying Stirling's formula (2) to the last summation, yields

$$\begin{aligned}\log\mathscr{A}(p,q)&=-p\gamma-\log(2\pi)\\&\quad+\lim_{n\to\infty}\left[-(2n+p+1)\log n+2n+\sum_{j=1}^{n}\log(j^2+pj+q)\right].\end{aligned} \tag{7}$$

Now we recall the infinite product formula of $\cosh z$:

$$\cosh z=\prod_{j=1}^{\infty}\left\{\left(1+\frac{4z^2}{(2j-1)^2\pi^2}\right)\right\}=\frac{e^z+e^{-z}}{2}. \tag{8}$$

If we set $z=\alpha\pi$ in (8) and take logarithms on both sides of the resulting equation, and use the following version of Stirling's formula (2):

$$\sum_{j=1}^{n}\log(2j-1)=\left(n+\frac{1}{2}\right)\log 2-n+n\log n+o(1)\quad (n\to\infty), \tag{9}$$

we find that

$$\log(e^{\alpha\pi}+e^{-\alpha\pi}) = \lim_{n\to\infty}\left[\sum_{j=1}^{n}\log\{(2j-1)^2+4\alpha^2\}+2n-2n\log(2n)\right]. \tag{10}$$

Combining (7) and (10), we obtain

$$\begin{aligned}\log\mathscr{A}(p,q) &= -p\gamma-\log(2\pi)+\log(e^{\alpha\pi}+e^{-\alpha\pi}) \\ &\quad+\lim_{n\to\infty}\left[-(p+1)\log n+\sum_{j=1}^{n}\log(4j^2+4pj+4q)\right. \\ &\qquad\left.-\sum_{j=1}^{n}\log\{(2j-1)^2+4\alpha^2\}\right].\end{aligned} \tag{11}$$

In a similar manner, if we consider the infinite product formula of $\sinh z$ instead of $\cosh z$ in (8), we also have

$$\begin{aligned}\log\mathscr{A}(p,q) &= -p\gamma-\log(2\beta\pi)+\log(e^{\beta\pi}-e^{-\beta\pi}) \\ &\quad+\lim_{n\to\infty}\left[-p\log n+\sum_{j=1}^{n}\log(j^2+pj+q)-\sum_{j=1}^{n}\log(j^2+\beta^2)\right].\end{aligned} \tag{12}$$

Finally, by setting

$$(p,q)=\left(1,\alpha^2+\frac{1}{4}\right) \quad \text{and} \quad (p,q)=(2,\beta^2+1)$$

in (11) and (12), respectively, we arrive at the desired formulas (3) and (4).

We note that the special case of (3) when $\alpha = 1/2$ leads to Wilf's formula (1). Other interesting special cases of (3) and (4) include all of the aforementioned results of Choi and Seo [2].

3. Further remarks and observations

Taking logarithms on both sides of (3) and (4), and using the Riemann Zeta function $\zeta(s)$ defined by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{1-2^{-s}}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^s} & (\Re(s)>1) \\ \displaystyle(1-2^{1-s})^{-1}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} & (\Re(s)>0;\, s\neq 1), \end{cases} \tag{13}$$

together with the Maclaurin expansion of $\log(1+x)$, we obtain the following interesting closed-form evaluation of two families of series involving the Riemann Zeta function:

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{\ell=0}^{n} \binom{n}{\ell} \left(\alpha^2 + \frac{1}{4}\right)^{\ell} \left\{ \zeta(n+\ell) - \sum_{k=1}^{j-1} \frac{1}{k^{n+\ell}} \right\} \tag{14}$$

$$= \sum_{k=1}^{j-1} \left[\frac{1}{k} + \left(\alpha^2 + \frac{1}{4}\right) \frac{1}{k^2} - \log\left(1 + \frac{1}{k} + \frac{\alpha^2 + 1/4}{k^2}\right) \right]$$

$$- \left(\alpha^2 + \frac{1}{4}\right) \zeta(2) + \log\left(\frac{2(e^{\alpha\pi} + e^{-\alpha\pi})}{(4\alpha^2 + 1)\pi e^{\gamma}} \right)$$

$$\left(j - 1 \in \boldsymbol{N}; \quad \alpha^2 < j^2 - j - \frac{1}{4}; \quad \alpha \in \boldsymbol{R} \right);$$

$$\sum_{n=2}^{\infty} (-1)^{n+1} \frac{2^n}{n} \sum_{\ell=0}^{n} \binom{n}{\ell} \left(\frac{\beta^2 + 1}{2} \right)^{\ell} \left\{ \zeta(n+\ell) - \sum_{k=1}^{j-1} \frac{1}{k^{n+\ell}} \right\} \tag{15}$$

$$= \sum_{k=1}^{j-1} \left[\frac{2}{k} + \frac{\beta^2 + 1}{k^2} - \log\left(1 + \frac{2}{k} + \frac{\beta^2 + 1}{k^2}\right) \right]$$

$$- (\beta^2 + 1)\zeta(2) + \log\left(\frac{e^{\beta\pi} - e^{-\beta\pi}}{2\beta(\beta^2 + 1)\pi e^{2\gamma}} \right)$$

$$(j - 2 \in \boldsymbol{N}; \quad \beta^2 < j^2 - 2j - 1; \quad \beta \in \boldsymbol{R}).$$

Just as it has been observed earlier (*cf.*, *e.g.*, [1], [3], and [4]; see also [5]), the subject of closed-form summation of series involving the Riemann Zeta function can be traced back to an over two-century-old theorem of Christian Goldbach (1690–1764).

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References

[1] V. S. Adamchik and H. M. Srivastava, Some series of the Zeta and related functions, Analysis (Munich), **18** (1998), 131–144.

[2] J. Choi and T. Y. Seo, Identities involving series of the Riemann Zeta function, Indian J. Pure Appl. Math., **30** (1999), 649–652.

[3] J. D. Shallit and K. Zikan, A theorem of Goldbach, Problems and Solutions, Amer. Math. Monthly, **93** (1986), 402–403.

[4] H. M. Srivastava, A unified presentation of certain classes of series of the Riemann Zeta function, Riv. Mat. Univ. Parma (4), **14** (1988), 1–23.

[5] H. M. Srivastava and J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, 2001.

[6] H. S. Wilf, Problem 10588, Problems and Solutions, Amer. Math. Monthly, **104** (1997), 456.

Department of Mathematics
College of Natural Sciences
Dongguk University
Kyongju 780-714
Korea
e-mail: junesang@mail.dongguk.ac.kr

Department of Mathematics
Ajou University
Suwon 442-749
Korea
e-mail: jslee@madang.ajou.ac.kr

Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada
e-mail: harimsri@math.uvic.ca