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ON EFFECTIVE DIVISORS ON SMOOTH PROJECTIVE SURFACES

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Abstract

Let X be a smooth projective surface defined over the complex number field and let D be an effective divisor on X. In this paper we will propose a special class of effective divisors which has some properties similar to that of the case where D is ample and we will study this divisor.

Introduction

Let X be a smooth projective variety defined over the complex number field and let L be a divisor on X. Then the pair (X, L) is called a *prepolarized manifold*. If L is ample, then (X, L) is called a *polarized manifold*.

In this paper we consider the case where $\dim X = 2$, and we study some special type of effective divisors.

In previous papers ([Fk1], [Fk2], [Fk3], [Fk5], [Fk6] and [Fk7]), we classified polarized surfaces (X, L) by using the value of $g(L)$, where $g(L)$ is the sectional genus of L, that is, $g(L) = 1 + (1/2)(K_X + L)L$. (Here K_X is the canonical divisor on X.) The details are as follows: If $h^0(L) > 0$, then we can prove that $g(L) \geq q(X)$ (see Lemma 1.2 in [Fk2]), where $q(X)$ is the irregularity of X, and we classified (X, L) with $h^0(L) > 0$ and $0 \leq g(L) - q(X) \leq 1$ (see [Fk1], [Fk2], [Fk3] and [Fk5]). Furthermore in [Fk6] and [Fk7], we classified (X, L) such that (X, L) satisfies one of the following:

(a) $g(L) = q(X) + m$ and $h^0(L) \geq m + 2$,

(b) $g(L) = q(X) + m$, $h^0(L) = m + 1$, and $\dim \mathrm{Bs}|L| \leq 0$,

where m is a non-negative integer.

When we classify (X, L) by the value of $g(L) - q(X)$, we need to study a lower bound for $K_X L$. So in [Fk4] we studied the intersection number $K_X L$. For example we obtained that $K_X L \geq 2q(X) - 4$ for any polarized surface (X, L) with $\kappa(X) \geq 0$ and $h^0(L) \geq 2$. The above results are useful to study projective surfaces.

But the author feels that in order to study projective surfaces more deeply, it is necessary to study more general effective divisors than ample effective divisors.

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So the author wants to find a class $\mathscr{C}$ of effective divisors on X which satisfies the following:

(1) Any effective divisor which is a member of the class $\mathscr{C}$ has properties similar to that of ample effective divisors.

(2) We can easily check whether an effective divisor D is a member of the class $\mathscr{C}$ or not.

(3) Any ample effective divisor is a member of the class $\mathscr{C}$.

If we can find the class $\mathscr{C}$ which satisfies the above three conditions, then it seems to be very useful to study projective surfaces.

As the first attempt, in [Fk4] we proposed a special effective divisor, which is called a CNNS-divisor (see Definition 1.1 below). We note that any ample effective divisor is a CNNS-divisor. If X is minimal and L is a CNNS-divisor, then L has properties similar to that of ample effective divisors. (For example, see [Fk4].) But when X is not minimal and L is a CNNS-divisor, L does not always have properties similar to that of ample effective divisors. For example, assume that $\pi : X \to X'$ is a birational morphism, where X' is a smooth projective surface. If L is ample, then $\mu_*(L)$ is ample and $K_X L \geq K_{X'}(\mu_*(L))$ is always true. If L is a CNNS-divisor, then so is $\mu_*(L)$, but $K_X L \geq K_{X'}(\mu_*(L))$ is not always true. So it needs to consider some special type of CNNS-divisors on X and to study these.

Hence in this paper we propose a new class of effective divisors and we study effective divisors of this class. We define a new class of effective divisors as follows:

(#) Let D be effective divisors on X such that $D = B + T_1 + \cdots + T_{n-1}$, where B is an effective divisor on X, and $T_0, T_1, \dots, T_{n-1}$ is a sequence of reduced effective divisors on X such that $(B; T_0, \dots, T_{n-1})$ is a generalized composite series with respect to B (see Definition 2.1).

Let D be an effective divisor on X such that D has the property (#) and B is a reduced CNNS-divisor. Then this effective divisor has properties similar to that of ample effective divisors. (For example, the sectional genus $g(D)$, the intersection number $K_X D$, and the vanishing of $h^i(-D)$.) We will study these in Section 2.

In Section 3, we prove that if D is a nef and big effective $\mathbf{Q}$-divisor on X, then $\lceil D \rceil$ has the property (#) such that B is a reduced CNNS-divisor. (See Theorem 3.1.) We also prove that if D is an s-connected effective divisor on X, then D has the property (#). (See Proposition 3.2.)

Theorem 3.1 determines a kind of a structure theorem of the Zariski decomposition for nef and big effective $\mathbf{Q}$-divisors on X, and is very useful to study nef and big effective $\mathbf{Q}$-divisors. For example, as an application of Theorem 3.1, we get that $g(\lceil D \rceil) \geq q(X)$ by Theorem 2.4 and Theorem 3.1, where D is a nef and big effective $\mathbf{Q}$-divisors. Furthermore we can classify (X, D) with $g(\lceil D \rceil) = 0$ (see Proposition 4.1). Here we note that Proposition 4.1 is a new result.

Here we note the following: when we study polarized surfaces (X, L), it is difficult to study (X, L) with $h^0(L) = 0$. But since any ample divisor is a nef and big effective $\boldsymbol{Q}$-divisor, we can expect that some results of this paper give one direction for studying polarized surfaces (X, L) with $h^0(L) = 0$, and we hope that some results in this paper become useful to study an ample divisor L with $h^0(L) = 0$.

We will study (X, D) with $g(\lceil D \rceil) = q(X)$ in a future paper.

We use the customary notation in algebraic geometry. In this paper we mainly study smooth projective surfaces defined over the complex number field.

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1. Preliminaries

Definition 1.1 (see Definition 4.3 in [Fk4]). Let X be a smooth projective surface and let D be an effective divisor on X. Then D is called a *CNNS-divisor* if the following conditions hold:

(1) D is connected.

(2) the intersection matrix $\|(C_i, C_j)\|_{i,j}$ of $D = \sum_i r_i C_i$ is not negative semidefinite.

Theorem 1.2. *Let X be a minimal smooth projective surface and let D be an effective CNNS-divisor on X such that one of the following conditions hold;*

(1) $\kappa(X) = 0, 1$,

(2) $\kappa(X) = 2$ *and* $h^0(D) \geq 2$.

Then $K_X D \geq 2q(X) - 4$.

Proof. (I) The case in which $\kappa(X) = 0$.

Then $q(X) \leq 2$. Since K_X is nef, we get that $K_X D \geq 0$. Hence $K_X D \geq 0 \geq 2q(X) - 4$.

(II) The case in which $\kappa(X) = 1$.

Let $f : X \to C$ be an elliptic fibration over a smooth curve C. If $g(C) \leq 1$, then $q(X) \leq 2$. Hence $K_X D \geq 0 \geq 2q(X) - 4$. Therefore we assume $g(C) \geq 2$. By the canonical bundle formula of elliptic fibrations, we get that

$$K_X D \geq (2g(C) - 2 + \chi(\mathcal{O}_X))DF$$

for a general fiber F of f. Since D is a CNNS-divisor on X, there exists a curve B such that B is not contained in a fiber of f. Thus we get that $DF \geq 1$. Hence

$$\begin{aligned} K_X D &\geq (2g(C) - 2 + \chi(\mathcal{O}_X))DF \\ &\geq 2g(C) - 2 + \chi(\mathcal{O}_X) \\ &\geq 2g(C) - 2 \\ &= 2g(C) + 2 - 4 \\ &\geq 2q(X) - 4. \end{aligned}$$

(III) The case in which $\kappa(X) = 2$ and $h^0(D) \geq 2$.

Let M be the movable part of $|D|$ and let Z be the fixed part of $|D|$. Then M is nef. If $M^2 > 0$, then M is nef and big. So we get that $K_X D \geq K_X M \geq 2q(X) - 2$ by Theorem 3.1 in [Fk4].

If $M^2 = 0$, then M is nef but not big. Moreover we get $\mathrm{Bs}|M| = \emptyset$. So we get a fiber space $f : X \to B$ defined by $|M|$, where B is a smooth projective curve. We remark that M is numerically equivalent to aF for a natural number a, where F is a general fiber of f.

If $g(B) = 0$, then

$$\begin{aligned} K_X D \geq K_X M &\geq K_X F \\ &= 2g(F) - 2 \\ &\geq 2q(X) - 2. \end{aligned}$$

If $g(B) \geq 1$, then

$$\begin{aligned} K_{X/B} D &\geq K_{X/B} M \\ &\geq 2g(F) - 2 \end{aligned}$$

because $K_{X/B}$ is nef by Arakelov's theorem (see [Be]). Therefore

$$K_X D \geq (2g(B) - 2)DF + 2g(F) - 2.$$

Since D is a CNNS-divisor, there exists a curve C such that C is not contained in a fiber of f but contained in D. So we obtain that $DF \geq 1$. Hence

$$\begin{aligned} K_X D &\geq 2g(B) - 2 + 2g(F) - 2 \\ &\geq 2q(X) - 4. \qquad \square \end{aligned}$$

THEOREM 1.3. *Let X be a minimal smooth surface of general type and let D be a CNNS-divisor with $h^0(D) = 1$ on X. If D is not of the following type* $(\star)$, *then* $K_X D \geq 2q(X) - 4$;

$(\star)$ $D = C_1 + \sum_{j \geq 2} r_j C_j$; $C_1^2 > 0$ *and the intersection matrix* $\|(C_j, C_k)\|_{j \geq 2, k \geq 2}$ *of* $\sum_{j \geq 2} r_j C_j$ *is negative semidefinite.*

Proof. See Theorem 4.5, Theorem 4.6 and Theorem 4.11 in [Fk4]. $\square$

DEFINITION 1.4 (see Definition 3.1 in [Mi]). Let X be a smooth projective surface and let D be an effective divisor on X. Then D is called *s-connected* (*in the sense of* [Mi]) if there exists a decomposition of D, $D = C_0 + C_1 + \cdots + C_r$ such that $(C_0 + \cdots + C_{i-1})C_i > 0$ for $i = 1, \ldots, r$, where C_i is an irreducible curve for any i.

Remark 1.4.1.

(1) The notion of s-connectedness in Definition 1.4 is different from the notion of m-connectedness in [BPV] (see p. 69 Definition in [BPV]), where m is a positive integer.

(2) If D is a 1-connected effective divisor, then D is s-connected. In particular if D is a nef and big effective divisor, then D is s-connected.

Proposition 1.5. *Let X be a smooth projective surface. An effective divisor D is not s-connected if and only if there exists a nontrivial decomposition $D = D_1 + D_2$ into effective divisors such that $D_1 C \leq 0$ for any irreducible component C of D_2.*

Proof. See Proposition 3.3 in [Mi]. □

Definition 1.6 (see p. 69 Definition in [BPV]). Let X be a smooth projective surface. Then an effective divisor D on X is said to be 1-*connected* if $D_1 D_2 > 0$ for any nonzero effective divisors D_1 and D_2 with $D = D_1 + D_2$.

Remark 1.6.1. If D is a reduced and connected effective divisor on X, then D is 1-connected.

Proposition 1.7. *Let X be a smooth projective surface and let D be an effective* 1*-connected divisor. Let $\pi : X \to X_1$ be the blowing down of a (-1)-curve E and we put $D_1 := \pi_*(D)$ in the sense of cycle theory. Then D_1 is effective and* 1*-connected. Furthermore if $D^2 > 0$, then $D_1^2 > 0$.*

Proof. We put $D = \pi^*(D_1) + aE$ for $a \in \boldsymbol{Z}$. Let $D_1 = D_{1,1} + D_{1,2}$ be a decomposition of effective divisors with $D_{1,1} \neq 0$ and $D_{1,2} \neq 0$. Then there exist integers a_1 and a_2 such that $\pi^*(D_{1,1}) + a_1 E$ and $\pi^*(D_{1,2}) + a_2 E$ are effective and

$$D = (\pi^*(D_{1,1}) + a_1 E) + (\pi^*(D_{1,2}) + a_2 E).$$

If $a_1 a_2 \geq 0$, then by assumption we get that

$$\begin{aligned} 0 &< (\pi^*(D_{1,1}) + a_1 E)(\pi^*(D_{1,2}) + a_2 E) \\ &= D_{1,1} D_{1,2} - a_1 a_2 \\ &\leq D_{1,1} D_{1,2}. \end{aligned}$$

If $a_1 a_2 < 0$, then we may assume that $a_1 > 0$ and $a_2 < 0$. Then we consider a decomposition

$$D = (\pi^*(D_{1,1})) + (\pi^*(D_{1,2}) + (a_1 + a_2)E).$$

Since $\pi^*(D_{1,2}) + a_2 E$ is effective, so is $\pi^*(D_{1,2}) + (a_1 + a_2)E$. Then

$$\begin{aligned} 0 &< (\pi^*(D_{1,1}))(\pi^*(D_{1,2}) + (a_1 + a_2)E) \\ &= D_{1,1} D_{1,2}. \end{aligned}$$

Therefore D_1 is 1-connected. On the other hand, $0 < D^2 = D_1^2 - a^2 \leq D_1^2$. □

Definition 1.8 (see Definition 1.9 in [Fk2]).

(1) Let X be a smooth projective surface and let D be a divisor on X. Then (X, D) is said to be *D-minimal* if $DE \neq 0$ for any (-1)-curve E on X.

(2) For any prepolarized surface (X, D), there exist a smooth projective surface X_0, a divisor D_0, and a birational morphism $\rho : X \to X_0$ such that $D = \rho^*(D_0)$ and (X_0, D_0) is D_0-minimal. Then we call (X_0, D_0) a *D-minimalization of* (X, D).

THEOREM 1.9. *Let X be a smooth projective surface and let D be an effective* 1*-connected divisor on X. Then there exist a smooth projective surface S, an effective* 1*-connected divisor D_S on S, a birational morphism $\pi : X \to S$, and $D = \pi^{-1}D_S + \sum_i a_i C_i$ for nonnegative integers a_i and smooth rational curves C_i with $C_i^2 \leq -1$ such that $g(D) = g(D_S)$ and one of the following holds*:
(1) $(S, D_S) \cong (\boldsymbol{P}^2, \mathcal{O}(1))$,
(2) $(S, D_S) \cong (\boldsymbol{P}^2, \mathcal{O}(2))$,
(3) (S, D_S) *is a scroll over a smooth curve*,
(4) $K_S + D_S$ *is nef*,
(5) D_S *is a smooth rational curve with* $D_S^2 \leq -2$,
where $\pi^{-1}D_S$ denotes the strict transform of D_S via π.

Proof. We put $X_0 := X$, and $D_0(0) := D$. First we take a $D_0(0)$-minimalization of $(X_0, D_0(0))$; $\pi_0 : X_0 \to X_1$, where X_1 is a smooth projective surface and π_0 is a birational morphism. Let $D_1 := (\pi_0)_*(D_0(0))$. For (X_1, D_1), if $K_{X_1} + D_1$ is nef, then we are done. So we assume that $K_{X_1} + D_1$ is not nef. Then there exists an irreducible curve C_1 such that $(K_{X_1} + D_1)C_1 < 0$.

If $K_{X_1}C_1 < 0$, then X_1 is isomorphic to $\boldsymbol{P}^2$, $\boldsymbol{P}^1$-bundle over a smooth curve, or X_1 has a (-1)-curve E_1.

If X_1 is the first two cases, then we are done.

If X_1 is the last case, then $K_{X_1}E_1 = -1$ and $D_1E_1 = 0$. But this is impossible because (X_1, D_1) is D_1-minimal.

If $K_{X_1}C_1 \geq 0$, then

$$\begin{aligned} 0 &> (K_{X_1} + C_1 + (D_1 - C_1))C_1 \\ &= 2g(C_1) - 2 + (D_1 - C_1)C_1. \end{aligned}$$

Since D_1 is 1-connected by Proposition 1.7, $(D_1 - C_1)C_1 > 0$. Hence $g(C_1) = 0$ and $(D_1 - C_1)C_1 = 1$. Furthermore $C_1^2 \leq -2$ because $g(C_1) = 0$ and $K_{X_1}C_1 \geq 0$. We put $D_1(1) := D_1 - C_1$. Then $D_1(1)$ is effective since $D_1C_1 = C_1^2 + 1 \leq -1$ and so $\operatorname{Supp} D_1 \supset C_1$.

CLAIM 1.9.1. *$D_1(1)$ is an effective* 1*-connected divisor.*

Proof. Let $D_1(1) = B_1 + B_2$ be a decomposition of $D_1(1)$ with $B_1 \neq 0$ and $B_2 \neq 0$, where B_1 and B_2 are effective divisors. Then $D_1 = D_1(1) + C_1 = B_1 + B_2 + C_1$. Since $1 = (D_1 - C_1)C_1 = (B_1 + B_2)C_1$, we may assume that $B_2C_1 \leq 0$. Then by 1-connectedness of D_1, we get that $0 < (B_1 + C_1)B_2 \leq B_1B_2$. This completes the proof of Claim 1.9.1. □

(Here we note that Claim 1.9.1 can be proved also by Appendix (A.4) Lemma in [CFL].)

Claim 1.9.2. $g(D_0(0)) = g(D_1(1))$.

Proof.

$$\begin{aligned} g(D_0(0)) &= 1 + \frac{1}{2}(K_X + D_0(0))D_0(0) \\ &= 1 + \frac{1}{2}(K_{X_1} + D_1)D_1 \\ &= 1 + \frac{1}{2}(K_{X_1} + D_1 - C_1)(D_1 - C_1) + \frac{1}{2}(K_{X_1}C_1) + \frac{1}{2}(-C_1^2 + 2D_1C_1) \\ &= g(D_1(1)) + \frac{1}{2}(K_X + 2D_1 - C_1)C_1. \end{aligned}$$

On the other hand, we get that

$$\begin{aligned} (K_X + 2D_1 - C_1)C_1 &= (K_X + D_1 + D_1 - C_1)C_1 \\ &= (K_X + C_1 + 2(D_1 - C_1))C_1 \\ &= -2 + 2 \\ &= 0. \end{aligned}$$

This completes the proof. □

Next we consider a pair $(X_i, D_i(i))$ for an effective 1-connected divisor $D_i(i)$ on X_i.

First we take a $D_i(i)$-minimalization of $(X_i, D_i(i))$; $\pi_i : X_i \to X_{i+1}$, where X_{i+1} is a smooth projective surface and π_i is a birational morphism. We put $D_{i+1} := (\pi_i)_*(D_i(i))$.

If $K_{X_{i+1}} + D_{i+1}$ is nef, then this is stopped.

If $K_{X_{i+1}} + D_{i+1}$ is not nef, then there exists an irreducible curve C_{i+1} such that $(K_{X_{i+1}} + D_{i+1})C_{i+1} < 0$.

If $K_{X_{i+1}}C_{i+1} < 0$, then X_{i+1} is isomorphic to $\boldsymbol{P}^2$, $\boldsymbol{P}^1$-bundle over a smooth curve, or X_{i+1} has a (-1)-curve E_{i+1}.

If the first two cases occur, then this is stopped. If X_{i+1} has a (-1)-curve E_{i+1}, then $K_{X_{i+1}}E_{i+1} = -1$ and $D_{i+1}E_{i+1} = 0$. But this is impossible because (X_{i+1}, D_{i+1}) is D_{i+1}-minimal.

If $K_{X_{i+1}}C_{i+1} \geq 0$, then

$$\begin{aligned} 0 &> (K_{X_{i+1}} + C_{i+1} + (D_{i+1} - C_{i+1}))C_{i+1} \\ &= 2g(C_{i+1}) - 2 + (D_{i+1} - C_{i+1})C_{i+1}. \end{aligned}$$

Since D_{i+1} is 1-connected by Proposition 1.7, $(D_{i+1} - C_{i+1})C_{i+1} > 0$. Hence

$g(C_{i+1}) = 0$ and $(D_{i+1} - C_{i+1})C_{i+1} = 1$. Furthermore $C_{i+1}^2 \le -2$ because $g(C_{i+1}) = 0$ and $K_{X_{i+1}}C_{i+1} \ge 0$. We put $D_{i+1}(i+1) := D_{i+1} - C_{i+1}$. Then $D_{i+1}(i+1)$ is effective since $D_{i+1}C_{i+1} = C_{i+1}^2 + 1 \le -1$ and so $\operatorname{Supp} D_{i+1} \supset C_{i+1}$. Then by the same argument as in the claim above, we can prove that $D_{i+1}(i+1)$ is 1-connected and $g(D_i(i)) = g(D_{i+1}(i+1))$.

Assume that $D_t(t) = 0$ for some t. Then D_t is a smooth rational curve. Since (X_t, D_t) is D_t-minimal, we get that (X_t, D_t) satisfies one of the above conditions in Theorem 1.9. This completes the proof of Theorem 1.9. □

Remark 1.9.3. We remark that $D_i^2 = (D_{i-1}(i-1))^2$ and

$$\begin{aligned} D_i(i)^2 = (D_i - C_i)^2 &= D_i^2 - 2D_iC_i + C_i^2 \\ &= D_i^2 - 2(D_i - C_i)C_i - C_i^2 \\ &\ge D_i^2 - 2 + 2 \\ &= D_i^2 \end{aligned}$$

for $i = 1, \dots, t$. Hence we obtain that if $D^2 > 0$, then $D_S^2 > 0$ and the type (5) in Theorem 1.9 is excluded.

THEOREM 1.10. *Let X be a smooth projective surface and let D be a nef and big $\boldsymbol{Q}$-divisor on X. Then $H^p(X, K_X + \lceil D \rceil) = 0$ for $p = 1, 2$.*

Proof. See Theorem 5.1 in [Sa]. (See also [Ka] and [V].) □

LEMMA 1.11. *Let X be a smooth projective surface and let D be an effective divisor on X such that D is numerically 1-connected. If $H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_D)$ is injective, then $h^1(-D) = 0$.*

Proof. We consider the following exact sequence:

$$\begin{aligned} 0 \to H^0(-D) \to H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_D) \\ \to H^1(-D) \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_D). \end{aligned}$$

Here we note that $h^0(-D) = 0$ and $h^0(\mathcal{O}_X) = 1$. By the assumption and p. 69 (12.3) Corollary in [BPV], we get that $h^0(\mathcal{O}_D) = 1$. Since $H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_D)$ is injective, we get that $h^1(-D) = 0$. □

2. Some properties of special effective divisors

Here we define the following notion which is a generalization of effective nef and big divisor.

DEFINITION 2.1. Let X be a smooth projective surface defined over the complex number field. Let B be an effective divisor on X, and let $T_0, T_1, \dots, T_{n-1}$ be a sequence of reduced effective divisors on X. We put $B_0 := B$ and $B_i := B_{i-1} + T_{i-1}$ for $i = 1, \dots, n$. Then $(B; T_0, \dots, T_{n-1})$ is called *a generalized composite series with respect to B* if $g(T_i) + B_i T_i - 1 \geq 0$ for any $i = 0, \dots, n-1$.

Next we study some properties of this.

Proposition 2.2. *Let X be a smooth projective surface defined over the complex number field. Let $(B; T_0, \dots, T_{n-1})$ be a generalized composite series with respect to B. We put $D := B + T_0 + \cdots + T_{n-1}$. Then $g(D) \geq g(B)$.*

Proof. First we get that

$$\begin{aligned} g(B_1) &= g(B_0 + T_0) \\ &= g(B_0) + g(T_0) + B_0 T_0 - 1 \\ &\geq g(B_0). \end{aligned}$$

In general, we can prove that

$$\begin{aligned} g(B_{i+1}) &= g(B_i + T_i) \\ &= g(B_i) + g(T_i) + B_i T_i - 1 \\ &\geq g(B_i). \end{aligned}$$

Therefore $g(D) \geq g(B_{n-1}) \geq \cdots \geq g(B_0) = g(B)$. This completes the proof of Proposition 2.2. □

Remark 2.3. We put $m_i = g(T_i) + B_i T_i - 1$. Then

$$g(D) = g(B) + \sum_{i=0}^{n-1} m_i.$$

By Proposition 2.2, we can prove the following theorem.

Theorem 2.4. *Let X be a smooth projective surface defined over the complex number field. Let $(B; T_0, \dots, T_{n-1})$ be a generalized composite series with respect to B. Assume that B is a reduced CNNS-divisor. Then $g(D) \geq q(X)$ for $D = B + T_0 + \cdots + T_{n-1}$.*

Proof. By Proposition 2.2 we can prove that $g(D) \geq g(B)$. Since B is a reduced and connected effective divisor, we get that $g(B) \geq 0$. So if $q(X) = 0$, then $g(D) \geq g(B) \geq 0 = q(X)$. So we may assume that $q(X) \geq 1$. Then let

$$\alpha(B) = \dim \operatorname{Ker}(H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_B)).$$

If $\alpha(B) > 0$, then by Lemma 1.3 in [Fk1] there exists a morphism $\beta : X \to A$ such that $\beta(B)$ is a point on A, where A is an abelian variety. But since B is a CNNS-divisor, this is impossible. Therefore $\alpha(B) = 0$, that is, $q(X) \leq h^1(\mathcal{O}_B)$. Since B is a reduced connected effective divisor, we get that $g(B) = h^1(\mathcal{O}_B)$. Hence $g(B) = h^1(\mathcal{O}_B) \geq q(X)$ and we get that $g(D) \geq g(B) \geq q(X)$. This completes the proof. □

LEMMA 2.5. *Let X be a minimal smooth projective surface with $\kappa(X) \geq 0$ and let $(B; T_0, \dots, T_{n-1})$ be a generalized composite series with respect to B. We put $D := B + T_0 + \cdots + T_{n-1}$. Then $K_X D \geq K_X B$.*

Proof. Since $\kappa(X) \geq 0$ and X is minimal, we get that K_X is nef. So this lemma can be easily proved. □

PROPOSITION 2.6. *Let X be a minimal smooth projective surface with $\kappa(X) \geq 0$. Let $(B; T_0, \dots, T_{n-1})$ be a generalized composite series with respect to B. We put $D := B + T_0 + \cdots + T_{n-1}$. Assume that B is a reduced CNNS-divisor such that B does not satisfy the following condition $(\star\star)$:*

$(\star\star)$ $h^0(B) = 1$ and there exists only one irreducible component B_1 of B such that $B_1^2 > 0$ and $B - B_1$ is negative semidefinite.

Then $K_X D \geq 2q(X) - 4$.

Proof. By Lemma 2.5 $K_X D \geq K_X B$. Hence it is sufficient to prove $K_X B \geq 2q(X) - 4$. But this is true by Theorem 1.2 and Theorem 1.3. □

Next we study the case where X is not minimal.

PROPOSITION 2.7. *Let X and S be smooth projective surfaces. Let $(B; T_0, \dots, T_{n-1})$ be a generalized composite series with respect to B on X. Let $\mu : X \to S$ be a blowing down of E on X. Assume that B is a reduced connected divisor with $\mu_*(B) \neq 0$ and T_i is connected for any i. We put $D := B + T_0 + \cdots + T_{n-1}$ and $D_S = \mu_*(D)$. Then $K_X D \geq K_S D_S$.*

Proof. Let $B'_0 := \mu_*(B_0) \neq 0$ and $T'_i := \mu_*(T_i)$ for $i = 0, \dots, n-1$.

CLAIM 2.7.1. *$B_i = \mu^*(B'_i) - aE$ for $a \geq 0$, where $B'_i = \mu_*(B_i)$.*

Proof of Claim 2.7.1. First we consider B_1.

If $T_0 = E$, then $B_0 T_0 \geq 1$ because $g(T_0) + B_0 T_0 - 1 \geq 0$. Hence we get that $B_0 = \mu^*(B'_0) - aE$ for $a \geq 1$. Therefore $B_1 = B_0 + T_0 = \mu^*(B'_0) - (a-1)E$ for $a - 1 \geq 0$. Hence $K_X B_1 \geq K_S B'_1 = K_S B'_0$, where $B'_1 = \mu_*(B_1)$. (We remark that in this case $B'_1 = \mu_*(B_1) = B'_0$.)

If $T_0 \neq E$, then $T_0 = \mu^*(T'_0) - bE$ for $b \geq 0$ because T_0 is reduced and connected. Since B_0 is reduced and connected, we get that $B_0 = \mu^*(B'_0) - aE$ for $a \geq 0$. Hence $B_1 = B_0 + T_0 = \mu^*(B'_0 + T'_0) - (a+b)E$ for $a + b \geq 0$.

For $i=k$, we assume that $B_k = \mu^*(B'_k) - a_k E$ for $a_k \geq 0$. We consider the case in which $i = k+1$.

If $T_k = E$, then $B_k E \geq 1$ because $g(T_k) + B_k T_{k-1} - 1 \geq 0$. Hence $B_k = \mu^*(B'_k) - aE$ for $a \geq 1$. Since $T_k = E$, we get that $B_{k+1} = B_k + T_k = \mu^*(B'_k) - (a-1)E$ for $a - 1 \geq 0$.

If $T_k \neq E$, then $T_k = \mu^*(T'_k) - cE$ for $c \geq 0$ because T_k is reduced and connected. Furthermore by assumption $B_k = \mu^*(B'_k) - a_k E$ for $a_k \geq 0$. Hence

$$\begin{aligned} B_{k+1} = B_k + T_k &= \mu^*(B'_k + T'_k) - (a_k + c)E \\ &= \mu^*(B'_{k+1}) - (a_{k+1})E \end{aligned}$$

for $a_{k+1} \geq 0$. Therefore this completes the proof of Claim 2.7.1. □

By this claim, we get that $D = \mu^*(D_S) - dE$ for $d \geq 0$. Therefore $K_X D \geq K_S D_S$. □

Proposition 2.8. *Let X and S be smooth projective surfaces, $(B; T_0, \dots, T_{n-1})$ a generalized composite series with respect to B on X. Assume that B is a reduced, connected, and effective divisor and T_i is connected for any i. Let $\mu : X \to S$ be a blowing down of a (-1)-curve E. Let $B' = \mu_*(B)$ and $T'_i = \mu_*(T_i)$, and assume that $T'_i \neq 0$ for some i and $B' \neq 0$. Then there exists a sequence of natural numbers $t_0, \dots, t_l$ with $0 \leq t_0 < \cdots < t_l \leq n-1$ such that $(B'; T'_{t_0}, \dots, T'_{t_l})$ is a generalized composite series with respect to B'.*

Proof. Let X and S be smooth projective surfaces and let $(B; T_0, \dots, T_{n-1})$ be a generalized composite series with respect to B. Let $\mu : X \to S$ be a blowing down of a (-1)-curve E. Recall $B_i = B_{i-1} + T_{i-1}$ and $B_0 := B$. We put $B'_i := \mu_*(B_i)$. Then $D_S := B'_n \geq B'_{n-1} \geq \cdots \geq B'_1 \geq B'_0$.

By reindexing we may assume that $B'_{k+1} \neq B'_k$ for any k. Then $\mu_*(T_k) \neq 0$, and $T_k = \mu^*(T'_k) - a_k E$ for $a_k \geq 0$. By Claim 2.7.1 we get that $B_k = \mu^*(B'_k) - b_k E$ with $b_k \geq 0$. Hence in this case

$$g(T_k) + B_k T_k - 1 = g(T'_k) - \frac{a_k^2 - a_k}{2} + B'_k T'_k - a_k b_k - 1.$$

Therefore

$$\begin{aligned} g(T'_k) + B'_k T'_k - 1 &= g(T_k) + B_k T_k - 1 + \frac{a_k^2 - a_k}{2} + a_k b_k \\ &\geq 0 \end{aligned}$$

This completes the proof of Proposition 2.8. □

Corollary 2.9. *Let X be a smooth projective surface with $\kappa(X) \geq 0$ and let $(B; T_0, \dots, T_{n-1})$ be a generalized composite series with respect to B. Assume that B is a reduced CNNS-divisor with $h^0(B) \geq 2$, and T_i is connected for any i. Then $K_X D \geq 2q(X) - 4$ for $D = B + T_0 + \cdots + T_{n-1}$.*

Proof. Let $\mu : X \to S$ be a minimalization of X. Let $D_S := \mu_*(D)$ and $B_S := \mu_*(B)$. By Proposition 2.7 $K_X D \geq K_S D_S$. By Lemma 2.5 and Proposition 2.8 we have $K_S D_S \geq K_S B_S$. By Claim 2.7.1 we get that B_S is a CNNS-divisor. Since $h^0(B_S) \geq h^0(B) \geq 2$, we obtain that $K_S B_S \geq 2q(S) - 4 = 2q(X) - 4$ by Theorem 1.2. Therefore we get the assertion. □

Next we consider a vanishing theorem.

THEOREM 2.10. *Let X be a smooth projective surface and let $(B; T_0, \ldots, T_{n-1})$ be a generalized composite series with respect to B. We put $T_i = \sum_{k=1}^{r_i} T_{i,k}$ for $i = 0, \ldots, n-1$, where $T_{i,k}$ is an irreducible and reduced divisor on X, and r_i is the number of irreducible components of T_i. We put $B_0 := B$ and $B_i := B_{i-1} + T_{i-1}$ for $i = 1, \ldots, n$. Assume that the following hold.*

(1) B is a reduced CNNS-divisor on X.

(2) $B_i T_{i,k} > 0$ for any integers i and k with $0 \leq i \leq n-1$ and $1 \leq k \leq r_i$. Then $h^1(K_X + D) = 0$ for $D = B + T_0 + \cdots + T_{n-1}$.

Proof. We put $B_{i,0} := B_i$ and $B_{i,k} := B_{i,k-1} + T_{i,k}$ for $1 \leq k \leq r_i$. Here we note that $B_{i,r_i} = B_{i+1,0}$.

We consider the following exact sequence:

$$0 \to \mathcal{O}(K_X + B_{i,k-1}) \to \mathcal{O}(K_X + B_{i,k-1} + T_{i,k}) \to \omega_{T_{i,k}} \otimes \mathcal{O}(B_{i,k-1})|_{T_{i,k}} \to 0,$$

where $\omega_{T_{i,k}}$ is the dualizing sheaf of $T_{i,k}$. Then we get that

$$H^1(K_X + B_{i,k-1}) \to H^1(K_X + B_{i,k}) \to H^1(\omega_{T_{i,k}} \otimes \mathcal{O}(B_{i,k-1})|_{T_{i,k}}).$$

We note that $T_{i,s} \neq T_{i,t}$ for $s \neq t$ because T_i is reduced by Definition 2.1. Hence by the assumption (2) above, we get that, for $k = 1, \ldots, r_i$,

$$B_{i,k-1} T_{i,k} = (B_i + T_{i,1} + \cdots + T_{i,k-1}) T_{i,k} > 0.$$

Since $T_{i,k}$ is irreducible and reduced, we get that $h^1(\omega_{T_{i,k}} \otimes \mathcal{O}(B_{i,k-1})|_{T_{i,k}}) = h^0(\mathcal{O}(-B_{i,k-1})|_{T_{i,k}}) = 0$ because $\deg(B_{i,k-1}|_{T_{i,k}}) = B_{i,k-1} T_{i,k} > 0$. Therefore

$$h^1(K_X + B_{i,k-1}) \geq h^1(K_X + B_{i,k})$$

for any integers i and k with $0 \leq i \leq n-1$ and $1 \leq k \leq r_i$.

By the assumption, B_0 is a reduced and connected effective divisor on X. Furthermore since B_0 is a CNNS-divisor, $H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_{B_0})$ is injective by Lemma 1.3 in [Fk1]. Hence by Lemma 1.11 and the Serre duality, we get that $h^1(K_X + B_0) = 0$.

Therefore we get that

$$\begin{aligned} 0 &= h^1(K_X + B_0) \\ &\geq h^1(K_X + B_{0,1}) \\ &\geq \cdots \\ &\geq h^1(K_X + B_{0,r_0}) \end{aligned}$$

$$\begin{aligned}
&= h^1(K_X + B_1) \\
&\geq \cdots \\
&\geq h^1(K_X + B_n) \\
&= h^1(K_X + D),
\end{aligned}$$

and this completes the proof of Theorem 2.10. □

By considering the above properties, it is natural to consider the following type:

($\star\star\star$) Let X be a smooth projective surface, and let $(B; T_0, \dots, T_{n-1})$ be a generalized composite series with respect to B. Assume that B is a reduced CNNS-divisor and T_i is connected for any $i = 0, \dots, n-1$.

Here we remark that if $D = B + T_0 + \cdots + T_{n-1}$ is a CNNS-divisor with $D_{\mathrm{red}} = B$, then B is a CNNS-divisor.

In the next section we will give some examples of ($\star\star\star$).

3. Some examples

Theorem 3.1. *Let X be a smooth projective surface. If D is a nef and big effective $\boldsymbol{Q}$-divisor, then there exists a generalized composite series with respect to B, $(B; T_0, \dots, T_{n-1})$ such that $\lceil D \rceil = B + T_0 + \cdots + T_{n-1}$, $(\lceil D \rceil)_{\mathrm{red}} = B$, and $(B; T_0, \dots, T_{n-1})$ satisfies ($\star\star\star$).*

Proof. Let D be an effective $\boldsymbol{Q}$-divisor on X such that D is nef and big. We put $D = \sum_i b_i D_i$ for $b_i \in \boldsymbol{Q}_{>0}$. Let

$$e_0 = \max_i \left\{ \frac{\lceil b_i \rceil - 1}{b_i} \right\}.$$

If $e_0 = 0$, then $\lceil D \rceil$ is a reduced divisor, and we put $B = \lceil D \rceil$ and $T_0 = \cdots = T_{n-1} = 0$.

So we assume that $e_0 \neq 0$. Then $\lceil D \rceil - \lceil e_0 D \rceil$ is a reduced effective divisor. We put

$$\overline{N}_0 = \lceil D \rceil - \lceil e_0 D \rceil, D(0) := D$$

and we put

$$D(1) := e_0 D(0).$$

In general, let

$$D(j) = \sum_i b_{j,i} D_i$$

and let

$$e_j = \max_i \left\{ \frac{\lceil b_{j,i} \rceil - 1}{b_{j,i}} \right\}.$$

Then $\lceil D(j) \rceil - \lceil e_j D(j) \rceil$ is a reduced effective divisor. Let

$$\overline{N}_j = \lceil D(j) \rceil - \lceil e_j D(j) \rceil$$

and we put

$$D(j+1) := e_j D(j).$$

We do this process repeatedly and we stop this process if $e_j = 0$. And we obtain that there exist a reduced effective divisor B and a sequence of reduced divisors $\overline{N}_0, \dots, \overline{N}_l$ such that $\lceil D \rceil = B + \overline{N}_0 + \cdots + \overline{N}_l$ and $(\lceil D \rceil)_{\mathrm{red}} = B = \lceil e_l \cdots e_0 D \rceil$.

Let

$$N_i := \overline{N}_{l-i},$$

and

$$N_i = \sum_{m=1}^{t_i} N_{i,m}$$

be a decomposition of connected component of N_i for $0 \leq i \leq l$, where t_i is a positive integer. We put $B_i = B_{i-1} + N_{i-1}$ and $B_0 = B$. Then by the choice of N_i, we get that $B_i = \lceil \beta_i D \rceil$ for $0 \leq i \leq l$, where $\beta_i = e_{l-i} \cdots e_0$ and $B_{l+1} = \lceil D \rceil$. Hence $h^1(K_X + B_i) = 0$ for any integer i with $0 \leq i \leq l+1$ by Theorem 1.10. On the other hand, there exists the following exact sequence for any integer i with $0 \leq i \leq l$

$$0 \to \mathcal{O}(K_X + B_i) \to \mathcal{O}(K_X + B_{i+1}) \to \mathcal{O}(K_X + B_{i+1})|_{N_i} \to 0.$$

Hence

$$H^1(K_X + B_{i+1}) \to H^1((K_X + B_{i+1})|_{N_i}) \to H^2(K_X + B_i)$$

is exact. Since $h^2(K_X + B_i) = 0$, we get that $h^1((K_X + B_{i+1})|_{N_i}) = 0$. Therefore $h^1((K_X + B_i + N_{i,m})|_{N_{i,m}}) = 0$ for any i, m because $N_{i,m} \cap N_{i,m'} = \emptyset$ for $m \neq m'$. Furthermore

$$\text{(A)} \qquad h^1\left(\left(K_X + B_i + \sum_{m=1}^{r} N_{i,m}\right)\Bigg|_{N_{i,r}}\right) = 0$$

for any $r = 1, \dots, t_i$.

Claim 3.1.1.

$$h^1\left(K_X + B_i + \sum_{m=1}^{r} N_{i,m}\right) = 0$$

for any $r = 1, \dots, t_i$.

Proof. We prove this by induction. Since $h^1(K_X+B_i)=0$ and $h^1((K_X+B_i+N_{i,1})|_{N_{i,1}})=0$ by (A), we get that $h^1(K_X+B_i+N_{i,1})=0$.

Assume that

$$h^1\left(K_X+B_i+\sum_{m=1}^{u}N_{i,m}\right)=0$$

for $1\leq u<t_i$. By using (A) we obtain that

$$h^1\left(K_X+B_i+\sum_{m=1}^{u+1}N_{i,m}\right)=0.$$

Hence we get the assertion of Claim 3.1.1. □

Let $B_{i,0}:=B_i$ and $B_{i,k}=B_{i,k-1}+N_{i,k}$ for $0\leq i\leq l$ and $1\leq k\leq t_i$, and $B_{i+1,0}=B_{i,t_i}$ for $0\leq i\leq l-1$. Here we note that $B_{l,t_l}=\lceil D\rceil$.

Then by Claim 3.1.1 we have $h^1(K_X+B_{i,k})=0$ for any integers i and k with $0\leq i\leq l$ and $0\leq k\leq t_i$. Since $h^2(K_X+B_{i,k})=0$, we get the following by the Riemann-Roch theorem:

$$\begin{aligned}h^0(K_X+B_{i,k})-h^0(K_X)&=g(B_{i,k})-q(X)\\&=g(B_{i,k-1})-q(X)+g(N_{i,k})+B_{i,k-1}N_{i,k}-1\end{aligned}$$

and

$$h^0(K_X+B_{i,k-1})-h^0(K_X)=g(B_{i,k-1})-q(X).$$

Therefore

$$h^0(K_X+B_{i,k})-h^0(K_X+B_{i,k-1})=g(N_{i,k})+B_{i,k-1}N_{i,k}-1.$$

On the other hand $h^0(K_X+B_{i,k})-h^0(K_X+B_{i,k-1})\geq 0$ by construction. Hence $g(N_{i,k})+B_{i,k-1}N_{i,k}-1\geq 0$.

Therefore $(B;N_{0,1},\dots,N_{0,t_0},N_{1,1},\dots,N_{1,t_1},\dots,N_{l,1},\dots,N_{l,t_l})$ is a generalized composite series with respect to B which satisfies $(\star\star\star)$. This completes the proof of Theorem 3.1. □

By Proposition 1.5 and the definition of an s-connected effective divisor, we can also prove the following result.

PROPOSITION 3.2. *Let X be a smooth projective surface. If D is an s-connected effective divisor, then there exists a generalized composite series with respect to B_0, $(B_0;T_0,\dots,T_{n-1})$ such that $D=B_0+T_0+\cdots+T_{n-1}$, $D_{\mathrm{red}}=B_0$, and T_i is irreducible for any i.*

Proof. Assume that D is s-connected. We put $D=\sum_i b_iD_i$. Let $B_0=D_{\mathrm{red}}$. Assume that $D\neq D_{\mathrm{red}}$. We put $B_0'=D-B_0$. Then by Proposition

1.5 there exists an irreducible component C_0' of B_0' such that $B_0C_0' > 0$. We put $B_1 = B_0 + C_0'$. If $D = B_1$, then this is stop. If $D \neq B_1$, then we put $B_1' = D - B_1$. Then by Proposition 1.5 there exists an irreducible component C_1' of B_1' such that $B_1C_1' > 0$. We put $B_2 = B_1 + C_1'$. For any i, if $D = B_i$, then this is stop. If $D \neq B_i$, then we put $B_i' = D - B_i$. Then by Proposition 1.5 there exists an irreducible component C_i' of B_i' such that $B_iC_i' > 0$. We put $B_{i+1} = B_i + C_i'$. We do this process repeatedly. So we get a generalized composite series with respect to D_{red}, $(D_{\mathrm{red}}; C_0', \dots, C_l')$, and $D = D_{\mathrm{red}} + C_0' + \cdots + C_l'$. This completes the proof of Proposition 3.2. □

4. Sectional genus of the round up of nef and big effective $\boldsymbol{Q}$-divisors

Here we consider the sectional genus of the round up of effective nef and big $\boldsymbol{Q}$-divisor D. Let X be a smooth projective surface. Then $g(\lceil D \rceil) \geq q(X)$ by Theorem 3.1 and Theorem 2.4 (or Theorem 1.10). So in particular $g(\lceil D \rceil) \geq 0$. Here we will classify (X, D) with $g(\lceil D \rceil) = 0$.

PROPOSITION 4.1. *Let X be a smooth projective surface and let D be a nef and big effective $\boldsymbol{Q}$-divisor on X. If $g(\lceil D \rceil) = 0$, then there exist a smooth projective surface S, an effective divisor D_S, and a birational morphism $\pi : X \to S$ such that $(\lceil D \rceil)_{\mathrm{red}} = \pi^{-1}D_S + \sum_i a_iC_i$ for nonnegative integers a_i and smooth rational curves C_i with $C_i^2 \leq -1$, $g(\lceil D \rceil) = g(D_S)$ and one of the following holds*:

(1) $(S, D_S) \cong (\boldsymbol{P}^2, \mathcal{O}(1))$,

(2) $(S, D_S) \cong (\boldsymbol{P}^2, \mathcal{O}(2))$,

(3) (S, D_S) *is a scroll over* $\boldsymbol{P}^1$,

(4) $(\lceil D \rceil)_{\mathrm{red}} = \sum C_i$, $C_iC_j \leq 1$ *for any $i \neq j$, and the dual graph of this is tree, where $\pi^{-1}D_S$ denotes the strict transform of D_S via π.*

Proof. By Theorem 3.1, there exists a generalized composite series with respect to B, $(B; T_0, \dots, T_{n-1})$ such that $\lceil D \rceil = B + T_0 + \cdots + T_{n-1}$, $(\lceil D \rceil)_{\mathrm{red}} = B$, and $(B; T_0, \dots, T_{n-1})$ satisfies $(\star\star\star)$. Let $B_i = B_{i-1} + T_{i-1}$ and $B_0 := B$. So we get that $g(\lceil D \rceil) \geq g(B_{n-1}) \geq \cdots \geq g(B_0)$. Since $g(B_0) \geq 0$, we get that $0 = g(\lceil D \rceil) = g(B_{n-1}) = \cdots = g(B_0)$ and $g(T_i) + B_iT_i - 1 = 0$. So we study (X, B_0) with $g(B_0) = 0$. Here we use Theorem 1.9 for (X, B_0). Then we get that there exist a smooth projective surface S, a birational morphism $\pi : X \to S$, and a reduced connected effective divisor D_S on S such that $g(B_0) = g(D_S)$ and $B_0 = \pi^{-1}D_S + \sum_i a_iC_i$ for nonnegative integers a_i and smooth rational curves C_i with $C_i^2 \leq -1$. Since $g(D_S) = g(B_0) = 0$, we get that $(K_S + D_S)D_S < 0$, and by Theorem 1.9 one of the following holds:

(I) $(S, D_S) \cong (\boldsymbol{P}^2, \mathcal{O}(1))$,

(II) $(S, D_S) \cong (\boldsymbol{P}^2, \mathcal{O}(2))$,

(III) (S, D_S) is a scroll over a smooth curve,

(IV) D_S is a smooth rational curve.

For the first two cases we find that $g(\lceil D \rceil) = g(D_S) = 0$.

If (S, D_S) is a scroll over a smooth curve C, then $g(D_S) = g(C)$. Hence $g(C) = 0$.

Next we study the last case. Since $g(D_S) = g(\lceil D \rceil) = g(B_0) = 0$, by the proof of Theorem 1.9 we get that $(\lceil D \rceil)_{\mathrm{red}} = \sum_i C_i$, where C_i is a smooth rational curve with $C_iC_j \leq 1$ for any $i \neq j$ and the dual graph of D is a tree. □

Question 4.2. If $\kappa(X) \geq 0$, then does there exist an example of the last case?

An answer of this question is YES.

Example 4.3. Let X be a smooth projective surface with $\kappa(X) = 1$. Assume that X is minimal. Then there exists an elliptic fibration $f : X \to C$. Furthermore we assume that $C = \boldsymbol{P}^1$, f has a section C_0, and f has a singular fiber of type I_n^*, II^*, III^*, or IV^*. The dual graph of any fiber of these types is a tree. Let F be one of its singular fiber. Then $C_0 + mF$ is a nef and big effective divisor for any large m. For large $N > 0$, we get that $D = (1/N)(C_0 + mF)$ is a nef and big $\boldsymbol{Q}$-divisor such that $\lceil D \rceil = (C_0 + mF)_{\mathrm{red}}$. Then $g(\lceil D \rceil) = 0$.

Example 4.4. Let $\mathscr{E}$ be a normalized vector bundle of rank 2 on $\boldsymbol{P}^1$ and let $p : \boldsymbol{P}(\mathscr{E}) \to \boldsymbol{P}^1$ be its projection. Let C_0 be a minimal section of p and F its fiber. (For the definition of the minimal section of p, see [Ha].) Let $e = -C_0^2$. Then $e \geq 0$. We put $B = 3C_0 + (3e+1)F$. Then $|2B| = |6C_0 + (6e+2)F|$. Since $|C_0 + eF|$ is base point free, we get that there exist smooth divisors $D_1, \dots, D_6$ such that $D_i \in |C_0 + eF|$ for any $i = 1, \dots, 6$. Then $D_iD_j = (C_0 + eF)^2 = e \geq 0$. Assume that $e = 1$. Let $T = \bigcup_{i<j}(D_i \cap D_j)$. Then T is a finite set with $T \neq \emptyset$. Let F_1 and F_2 be fibers of p such that $F_i \cap T = \emptyset$ for $i = 1, 2$. On the other hand, $(C_0 + F)C_0 = 0$. So $C_0 \cap D_i = \emptyset$ for any i. Let $\{x_{i,j}\} = F_i \cap D_j$. Here we consider a double covering branched at $D_1 + \dots + D_6 + F_1 + F_2$. Since $T \neq \emptyset$ and $F_i \cap D_j \neq \emptyset$, in order to make a double covering between smooth projective surfaces we take the canonical resolution of the double covering. (See Section 2 in [Ho].)

Here we take the minimal even resolution of $D_1 + \dots + D_6 + F_1 + F_2$; $\mu : \bar{P} \to \boldsymbol{P}(\mathscr{E})$. Then $\mu^*(C_0 + F_i)$ is composed with rational curves for $i = 1, 2$. Moreover, $(\mu^*(C_0 + F_i))_{\mathrm{red}}$ is a simple normal crossing divisor and the dual graph of $(\mu^*(C_0 + F_i))_{\mathrm{red}}$ is a tree. Then we get a double covering $\pi : X \to \bar{P}$ whose branch locus is the strict transform of $D_1 + \dots + D_6 + F_1 + F_2$ via μ, where X is a smooth projective surface. By construction we get that $\kappa(X) = 2$ and $q(X) = 0$. Furthermore $\pi^* \circ \mu^*(C_0 + F_i)$ is nef and big, $g((\pi^* \circ \mu^*(C_0 + F_i))_{\mathrm{red}}) = 0$, and

$$\left\lceil \frac{1}{2}\pi^* \circ \mu^*(C_0 + F_i) \right\rceil = (\pi^* \circ \mu^*(C_0 + F_i))_{\mathrm{red}}.$$

This is an example.

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