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NEWTON-PUISEUX APPROXIMATION AND ŁOJASIEWICZ EXPONENTS

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Abstract

We give a process to construct what we call the *Newton-Puiseux approximation* for a system of germs (at the origin and at infinity) and indicate how the Newton-Puiseux approximation may be used to obtain formulas for the Łojasiewicz exponents.

1. Introduction

1. Let $F := (f_1, f_2, \dots, f_n) : (\boldsymbol{K}^2, 0) \to (\boldsymbol{K}^n, 0)$ be a germ of mapping of two variables, where $\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{K} = \boldsymbol{C}$. We define the *Łojasiewicz exponent* $\mathscr{L}(F)$ of the germ F to be the greatest lower bound of the set of all real $\alpha > 0$ which satisfy the following condition: there exist positive constants c and ρ such that

$$\max_{l=1,2,\dots,n} |f_l(x,y)| \ge c\|(x,y)\|^{\alpha}, \quad \text{for } (x,y) \in B_\rho,$$

where B_ρ is the ball centered at $(0,0)$ with radius ρ.

In this paper, we first give a process to construct what we call the *Newton-Puiseux approximation* of the germ F. This process (1) either yields all common non-constant factors of the real (or complex) analytic functions $f_1, f_2, \dots, f_n$ in a suitable neighbourhood of the origin; (2) or else, after a finite number of steps, shows that $(0,0)$ is a common isolated zero of the functions $f_1, f_2, \dots, f_n$. In the latter case, we apply the Newton-Puiseux approximation to obtain a formula for the Łojasiewicz exponent $\mathscr{L}(F)$, where F is a germ of real analytic, complex analytic or smooth mapping.

For the case where F is a germ of real (or complex) analytic mapping, $\mathscr{L}(F)$ is finite if and only if F has an isolated zero at $(0,0)$. In the case F is a germ of smooth mapping it is well-known [13] that the following three statements are equivalent

(i) The Łojasiewicz exponent $\mathscr{L}(F)$ is finite.

(ii) The inclusion $\mathfrak{m}^\infty \subset (F)$ holds, where $\mathfrak{m}^\infty$ is the ideal of all flat germs

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of the ring $\mathscr{E}_2$ of germs of smooth functions on $(\boldsymbol{R}^2, 0)$, and (F) is the ideal generated by germs $f_1, f_2, \ldots, f_n$.

(iii) The ring $\mathscr{E}_2/(F)$ is Noether.

Thus, the result of this paper also provides a way to check whether the above statements (i) and (ii) hold.

In the case $n = 1$ and $F = f_1$ is a germ of real analytic function, an exact formula for $\mathscr{L}(f_1)$ was given by Kuo [10]. For the case $n = 2$ and $F = (f_1, f_2)$ is a germ of complex analytic mapping, some results for $\mathscr{L}(F)$ obtained in [11], [3], [14]. Their proofs depend heavily on the use of the Newton-Puiseux expansions of the each components of F. On the other hand, our method is based on the Newton-Puiseux approximation of the germ $F = (f_1, f_2, \ldots, f_n)$. We have learnt this idea from [8] (see Appendix).

2. We next suppose that $F := (f_1, f_2, \ldots, f_n) : \boldsymbol{K}^2 \to \boldsymbol{K}^n$ is a polynomial mapping of two variables. The *Łojasiewicz exponent at infinity* of F, denoted by $\mathscr{L}_\infty(F)$, is defined to be the least upper bound of the set of all real α such that

$$\max_{l=1,2,\ldots,n} |f_l(x, y)| \geq c\|(x, y)\|^\alpha$$

for sufficiently large $\|(x, y)\|$ and for $c > 0$. If the set of all the exponents is empty we put $\mathscr{L}_\infty(F) = -\infty$.

In the case $n = 2$, Hà [6] gave an exact formula for the Łojasiewicz exponent at infinity, $\mathscr{L}_\infty(\operatorname{grad} f)$, of the gradient of a complex polynomial f, and he showed a link between $\mathscr{L}_\infty(\operatorname{grad} f)$ and the singularities at infinity of f. In the papers [4], [5] Chadzynski and Krasinski described the Łojasiewicz exponent at infinity of a polynomial mapping $F : \boldsymbol{C}^2 \to \boldsymbol{C}^2$, and they obtained a characterization of a component of a polynomial automorphism of $\boldsymbol{C}^2$ from a characterization of $\mathscr{L}_\infty(F)$. Recently, Lenarcik [12] gave an estimation of $\mathscr{L}_\infty(f_1, f_2)$ in terms of the Newton polygons of polynomials f_1, f_2; while for non-degenerate polynomials, the equality was obtained.

3. This paper is organized as follows. In Section 2 we shall describe the Newton-Puiseux approximation and apply it to obtain a formula for the Łojasiewicz exponent of a germ. In Section 3, we shall construct the Newton-Puiseux approximation for a polynomial mapping in a neighbourhood of infinity and we then indicate how this method may be used to give the Łojasiewicz exponent at infinity. For the non-degenerate case, the Łojasiewicz exponents (at the origin and at infinity) are obtained in terms of the Newton polygon of F (Corollary 1 and 2).

2. Newton-Puiseux approximation

We shall only consider for the case where $F = (f_1, f_2, \ldots, f_n) : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^n, 0)$ is a germ of real analytic mapping. A slight change in the proof actually shows that the proposed process also holds in the case where F is a germ of complex analytic (or smooth) mapping (the details are left to the reader).

Let $m_0 := \min_{l=1,2,\dots,n} O(f_l)$, where $O(f_l)$ is the order of f_l. Then we can write

$$f_l(x,y) = f_{l,m_0}(x,y) + f_{l,m_0+1}(x,y) + \cdots, \quad l = 1,2,\dots,n,$$

where $f_{l,i}(x,y)$ are homogeneous polynomials of degree i.

Case 1. If the algebraic equations

$$f_{1,m_0}(k,1) = 0, \; f_{2,m_0}(k,1) = 0, \dots, f_{n,m_0}(k,1) = 0, \tag{1}$$

have no common real finite or infinite roots then it easy to check that the functions $f_1, f_2, \dots, f_n$ have no common (real) tangent lines. The process is finished, $\mathscr{L}(F) = m_0$.

Case 2. We will examine the case where (1) has one or several common real solutions, say, $k_1, k_2, \dots, k_s$. Then the polynomial functions

$$f_{1,m_0}(x,y), \quad f_{2,m_0}(x,y), \dots, f_{n,m_0}(x,y)$$

vanish on the following $2s$ rays:

$$L_1 : \{x = k_1 y, \, y \geq 0\}; \quad L_2 : \{x = k_1 y, \, y \leq 0\};$$

$$\cdots$$

$$L_{2s-1} : \{x = k_s y, \, y \geq 0\}; \quad L_{2s} : \{x = k_s y, \, y \leq 0\};$$

(if among the solutions $k_1, k_2, \dots, k_s$ there is a solution $k = \infty$, the rays $y = 0$ $(x \geq 0)$ and $y = 0$ $(x \leq 0)$ correspond to this solution). We shall refer to the rays L_σ as the *degeneracy rays* of the germ F.

We will denote by $\Gamma_\sigma(\varepsilon)$ $(\sigma = 1,2,\dots,2s)$ the set of points which lie inside the angle of 2ε radians, whose bisector is L_σ; and by $\mathscr{L}(F; L_\sigma)$ the greatest lower bound of the set of all $\alpha > 0$ such that the following inequality holds

$$\max_{l=1,2,\dots,n} |f_l(x,y)| \geq c\|(x,y)\|^\alpha, \quad (x,y) \in \Gamma_\sigma(\varepsilon) \cap B_\rho,$$

for some $c, \rho > 0$. We call $\mathscr{L}(F; L_\sigma)$ *the Łojasiewicz exponent of the germ F* with respect to the ray L_σ.

Resolution of degeneracy of rays. We will now examine the case where the polynomials f_{l,m_0} vanish on the $2s$ rays L_σ, $\sigma = 1,2,\dots,2s$.

The process described below, (1) either yields all common non-constant factors of the functions $f_1, f_2, \dots, f_n$; or else (2), after a finite number of steps, enables us to calculate the Łojasiewicz exponents $\mathscr{L}(F; L_\sigma)$ of the germ F with respect to the rays L_σ, and hence the Łojasiewicz exponent $\mathscr{L}(F)$.

Fix $\sigma \in \{1,2,\dots,2s\}$. Assume that the degeneracy ray under consideration, L_σ, coincides with the positive half of the y-axis, $y \geq 0$. This assumption does not affect the generality of our argument, since we could in any case perform a linear change of variables by an orthogonal matrix which transforms the ray L_σ into the positive half of the y-axis. The transformed germ can be expressed, of course, in a form similar to the germ F; moreover, the Łojasiewicz exponent $\mathscr{L}(F; L_\sigma)$ of the germ F with respect to the ray L_σ will remain unchanged.

Let us denote by $a^l_{i,j}$ the coefficients of $x^i y^j$ in the expansion of functions $f_l(x, y)$ in powers of x and y. Let

$$f_{l,m_0}(x, y) = a^l_{m_0,0} x^{m_0} + \cdots + a^l_{\mu, m_0-\mu} x^\mu y^{m_0-\mu}, \quad l = 1, 2, \ldots, n,$$

where at least one of the numbers $a^l_{\mu, m_0-\mu}$, $l = 1, 2, \ldots, n$, is different from zero.

CASE 2.1. If there is no term $a^l_{i,j} x^i y^j$ such that $i < \mu$ and $j > m_0 - \mu$ then it is obvious that the functions $f_1, f_2, \ldots, f_n$ have a common factor x^μ. The process is finished.

CASE 2.2. Otherwise, we put

$$r_0 = \min\left\{ \frac{j - m_0 + \mu}{\mu - i} \mid i < \mu, i + j > m_0 \right\}$$

and let

$$\psi_{l,\sigma}(x, y) = \sum_{(i,j)} a^l_{i,j} x^i y^j,$$

the summation being taken over all values of (i, j) for which $ir_0 + j = m_0 + \mu(r_0 - 1)$.

CASE 2.2.1. If the algebraic equations

$$\psi_{1,\sigma}(k, 1) = 0, \ \ \psi_{2,\sigma}(k, 1) = 0, \ldots, \psi_{n,\sigma}(k, 1) = 0,$$

have no common real solutions, then the process of resolution of degeneracy of the ray L_σ is finished.

CASE 2.2.2. Let us pass to the case where the polynomials $\psi_{l,\sigma}(k, 1)$, $l = 1, 2, \ldots, n$, have common real solutions $k_{\sigma,1}, k_{\sigma,2}, \ldots, k_{\sigma,s(\sigma)}$.

Let $r_0 = p/q$, where p and q are two relatively prime numbers, $p > q$. The set $\{m_0, \mu, r_0, p, q\}$ is called *the characteristics* of the ray L_σ.

Fix $t \in \{1, 2, \ldots, s(\sigma)\}$. We shall change the variables

$$x = y_1^p(k_{\sigma,t} + x_1), \quad y = y_1^q,$$

and consider the functions

$$g_{l,t}(x_1, y_1) = y_1^{-[m_0 q + \mu(p-q)]} f_l(y_1^p(k_{\sigma,t} + x_1), y_1^q), \quad l = 1, 2, \ldots, n,$$

on the positive half of the y_1-axis, $y_1 \geq 0$. We shall refer to the ray

$$L_{\sigma,t} = \{x_1 = 0, y_1 \geq 0\}$$

as *degeneracy* of the germ $G_t := (g_{1,t}, g_{2,t}, \ldots, g_{n,t})$. It is easy to check that if the germ F has an isolated zero at $(0, 0)$, then so does G_t.

The reasoning expounded above can be applied again to the degeneracy rays $L_{\sigma,t}$, and so on. If the germ F has $(0,0)$ as an isolated zero in some neighbourhood of $(0,0)$ then after a finite number of recursive applications of the above process we can calculate the Łojasiewicz exponent of the germ F with respect to the ray L_σ. We will clarify this in the following. Let us call the above passage from F to G_t a *Newton-Puiseux approximation*.

PROPOSITION 1. *If the germ F has an isolated zero at $(0,0)$ then every sequence of recursive Newton-Puiseux approximations, beginning with F, is finite.*

Proof. Suppose that an infinite sequence of degeneracy $L^{(k)}$, $(k = 1, 2, \ldots)$, of successively constructed germs, had been built up. We shall derive a contradiction. The equation of $L^{(k)}$ in the coordinate system (x, y) would have the form

$$x = x(y) = c_1 y^{\gamma_1} + c_2 y^{\gamma_1+\gamma_2} + \cdots,$$

where $c_i \neq 0$ and γ_i are positive rational numbers having a common denominate. It is found that this series coincides with one of the series obtained with the help of Newton-Puiseux's method applied to functions $f_1, f_2, \ldots, f_n$ (see [2], [15]). In accordance with a well-known theorem ([1], [2]) this series is convergent and defines the general solution of the system of equations $f_l(x, y) = 0$, $l = 1, 2, \ldots, n$, which contradicts the fact that the functions $f_1, f_2, \ldots, f_n$ vanish simultaneously only at 0. □

We now suppose that the germ F has an isolated zero at $(0,0)$. Then the process of resolution of degeneracy of L_σ gives us a tree T_σ whose vertices correspond to the degeneracy rays of successively constructed germs, and the root of T_σ corresponds to L_σ. We will denote by $L_\sigma^{(1)}, L_\sigma^{(2)}, \ldots, L_\sigma^{(d(\sigma))}$ the degeneracy rays with respect to the leaves of the tree T_σ. The equation of $L_\sigma^{(i)}$ in the coordinate system (x, y) would have the form of a certain finite sum in fractional powers of y:

$$x := \lambda_{\sigma,i}(y) = c_1 y^{\gamma_1} + c_2 y^{\gamma_1+\gamma_2} + \cdots + c_k y^{\gamma_1+\gamma_2+\cdots+\gamma_k}, \quad 1 \gg y \geq 0,$$

with $c_i \neq 0$, $\gamma_i > 0$. We call γ_1 the *valuation* of the curve $x = \lambda_{\sigma,i}(y)$ and denote it by $\mathrm{val}(\lambda_{\sigma,i}) = \gamma_1$.

THEOREM 1. *Suppose that the germ F has an isolated zero at $(0,0)$. Then*

$$\mathscr{L}(F) = \max\left[m_0, \max_\sigma \max_i \ \mathrm{val}(F(\lambda_{\sigma,i}(y), y))\right].$$

Proof. If Case 1 holds, $\mathscr{L}(F) = m_0$. Conversely, it is evident that

$$\mathscr{L}(F) = \max_{\sigma=1,\ldots,2s} \mathscr{L}(F; L_\sigma),$$

since the germ F has an isolated zero at $(0,0)$.

Let $\Pi(\varepsilon)$ be the closure of the set of points which do not lie in $\bigcup_{\sigma=1}^{2s} \Gamma_\sigma(\varepsilon)$. The vector $(f_{1,m_0}(x,y), f_{2,m_0}(x,y), \ldots, f_{n,m_0}(x,y))$ are different from zero at the non-zero points of $\Pi(\varepsilon)$. Therefore, there are numbers $\rho > 0$ and $c > 0$ such that for $(x,y) \in \Pi(\varepsilon) \cap B_\rho$ the following inequality holds

$$\max_{l=1,2,\ldots,n} |f_{l,m_0}(x,y)| \geq c\|(x,y)\|^{m_0};$$

and hence

$$\max_{l=1,2,\ldots,n} |f_l(x,y)| \geq c\|(x,y)\|^{m_0}, \quad (x,y) \in \Pi(\varepsilon) \cap B_\rho.$$

Since $\max_{l=1,2,\ldots,n}|f_{l,m_0}(x,y)|$ does not vanish at those points of $\Gamma_\sigma(\varepsilon)$ which do not lie on L_σ, the following inequality takes place

(2) $$\max_{l=1,2,\ldots,n} |f_{l,m_0}(x,y)| \geq c'|x|^\mu \|(x,y)\|^{m_0-\mu} \quad (c' > 0; (x,y) \in \Gamma_\sigma(\varepsilon) \cap B_{\rho'}).$$

We shall denote by $\chi_l^1(x,y)$ the sum of those terms in the expansion of $f_l(x,y)$ in powers of x and y, for which $i+j > m_0$ and $j \leq m_0 - \mu$; and by $\chi_l^2(x,y)$ the sum of all remaining terms which satisfy the condition $i+j > m_0$. We have

$$\max_{l=1,2,\ldots,n} |\chi_l^1(x,y)| = \max_{l=1,2,\ldots,n} \left|\sum_{i,j} a_{i,j}^l x^i y^j\right| \leq \max_{l=1,2,\ldots,n} \left[\sum_{i,j} |a_{i,j}^l|\, |x|^{i-\mu} y^j\right] |x|^\mu$$

$$\leq \max_{l=1,2,\ldots,n} \left[\sum_{i,j} |a_{i,j}^l| (\tan\varepsilon)^{i-\mu} y^{i+j-\mu}\right] |x|^\mu$$

for $(x,y) \in \Gamma_\sigma(\varepsilon)$ near $(0,0)$. This implies

(3) $$\max_{l=1,2,\ldots,n} |\chi_l^1(x,y)| = o[|x|^\mu y^{m_0-\mu}].$$

Similarly, when $(x,y) \notin R_\sigma(\eta) := \{(x,y) \in \Gamma_\sigma(\varepsilon) \mid |x| \leq \eta y^{r_0}\}$ we get

$$\max_{l=1,2,\ldots,n} |\chi_l^2(x,y)| = \max_{l=1,2,\ldots,n} \left|\sum_{i,j} a_{i,j}^l x^i y^j\right| \leq \max_{l=1,2,\ldots,n} \left[\sum_{i,j} |a_{i,j}^l|\, |x|^i y^{j-m_0+\mu}\right] y^{m_0-\mu}$$

$$\leq \max_{l=1,2,\ldots,n} \left[\sum_{i,j} |a_{i,j}^l| \eta^{-(j-m_0+\mu)/r_0} |x|^{i+(j-m_0+\mu)/r_0}\right] y^{m_0-\mu}.$$

Hence, by the definition of r_0,

(4) $$\max_{l=1,2,\ldots,n} |\chi_l^2(x,y)| \leq \delta(\eta)|x|^\mu y^{m_0-\mu}, \quad (x,y) \notin R_\sigma(\eta), (x,y) \text{ near } (0,0),$$

where

$$\delta(\eta) = \max_{l=1,2,\ldots,n} \left[\sum_{i,j} |a_{i,j}^l| \eta^{-(j-m_0+\mu)/r_0}\right] \to 0$$

as $\eta \to \infty$.

We conclude from (2), (3) and (4) that there are positive numbers η_0, ρ_1, c_1 such that the following inequality holds for all $(x, y) \in [\Gamma_\sigma(\varepsilon) \backslash R_\sigma(\eta)] \cap B_{\rho_1}$ $(\eta \geq \eta_0)$:

(5) $$\max_l |f_l(x, y)| \geq c_1 |x|^\mu y^{m_0 - \mu} \geq c_1 \eta^{\mu r_0} y^{m_0 + \mu(r_0 - 1)}.$$

On the other hand, it is easily seen that

(6) $$|f_l(x, y) - \psi_{l,\sigma}(x, y)| = o(y^{m_0 + \mu(r_0 - 1)}), \quad l = 1, 2, \ldots, n,$$

is valid on $R_\sigma(\eta)$.

If Case 2.2.1 holds then there are positive numbers ρ_2, c_2 such that

$$\max_{l=1,2,\ldots,n} |\psi_{l,\sigma}(x, y)| \geq c_2 y^{m_0 + \mu(r_0 - 1)}$$

for every $(x, y) \in R_\sigma(\eta) \cap B_{\rho_2}$. Hence, from (6), it follows that

$$\mathscr{L}(F; L_\sigma) = m_0 + \mu(r_0 - 1).$$

Suppose now that Case 2.2.2 holds. We consider the germ F in "the horn neighbourhood" (see [10], [11]) $H_{r_0}(t, w)$ of the curve $x = k_{\sigma,t} y^{r_0}$, $0 \leq y \ll 1$, where

$$H_{r_0}(t, w) = \{(x, y) \in R_\sigma(\eta) \mid |x - k_{\sigma,t} y^{r_0}| < w y^{r_0}\}$$

with $0 < w \ll 1$.

For all $(x, y) \in H_{r_0}(t, w)$ we have

(7) $$\begin{aligned} \max_{l=1,2,\ldots,n} |f_l(x, y)| &= \max_{l=1,2,\ldots,n} y_1^{m_0 q + \mu(p-q)} |g_{l,t}(x_1, y_1)| \\ &\geq c_3 y_1^{m_0 q + \mu(p-q) + \mathscr{L}(G_t; L_{\sigma,t})} \\ &= c_3 y^{m_0 + \mu(r_0 - 1) + \mathscr{L}(G_t; L_{\sigma,t})/q} \quad (c_3 > 0). \end{aligned}$$

It is also easy to see that there are positive constants c_4, ρ_4 such that

(8) $$\max_l |\psi_{l,\sigma}(x, y)| \geq c_4 y^{m_0 + \mu(r_0 - 1)}$$

for $(x, y) \in [R_\sigma(\eta) \backslash \bigcup_{t=1}^{s(\sigma)} H_{r_0}(t, w)] \cap B_{\rho_4}$.

From inequalities (5), (6), (7) and (8), we deduce that

$$\mathscr{L}(F; L_\sigma) = m_0 + \mu(r_0 - 1) + \max_{t=1,2,\ldots,s(\sigma)} \frac{\mathscr{L}(G_t; L_{\sigma,t})}{q}.$$

By solving the above recurrence equation, we obtain

$$\mathscr{L}(F; L_\sigma) = \max_i \operatorname{val}(F(\lambda_{\sigma,i}(y), y)).$$

The proof is complete. □

Example ([9]). Consider $F(x, y) = \operatorname{grad}(x^3 - 3xy^3) = (3x^2 - 3y^3, -9xy^2)$. In our case $m_0 = 2$, and the initial forms $f_{1,2}(x, y) = 3x^2$ and $f_{2,2}(x, y) = 0$ of F vanish on two rays

$$L_1 : \{x := \lambda_1(y) = 0, y \geq 0\}; \quad L_2 : \{x := \lambda_2(y) = 0, y \leq 0\}.$$

It is easily seen that $\mu = 2$, $r_0 = 3/2$, and

$$\psi_{1,\sigma}(x, y) = 3x^2 - 3y^3, \quad \psi_{2,\sigma}(x, y) = 0, \quad \sigma = 1, 2.$$

The system of equations

$$\psi_{1,1}(k, 1) = 0, \quad \psi_{2,1}(k, 1) = 0$$

has two real solutions $k_{1,1} = 1$, $k_{1,2} = -1$. Fix $t \in \{1, 2\}$. To examine the ray L_1 we perform the change of variables $(p = 3, q = 2)$:

$$x = y_1^3(k_{1,t} + x_1), \quad y = y_1^2.$$

Let us write the components of G_t as follows

$$\begin{aligned} G_t(x_1, y_1) &= y_1^{-[m_0 q + \mu(p-q)]} F[y_1^3(k_{1,t} + x_1), y_1^2], \\ &= (3x_1^2 + 6k_{1,t}x_1, -9k_{1,t}y_1 - 9x_1y_1). \end{aligned}$$

The initial forms of components of G_t are

$$(6k_{1,t}x_1, -9k_{1,t}y_1).$$

This germ does not vanish on $L_{1,t} := \{x_1 = 0, y_1 \geq 0\}$, and so, the process of resolution of degeneracy of $L_{1,t}$ is finished; and the equation of $L_{1,t}$ in the coordinate system (x, y) is of the form

$$L_{1,t} : \{x := \lambda_{1,t}(y) = k_{1,t}y^{3/2}, y \geq 0\}.$$

On the other hand, the system of equations

$$\psi_{1,2}(k, -1) = 0, \quad \psi_{2,2}(k, -1) = 0$$

has no real solutions. Hence the process of resolution of degeneracy of L_2 also terminates.

Theorem 1 now implies

$$\mathscr{L}(F) = \max(\mathrm{val}(F(\lambda_2(y), y)), \quad \max_{t=1,2} \mathrm{val}(F(\lambda_{1,t}(y), y))) = \max\left(3, \frac{7}{2}\right) = \frac{7}{2}.$$

The case of non-degenerate germs. Let $a_{i,j}^l$ denote the coefficients of $x^i y^j$ in the expansion of $f_l(x, y)$ in powers of x and y, i.e.

$$f_l(x, y) = \sum_{i+j \geq 0} a_{i,j}^l x^i y^j, \quad l = 1, 2, \ldots, n.$$

Let $\mathrm{supp}(F) := \{(i, j) \in \boldsymbol{N}^2 \mid \text{there is } l \text{ such that } a_{i,j}^l \neq 0\}$. The *Newton polygon* $N(F)$ is the set of compact faces of the boundary of the convex hull of $[\mathrm{supp}(F) + (\boldsymbol{R}^+)^2]$. We call F *convenient* if there are vertices of the Newton polygon $N(F)$ which lie on the axes $x = 0$ and $y = 0$. For any edge e of the Newton polygon $N(F)$, let $\psi_{l,e}(x, y)$ be the sum all monomials $a^l_{i,j} x^i y^j$ in f_l such that $(i, j) \in e$. The convenient germ F is called *non-degenerate* if for every edge e of the Newton polygon $N(F)$ one has the following: the algebraic equations

$$\psi_{l,e}(x, y) = 0, \quad l = 1, 2, \ldots, n, \tag{9}$$

have no common real solutions in $(\boldsymbol{R}\backslash\{0\}) \times (\boldsymbol{R}\backslash\{0\})$. One can check that the non-degenerate condition is generic in the sense of Kouchnirenko (cf. [7]).

Corollary 1. *Suppose that the germ F is convenient and non-degenerate, and let $(a, 0)$ and $(0, b)$ be the vertices of $N(F)$ which lie on the axes. Then*

$$\mathscr{L}(F) = \max(a, b).$$

Proof. Since the germ F is non-degenerate, the equations

$$f_{1,m_0}(k, 1) = 0, \ f_{2,m_0}(k, 1) = 0, \ldots, f_{n,m_0}(k, 1) = 0$$

have common real solutions $k = 0$ and $k = \infty$. Let

$$L_1 = \{x = 0, y \geq 0\}, \quad L_2 = \{x = 0, y \leq 0\},$$
$$L_3 = \{y = 0, x \geq 0\}, \quad L_4 = \{y = 0, x \leq 0\}.$$

Consider L_1. Let $\{m_0, \mu, r_0, p, q\}$ denote the characteristics of L_1. If the segment e, which joints $(0, b)$ and $(\mu, m_0 - \mu)$, belongs to the Newton polygon $N(F)$, then $k = 0$ is not a solution to the system of equations (9) and so

$$\mathscr{L}(F; L_1) = b.$$

Otherwise, because the germ F is non-degenerate, (9) only has a solution $k = 0$. By the change of variables $x = y_1^p x_1$, $y = y_1^q$, there is only one degeneracy ray

$$L_{1,1} = \{x_1 = 0, y_1 \geq 0\}$$

that needs to be considered. Let

$$g_{l,1}(x_1, y_1) = y_1^{-[m_0 q + \mu(p-q)]} f_l(y_1^p x_1, y_1^q), \quad l = 1, 2, \ldots, n.$$

It is easy to check that $G_1 := (g_{1,1}, g_{2,1}, \ldots, g_{n,1})$ is non-degenerate and the point $(0, bq - m_0 q - \mu(p - q))$, which lies on the axis $x_1 = 0$, belongs to the Newton polygon $N(G_1)$. Moreover, the number of vertices of $N(G_1)$ is smaller that of $N(F)$. By Theorem 1 and by induction on the number of vertices of the Newton polygon $N(F)$, we can show that

$$\begin{aligned}
\mathscr{L}(F; L_1) &= m_0 + \mu(r_0 - 1) + \frac{\mathscr{L}(G_1; L_{1,1})}{q} \\
&= m_0 + \mu(r_0 - 1) + \frac{bq - m_0 q - \mu(p - q)}{q} \\
&= b.
\end{aligned}$$

It follows from a similar argument that, $\mathscr{L}(F; L_2) = b$ and $\mathscr{L}(F; L_3) = \mathscr{L}(F; L_4) = a$. Hence

$$\mathscr{L}(F) = \max(a, b).$$

□

3. Newton-Puiseux approximation at infinity

We now suppose that $F = (f_1, f_2, \ldots, f_n) : \boldsymbol{K}^2 \to \boldsymbol{K}^n$ is a polynomial mapping of two variables. The aim of this section is to construct *the Newton-Puiseux approximation at infinity* of F. From this method we immediately obtain a way of calculating the Łojasiewicz exponent $\mathscr{L}_\infty(F)$ of F.

The proofs of the results in this section are done by the same method as in Section 2. Hence we shall only describe the Newton-Puiseux approximation at infinity of F. Furthermore we will only consider the case where F is a real polynomial mapping. Similar results can be obtained for complex polynomial mappings.

Let $d := \max_{l=1,2,\ldots,n} \deg(f_l)$, where $\deg(f_l)$ is the degree of f_l. Then we can write

$$f_l(x, y) = f_{l,d}(x, y) + f_{l,d-1}(x, y) + \cdots, \quad l = 1, 2, \ldots, n,$$

where $f_{l,i}(x, y)$ are homogeneous polynomials of degree i.

Case 1. If the algebraic equations

$$f_{1,d}(k, 1) = f_{2,d}(k, 1) = \cdots = f_{n,d}(k, 1) = 0, \tag{10}$$

have no common real finite or infinite roots, then $\#F^{-1}(0) = \emptyset$ and $\mathscr{L}_\infty(F) = d$. The algorithm is finished.

Case 2. Otherwise, let $k_1, k_2, \ldots, k_s$ be common real roots of (10). Let

$$L_1 : \{x = k_1 y, y \gg 0\}; \quad L_2 : \{x = k_1 y, y \ll 0\};$$

$$\ldots$$

$$L_{2s-1} : \{x = k_s y, y \gg 0\}; \quad L_{2s} : \{x = k_s y, y \ll 0\};$$

We shall refer to L_σ as *degeneracy rays at infinity* of F.

We will denote by $\Gamma_\sigma(\varepsilon)$ $(\sigma = 1, 2, \ldots, 2s)$ the set of points which lie inside the angle of 2ε radians, whose bisector is L_σ; and by $\mathscr{L}_\infty(F; L_\sigma)$ the smallest upper bound of the set of all real α such that the following inequality holds

$$\max_{l=1,2,\ldots,n} |f_l(x, y)| \geq c\|(x, y)\|^\alpha, \quad (x, y) \in \Gamma_\sigma(\varepsilon) \backslash B_\rho,$$

for some $c > 0$, $\rho \gg 0$. We call $\mathscr{L}_\infty(F; L_\sigma)$ the *Łojasiewicz exponent at infinity of* F with respect to the ray L_σ.

Fix $\sigma \in \{1, 2, \ldots, 2s\}$. Assume that $L_\sigma = \{x = 0, y \gg 0\}$.

Let us denote by $a^l_{i,j}$ the coefficients of x^iy^j in $f_l(x,y)$. Let

$$f_{l,d}(x,y) = a^l_{d,0}x^d + \cdots + a^l_{\mu,d-\mu}x^\mu y^{d-\mu}, \quad l = 1,2,\ldots,n,$$

where the vector $(a^1_{\mu,d-\mu}, a^2_{\mu,d-\mu}, \ldots, a^n_{\mu,d-\mu})$ is not zero.

Case 2.1. If there is no term $a^l_{i,j}x^iy^j$ such that $i < \mu$ then $\#F^{-1}(0) = \infty$; the process of resolution of degeneracy at infinity of L_σ terminates.

Case 2.2. Otherwise, let

$$r_0 = \max\left\{\frac{j-d+\mu}{\mu - i} \mid i < \mu\right\}$$

and

$$\psi_{l,\sigma}(x,y) = \sum_{(i,j)} a^l_{i,j}x^iy^j,$$

the summation being taken over all values of (i,j) for which $ir_0 + j = d + \mu(r_0 - 1)$.

Case 2.2.1. If the system of algebraic equations

$$\psi_{1,\sigma}(k,1) = 0, \ \ \psi_{2,\sigma}(k,1) = 0, \ldots, \psi_{n,\sigma}(k,1) = 0,$$

has no real solutions, then the process of resolution of degeneracy at infinity of L_σ terminates and $\mathscr{L}_\infty(F; L_\sigma) = d + \mu(r_0 - 1)$.

Case 2.2.2. Otherwise, suppose that $k_{\sigma,1}, k_{\sigma,2}, \ldots, k_{\sigma,s(\sigma)}$ are the common real roots of the polynomials $\psi_{l,\sigma}(k,1)$, $l = 1,2,\ldots,n$.

Let $r_0 = p/q < 1$, where p and q are two relatively prime numbers, $q > 0$. Fix $t \in \{1,2,\ldots,s(\sigma)\}$. Consider the change the variables

$$x = y_1^p(k_{\sigma,t} + x_1), \quad y = y_1^q,$$

and the functions

$$g_{l,t}(x_1,y_1) = f_l(y_1^p(k_{\sigma,t} + x_1), y_1^q), \quad l = 1,2,\ldots,n,$$

on the positive half of the y_1-axis, $y_1 \gg 0$. We shall refer to

$$L_{\sigma,t} = \{x_1 = 0, y_1 \gg 0\}$$

as *degeneracy at infinity* of the mapping $G_t := (g_{1,t}, g_{2,t}, \ldots, g_{n,t})$.

Resolution of degeneracy at infinity of $L_{\sigma,t}$. We first note that, for each $x_1 \in \boldsymbol{R}$ the function $g_{l,t}(x_1,y_1)$ is a Laurent series in y_1 with a finite number of terms of positive degrees. We write

$$g_{l,t}(x_1,y_1) = \sum_{i,j} b^l_{i,j}x_1^iy_1^j$$

and put $\mathrm{supp}(G_t) = \{(i,j) \mid \exists l, b^l_{i,j} \neq 0\}$. Let $(d_0, \mu') \in \mathrm{supp}(G_t)$ denote the point satisfying the following conditions

$$d_0 = \max\{j \mid (i,j) \in \mathrm{supp}(G_t)\},$$
$$\mu' = \min\{i \mid (i, d_0) \in \mathrm{supp}(G_t)\}.$$

Case 2.2.2(a). If there does not exist a point $(i,j) \in \mathrm{supp}(G_t)$ such that $i < \mu'$, $j < d_0$ then $\#F^{-1}(0) = \infty$ and the process of resolution of degeneracy at infinity of $L_{\sigma,t}$ is finished.

Case 2.2.2(b). Otherwise, let

$$r'_0 = \min\left\{\frac{d_0 - j}{\mu' - i} \mid i < \mu', j < d_0\right\}$$

and

$$\varphi_l(x_1, y_1) = \sum_{(i,j)} b^l_{i,j} x_1^i y_1^j,$$

the summation being taken over all values of (i,j) for which $j - ir'_0 = d_0 - \mu' r'_0$.

If the system of equations $\varphi_l(k, 1) = 0$, $l = 1, 2, \ldots, n$, has no real solutions, then the process of resolution of degeneracy at infinity of $L_{\sigma,t}$ terminates and $\mathscr{L}_\infty(G_t; L_{\sigma,t}) = d_0 + \mu'(r'_0 - 1)$.

Otherwise, suppose that $k_{\sigma,t,1}, k_{\sigma,t,2}, \ldots, k_{\sigma,t,s(\sigma,t)}$ are the common real roots of $\varphi_l(k,1) = 0$. Let $r'_0 = p'/q' > 0$, where p' and q' are two relatively prime numbers. Fix $u \in \{1, 2, \ldots, s(\sigma,t)\}$. Consider the change the variables

$$x_1 = y_2^{-p'}(k_{\sigma,t,u} + x_2), \quad y_1 = y_2^{q'},$$

and the functions

$$h_{l,t,u}(x_2, y_2) = g_{l,t}[y_2^{-p'}(k_{\sigma,t,u} + x_2), y_2^{q'}], \quad l = 1, 2, \ldots, n,$$

on the positive half of the y_2-axis, $y_2 \gg 0$. We shall refer to

$$L_{\sigma,t,u} = \{x_2 = 0, y_2 \gg 0\}$$

as *degeneracy at infinity* of the mapping $H_{t,u} := (h_{1,t,u}, h_{2,t,u}, \ldots, h_{n,t,u})$.

We now repeat the process of resolution of degeneracy at infinity of $L_{\sigma,t}$. If $F^{-1}(0) < \infty$ then after passing a finite number of steps from G_t to $H_{t,u}$ we can calculate the Łojasiewicz exponent at infinity of G with respect to $L_{\sigma,t}$; and this gives $\mathscr{L}_\infty(F; L_\sigma)$. Moreover, the process of resolution of degeneracy at infinity of L_σ gives us a tree T_σ whose vertices are correspond to the degeneracy rays of successively constructed mapping, and the root of T_σ is correspond to L_σ. Denote by $L_\sigma^{(1)}, L_\sigma^{(2)}, \ldots, L_\sigma^{(d(\sigma))}$ the degeneracy rays with respect to the leaves of T_σ. Assume that $x = \lambda_{\sigma,i}(y)$ $(y \gg 0)$, $i = 1, 2, \ldots, d(\sigma)$, are the equations of $L_\sigma^{(i)}$ in the coordinate system (x, y).

With exactly the same method as in Theorem 1, we can prove the following

THEOREM 2. *Suppose that F is a polynomial mapping satisfying $\#F^{-1}(0) < \infty$. Then*

$$\mathscr{L}_\infty(F) = \min\Big(d, \min_\sigma \min_i \operatorname{val}(F(\lambda_{\sigma,i}(y), y))\Big).$$

Example ([6]). Consider the polynomial mapping

$$F(x, y) = \operatorname{grad}(3x - x^3 y) = (3 - 3x^2 y, -x^3).$$

In our case $d = 3$, and the polynomials $f_{1,d}(x, y) = -3x^2 y$ and $f_{2,d}(x, y) = -x^3$ vanish on two rays

$$L_1 : \{x := \lambda_1(y) = 0, y \gg 0\}; \quad L_2 : \{x := \lambda_2(y) = 0, y \ll 0\}.$$

We shall examine the rays L_1 and L_2 simultaneously. It is easily seen that $\mu = 2$, $r_0 = -1/2$ and

$$\psi_{1,\sigma}(x, y) = 3 - 3x^2 y, \quad \psi_{2,\sigma}(x, y) = 0, \quad \sigma = 1, 2.$$

The system of equations

$$\psi_{1,1}(k, 1) = 0, \quad \psi_{2,1}(k, 1) = 0$$

has two real solutions $k_{1,1} = 1$, $k_{1,2} = -1$. Fix $t \in \{1, 2\}$. We perform the change of variables $(p = -1, q = 2)$:

$$x = y_1^{-1}(k_{1,t} + x_1), \quad y = y_1^2.$$

Let us write the components of G_t as follows

$$G_t(x_1, y_1) = F[y_1^{-1}(k_{1,t} + x_1), y_1^2],$$
$$= (-6k_{1,t}x_1 - 3x_1^2, -k_{1,t}y_1^{-3} - 3x_1 y_1^{-3} - 3k_{1,t}x_1^2 y_1^{-3} - x_1^3 y_1^{-3}).$$

It follows that $d_0 = 0$, $\mu' = 1$, $r_0' = 3$, $p' = 3$, $q' = 1$ and

$$\varphi_1(x_1, y_1) = -6k_{1,t}x_1, \quad \varphi_2(x_1, y_1) = -k_{1,t}y_1^{-3}.$$

It is obvious that the system of equations $\varphi_1(k, 1) = \varphi_2(k, 1) = 0$ has no real solutions. Therefore the process of resolution of degeneracy at infinity of $L_{1,t} := \{x_1 = 0, y_1 \gg 0\}$ stops. Moreover, the equation of $L_{1,t}$ in the coordinate system (x, y) is

$$L_{1,t} : \{x := \lambda_{1,t}(y) = k_{1,t}y^{-1/2}, y \gg 0\}.$$

On the other hand, the system of equations

$$\psi_{1,2}(k, -1) = 0, \quad \psi_{2,2}(k, -1) = 0$$

has no real solutions. It follows that the process of resolution of degeneracy at infinity of L_2 also stops.

Theorem 2 now yields

$$\mathscr{L}_\infty(F) = \min(\mathrm{val}(F(\lambda_2(y), y)), \quad \min_{t=1,2} \mathrm{val}(F(\lambda_{1,t}(y), y))) = \min\left(0, -\frac{3}{2}\right) = -\frac{3}{2}.$$

The case with non-degenerate infinity. Let $a^l_{i,j}$ denote the coefficients of x^iy^j in $f_l(x, y)$:

$$f_l(x, y) = \sum_{i+j\geq 0} a^l_{i,j}x^iy^j, \quad l = 1, 2, \ldots, n.$$

We call the polynomial mapping $F = (f_1, f_2, \ldots, f_n)$ *convenient* if $F(x, 0)$ and $F(0, y)$ are non-zero polynomial mappings in $\boldsymbol{R}[x, y]$. We will denote by $\Delta(F)$ the convex hull of the set

$$\{(0, 0)\} \cup \{(i, j) \mid \text{there is } l \text{ such that } a^l_{i,j} \neq 0\}.$$

The *Newton polygon at infinity* $N_\infty(F)$ consists of all the boundary edges of $\Delta(F)$ which are not contained in two axes. If $e \in N_\infty(F)$ then we let $\psi_{l,e}(x, y)$ be the sum all monomials $a^l_{i,j}x^iy^j$ in f_l such that $(i, j) \in e$. The convenient mapping F is *non-degenerate at infinity* if for any $e \in N_\infty(F)$, the system of equations

$$\psi_{1,e}(x, y) = 0, \ \psi_{2,e}(x, y) = 0, \ldots, \psi_{n,e}(x, y) = 0,$$

has no solutions in $(\boldsymbol{R}\backslash\{0\}) \times (\boldsymbol{R}\backslash\{0\})$.

In particular, Theorem 2 also has a simple geometrical meaning as follows.

Corollary 2. *Suppose that the polynomial mapping F is convenient and non-degenerate at infinity and let $(a, 0)$ and $(0, b)$ be the vertices of $N_\infty(F)$ which lay on the axes. Then*

$$\mathscr{L}_\infty(F) = \min(a, b).$$

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