Deformation for coupled Kähler–Einstein metrics

By Satoshi Nakamura

Abstract. The notion of coupled Kähler–Einstein metrics was introduced recently by Hultgren–Witt Nyström. In this paper we discuss deformation of a coupled Kähler–Einstein metric on a Fano manifold. We obtain a necessary and sufficient condition for a coupled Kähler–Einstein metric to be deformed to another coupled Kähler–Einstein metric for a Fano manifold admitting non-trivial holomorphic vector fields. In addition we also discuss deformation for a coupled Käher–Einstein metric on a Fano manifold when the complex structure varies.

1. Introduction.

The existence problem for canonical Kähler metrics for polarized manifolds is one of the central topics in Kähler geometry. In particular, Kähler–Einstein metrics for Fano manifolds have been discussed by many experts from 1980's. In 2015, Chen–Donaldson–Sun [**1**], [**2**], [**3**] and Tian [**19**] showed that a Fano manifold admits a Kähler–Einstein metric if and only if it satisfies K-polystability which is an algebro-geometric condition originating in the geometric invariant theory.

Some generalizations of Kähler–Einstein metrics for Fano manifolds have also been discussed. In this paper, we focus on coupled Kähler–Einstein metrics introduced recently by Hultgren–Witt Nyström [**10**]. Let X be an n-dimensional Fano manifold. A decomposition of the first Chern class $2\pi c_1(X)$ is a sum

$$2\pi c_1(X) = \alpha_1 + \cdots + \alpha_N$$

where each α_i is a Kähler class. For a Kähler metric $\omega_i \in \alpha_i$, the pair $(\omega_i)_{i=1}^N$ is called the *coupled Kähler–Einstein metric* for the decomposition $(\alpha_i)_{i=1}^N$ if it satisfies the system

$$\mathrm{Ric}(\omega_1) = \cdots = \mathrm{Ric}(\omega_N) = \sum_{i=1}^N \omega_i.$$

Note that the ordinary Kähler–Einstein metric $\omega_{\mathrm{KE}} \in 2\pi c_1(X)$ can be seen as a trivial coupled Kähler–Einstein metric. Indeed, for fixed $\lambda_i \in \mathbb{R}_{>0}$ satisfying $\sum_{i=1}^N \lambda_i = 1$, the pair $(\lambda_i \omega_{\mathrm{KE}})_{i=1}^N$ defines a coupled Kähler–Einstein metric for the decomposition $(\lambda_i 2\pi c_1(X))_{i=1}^N$.

Coupled Kähler–Einstein metrics do not always exist for any decomposition $(\alpha_i)_{i=1}^N$. In fact, the Matsushima-type obstruction theorem and the Futaki-type obstruction theorem are known. The Matsushima-type obstruction theorem states that if $(\omega_i)_{i=1}^N$ is a coupled Kähler–Einstein metric, then the identity component of the holomorphic automorphism group $\mathrm{Aut}_0(X)$ is the complexification of the identity component of the isometry group $\mathrm{Isom}_0(X, \omega_1)$ of ω_1, and in particular, it is then reductive [10], [6]. Note that, in this case, $\mathrm{Isom}_0(X, \omega_1) = \mathrm{Isom}_0(X, \omega_2) = \cdots = \mathrm{Isom}_0(X, \omega_N)$ as pointed out in [17]. On the other hand, for the Futaki-type obstruction, the Futaki-type invariant is defined in the following way. First define a function $f_i \in C^\infty(X, \mathbb{R})$ for each $i = 1, 2, \ldots, N$ as

$$\mathrm{Ric}(\omega_i) - \sum_{j=1}^N \omega_j = \sqrt{-1}\partial\bar{\partial}f_i \quad \text{and} \quad \int_X (1 - e^{f_i})\omega_i^n = 0. \tag{1}$$

In this paper, we call the pair $(f_i)_{i=1}^N$ *the Ricci potential* for $(\omega_i)_{i=1}^N$. Note that the pair $(\omega_i)_{i=1}^N$ defines a coupled Kähler–Einstein metric if and only if every Ricci potential f_i vanishes. For any holomorphic vector field V on X, *the Futaki-type invariant* Fut_c for the decomposition $(\alpha_i)_{i=1}^N$ is defined by

$$\mathrm{Fut}_c(V) = \sum_{i=1}^N \int_X H_i(1 - e^{f_i})\frac{\omega_i^n}{\int_X \omega_i^n} \tag{2}$$

where H_i is the potential function for V, that is, $i_V \omega_i = \sqrt{-1}\,\bar{\partial}H_i$. Following [10], [6], [4], the value $\mathrm{Fut}_c(V)$ is independent of the choice of metrics $\omega_i \in \alpha_i$. In particular, if there exists a coupled Kähler–Einstein metric for $(\alpha_i)_{i=1}^N$, then the invariant Fut_c must vanish identically. Recently Futaki–Zhang [7] showed a residue formula to compute this invariant.

In the same spirit of theory for ordinary Kähler–Einstein metrics for Fano manifolds, Hultgren–Witt Nyström [10] established fundamental properties for coupled Kähler–Einstein metrics. They showed a uniqueness theorem and a stability theorem which states that the existence of a coupled Kähler–Einstein metric implies an algebro-geometric stability condition when $\alpha_i = 2\pi c_1(L_i)$ for some ample line bundle L_i over X. The work of Hultgren–Witt Nyström has raised much interest in the study of coupled Kähler–Einstein metrics. In particular, the existence theorem was developed. Pingali [15] and Takahashi [17] introduced a continuity method and a Ricci iteration method respectively to construct coupled Kähler–Einstein metrics. Hultgren [9] developed a detailed study for the existence of such metrics on toric Fano manifolds, and Delcroix–Hultgren [5] extended it to more general settings. Futaki–Zhang [6] introduced Sasakian analogue. Datar–Pingali [4] introduced the notion of coupled constant scalar curvature Kähler metrics which is a generalization of coupled Kähler–Einstein metrics, and also introduced a framework of geometric invariant theory for them. Takahashi [18] introduced a geometric quantization.

Now we state results in this paper. As an existence theorem for non-trivial coupled Kähler–Einstein metrics, Hultgren–Witt Nyström [10] proved the following:

THEOREM 1.1 ([10, Theorem C]). *Let X be a Fano Kähler–Einstein manifold without non-trivial holomorphic vector fields. Fix positive real constants $\lambda_i > 0$ satisfying*

$\sum_{i=1}^{N} \lambda_i = 1$. *Then for any real closed* $(1,1)$*-forms* $\eta_1, \ldots, \eta_N$ *satisfying* $\sum_{i=1}^{N}[\eta_i] = 0$, *there exists a coupled Kähler–Einstein metric for the decomposition* $(2\pi\lambda_i c_1(X) + t[\eta_i])_{i=1}^{N}$ *for* $0 < t \ll 1$.

The main purpose of this paper is to extend the above theorem from view points of (i) the case when a Fano manifold admits continuous automorphism group and (ii) the case of deformation for a non-trivial coupled Kähler–Einstein metric. We follow the strategy of deformation theory for constant scalar curvature Kähler metrics [**13**], [**12**] and extremal Kähler metrics [**16**]. Let X be a Fano manifold admitting a coupled Kähler–Einstein metric $(\theta_i)_{i=1}^{N}$ for a decomposition $(\alpha_i)_{i=1}^{N}$. Take a Kähler metric ω_0 defined by $\mathrm{Ric}(\omega_0) = \sum_{i=1}^{N} \theta_i$. We define

$$U = \left\{ \eta = (\eta_1, \ldots, \eta_N) \,\middle|\, \eta_i \text{ is an } \mathbb{R}\text{-valued } \theta_i\text{-harmonic } (1,1)\text{-form s.t. } \sum_{i=1}^{N}[\eta_i] = 0 \right\} \quad (3)$$

and define $U_0 = \{\eta \in U \mid \|\eta\|_{\omega_0} = 1\}$. The following is the main result of this paper.

THEOREM 1.2. *There exist* $\varepsilon_0 > 0$ *and a smooth function* $\mathcal{F} : [0, \varepsilon_0) \times U_0 \to \mathbb{R}$ *such that if* $\eta \in U_0$ *satisfies* $\mathcal{F}(t, \eta) = 0$ *for some* $t \in [0, \varepsilon_0)$, *then there exists a coupled Kähler–Einstein metric for the decomposition* $(\alpha_i + t[\eta_i])_{i=1}^{N}$.

Moreover, if $\mathrm{tr}_{\theta_i} \eta_i = 0$ *for all* i, *then the leading term of the asymptotic expansion of the function* $\mathcal{F}(t, \eta)$ *around* $t = 0$ *is at least of order* 2 *(see Proposition 3.1). If* $\mathrm{tr}_{\theta_i} \eta_i = 0$ *for all* i *and if* $(\theta_i)_{i=1}^{N}$ *is a trivial coupled Kähler–Einstein metric* $(\lambda_i \omega_{\mathrm{KE}})_{i=1}^{N}$, *then the leading term is at least of order* 4 *(see Proposition 3.3). Furthermore these leading coefficients are described explicitly in terms of initial data (see Propositions 3.1 and 3.3 for the explicit description of these leading coefficients).*

This theorem is divided to Theorem 2.5, Proposition 3.1 and Proposition 3.3. The function $\mathcal{F}$ in Theorem 1.2 tells us in which directions we can find a coupled Kähler–Einstein metric. The function $\mathcal{F}$ is in fact given by the Futaki-type invariant $\mathrm{Fut}_c(V_{t,\eta})$ for the decomposition $(\theta_i + t[\eta_i])_{i=1}^{N}$, where $V_{t,\eta}$ is a holomorphic vector field depending on $t \in [0, \varepsilon_0)$ and $\eta \in U_0$. Therefore Theorem 1.2 is a generalization of Theorem 1.1.

In view of Theorem 1.2, it is natural to discuss the case when $\mathcal{F}(t, \eta) \neq 0$ for some small $t \neq 0$. In this case there exists no coupled Kähler–Einstein metric for the decomposition $(\alpha_i + t[\eta_i])_{i=1}^{N}$. However, under the assumption that the function $\mathcal{F}$ has an asymptotic expansion of some order at $t = 0$, we can construct an almost coupled Kähler–Einstein metric in the following sense:

COROLLARY 1.3. *Suppose the function* $\mathcal{F}$ *in Theorem 1.2 has an asymptotic expansion* $\mathcal{F}(t, \eta) = a_1(\eta)t + a_2(\eta)t^2 + \cdots$ *as* $t \to 0$ *with* $a_1(\eta) = a_2(\eta) = \cdots = a_m(\eta) = 0$ *for some* $\eta \in U_0$ *and for some positive integer* m. *Then there exist* $\varepsilon_0, C > 0$ *such that for any* $i = 1, 2, \ldots, N$ *and any* $t \in (0, \varepsilon_0)$, *there exists a Kähler metric* $\omega_i(t, \eta)$ *in* $\alpha_i + t[\eta_i]$ *satisfying*

$$\left\| 1 - e^{f_i(t,\eta)} \right\|_{C^l(X,\mathbb{R})} \leq C t^{(m+1)/2},$$

where $(f_i(t, \eta))_{i=1}^{N}$ *is the Ricci potential for* $(\omega_i(t, \eta))_{i=1}^{N}$, *and* l *is some positive integer.*

The same technique as in Theorem 1.2 allows us to discuss the deformation of a coupled Kähler–Einstein metric on a Fano manifold when the complex structure varies. Let (X, J) be an n-dimensional Fano manifold with a complex structure admitting a coupled Kähler–Einstein metric $(\theta_i)_{i=1}^N$. Consider a complex deformation $(J(t), (\theta_i(t))_{i=1}^N)$ of $(J, (\theta_i)_{i=1}^N)$ satisfying $\sum_{i=1}^N [\theta_i(t)] = 2\pi c_1(X, J(t))$. In general, the action of $\mathrm{Isom}_0(X, \theta_1)$ may not extend to $(J(t), (\theta_i(t))_{i=1}^N)$ for $t \neq 0$. Based on a work of Rollin–Simanca–Tipler [16] in the context of complex deformation theory of extremal Kähler metrics, we assume that a compact connected subgroup $G' \subset \mathrm{Isom}_0(X, \theta_1)$ acts holomorphically on $(J(t), (\theta_i(t))_{i=1}^N)$. We denote by $B_{G'}$ the space of all such complex deformations. Let $W_{G'}^{l+2,2}(X)$ be the subspace of G'-invariant real functions in the Sobolev space $W^{l+2,2}(X)$. Define an operator $\mathbb{L} : (W^{l+2,2}(X))^N \to (W^{l,2}(X))^N$ as follows:

$$
\mathbb{L}(u_1, \ldots, u_N) = \begin{pmatrix} \Delta_{\theta_1} u_1 + \sum_{j=1}^N u_j - \int_X \sum_{j=1}^N u_j \left(\omega_0^n / \int_X \omega_0^n \right) \\ \vdots \\ \Delta_{\theta_N} u_N + \sum_{j=1}^N u_j - \int_X \sum_{j=1}^N u_j \left(\omega_0^n / \int_X \omega_0^n \right) \end{pmatrix}, \tag{4}
$$

where Δ_{θ_i} is the negative Laplacian for θ_i. Let $\mathcal{H}_{\mathfrak{z}'}$ be the space of all functions $(u_1, \ldots, u_N) \in (C_{G'}^\infty(X; \mathbb{R}))^N$ such that $\int_X u_i \theta_i^n = 0$ for each i and $\mathrm{grad}_{\theta_1} u_1 = \cdots = \mathrm{grad}_{\theta_N} u_N = V$ for some holomorphic vector field V on (X, J) corresponding to an element in $\mathfrak{z}'$, where $\mathfrak{z}'$ is the center of $\mathrm{Lie}(G')$. Then we have

THEOREM 1.4. *Let $(X, J, (\theta_i)_{i=1}^N)$ be a Fano manifold with a complex structure admitting a coupled Kähler–Einstein metric satisfying*

$$
\mathrm{Ker}\, \mathbb{L} \cap \left(W_{G'}^{l+2,2}(X) \right)^N \subset \mathbb{R}^N \oplus \mathcal{H}_{\mathfrak{z}'}. \tag{5}
$$

For any $(J(t), (\theta_i(t))_{i=1}^N) \in B_{G'}$, there exist $\varepsilon_0 > 0$ and a smooth function $\mathcal{G} : [0, \varepsilon_0) \to \mathbb{R}$ such that if $\mathcal{G}(t) = 0$ for some $t \in [0, \varepsilon_0)$, then there exists a coupled Kähler–Einstein metric for the decomposition $([\theta_i(t)])_{i=1}^N$.

We can prove a corresponding result as Corollary 1.3 and an asymptotic expansion for $\mathcal{G}$ as in Theorem 1.2 in the complex deformation setting. Since these results will not be used in this paper, we however omit these proof.

The condition (5) is an analogue of a condition introduced by Li [12] in the context of complex deformation theory for constant scalar curvature Kähler metrics. According to [12], Li's original condition coincides with the non-degeneracy condition for the relative Futaki invariant introduced by Rollin–Simanca–Tipler [16] in the context of the complex deformation theory of extremal Kähler metrics. In our case, the condition (5) is expected to coincide with the non-degeneracy in the sense of Rollin–Simanca–Tipler [16] for "the relative Futaki-type invariant" which should be an obstruction to the existence of "the coupled Mabuchi soliton" defined as the holomorphy of the vector field $\mathrm{grad}_{\omega_1}(1 - e^{f_1}) = \cdots = \mathrm{grad}_{\omega_N}(1 - e^{f_N})$. Here $(f_i)_{i=1}^N$ is the Ricci potential for $(\omega_i)_{i=1}^N$. Note that original Mabuchi solitons [14] (see also [8]) can be seen as the counterpart of extremal Kähler metrics on Fano manifolds. Thus the condition (5) is expected to apply to the deformation theory of the coupled Mabuchi soliton. We leave its study including studies of the coupled Mabuchi soliton and the relative Futaki-type invariant as the future works.

This paper is organized as follows: In Section 2, an operator to deform a coupled Kähler–Einstein metric by the implicit function theorem is introduced, and the first part of Theorem 1.2 and Corollary 1.3 are proved. In Section 3, an asymptotic expansion of the function $\mathcal{F}$ in Theorem 1.2 is calculated to complete the proof of Theorem 1.2. In Section 4, the technique used in Section 2 is applied to prove Theorem 1.4.

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2.   Deformation for coupled Kähler–Einstein metrics.

In this section we prove Theorem 1.2. Let X be an n-dimensional Fano manifold admitting a coupled Kähler–Einstein metric $(\theta_i)_{i=1}^N$ for a decomposition $(\alpha_i)_{i=1}^N$, and G be the identity component of the isometry group $\mathrm{Isom}_0(X, \theta_1)$. Then $\mathrm{Isom}_0(X, \theta_1) = \cdots = \mathrm{Isom}_0(X, \theta_N)$ by [17, Lemma 2.2]. Fix a Kähler metric ω_0 such that $\mathrm{Ric}(\omega_0) = \sum_{i=1}^N \theta_i$. The normalized volume form $\omega_0^n / \int_X \omega_0^n$ is equal to the others $\theta_1^n / \int_X \theta_1^n = \cdots = \theta_N^n / \int_X \theta_N^n$, which comes from the definition of the coupled Kähler–Einstein metric. For any $\eta = (\eta_1, \ldots, \eta_N) \in U_0$, note that every η_i is automatically G-invariant. Let $W_G^{l+2,2}(X)$ be the subspace of real $(l+2)$-th Sobolev space $W^{l+2,2}(X)$, whose elements are G-invariant. Note $W_G^{l+2,2}(X) \subset C^m(X; \mathbb{R})$ if $l+2 > n+m$ by the Sobolev embedding theorem. We can take a neighborhood $\mathcal{U}_{l+2} \subset (W_G^{l+2,2}(X))^N$ at the origin to assume that there exists $\varepsilon > 0$ such that $\omega_i(t, \phi_i) := \theta_i + t\eta_i + \sqrt{-1}\partial\bar{\partial}\phi_i$ defines a Kähler metric for any $t \in [0, \varepsilon)$, any $\eta \in U_0$, any $(\phi_i)_{i=1}^N \in \mathcal{U}_{l+2}$ and each $i = 1, 2, \ldots, N$. For $\Phi = (\phi_i)_{i=1}^N \in \mathcal{U}_{l+2}$, we denote by $(f_i(t, \Phi))_{i=1}^N$ the Ricci potential for $(\omega_i(t, \phi_i))_{i=1}^N$.

In order to construct a coupled Kähler–Einstein metric for the decomposition $(\alpha_i + t[\eta_i])_{i=1}^N$, we consider the following operator $\mathbb{F} = (F_1, F_2, \ldots, F_{2N}) : [0, \varepsilon) \times \mathcal{U}_{l+2} \to (W_G^{l,2})^N \times \mathbb{R}^N$:

$$F_k(t, \Phi) = \begin{cases} 1 - e^{f_i(t,\Phi)} & (k = 1, 2, \ldots, N) \\ \log\big(1/\int_X \omega_0^n\big) \int_X e^{-\sum_{j=1}^N \phi_j} \omega_0^n & (k = N+1) \\ \int_X \phi_k \omega_0^n & (k = N+2, N+3, \ldots, 2N). \end{cases} \tag{6}$$

Note that $(\omega_i(t, \phi_i))_{i=1}^N$ defines a coupled Kähler–Einstein metric if and only if $\mathbb{F}(t, (\phi_i + c_i)_{i=1}^N) = 0$ for some constants $c_i \in \mathbb{R}$, and note also $\mathbb{F}(0, 0) = 0$.

REMARK 2.1.   Hultgren–Witt Nyström [10] introduced another operator to prove Theorem 1.1. See [10] for more detail. However it is technically natural to use our operator $\mathbb{F}$ from view points of the Futaki-type invariant. The operator $\mathbb{F}$ was inspired by a generalization of Kähler–Einstein metrics introduced by Mabuchi [14] (see also [8]) which is called the generalized Kähler–Einstein metric or the Mabuchi soliton in the literature.

Consider the equation $\delta_\Phi \mathbb{F}(0, 0) = 0$ for a variation $(\delta\phi_1, \ldots, \delta\phi_N) \in T_{(0,0)}(\{0\} \times \mathcal{U}_{l+2})$ to apply the implicit function theorem, where $\delta_\Phi \mathbb{F}(0, 0)$ stands for the derivative along Φ-direction at $(t, \Phi) = (0, 0)$. This is equivalent to the following equations:

$$\Delta_{\theta_i}\delta\phi_i + \sum_{j=1}^{N}\delta\phi_j = 0 \quad \text{and} \quad \int_X \delta\phi_i\omega_0^n = 0 \quad \text{for} \quad i = 1, 2, \ldots, N, \tag{7}$$

where Δ_{θ_i} is the negative Laplacian for θ_i. To see this, we prove the following:

LEMMA 2.2.   *The variation of the Ricci potential $f_i(t,\Phi)$ along $(\delta\phi_1,\ldots,\delta\phi_N) \in T_0\mathcal{U}_{l+2}$ at $(t,\Phi) = (0,0)$ is*

$$\delta_\Phi f_i = -\Delta_{\theta_i}\delta\phi_i - \sum_{j=1}^{N}\delta\phi_j + \int_X \sum_{j=1}^{N}\delta\phi_j \frac{\omega_0^n}{\int_X \omega_0^n}.$$

PROOF.   The derivation of the first equation in (1) shows $\delta_\Phi f_i = -\Delta_{\theta_i}\delta\phi_i - \sum_{j=1}^{N}\delta\phi_j + C$ for some constant C. The constant C is equal to $\int_X \sum_{j=1}^{N}\delta\phi_j(\theta_i^n/\int_X \theta_i^n) = \int_X \sum_{j=1}^{N}\delta\phi_j(\omega_0^n/\int_X \omega_0^n)$ by the derivation of the second equation in (1).   □

If the above equation (7) has only the trivial solution, the implicit function theorem can be applied. However it has nontrivial solutions in general by the following result. Recall an operator $\mathbb{L} : (W^{l+2,2}(X))^N \to (W^{l,2}(X))^N$ defined by

$$\mathbb{L}(u_1,\ldots,u_N) = \begin{pmatrix} \Delta_{\theta_1}u_1 + \sum_{j=1}^{N}u_j - \int_X \sum_{j=1}^{N}u_j\left(\omega_0^n/\int_X \omega_0^n\right) \\ \vdots \\ \Delta_{\theta_N}u_N + \sum_{j=1}^{N}u_j - \int_X \sum_{j=1}^{N}u_j\left(\omega_0^n/\int_X \omega_0^n\right) \end{pmatrix}.$$

LEMMA 2.3 (a specific situation in [**18**, Proposition 2.4]).   *The kernel* $\operatorname{Ker}\mathbb{L}$ *is equal to*

$$\left\{(u_1,\ldots,u_N) \in (C^\infty(X;\mathbb{R}))^N \mid \operatorname{grad}_{\theta_1}u_1 = \cdots = \operatorname{grad}_{\theta_N}u_N =: V \text{ and } V \text{ is holomorphic}\right\},$$

where $\operatorname{grad}_{\theta_i}u_i$ *is a type $(1,0)$ gradient vector field on X defined by* $i_{(\operatorname{grad}_{\theta_i}u_i)}\theta_i = \sqrt{-1}\,\overline{\partial}u_i$.

Therefore we modify the operator $\mathbb{F}$ to apply the implicit function theorem. Let $\mathfrak{g}$ be the Lie algebra of G. Since X is Fano, $\mathfrak{g}$ is nothing but the ideal of killing vector fields with zeros. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$. For any G-invariant Kähler metric ω and for any $\xi \in \mathfrak{z}$, the holomorphic vector field $V = J\xi + \sqrt{-1}\xi$ defines a smooth real-valued G-invariant function u satisfying

$$i_V\omega = \sqrt{-1}\,\overline{\partial}u \quad \text{and} \quad \int_X u\omega^n = 0,$$

where J is a fixed complex structure of X. The function u is called *the holomorphic potential* of V with respect to ω. For the holomorphic potentials u_i of V with respect to θ_i, we call $\boldsymbol{u} = (u_1,\ldots,u_N)$ *the holomorphic potential vector* of V with respect to the coupled Kähler–Einstein metric $(\theta_i)_{i=1}^N$. Let $\mathcal{H}_{\mathfrak{z}}$ be the space of holomorphic potential vectors corresponding to elements in $\mathfrak{z}$ with respect to $(\theta_i)_{i=1}^N$, endowed with the induced L^2-inner product $\langle\!\langle\boldsymbol{u},\boldsymbol{v}\rangle\!\rangle_{\omega_0} = \int_X \langle\boldsymbol{u},\boldsymbol{v}\rangle\omega_0^n/\int_X \omega_0^n$ from $(W_G^{l+2,2}(X))^N$ where $\langle\boldsymbol{u},\boldsymbol{v}\rangle$ is the

pointwise inner product. Note that $\mathbb{R}^N \oplus \mathcal{H}_{\mathfrak{z}}$ is nothing but the space of G-invariant kernels $\operatorname{Ker} \mathbb{L} \cap (W_G^{l+2,2}(X))^N$. Note also that the operator $\mathbb{L}$ is self-adjoint with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\omega_0}$ (see [**18**, Proposition 2.4] for more details). The inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\omega_0}$, the Sobolev space $(W_G^{l+2,2}(X))^N$ is decomposed as $\mathbb{R}^N \oplus \mathcal{H}_{\mathfrak{z}} \oplus \mathcal{H}_{\mathfrak{z},l+2}^{\perp}$, where $\mathcal{H}_{\mathfrak{z},l+2}^{\perp}$ is the orthogonal complement. We fix an orthonormal basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_d$ of $\mathbb{R}^N \oplus \mathcal{H}_{\mathfrak{z}}$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\omega_0}$. Let $\pi_{\mathfrak{z}}^{\perp} : (W_G^{l+2,2}(X))^N \to \mathcal{H}_{\mathfrak{z},l+2}^{\perp}$ be the orthogonal projection.

In these setting, we define a modified operator $\tilde{\mathbb{F}} = (\tilde{F}_1, \dots, \tilde{F}_{2N}) : [0, \varepsilon) \times (\mathcal{U}_{l+2} \cap \mathcal{H}_{\mathfrak{z},l+2}^{\perp}) \to \mathcal{H}_{\mathfrak{z},l}^{\perp} \times \mathbb{R}^N$ as follows:

$$\begin{cases} (\tilde{F}_1, \dots, \tilde{F}_N) = \pi_{\mathfrak{z}}^{\perp}(F_1, \dots, F_N) \\ \tilde{F}_{N+i} \quad\quad = F_{N+i} \quad (i = 1, 2, \dots, N). \end{cases} \tag{8}$$

Then the equation $\delta_{\Phi}\tilde{\mathbb{F}}(0,0) = 0$ for a variation $(\delta\phi_1, \dots, \delta\phi_N) \in T_{(0,0)}(\{0\} \times (\mathcal{U}_{l+2} \cap \mathcal{H}_{\mathfrak{z},l+2}^{\perp}))$ is equivalent to

$$\begin{cases} \pi_{\mathfrak{z}}^{\perp} \circ \mathbb{L}(\delta\phi_1, \dots, \delta\phi_N) = 0 \\ \int_X \delta\phi_i \omega_0^n = 0 \quad \text{for} \quad i = 1, 2, \dots, N. \end{cases} \tag{9}$$

This equation has the only solution $(\delta\phi_1, \dots, \delta\phi_N) = (0, \dots, 0)$, since $(\delta\phi_1, \dots, \delta\phi_N) \in \mathcal{H}_{l+2,\mathfrak{z}}^{\perp}$ and $\mathbb{L}(\delta\phi_1, \dots, \delta\phi_N) \in \mathcal{H}_{l,\mathfrak{z}}^{\perp}$. Therefore, by the implicit function theorem, for small $t > 0$ there exists $(\phi_i(t,\eta))_{i=1}^N \in \mathcal{U}_{l+2} \cap \mathcal{H}_{\mathfrak{z},l+2}^{\perp}$ such that $\tilde{\mathbb{F}}(t, (\phi_i(t,\eta))_{i=1}^N) = 0$. More precisely, we have

LEMMA 2.4. *There exists $\varepsilon_0 > 0$ such that for any $t \in [0, \varepsilon_0)$ and for any $\eta \in U_0$, we have a pair of Kähler potentials $(\phi_i(t,\eta))_{i=1}^N \in \mathcal{U}_{l+2} \cap \mathcal{H}_{\mathfrak{z},l+2}^{\perp}$ and functions $c(t,\eta) := (c_p(t,\eta))_{p=1}^d : [0, \varepsilon_0) \times U_0 \to \mathbb{R}^d$ satisfying*

$$\left(1 - e^{f_1(t,\eta)}, \dots, 1 - e^{f_N(t,\eta)}\right) = \sum_{p=1}^d c_p(t,\eta)\boldsymbol{v}_p, \tag{10}$$

where $(f_i(t,\eta))_{i=1}^N$ denotes the Ricci potential for the Kähler metrics $(\theta_i + t\eta_i + \sqrt{-1}\partial\bar{\partial}\phi_i)_{i=1}^N$. Moreover there exists $C > 0$ such that for any $t \in [0, \varepsilon_0)$, any $\eta \in U_0$ and each $i = 1, 2, \dots, N$, $\|\phi_i(t,\eta)\|_{W_G^{l+2,2}} \le C\varepsilon_0$ and $\|c(t,\eta)\|_{\mathrm{Euc}} := \{\sum_{p=1}^d c_p(t,\eta)^2\}^{1/2} \le C\varepsilon_0$.

Now we define the function $\mathcal{F}$ as in Theorem 1.2. Let $V(t,\eta)$ be the holomorphic vector field on X corresponding to $\sum_{p=1}^d c_p(t,\eta)\boldsymbol{v}_p \in \mathbb{R}^N \oplus \mathcal{H}_{\mathfrak{z}}$ in Lemma 2.4, that is, $V(t,\eta) = \operatorname{grad}_{\theta_1}(1 - e^{f_1(t,\eta)}) = \cdots = \operatorname{grad}_{\theta_N}(1 - e^{f_N(t,\eta)})$. Let $H_i(t,\eta)$ be the holomorphic potential for $V(t,\eta)$ with respect to the Kähler metric $\omega_i(t,\eta) := \theta_i + t\eta_i + \sqrt{-1}\partial\bar{\partial}\phi_i(t,\eta)$, that is,

$$i_{V(t,\eta)}\omega_i(t,\eta) = \sqrt{-1}\,\bar{\partial}H_i(t,\eta) \quad \text{and} \quad \int_X H_i(t,\eta)\omega_i(t,\eta)^n = 0 \tag{11}$$

for each i. Then we introduce a function $\mathcal{F} : [0, \varepsilon_0) \times U_0 \to \mathbb{R}$ as follows:

$$\mathcal{F}(t,\eta) = \int_X \sum_{i=1}^N H_i(t,\eta)(1 - e^{f_i(t,\eta)}) \frac{\omega_i(t,\eta)^n}{\int_X \omega_i(t,\eta)^n}.$$

This function is nothing but the Futaki-type invariant $\mathrm{Fut}_c(V(t,\eta))$ for the holomorphic vector field $V(t,\eta)$ with respect to the decomposition $([\omega_i(t,\eta)])_{i=1}^N$.

Now we prove the first part of Theorem 1.2.

THEOREM 2.5.   *There exists $\varepsilon_0 > 0$ such that if $\eta \in U_0$ satisfies $\mathcal{F}(t,\eta) = 0$ for some $t \in [0,\varepsilon_0)$, then there exists a coupled Kähler–Einstein metric for the decomposition $(\alpha_i + t[\eta_i])_{i=1}^N$.*

PROOF.   It suffice to show that the vector $c(t,\eta) = (c_p(t,\eta))_{p=1}^d$ in Lemma 2.4 vanishes when $\mathcal{F}(t,\eta) = 0$.

We first claim that there exists a constant $C > 0$ independent of t and η satisfying

$$\big\|1 - e^{f_i(t,\eta)} - H_i(t,\eta)\big\|_{L^2(\omega_0)} \leq C\varepsilon_0 \|c(t,\eta)\|_{\mathrm{Euc}} \tag{12}$$

for each i (here the L^2-norm is defined with respect to the measure $\omega_0^n / \int_X \omega_0^n$). Indeed, by definition of the holomorphic vector field $V(t,\eta)$ and by (11), we have

$$\sqrt{-1}\,\overline{\partial}\big(1 - e^{f_i(t,\eta)} - H_i(t,\eta)\big) = i_{V(t,\eta)}\big(t\eta_i + \sqrt{-1}\partial\overline{\partial}\phi_i(t,\eta)\big)$$

$$= \sum_{p=1}^d c_p(t,\eta) i_{V_p}\big(t\eta_i + \sqrt{-1}\partial\overline{\partial}\phi_i(t,\eta)\big),$$

where V_p denotes the gradient holomorphic vector field corresponding to $\boldsymbol{v}_p \in \mathbb{R}^N \oplus \mathcal{H}_{\mathfrak{z}}$. Since $\|\phi_i(t,\eta)\|_{W_G^{l+2,2}} \leq C\varepsilon_0$ for any $t \in [0,\varepsilon_0)$ by Lemma 2.4, we have

$$\big|\Delta_{\omega_0}\big(1 - e^{f_i(t,\eta)} - H_i(t,\eta)\big)\big| = \left| \sum_{p=1}^d c_p(t,\eta)\mathrm{tr}_{\omega_0}\big\{\partial\big(i_{V_p}(t\eta_i + \sqrt{-1}\partial\overline{\partial}\phi_i(t,\eta))\big)\big\} \right|$$

$$\leq C\varepsilon_0 \|c(t,\eta)\|_{\mathrm{Euc}}.$$

Then the eigenvalue decomposition for Δ_{ω_0} and the normalization condition in (11) shows the estimate (12).

Now we estimate the norm $\|c(t,\eta)\|_{\mathrm{Euc}}$. Since $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ is an orthonormal basis of $\mathbb{R}^N \oplus \mathcal{H}_{\mathfrak{z}}$ and since the equation (10) holds, we have

$$\|c(t,\eta)\|_{\mathrm{Euc}}^2 = \left\langle\!\!\left\langle \sum_{p=1}^d c_p(t,\eta)\boldsymbol{v}_p, \sum_{q=1}^d c_q(t,\eta)\boldsymbol{v}_q \right\rangle\!\!\right\rangle_{\omega_0}$$

$$= \int_X \sum_{i=1}^N (1 - e^{f_i(t,\eta)})^2 \frac{\omega_0^n}{\int_X \omega_0^n}$$

$$\leq C \int_X \sum_{i=1}^N (1 - e^{f_i(t,\eta)})^2 \frac{\omega_i(t,\eta)^n}{\int_X \omega_i(t,\eta)^n}$$

where $C > 0$ is a constant independent of t and η. Here we used the fact that there exists a constant $C > 0$ independent of t and η satisfying the estimate for the density $C^{-1} \leq (\omega_i(t,\eta)^n / \int_X \omega_i(t,\eta)^n)(\omega_0^n / \int_X \omega_0^n)^{-1} \leq C$ which comes from the estimate $\|\phi_i(t,\eta)\|_{W_G^{l+2,2}} \leq C\varepsilon_0$ in Lemma 2.4. By the assumption $\mathcal{F}(t,\eta) = 0$, the inequality (12) and the Cauchy–Schwarz inequality, we thus have

$$\int_X \sum_{i=1}^N (1 - e^{f_i(t,\eta)})^2 \frac{\omega_i(t,\eta)^n}{\int_X \omega_i(t,\eta)^n} = \int_X \sum_{i=1}^N \big(1 - e^{f_i(t,\eta)} - H_i(t,\eta)\big)(1 - e^{f_i(t,\eta)}) \frac{\omega_i(t,\eta)^n}{\int_X \omega_i(t,\eta)^n}$$

$$\leq C \sum_{i=1}^N \big\|1 - e^{f_i(t,\eta)} - H_i(t,\eta)\big\|_{L^2(\omega_0)} \big\|1 - e^{f_i(t,\eta)}\big\|_{L^2(\omega_0)}$$

$$\leq C\varepsilon_0 \|c(t,\eta)\|_{\mathrm{Euc}} \sqrt{N} \left(\sum_{i=1}^N \big\|1 - e^{f_i(t,\eta)}\big\|_{L^2(\omega_0)}^2 \right)^{1/2}$$

$$\leq C\varepsilon_0 \|c(t,\eta)\|_{\mathrm{Euc}}^2,$$

where each C is again a positive constant independent of t and η, and we again used the estimate $C^{-1} \leq (\omega_i(t,\eta)^n / \int_X \omega_i(t,\eta)^n)(\omega_0^n / \int_X \omega_0^n)^{-1} \leq C$ in the first inequality. Therefore if $\varepsilon_0 > 0$ is small enough then $c(t,\eta) = 0$. This completes the proof. $\square$

Using the same technique as in the previous proof we prove Corollary 1.3.

PROOF OF COROLLARY 1.3. It follows from the assumption that $|\mathcal{F}(t,\eta)| \leq Ct^{m+1}$ for any $t \in [0,\varepsilon_0)$. By the same calculation as the last estimate in the previous proof, the following holds:

$$\left| \mathcal{F}(t,\eta) - \sum_{i=1}^N \int_X (1 - e^{f_i(t,\eta)})^2 \frac{\omega_i^n(t,\eta)}{\int_X \omega_i^n(t,\eta)} \right| \leq C\varepsilon_0 \|c(t,\eta)\|_{\mathrm{Euc}}^2.$$

Since $\|\phi_i(t,\eta)\|_{W_G^{l+2,2}} \leq C\varepsilon_0$ by Lemma 2.4, then $\sum_{i=1}^N \int_X (1 - e^{f_i(t,\eta)})^2 (\omega_i^n(t,\eta) / \int_X \omega_i^n(t,\eta)) \leq C\|c(t,\eta)\|_{\mathrm{Euc}}^2$. Thus $\|c(t,\eta)\|_{\mathrm{Euc}}^2 \leq Ct^{m+1}$ after perhaps replacing the constant ε_0 with a smaller one. In view of the equation (10), we finally have

$$\big\|1 - e^{f_i(t,\eta)}\big\|_{C^l(X)}^2 \leq C\big\|c(t,\eta)\big\|_{\mathrm{Euc}}^2 \leq Ct^{m+1}.$$

This completes the proof of Corollary 1.3. $\square$

3. The asymptotic expansion of the function $\mathcal{F}$.

We use same notation as in the previous section to prove the second part of Theorem 1.2. Namely we determine the leading term of the asymptotic expansion of $\mathcal{F}(t,\eta)$ at $t = 0$ under the technical assumption $\mathrm{tr}_{\theta_i}\eta_i = 0$ for $i = 1, 2, \ldots, N$.

Let h_η be a smooth function defined by $\sum_{j=1}^N \eta_j = \sqrt{-1}\partial\bar{\partial}h_\eta$ and $\int_X h_\eta \omega_0^n = 0$. Define $\boldsymbol{h}_\eta = (h_\eta, \ldots, h_\eta) \in (C_G^\infty(X;\mathbb{R}))^N$. Let $\pi_{\mathfrak{z}}$ be the L^2-projection from $(W_G^{l,2})^N$ to $\mathbb{R}^N \oplus \mathcal{H}_{\mathfrak{z}}$.

PROPOSITION 3.1.   *Suppose* $\mathrm{tr}_{\theta_i}\eta_i = 0$ *for* $i = 1, 2, \ldots, N$. *Then*

$$\mathcal{F}(t, \eta) = t^2 \int_X |\pi_{\mathfrak{z}}(\boldsymbol{h}_\eta)|^2 \frac{\omega_0^n}{\int_X \omega_0^n} + \mathcal{O}(t^3) \quad as \quad t \to 0.$$

REMARK 3.2.   It follows from the proof of Proposition 3.1 that the first derivative $\mathcal{F}'(0, \eta)$ with respect to t vanishes without the assumption $\mathrm{tr}_{\theta_i}\eta_i = 0$. This assumption is used for making the second derivative $\mathcal{F}''(0, \eta)$ simple to analyze.

PROOF.   It is easy to see $\mathcal{F}(0, \eta)$ and the first derivative $\mathcal{F}'(0, \eta)$ with respect to t vanishes by $H_i(0, \eta) = 0$ and $f_i(0, \eta) = 0$. We describe the second derivative $\mathcal{F}''(0, \eta)$ in terms of the initial data. It is also easy to see

$$\mathcal{F}''(0, \eta) = -2 \sum_{i=1}^N \int_X H_i'(0, \eta) f_i'(0, \eta) \frac{\theta_i^n}{\int_X \theta_i^n}$$

$$= -2 \sum_{i=1}^N \int_X H_i'(0, \eta) f_i'(0, \eta) \frac{\omega_0^n}{\int_X \omega_0^n}.$$

Here we used the equalities $\omega_0^n / \int_X \omega_0^n = \theta_1^n / \int_X \theta_1^n = \cdots = \theta_N^n / \int_X \theta_N^n$.

The derivative of the defining equation of the holomorphic potential $H_i(t, \eta)$ in (11) shows

$$\sqrt{-1}\,\overline{\partial} H_i'(0, \eta) = i_{V'(0,\eta)}\omega_i(0, \eta) + i_{V(0,\eta)}\omega_i'(0, \eta)$$

$$= -\sqrt{-1}\,\overline{\partial} f_i'(0, \eta).$$

Then $H_i'(0, \eta) = -f_i'(0, \eta)$ by equations $\int_X H_i'(0, \eta)\theta_i^n = \int_X -f_i'(0, \eta)\theta_i^n = 0$ which come from the derivative of the normalization conditions for $H_i(t, \eta)$ and $f_i(t, \eta)$.

On the other hand, the derivative of the defining equation of the Ricci potential $f_i(t, \eta)$ as in (1) together with the assumption $\mathrm{tr}_{\theta_i}\eta_i = 0$ shows

$$f_i'(0, \eta) = -\Delta_{\theta_i}\phi_i'(0, \eta) - h_\eta - \sum_{j=1}^N \phi_j'(0, \eta) + C$$

for some constant C. Since this constant C is equal to $\int_X \sum_{j=1}^N \delta\phi_j(\theta_i^n / \int_X \theta_i^n) = \int_X \sum_{j=1}^N \delta\phi_j(\omega_0^n / \int_X \omega_0^n)$ by normalization conditions for $f_i'(0, \eta)$ and h_η, then

$$(f_1'(0, \eta), \ldots, f_N'(0, \eta)) = -\mathbb{L}(\phi_1'(0, \eta), \ldots, \phi_N'(0, \eta)) - \boldsymbol{h}_\eta. \tag{13}$$

Since the equation (10) shows $\pi_{\mathfrak{z}}(f_1'(0, \eta), \ldots, f_N'(0, \eta)) = (f_1'(0, \eta), \ldots, f_N'(0, \eta))$ and since $\mathbb{L}(\phi_1'(0, \eta), \ldots, \phi_N'(0, \eta)) \in \mathcal{H}_{l,\mathfrak{z}'}^\perp$, therefore $(f_1'(0, \eta), \ldots, f_N'(0, \eta)) = -\pi_{\mathfrak{z}}(\boldsymbol{h}_\eta)$ by (13). This completes the proof.   □

In the above proposition, if the initial coupled Kähler–Einstein metric $(\theta_i)_{i=1}^N$ is trivial, that is, if there exists positive constants $(\lambda_i)_{i=1}^N$ satisfying $\sum_i \lambda_i = 1$ and $\theta_i = \lambda_i \omega_{\mathrm{KE}}$ for all i, then the coefficient $\int_X |\pi_{\mathfrak{z}}(\boldsymbol{h}_\eta)|^2 \omega_0^n / \int_X \omega_0^n$ in the asymptotic expansion

vanishes. Indeed $\Delta_{\omega_{\mathrm{KE}}} h_\eta = \sum_i \mathrm{tr}_{\theta_i} \eta_i / \lambda_i = 0$, and thus $\boldsymbol{h}_\eta = 0$ by the normalization condition of h_η. We show the following proposition to end this section. Define

$$\boldsymbol{I}_\eta = \left(|\eta_1|^2_{\theta_1} - \int_X |\eta_1|^2_{\theta_1} \frac{\theta_1^n}{\int_X \theta_1^n}, \ldots, |\eta_N|^2_{\theta_N} - \int_X |\eta_N|^2_{\theta_N} \frac{\theta_N^n}{\int_X \theta_N^n} \right) \in (C_G^\infty(X; \mathbb{R}))^N.$$

PROPOSITION 3.3. *Suppose* $\mathrm{tr}_{\theta_i} \eta_i = 0$ *for* $i = 1, 2, \ldots, N$. *Suppose also that there exist a Kähler–Einstein metric* ω_{KE} *and positive constants* $(\lambda_i)_{i=1}^N$ *satisfying* $\sum_{j=1}^N \lambda_j = 1$ *and* $\theta_i = \lambda_i \omega_{\mathrm{KE}}$ *for* $i = 1, 2, \ldots, N$. *Then*

$$\mathcal{F}(t, \eta) = \frac{t^4}{4} \int_X |\pi_{\mathfrak{z}}(\boldsymbol{I}_\eta)|^2 \frac{\omega_{\mathrm{KE}}^n}{\int_X \omega_{\mathrm{KE}}^n} + \mathcal{O}(t^5) \quad as \quad t \to 0.$$

PROOF. It is easy to see the third derivative $\mathcal{F}^{(3)}(0, \eta)$ with respect to t vanishes by formulas $H_i(0, \eta) = f_i(0, \eta) = 0$ and $H_i'(0, \eta) = f_i'(0, \eta) = 0$ which come from the proof of Proposition 3.1 and $\boldsymbol{h}_\eta = 0$. In the following we describe the fourth derivative $\mathcal{F}^{(4)}(0, \eta)$ in terms of the initial data. A direct calculation with the above formulas shows

$$\mathcal{F}^{(4)}(0, \eta) = -6 \sum_{i=1}^N \int_X H_i''(0, \eta) f_i''(0, \eta) \frac{\omega_{\mathrm{KE}}^n}{\int_X \omega_{\mathrm{KE}}^n}.$$

By the second derivative of the defining equation of $H_i(t, \eta)$,

$$\sqrt{-1}\,\bar{\partial} H_i''(0, \eta) = i_{V''(0,\eta)} \omega_i(0, \eta) + 2 i_{V'(0,\eta)} \omega_i'(0, \eta) + i_{V(0,\eta)} \omega_i''(0, \eta)$$
$$= -\sqrt{-1}\,\bar{\partial} f_i''(0, \eta).$$

Then $H_i''(0, \eta) = -f_i''(0, \eta)$, since $\int_X H_i''(0, \eta) \theta_i^n = \int_X -f_i''(0, \eta) \theta_i^n = 0$ given by the second derivative of normalization conditions for $H_i(t, \eta)$ and $f_i(t, \eta)$.

Also observe $\phi_i'(0, \eta) = 0$. Indeed formulas $f_i'(0, \eta) = 0$ and $\boldsymbol{h}_\eta = 0$ and the equation (13) show $\phi_i'(0, \eta)$ is constant, and its constant is equals to 0 by $\int_X \phi_i'(0, \eta) \omega_0^n = 0$ which come from the derivative of the condition $\tilde{F}_k(t, (\phi_i(t, \eta))_{i=1}^N) = 0$ for $k = N + 1, \ldots, 2N$ in the modified operator (8).

The second derivative of the defining equation of $f_i(t, \eta)$ together with the formula $\phi_i'(0, \eta) = 0$ and the assumption $\mathrm{tr}_{\theta_i} \eta_i = 0$ shows

$$f_i''(0, \eta) = \frac{d}{dt}\Big|_{t=0} \left(-\mathrm{tr}_{\omega_i(t,\eta)}\big(\eta_i + \sqrt{-1}\partial\bar{\partial}\phi_i'(t, \eta)\big) - h_\eta - \sum_{j=1}^N \phi_j'(t, \eta) \right) + C$$

$$= |\eta_i|^2_{\theta_i} - \Delta_{\theta_i} \phi_i''(0, \eta) - \sum_{j=1}^N \phi_j''(0, \eta) + C$$

for some constant C. Then $C = \int_X \sum_{j=1}^N \phi_j''(0, \eta)(\omega_0^n / \int_X \omega_0^n) - \int_X |\eta_1|^2_{\theta_1} (\theta_1^n / \int_X \theta_1^n)$ by the normalization $\int_X f_i''(0, \eta) \theta_i^n = \int_X f_i''(0, \eta) \omega_0^n = 0$. Thus we have

$$\big(f_1''(0, \eta), \ldots, f_N''(0, \eta)\big) = -\mathbb{L}\big(\phi_1''(0, \eta), \ldots, \phi_N''(0, \eta)\big) + \boldsymbol{I}_\eta.$$

In view of the equation (10), $\pi_{\mathfrak{z}}(f_1''(0,\eta),\ldots,f_N''(0,\eta)) = (f_1''(0,\eta),\ldots,f_N''(0,\eta))$. Therefore $(f_1''(0,\eta),\ldots,f_N''(0,\eta)) = \pi_{\mathfrak{z}}(\boldsymbol{I}_\eta)$. This completes the proof. $\square$

4. Deformation for complex structure and coupled Kähler–Einstein metrics.

In this section we consider the deformation of a coupled Kähler–Einstein metric on a Fano manifold under the deformation of the complex structure by applying the technique used in Section 2. Let (X, J) be a Fano manifold with a complex structure admitting a coupled Kähler–Einstein metric $(\theta_i)_{i=1}^N$. As in Section 2 we fix a Kähler metric ω_0 satisfying $\mathrm{Ric}(\omega_0) = \sum_{i=1}^N \theta_i$. Consider a smooth family of complex structures $J(t)$ with $J(0) = J$. Kodaira–Spencer [11] showed that there existed a smooth family of compatible Kähler metrics $\theta_i(t)$ compatible with $J(t)$ for small $t > 0$ satisfying $\theta_i(0) = \theta_i$. For our purpose, we only consider smooth families $J(t)$ and $(\theta_i(t))_{i=1}^N$ satisfying $\sum_{i=1}^N [\theta_i(t)] = 2\pi c_1(X, J(t))$. In this paper, such pair $(J(t), (\theta_i(t))_{i=1}^N)$ is called complex deformation of $(J, (\theta_i)_{i=1}^N)$. We ask whether there exists a coupled Kähler–Einstein metric for the decomposition $([\theta_i(t)])_{i=1}^N$.

Let G be the identity component of the isometry group of the Kähler metric θ_i. Recall the identity component of the automorphism group of (X, J) is the complexification of G. The action of G may not extend to $(X, J(t))$ in general. Based on the idea of Rollin–Simanca–Tipler [16], we assume that there exists a compact connected subgroup G' of G such that the action of G' extends holomorphically on the complex deformation $(J(t), (\theta_i(t))_{i=1}^N)$. Let $B_{G'}$ be the space of complex deformations of $(J, (\theta_i)_{i=1}^N)$ admitting a holomorphic G'-action. Let $\mathfrak{g}'$ be the Lie algebra of G', and $\mathfrak{z}'$ be the center of $\mathfrak{g}'$. As in Section 2, let $\mathcal{H}_{\mathfrak{z}'} \subset (W_{G'}^{l+2,2}(X))^N$ be the space of holomorphic potential vectors corresponding to elements in $\mathfrak{z}'$ with respect to $(\theta_i)_{i=1}^N$, and fix an orthonormal basis $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_d$ of $\mathbb{R}^N \oplus \mathcal{H}_{\mathfrak{z}'}$ with respect to $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\omega_0}$. Take the orthogonal decomposition $(W_{G'}^{l+2,2}(X))^N = \mathbb{R}^N \oplus \mathcal{H}_{\mathfrak{z}'} \oplus \mathcal{H}_{\mathfrak{z},l+2}^\perp$, and define $\pi_{\mathfrak{z}'}^\perp$ as the projection $(W_{G'}^{l+2,2}(X))^N \to \mathcal{H}_{\mathfrak{z},l+2}^\perp$.

Fix $(J(t), (\theta_i(t))_{i=1}^N) \in B_{G'}$. For a neighborhood $\mathcal{U}_{l+2} \subset (W_{G'}^{l+2,2}(X))^N$ at the origin, it is possible to assume there exists $\varepsilon > 0$ such that $\theta_i(t) + \sqrt{-1}\partial_t\overline{\partial}_t\phi_i$ defines a Kähler metric for any $t \in [0,\varepsilon)$, any $(\phi_i)_{i=1}^N \in \mathcal{U}_{l+2}$ and each i, where $\overline{\partial}_t := (1/2)(d - \sqrt{-1}J(t)d)$ and ∂_t is its complex conjugate. For $\Phi = (\phi_i)_{i=1}^N \in \mathcal{U}_{l+2}$, we denote by $(f_i(t,\Phi))_{i=1}^N$ the Ricci potential for $(\theta_i(t) + \sqrt{-1}\partial_t\overline{\partial}_t\phi_i)_{i=1}^N$. In order to construct a coupled Kähler–Einstein metric for the decomposition $([\theta_i(t)])_{i=1}^N$, consider an operator $\tilde{\mathbb{F}} : [0,\varepsilon) \times (\mathcal{U}_{l+2} \cap \mathcal{H}_{\mathfrak{z}',l+2}^\perp) \to \mathcal{H}_{\mathfrak{z}',l}^\perp \times \mathbb{R}^N$ defined by

$$\begin{cases} (\tilde{F}_1,\ldots,\tilde{F}_N) = \pi_{\mathfrak{z}'}^\perp(F_1,\ldots,F_N) \\ \tilde{F}_{N+i} \qquad = F_{N+i} \quad (i = 1, 2, \ldots, N), \end{cases} \tag{14}$$

where each F_k is defined by the same manner as (6). Then the implicit function theorem shows the following:

LEMMA 4.1.   *Suppose* $\mathrm{Ker}\,\mathbb{L} \cap (W_{G'}^{l+2,2}(X))^N \subset \mathbb{R}^N \oplus \mathcal{H}_{\mathfrak{z}'}$. *For any* $(J(t), (\theta_i(t))_{i=1}^N) \in B_{G'}$, *there exists* $\varepsilon_0 > 0$ *such that for any* $t \in [0,\varepsilon_0)$, *we have Kähler*

potentials $(\phi_i(t))_{i=1}^N \in \mathcal{U}_{l+2} \cap \mathcal{H}_{\mathfrak{z}',l+2}^\perp$ *and functions* $c(t) := (c_1(t), \ldots, c_d(t)) : [0, \varepsilon_0) \to \mathbb{R}^d$ *satisfying*

$$\left(1 - e^{f_1(t)}, \ldots, 1 - e^{f_N(t)}\right) = \sum_{p=1}^d c_p(t) \boldsymbol{v}_p, \tag{15}$$

where $(f_i(t))_{i=1}^N$ *denotes the Ricci potential for the Kähler metrics* $(\theta_i + \sqrt{-1}\partial_t \overline{\partial}_t \phi_i(t))_{i=1}^N$. *Moreover there exists* $C > 0$ *such that for each* $i = 1, 2, \ldots, N$ *and for any* $t \in [0, \varepsilon_0)$, *we have* $\|\phi_i(t)\|_{W_{G'}^{l+2,2}} \le C\varepsilon_0$ *and* $\|c(t)\|_{\mathrm{Euc}} \le C\varepsilon_0$.

PROOF. By the same calculation as in Lemma 2.2, the equation $\delta_\Phi \widetilde{\mathbb{F}}(0,0) = 0$ for a variation $(\delta\phi_1, \ldots, \delta\phi_N) \in T_{(0,0)}(\{0\} \times (\mathcal{U}_{l+2} \cap \mathcal{H}_{\mathfrak{z}',l+2}^\perp))$ is given by

$$\begin{cases} \pi_{\mathfrak{z}'}^\perp \circ \mathbb{L}(\delta\phi_1, \ldots, \delta\phi_N) = 0 \\ \int_X \delta\phi_i \omega_0^n = 0 \quad \text{for} \quad i = 1, 2, \ldots, N. \end{cases} \tag{16}$$

Therefore the linearized operator $\delta_\Phi \widetilde{\mathbb{F}}(0,0) : T_{(0,0)}(\{0\} \times (\mathcal{H}_{\mathfrak{z}',l+2}^\perp \cap \mathcal{U}_{l+2})) \to T_{(0,0)}(\mathcal{H}_{\mathfrak{z}',l}^\perp \times \mathbb{R}^N)$ is invertible if and only if the condition $\mathrm{Ker}\,\mathbb{L} \cap (W_{G'}^{l+2,2}(X))^N \subset \mathbb{R}^N \oplus \mathcal{H}_{\mathfrak{z}'}$ is satisfied. $\square$

Now we define the function $\mathcal{G} : [0, \varepsilon_0) \to \mathbb{R}$ in Theorem 1.4. Under the assumption $\mathrm{Ker}\,\mathbb{L} \cap (W_{G'}^{l+2,2}(X))^N \subset \mathbb{R}^N \oplus \mathcal{H}_{\mathfrak{z}'}$, we have $(\phi_i(t))_{i=1}^N \in \mathcal{U}_{l+2} \cap \mathcal{H}_{\mathfrak{z}',l+2}^\perp$ and $c(t) := (c_1(t), \ldots, c_d(t)) : [0, \varepsilon_0) \to \mathbb{R}^d$ as in Lemma 4.1. Let ξ_p be the killing vector field in $\mathfrak{z}'$ corresponding to $\boldsymbol{v}_p \in \mathbb{R}^N \oplus \mathcal{H}_{\mathfrak{z}'}$. For $p = 1, \ldots d$, the vector field $V_p(t) := J(t)\xi_p + \sqrt{-1}\xi_p$ is holomorphic on $(X, J(t))$ since $(J(t), (\theta_i(t))_{i=1}^N) \in B_{G'}$. We define $V(t) := \sum_{p=1}^d c_p(t) V_p(t)$. The holomorphic potential $H_i(t)$ for $V(t)$ with respect to $\omega_i(t) := \theta_i(t) + \sqrt{-1}\partial_t \overline{\partial}_t \phi_i(t)$ is defined by

$$i_{V(t)}\omega_i(t) = \sqrt{-1}\,\overline{\partial}_t H_i(t) \quad \text{and} \quad \int_X H_i(t)\omega_i(t)^n = 0. \tag{17}$$

Then we define $\mathcal{G} : [0, \varepsilon_0) \to \mathbb{R}$ as follows;

$$\mathcal{G}(t) = \int_X \sum_{i=1}^N H_i(t)(1 - e^{f_i(t)}) \frac{\omega_i(t)^n}{\int_X \omega_i(t)^n},$$

where $(f_i(t))_{i=1}^N$ is the Ricci potential for $(\theta_i(t) + \sqrt{-1}\partial_t \overline{\partial}_t \phi_i(t))_{i=1}^N$.

Since Theorem 1.4 is proved by the same way as in the proof of Theorem 2.5 together with the following lemma, we omit the proof of it.

LEMMA 4.2. *If* $(J(t), (\theta_i(t))_{i=1}^N) \in B_{G'}$ *satisfies*

$$\|J(t) - J\|_{C^1(X,\omega_0)} \le C\varepsilon_0, \tag{18}$$

then there exists $C' > 0$ *such that for any* $t \in [0, \varepsilon_0)$ *and each* i,

$$\big\|1 - e^{f_i(t)} - H_i(t)\big\|_{L^2(\omega_0)} \le C'\varepsilon_0\|c(t)\|_{\mathrm{Euc}}.$$

PROOF.   First we define $\hat{V}(t) = \sum_{p=1}^{d} c_p(t)V_p(0)$ as a holomorphic vector field on (X, J). In view of the equation (15), for each i, it satisfies

$$i_{\hat{V}(t)}\theta_i = \sqrt{-1}\,\bar{\partial}_0(1 - e^{f_i(t)}).$$

Together with (17), we have

$$\sqrt{-1}\,\bar{\partial}_0\big(1 - e^{f_i(t)} - H_i(t)\big) = i_{\hat{V}(t)}\theta_i - i_{V(t)}\omega_i(t) + \sqrt{-1}(\bar{\partial}_t - \bar{\partial}_0)H_i(t). \qquad (19)$$

The estimates $\|\omega_i(t) - \theta_i\|_{W^{l+2,2}} \le C\varepsilon_0$ given in Lemma 4.1 and

$$\|(V_p(t) - V_p(0))\|_{C^0} + \|\partial_0(V_p(t) - V_p(0))\|_{C^0} = \|(J(t) - J)\xi_p\|_{C^0} + \|\partial_0(J(t) - J)\xi_p\|_{C^0}$$
$$\le C\varepsilon_0$$

show

$$\big\|\partial_0(i_{\hat{V}(t)}\theta_i - i_{V(t)}\omega_i(t))\big\|_{C^0} = \left\|\sum_{p=1}^{d} c_p(t)\partial_0(i_{V_p(0)}\theta_i - i_{V_p(t)}\omega_i(t))\right\|_{C^0}$$

$$\le \|c(t)\|_{\mathrm{Euc}}\Bigg\{\sum_{p=1}^{d}\big(\big\|i_{\partial_0(V_p(0) - V_p(t))}\theta_i\big\|_{C^0}$$

$$+ \big\|i_{\partial_0 V_p(t)}(\theta_i - \omega_i(t))\big\|_{C^0}$$

$$+ \big\|i_{(V_p(0) - V_p(t))}\partial_0\theta_i\big\|_{C^0}$$

$$+ \big\|i_{V_p(t)}\partial_0(\theta_i - \omega_0(t))\big\|_{C^0}\big)^2\Bigg\}^{1/2}$$

$$\le C\varepsilon_0\|c(t)\|_{\mathrm{Euc}}. \qquad (20)$$

We next estimate the holomorphic potential $H_i(t)$. Since

$$\Delta_{\omega_i(t)}H_i(t) = \mathrm{tr}_{\omega_i(t)}\sqrt{-1}\partial_t\bar{\partial}_t H_i(t) = \sum_{p=1}^{d} c_p(t)\mathrm{tr}_{\omega_i(t)}\partial_t(i_{V_p(t)}\omega_i(t))$$

and since $\|\omega_i(t) - \theta_i\|_{C^{2,\alpha}} \le C\varepsilon_0$ (if the exponent l is taken sufficiently large in Lemma 4.1), then there exists $C > 0$ independent t such that $\|H_i(t)\|_{C^2} \le C\|c(t)\|_{\mathrm{Euc}}$. Thus

$$\big|\partial_0(\bar{\partial}_t - \partial_0)H_i(t)\big| = \frac{1}{2}\big|\partial_0(J_t - J)dH_i(t)\big| \le C\varepsilon_0\|c(t)\|_{\mathrm{Euc}}. \qquad (21)$$

Therefore, by (19), (20) and (21), we have $|\Delta_{\omega_0}(1 - e^{f_i(t)} - H_i(t))| \le C\varepsilon_0\|c(t)\|_{\mathrm{Euc}}$, and the eigenvalue decomposition for Δ_{ω_0} and the normalization conditions for $f_i(t)$ and $H_i(t)$ shows $\|1 - e^{f_i(t)} - H_i(t)\|_{L^2(\omega_0)} \le C'\varepsilon_0\|c(t)\|_{\mathrm{Euc}}$. This completes the proof. $\square$

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Satoshi NAKAMURA

Department of Mathematics
Saitama University
255 Shimo-Okubo, Sakura-Ku
Saitama 380-8570, Japan
E-mail: nakamura1990@mail.saitama-u.ac.jp

Current address:
Institute of Mathematics
University of Tsukuba
Tsukuba 305-8571, Japan
E-mail: s.nakamura@math.tsukuba.ac.jp