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Left orderable surgeries of double twist knots

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Abstract. A rational number r is called a left orderable slope of a knot $K \subset S^3$ if the 3-manifold obtained from S^3 by r-surgery along K has left orderable fundamental group. In this paper we consider the double twist knots $C(k, l)$ in the Conway notation. For any positive integers m and n, we show that if K is a double twist knot of the form $C(2m, -2n)$, $C(2m+1, 2n)$ or $C(2m+1, -2n)$ then there is an explicit unbounded interval I such that any rational number $r \in I$ is a left orderable slope of K.

1. Introduction.

The motivation of this paper is the L-space conjecture of Boyer, Gordon and Watson [**BGW**] which states that an irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left orderable. Here a rational homology 3-sphere Y is an L-space if its Heegaard Floer homology $\widehat{\mathrm{HF}}(Y)$ has rank equal to the order of $H_1(Y; \mathbb{Z})$, and a non-trivial group G is left orderable if it admits a total ordering $<$ such that $g < h$ implies $fg < fh$ for all elements f, g, h in G. A knot K in S^3 is called an L-space knot if it admits a positive Dehn surgery yielding an L-space. It is known that non-torus alternating knots are not L-space knots, see [**OS**]. In view of the L-space conjecture, this would imply that any non-trivial Dehn surgery along a non-torus alternating knot produces a 3-manifold with left orderable fundamental group.

A rational number r is called a left orderable slope of a knot $K \subset S^3$ if the 3-manifold obtained from S^3 by r-surgery along K has left orderable fundamental group. As mentioned above, one would expect that any rational number is a left orderable slope of any non-torus alternating knot. It is known that any rational number $r \in (-4, 4)$ is a left orderable slope of the figure eight knot, and any rational number $r \in [0, 4]$ is a left orderable slope of the hyperbolic twist knot 5_2, see [**BGW**] and [**HTe2**] respectively. Consider the double twist knot $C(k, l)$ in the Conway notation as in Figure 1, where k, l denote the numbers of horizontal half-twists with sign in the boxes. Here the sign of  is positive in the box k and is negative in the box l. Then the following results were shown in [**HTe1**], [**Tr**] by using continuous families of hyperbolic $\mathrm{SL}_2(\mathbb{R})$-representations of knot groups. If m, n are integers ≥ 1, any rational number $r \in (-4n, 4m)$ is a left orderable slope of $C(2m, 2n)$. If m, n are integers ≥ 2 then any rational number $r \in [0, \max\{4m, 4n\})$ is a left orderable slope of $C(2m, -2n)$ and any rational number $r \in [0, 4]$ is a left orderable slope of both $C(2m, -2)$ and $C(2, -2n)$. Note that $C(2, 2)$ is

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the figure eight knot and $C(4,-2)$ is the twist knot 5_2. Moreover $C(2,-2)$ is the trefoil knot, which is the $(2,3)$-torus knot.

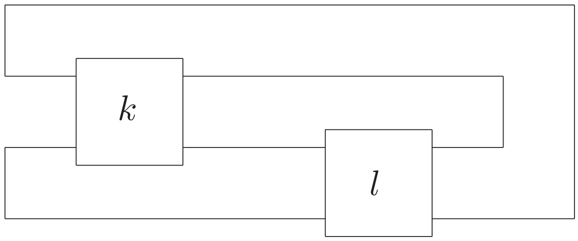

Figure 1. The double twist knot/link $C(k,l)$ in the Conway notation.

In this paper, by using continuous families of elliptic $\mathrm{SL}_2(\mathbb{R})$-representations of knot groups we extend the range of left orderable slopes of $C(2m,-2n)$. Moreover, we also give left orderable slopes of $C(2m+1,\pm 2n)$.

THEOREM 1. *Suppose K is a double twist knot of the form $C(2m,-2n)$, $C(2m+1,2n)$ or $C(2m+1,-2n)$ in the Conway notation for some positive integers m and n. Let*

$$\mathrm{LO}_K = \begin{cases} (-\infty,1) & \text{if } K = C(2m,-2n), \\ (-\infty,2n-1) & \text{if } K = C(2m+1,2n), \\ (3-2n,\infty) & \text{if } K = C(2m+1,-2n) \text{ and } n \geq 2. \end{cases}$$

Then any rational number $r \in \mathrm{LO}_K$ is a left orderable slope of K.

Combining this with results in [**HTe1**], [**Tr**], we conclude that if m and n are integers ≥ 2 then any rational number $r \in (-\infty, \max\{4m,4n\})$ is a left orderable slope of $C(2m,-2n)$ and any rational number $r \in (-\infty,4]$ is a left orderable slope of both $C(2m,-2)$ and $C(2,-2n)$. In the subsequent paper [**KTT**] we will use continuous families of hyperbolic $\mathrm{SL}_2(\mathbb{R})$-representations of knot groups to extend the range of left orderable slopes of $C(2m+1,-2n)$. More specifically, we will show that any rational number $r \in (-4n,4m)$ is a left orderable slope of $C(2m+1,-2n)$ detected by hyperbolic $\mathrm{SL}_2(\mathbb{R})$-representations of the knot group.

We remark that in the case of $C(2m+1,\pm 2n)$, where m and n are positive integers, Gao [**Ga**] independently obtains similar results. She proves a weaker result that any rational number $r \in (-\infty,1)$ is a left orderable slope of $C(2m+1,2n)$ and a stronger result that any rational number $r \in (-4n,\infty)$ is a left orderable slope of $C(2m+1,-2n)$.

As in [**BGW**], [**CD**], [**HTe1**], [**HTe2**], [**Tr**] the proof of Theorem 1 is based on the existence of continuous families of elliptic $\mathrm{SL}_2(\mathbb{R})$-representations of the knot groups of double twist knots $C(2m,-2n)$ and $C(2m+1,\pm 2n)$ into $\mathrm{SL}_2(\mathbb{R})$ and the fact that $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, which is the universal covering group of $\mathrm{SL}_2(\mathbb{R})$, is a left orderable group.

This paper is organized as follows. In Section 1, we study certain real roots of the Riley polynomial of double twist knots $C(k,-2p)$, whose zero locus describes all non-abelian representations of the knot group into $\mathrm{SL}_2(\mathbb{C})$. In Section 2, we prove Theorem 1.

2. Real roots of the Riley polynomial.

For a knot K in S^3, let $G(K)$ denote the knot group of K which is the fundamental group of the complement of an open tubular neighborhood of K.

Consider the double twist knot/link $C(k,l)$ in the Conway notation as in Figure 1, where k,l are integers such that $|kl| \geq 3$. Note that $C(k,l)$ is the rational knot/link corresponding to continued fraction $k+1/l$. It is easy to see that $C(k,l)$ is the mirror image of $C(l,k) = C(-k,-l)$. Moreover, $C(k,l)$ is a knot if kl is even and is a two-component link if kl is odd. In this paper, we only consider knots and so we can assume that $k>0$ and $l=-2p$ is even.

Note that $C(k,-2p)$ is the mirror image of the double twist knot $J(k,2p)$ in [**HS**]. Then, by [**HS**], the knot group of $C(k,-2p)$ has a presentation

$$G(C(k,-2p)) = \langle a, b \mid aw^p = w^p b \rangle$$

where a,b are meridians and

$$w = \begin{cases} (ab^{-1})^m (a^{-1}b)^m & \text{if } \ k = 2m, \\ (ab^{-1})^m ab (a^{-1}b)^m & \text{if } \ k = 2m+1. \end{cases}$$

Moreover, the canonical longitude of $C(k,-2p)$ corresponding to the meridian $\mu = a$ is $\lambda = (w^p (w^p)^* a^{-2\varepsilon})^{-1}$, where $\varepsilon = 0$ if $k=2m$ and $\varepsilon = 2p$ if $k = 2m+1$. Here, for a word u in the letters a,b we let u^* be the word obtained by reading v backwards.

Suppose $\rho : G(C(k,-2p)) \to \mathrm{SL}_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} M & 0 \\ 2-y & M^{-1} \end{bmatrix} \tag{2.1}$$

where $(M,y) \in \mathbb{C}^2$ satisfies the matrix equation $\rho(aw^p) = \rho(w^p b)$. It is known that this matrix equation is equivalent to a single polynomial equation $R_{C(k,-2p)}(x,y) = 0$, where $x = (\operatorname{tr} \rho(a))^2$ and $R_K(x,y)$ is the Riley polynomial of K, see [**Ri**]. This polynomial can be described via the Chebychev polynomials as follows.

Let $\{S_j(v)\}_{j \in \mathbb{Z}}$ be the Chebychev polynomials in the variable v defined by $S_0(v) = 1$, $S_1(v) = v$ and $S_j(v) = vS_{j-1}(v) - S_{j-2}(v)$ for all integers j. Note that $S_j(v) = -S_{-j-2}(v)$ and $S_j(\pm 2) = (\pm 1)^j (j+1)$. Moreover, we have $S_j(v) = (s^{j+1} - s^{-(j+1)})/(s - s^{-1})$ for $v = s + s^{-1} \neq \pm 2$. Using this identity one can prove the following.

Lemma 2.1. *For any integer j and any positive integer n we have*

(1) $S_j^2(v) - vS_j(v)S_{j-1}(v) + S_{j-1}^2(v) = 1$.

(2) $S_n(v) - S_{n-1}(v) = \displaystyle\prod_{j=1}^{n} \Big(v - 2\cos \frac{(2j-1)\pi}{2n+1} \Big)$.

(3) $S_n(v) + S_{n-1}(v) = \displaystyle\prod_{j=1}^{n} \Big(v - 2\cos \frac{2j\pi}{2n+1} \Big)$.

(4) $S_n(v) = \prod_{j=1}^{n}\Big(v - 2\cos\frac{j\pi}{n+1}\Big)$.

The Riley polynomial of $C(k,-2p)$, whose zero locus describes all non-abelian representations of the knot group of $C(k,-2p)$ into $\mathrm{SL}_2(\mathbb{C})$, is

$$R_{C(k,-2p)}(x,y) = S_p(t) - zS_{p-1}(t)$$

where

$$t = \operatorname{tr}\rho(w) = \begin{cases} 2 + (y+2-x)(y-2)S^2_{m-1}(y) & \text{if } k = 2m,\\ 2 - (y+2-x)(S_m(y) - S_{m-1}(y))^2 & \text{if } k = 2m+1,\end{cases}$$

and

$$z = \begin{cases} 1 + (y+2-x)S_{m-1}(y)(S_m(y) - S_{m-1}(y)) & \text{if } k = 2m,\\ 1 - (y+2-x)S_m(y)(S_m(y) - S_{m-1}(y)) & \text{if } k = 2m+1.\end{cases}$$

Moreover, for the representation $\rho : G(C(k,-2p)) \to \mathrm{SL}_2(\mathbb{C})$ of the form (2.1) the image of the canonical longitude $\lambda = (w^p(w^p)^* a^{-2\varepsilon})^{-1}$ has the form $\rho(\lambda) = \left[\begin{smallmatrix} L & * \\ 0 & L^{-1}\end{smallmatrix}\right]$, where

$$L = -\frac{M^{-1}(S_m(y) - S_{m-1}(y)) - M(S_{m-1}(y) - S_{m-2}(y))}{M(S_m(y) - S_{m-1}(y)) - M^{-1}(S_{m-1}(y) - S_{m-2}(y))} \quad \text{if } k = 2m$$

and

$$L = -M^{4p}\frac{M^{-1}S_m(y) - MS_{m-1}(y)}{MS_m(y) - M^{-1}S_{m-1}(y)} \quad \text{if } k = 2m+1.$$

See e.g. [**Pe**], [**Tr**].

Lemmas (2.2)–(2.4) below describe continuous families of real roots of the Riley polynomials of the double twist knots $C(2m,-2n)$, $C(2m+1,2n)$ and $C(2m+1,-2n)$ respectively, where m and n are positive integers.

LEMMA 2.2. *There exists a continuous real function $y : [4 - 1/(mn), 4] \to [2,\infty)$ in the variable x such that*

- *$y(4 - 1/(mn)) = 2$ and*
- *$R_{C(2m,-2n)}(x, y(x)) = 0$ for all $x \in [4 - 1/(mn), 4]$.*

PROOF. Let $K = C(2m,-2n)$. We have $R_K(x,y) = S_n(t) - zS_{n-1}(t)$ where

$$t = 2 + (y+2-x)(y-2)S^2_{m-1}(y),$$
$$z = 1 + (y+2-x)S_{m-1}(y)(S_m(y) - S_{m-1}(y)).$$

Consider real numbers $x \in [4 - 1/(mn), 4]$ and $y \in [2,\infty)$. Since $y \geq 2 \geq x - 2$, we have $t \geq 2$ and $z \geq 1$. This implies that $zS_{n-1}(t) - S_{n-2}(t) \geq S_{n-1}(t) - S_{n-2}(t) > 0$, by Lemma 2.1. The equation $R_K(x,y) = 0$ is then equivalent to

$$\big(S_n(t) - zS_{n-1}(t)\big)\big(S_{n-2}(t) - zS_{n-1}(t)\big) = 0. \tag{2.2}$$

Let $P(x,y)$ denote the left hand side of equation (2.2). By Lemma 2.1, we have $S_n^2(t) - tS_n(t)S_{n-1}(t) + S_{n-1}^2(t) = 1$. This can be written as $S_n(t)S_{n-2}(t) = S_{n-1}^2(t) - 1$. From this and $S_n(t) + S_{n-2}(t) = tS_{n-1}(t)$ we get

$$P(x,y) = (z^2 - tz + 1)S_{n-1}^2(t) - 1.$$

By a direct calculation, using $S_m^2(y) + S_{m-1}^2(y) - yS_m(y)S_{m-1}(y) = 1$, we have

$$\begin{aligned}
&z^2 - tz + 1 \\
&\quad = (z-1)^2 - (t-2)z \\
&\quad = (y+2-x)^2 S_{m-1}^2(y)(S_m(y) - S_{m-1}(y))^2 \\
&\qquad - (y+2-x)(y-2)S_{m-1}^2(y)\big[1 + (y+2-x)S_{m-1}(y)(S_m(y) - S_{m-1}(y))\big] \\
&\quad = (y+2-x)S_{m-1}^2(y)\big[4 - x + (y+2-x)(y-2)S_{m-1}^2(y)\big] \\
&\quad = (y+2-x)S_{m-1}^2(y)(t+2-x).
\end{aligned}$$

Hence $P(x,y) = (y+2-x)S_{m-1}^2(y)(t+2-x)S_{n-1}^2(t) - 1$.

By Lemma 2.1(4), for any positive integer l the Chebychev polynomial $S_l(v) = \prod_{j=1}^{l}(v - 2\cos(j\pi/(l+1)))$ is a strictly increasing function in $v \in [2,\infty)$. This implies that, for a fixed real number $x \in [4 - 1/(mn), 4]$, the polynomials $t = 2 + (y+2-x)(y-2)S_{m-1}^2(y) \geq 2$ and $P(x,y) = (y+2-x)S_{m-1}^2(y)(t+2-x)S_{n-1}^2(t) - 1$ are strictly increasing functions in $y \in [2,\infty)$. Note that $\lim_{y\to\infty} P(x,y) = \infty$ and

$$\lim_{y\to 2^+} P(x,y) = P(x,2) = (4-x)^2 m^2 n^2 - 1 \leq 0.$$

Hence there exists a unique real number $y(x) \in [2,\infty)$ such that $P(x,y(x)) = 0$. Since $P(4 - 1/mn, 2) = 0$ we have $y(4 - 1/mn) = 2$. Finally, by the implicit function theorem $y = y(x)$ is a continuous function in $x \in [4 - 1/(mn), 4]$. $\square$

LEMMA 2.3. *There exists a continuous real function $x : [2,\infty) \to (4\cos^2((2n-1)\pi/(4n+2)), \infty)$ in the variable y such that*

- $x(2) < 4\cos^2 \dfrac{(2n-2)\pi}{4n+2}$,
- $\lim_{y\to\infty} x(y) = \infty$ *and*
- $R_{C(2m+1,2n)}(x(y), y) = 0$ *for all* $y \in [2,\infty)$.

PROOF. Let $K = C(2m+1, 2n)$. We have $R_K(x,y) = S_{-n}(t) - zS_{-n-1}(t)$ where

$$\begin{aligned}
t &= 2 - (y+2-x)(S_m(y) - S_{m-1}(y))^2, \\
z &= 1 - (y+2-x)S_m(y)(S_m(y) - S_{m-1}(y)).
\end{aligned}$$

Note that $R_K(x,y) = (t-z)S_{-n-1}(t) - S_{-n-2}(t) = S_n(t) - (t-z)S_{n-1}(t)$.

By Lemma 2.1 we have

$$S_n(t) - S_{n-1}(t) = \prod_{j=1}^{n}\Big(t - 2\cos\frac{(2j-1)\pi}{2n+1}\Big),$$
$$S_n(t) + S_{n-1}(t) = \prod_{j=1}^{n}\Big(t - 2\cos\frac{2j\pi}{2n+1}\Big).$$

Let $t_j = 2\cos(j\pi/(2n+1))$ for $j = 1, \dots, 2n$. By writing $t_{2j-1} = e^{i\theta} + e^{-i\theta}$ where $\theta = (2j-1)\pi/(2n+1)$, we have

$$\begin{aligned} S_n(t_{2j-1}) &= \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin\big((2j-1)(n+1)\pi/(2n+1)\big)}{\sin\big((2j-1)\pi/(2n+1)\big)} \\ &= \frac{\sin\big(j\pi - \pi/2 + (2j-1)\pi/2(2n+1)\big)}{\sin(2j-1)\pi/(2n+1)} = (-1)^{j-1}\frac{\cos\big((2j-1)\pi/2(2n+1)\big)}{\sin\big((2j-1)\pi/(2n+1)\big)}. \end{aligned}$$

This implies that $(-1)^{j-1}S_n(t_{2j-1}) > 0$. Similarly, $(-1)^j S_n(t_{2j}) > 0$.

Fix a real number $y \geq 2$. Let $s_j(y) = y + 2 - (2 - t_j)/(S_m(y) - S_{m-1}(y))^2$ for $j = 1, \dots, 2n$. We also let $s_0 = y + 2$. Since $-2 < t_{2n} < \cdots < t_1 < 2$ we have $s_{2n}(y) < \cdots < s_1(y) < y + 2 = s_0(y)$. At $x = s_{2j-1}(y)$ we have $t = t_{2j-1}$ and so $S_n(t) = S_{n-1}(t)$. This implies that

$$\begin{aligned} R_K(s_{2j-1}(y), y) &= (1 - (t - z))S_n(t_{2j-1}) \\ &= -(y + 2 - s_{2j-1}(y))S_{m-1}(y)(S_m(y) - S_{m-1}(y))S_n(t_{2j-1}). \end{aligned}$$

Since $y \geq 2$, by Lemma 2.1 we have $S_m(y) - S_{m-1}(y) \geq S_m(2) - S_{m-1}(2) = 1$ and $S_{m-1}(y) \geq S_{m-1}(2) = m$. Hence $(-1)^j R_K(s_{2j-1}(y), y) > 0$.

Similarly, for $1 \leq j \leq n$ we have

$$\begin{aligned} R_K(s_{2j}(y), y) &= (1 + t - z)S_n(t_{2j}) \\ &= \big[2 + (y + 2 - s_{2j-1}(y))S_{m-1}(y)(S_m(y) - S_{m-1}(y))\big]S_n(t_{2j}), \end{aligned}$$

which implies that $(-1)^j R_K(s_{2j}(y), y) > 0$.

For each $1 \leq j \leq n - 1$, since

$$R_K(s_{2j+1}(y), y)R_K(s_{2j}(y), y) < 0$$

there exists $x_j(y) \in (s_{2j+1}(y), s_{2j}(y))$ such that $R_K(x_j(y), y) = 0$. Since

$$R_K(s_0(y), y) = R_K(y + 2, y) = 1$$

and $R_K(s_1(y), y) < 0$ there exists $x_0(y) \in (s_1(y), s_0(y))$ such that $R_K(x_0(y), y) = 0$.

Since $R_K(x, y) = zS_{n-1}(t) - S_{n-2}(t)$, we see that $R_K(x, y)$ is a polynomial of degree n in x for each fixed real number $y \geq 2$. This polynomial has exactly n simple real roots $x_0(y), \dots, x_{n-1}(y)$ satisfying $x_{n-1}(y) < \cdots < x_0(y) < y + 2$, hence the implicit function theorem implies that each $x_j(y)$ is a continuous function in $y \geq 2$.

By letting $x(y) = x_{n-1}(y)$ for $y \geq 2$, we have $R_K(x(y), y) = 0$. Moreover, since

$$x(y) > s_{2n-1}(y) = y + 2 - \frac{2 - 2\cos\big((2n-1)\pi/(2n+1)\big)}{(S_m(y) - S_{m-1}(y))^2}$$

we have $\lim_{y\to\infty} x(y) = \infty$ and $x(y) > 4 - (2 - 2\cos((2n-1)\pi/(2n+1))) = 4\cos^2((2n-1)\pi/(4n+2))$ for $y \geq 2$.

Finally, since $x(y) < s_{2n-2}(y)$ for all $y \geq 2$ we have $x(2) < s_{2n-2}(2) = 4\cos^2((2n-2)\pi/(4n+2))$. □

LEMMA 2.4. *Suppose $n \geq 2$. Then there exists a continuous real function $x : [2,\infty) \to (4\cos^2((2n-1)\pi/(4n+2)), \infty)$ in the variable y such that*

- $x(2) < 4\cos^2 \dfrac{(2n-3)\pi}{4n+2}$,
- $\lim_{y\to\infty} x(y) = \infty$ *and*
- $R_{C(2m+1,-2n)}(x(y), y) = 0$ *for all* $y \in [2,\infty)$.

PROOF. Let $K = C(2m+1, -2n)$. We have $R_K(x,y) = S_n(t) - zS_{n-1}(t)$ where

$$t = 2 - (y + 2 - x)(S_m(y) - S_{m-1}(y))^2,$$
$$z = 1 - (y + 2 - x)S_m(y)(S_m(y) - S_{m-1}(y)).$$

Fix a real number $y \geq 2$. Choose t_j and $s_j(y)$ for $1 \leq j \leq 2n$ as in Lemma 2.3. Since

$$\begin{aligned} R_K(s_{2j-1}(y), y) &= (1 - z)S_n(t_{2j-1}) \\ &= (y + 2 - s_{2j-1}(y))S_m(y)(S_m(y) - S_{m-1}(y))S_n(t_{2j-1}), \end{aligned}$$

we have $(-1)^{j-1}R_K(s_{2j-1}(y), y) > 0$. Hence, there exists $x_j(y) \in (s_{2j+1}(y), s_{2j-1}(y))$ such that $R_K(x_j(y), y) = 0$ for each $1 \leq j \leq n-1$.

By writing $R_K(x,y) = (t - z)S_{n-1}(t) - S_{n-2}(t)$ and noting that

$$t - z = 1 + (y + 2 - x)(S_m(y) - S_{m-1}(y))S_{m-1}(y),$$

we see that $R_K(x,y)$ is a polynomial of degree n in x with negative highest coefficient for each fixed real number $y \geq 2$. Since $\lim_{x\to\infty} R_K(x,y) = -\infty$ and $R_K(y+2, y) = 1$, there exists $x_0(y) \in (y+2, \infty)$ such that $R_K(x_0(y), y) = 0$. For a fixed real number $y \geq 2$, the polynomial $R_K(x,y)$ of degree n in x has exactly n simple real roots $x_0(y), \dots, x_{n-1}(y)$ satisfying $x_{n-1}(y) < \cdots < x_1(y) < y + 2 < x_0(y)$, hence the implicit function theorem implies that each $x_j(y)$ is a continuous function in $y \geq 2$.

By letting $x(y) = x_{n-1}(y)$ for $y \geq 2$, we have $R_K(x(y), y) = 0$. Moreover, since

$$x(y) > s_{2n-1}(y) = y + 2 - \frac{2 - 2\cos\big((2n-1)\pi/(2n+1)\big)}{(S_m(y) - S_{m-1}(y))^2}$$

we have $\lim_{y\to\infty} x(y) = \infty$ and $x(y) > 4 - (2 - 2\cos((2n-1)\pi/(2n+1))) = 4\cos^2((2n-1)\pi/(4n+2))$ for $y \geq 2$.

Finally, since $x(y) < s_{2n-3}(y)$ for all $y \geq 2$ we have $x(2) < s_{2n-3}(2) = 4\cos^2((2n-3)\pi/(4n+2))$. □

3. Proof of Theorem 1.

Suppose K is a double twist knot of the form $C(2m,-2n)$, $C(2m+1,2n)$ or $C(2m+1,-2n)$ in the Conway notation for some positive integers m and n. Let X be the complement of an open tubular neighborhood of K in S^3, and X_r the 3-manifold obtained from S^3 by r-surgery along K. Recall that

$$\mathrm{LO}_K = \begin{cases} (-\infty,1) & \text{if } K = C(2m,-2n), \\ (-\infty,2n-1) & \text{if } K = C(2m+1,2n), \\ (3-2n,\infty) & \text{if } K = C(2m+1,-2n) \text{ and } n \geq 2. \end{cases}$$

An element of $\mathrm{SL}_2(\mathbb{R})$ is called elliptic if its trace is a real number in $(-2,2)$. A representation $\rho : \mathbb{Z}^2 \to \mathrm{SL}_2(\mathbb{R})$ is called elliptic if the image group $\rho(\mathbb{Z}^2)$ contains an elliptic element of $\mathrm{SL}_2(\mathbb{R})$. In which case, since $\mathbb{Z}^2$ is an abelian group every non-trivial element of $\rho(\mathbb{Z}^2)$ must also be elliptic.

Using Lemmas 2.2–2.4 we first prove the following.

Proposition 3.1. *For each rational number $r \in \mathrm{LO}_K \setminus \{0\}$ there exists a representation $\rho : \pi_1(X_r) \to \mathrm{SL}_2(\mathbb{R})$ such that $\rho\big|_{\pi_1(\partial X)} : \pi_1(\partial X) \cong \mathbb{Z}^2 \to \mathrm{SL}_2(\mathbb{R})$ is an elliptic representation.*

Proof. We first consider the case $K = C(2m,-2n)$. Let $\theta_0 = \arccos\sqrt{1-1/(4mn)}$. For $\theta \in (0,\theta_0) \cup (\pi-\theta_0,\pi)$ we let $x = 4\cos^2\theta$. Then $x \in (4-1/(mn),4)$. Consider the continuous real function

$$y : [4-1/(mn),4] \to [2,\infty)$$

in Lemma 2.2. Let $M = e^{i\theta}$. Then $x = 4\cos^2\theta = (M+M^{-1})^2$. Since $R_K(x,y(x)) = 0$ there exists a non-abelian representation $\rho : \pi_1(X) \to \mathrm{SL}_2(\mathbb{C})$ such that

$$\rho(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} M & 0 \\ 2-y(x) & M^{-1} \end{bmatrix}.$$

Note that x is the square of the trace of a meridian. Moreover, the image of the canonical longitude λ corresponding to the meridian $\mu = a$ has the form $\rho(\lambda) = \left[\begin{smallmatrix} L & * \\ 0 & L^{-1} \end{smallmatrix}\right]$, where

$$L = -\frac{M^{-1}\alpha - M\beta}{M\alpha - M^{-1}\beta}$$

and $\alpha = S_m(y(x)) - S_{m-1}(y(x))$, $\beta = S_{m-1}(y(x)) - S_{m-2}(y(x))$. Note that $\alpha > \beta > 0$, since $y(x) > 2$.

It is easy to see that $|L| = \sqrt{L\bar{L}} = 1$, where $\bar{L}$ denotes the complex conjugate of L. Moreover, by a direct calculation, we have

$$\mathrm{Re}(L) = \big(2\alpha\beta - (\alpha^2+\beta^2)\cos 2\theta\big)/|M\alpha - M^{-1}\beta|^2,$$
$$\mathrm{Im}(L) = (\alpha^2-\beta^2)\sin 2\theta/|M\alpha - M^{-1}\beta|^2.$$

Note that $\mathrm{Im}(L) > 0$ if $\theta \in (0,\theta_0)$ and $\mathrm{Im}(L) < 0$ if $\theta \in (\pi - \theta_0, \pi)$. Let

$$\varphi(\theta) = \begin{cases} \arccos\left[\big(2\alpha\beta - (\alpha^2+\beta^2)\cos 2\theta\big) \,/\, \left|e^{i\theta}\alpha - e^{-i\theta}\beta\right|^2\right] & \text{if } \theta \in (0,\theta_0), \\ -\arccos\left[\big(2\alpha\beta - (\alpha^2+\beta^2)\cos 2\theta\big) \,/\, \left|e^{i\theta}\alpha - e^{-i\theta}\beta\right|^2\right] & \text{if } \theta \in (\pi-\theta_0,\pi). \end{cases}$$

Then $L = e^{i\varphi(\theta)}$. Note that $\varphi(\theta) \in (0,\pi)$ if $\theta \in (0,\theta_0)$ and $\varphi(\theta) \in (-\pi, 0)$ if $\theta \in (\pi - \theta_0, \pi)$.

The function $f(\theta) := -\varphi(\theta)/\theta$ is a continuous function on each of the intervals $(0,\theta_0)$ and $(\pi-\theta_0,\pi)$. As $\theta \to 0^+$ we have $M \to 1$ and $L = -(M^{-1}\alpha - M\beta)/(M\alpha - M^{-1}\beta) \to -1$, so $\varphi(\theta) \to \pi$. As $\theta \to \theta_0^-$ we have $x \to 4 - 1/(mn)$, $y(x) \to 2$ and $\alpha, \beta \to 1$, so $L = -(M^{-1}\alpha - M\beta)/(M\alpha - M^{-1}\beta) \to 1$ and $\varphi(\theta) \to 0$. This implies that

$$\lim_{\theta \to 0^+} -\frac{\varphi(\theta)}{\theta} = -\infty \qquad \text{and} \qquad \lim_{\theta \to \theta_0^-} -\frac{\varphi(\theta)}{\theta} = 0.$$

Hence the image of $f(\theta)$ on the interval $(0,\theta_0)$ contains the interval $(-\infty, 0)$.

Similarly, since

$$\lim_{\theta \to (\pi-\theta_0)^+} -\frac{\varphi(\theta)}{\theta} = 0 \qquad \text{and} \qquad \lim_{\theta \to \pi^-} -\frac{\varphi(\theta)}{\theta} = 1,$$

the image of $f(\theta)$ on the interval $(\pi - \theta_0, \pi)$ contains the interval $(0,1)$.

Suppose $r = p/q$ is a rational number such that $r \in (-\infty, 0) \cup (0,1)$. Then $r = f(\theta) = -\varphi(\theta)/\theta$ for some $\theta \in (0,\theta_0) \cup (\pi - \theta_0, \pi)$. Since $M^p L^q = e^{i(p\theta + q\varphi(\theta))} = 1$, we have $\rho(\mu^p \lambda^q) = I$. This means that the non-abelian representation $\rho : \pi_1(X) \to \mathrm{SL}_2(\mathbb{C})$ extends to a representation $\rho : \pi_1(X_r) \to \mathrm{SL}_2(\mathbb{C})$. Finally, since $2 - y(x) < 0$, a result in [**Kh**, p.786] implies that ρ can be conjugated to an $\mathrm{SL}_2(\mathbb{R})$-representation. Note that the restriction of this representation to the peripheral subgroup $\pi_1(\partial X)$ of the knot group is an elliptic representation. This completes the proof of Proposition 3.1 for $K = C(2m, -2n)$.

We now consider the case $K = C(2m+1, 2n)$. Consider the continuous real function

$$x : [2, \infty) \to \left(4\cos^2 \frac{(2n-1)\pi}{4n+2}, \infty\right)$$

in Lemma 2.3. Since $x(2) < 4\cos^2((2n-2)\pi/(4n+2))$ and $\lim_{y\to\infty} x(y) = \infty$, there exists $y^* > 2$ such that $x(y^*) = 4$ and $4\cos^2((2n-1)\pi/(4n+2)) < x(y) < 4$ for all $y \in [2, y^*)$.

For each $y \in [2, y^*)$ we let $\theta(y) = \arccos(\sqrt{x(y)}/2)$. Then $\theta(2) > (2n-2)\pi/(4n+2)$, and for $y \in [2, y^*)$ we have $0 < \theta(y) < (2n-1)\pi/(4n+2)$ and $x(y) = 4\cos^2\theta(y)$. Since $R_K(x(y), y) = 0$ there exists a non-abelian representation $\rho : \pi_1(X) \to \mathrm{SL}_2(\mathbb{C})$ such that

$$\rho(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} M & 0 \\ 2-y & M^{-1} \end{bmatrix},$$

where $M = e^{i\theta(y)}$. Moreover, the image of the canonical longitude λ corresponding to the meridian $\mu = a$ has the form $\rho(\lambda) = \left[\begin{smallmatrix} L & * \\ 0 & L^{-1} \end{smallmatrix}\right]$, where

$$L = -M^{-4n}\frac{M^{-1}\gamma - M\delta}{M\gamma - M^{-1}\delta}$$

and $\gamma = S_m(y)$, $\delta = S_{m-1}(y)$. Note that $\gamma > \delta > 0$, since $y > 2$.

As in the previous case, we write $L = e^{i\varphi(y)}$ where

$$\varphi(y) = (2n-2)\pi - 4n\theta(y) + \arccos\left[\left(2\gamma\delta - (\gamma^2+\delta^2)\cos 2\theta(y)\right) / \left|e^{i\theta(y)}\gamma - e^{-i\theta(y)}\delta\right|^2\right].$$

Since $(2n-2)\pi/(4n+2) < \theta(2) < (2n-1)\pi/(4n+2)$ we have $-2\pi/(2n+1) < \varphi(2) < 2\pi - 3\pi/(2n+1)$.

As $y \to 2^+$, ρ approaches a reducible representation and so $L \to 1$, $\varphi(y) \to \varphi(2) = k2\pi$ for some integer k. Since $-2\pi/(2n+1) < \varphi(2) < 2\pi - 3\pi/(2n+1)$, we must have $\varphi(2) = 0$. As $y \to (y^*)^-$, we have $x(y) \to 4$, $M \to 1$, $L = -M^{-4n}(M^{-1}\gamma - M\delta)/(M\gamma - M^{-1}\delta) \to -1$ and hence $\theta(y) \to 0^+$, $\varphi(y) \to (2l-1)\pi$ for some integer l. Since

$$\begin{aligned}(2l-1)\pi &= \lim_{y\to(y^*)^-} (2n-2)\pi - 4n\theta(y) \\ &\quad + \arccos\left[\left(2\gamma\delta - (\gamma^2+\delta^2)\cos 2\theta(y)\right) / \left|e^{i\theta(y)}\gamma - e^{-i\theta(y)}\delta\right|^2\right] \\ &= \lim_{y\to(y^*)^-} (2n-2)\pi \\ &\quad + \arccos\left[\left(2\gamma\delta - (\gamma^2+\delta^2)\cos 2\theta(y)\right) / \left|e^{i\theta(y)}\gamma - e^{-i\theta(y)}\delta\right|^2\right],\end{aligned}$$

we have $(2n-2)\pi \le (2l-1)\pi \le (2n-1)\pi$. This implies that $2l-1 = 2n-1$ and $\varphi(y) \to (2n-1)\pi$ as $y \to (y^*)^-$. Hence the image of $g(y) := -\varphi(y)/\theta(y)$ on the interval $(2, y^*)$ contains the interval $(-\infty, 0)$.

Similarly, with $\theta_1(y) = \pi - \theta(y)$ we have $x(y) = 4\cos^2(\theta_1(y))$ and hence for each $y \in [2, y^*)$ there exists a non-abelian representation $\rho_1 : \pi_1(X) \to \mathrm{SL}_2(\mathbb{C})$ such that

$$\rho_1(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho_1(b) = \begin{bmatrix} M & 0 \\ 2-y & M^{-1} \end{bmatrix},$$

where $M = e^{i\theta_1(y)}$. Moreover, the image of the canonical longitude λ corresponding to the meridian $\mu = a$ has the form $\rho_1(\lambda) = \left[\begin{smallmatrix} L & * \\ 0 & L^{-1} \end{smallmatrix}\right]$, where $L = e^{i\varphi_1(y)}$ and

$$\begin{aligned}\varphi_1(y) &= -(2n-2)\pi + 4n\pi - 4n\theta_1(y) \\ &\quad - \arccos\left[\left(2\gamma\delta - (\gamma^2+\delta^2)\cos 2\theta_1(y)\right) / \left|e^{i\theta_1(y)}\gamma - e^{-i\theta_1(y)}\delta\right|^2\right] \\ &= -(2n-2)\pi + 4n\theta(y) \\ &\quad - \arccos\left[\left(2\gamma\delta - (\gamma^2+\delta^2)\cos 2\theta_1(y)\right) / \left|e^{i\theta_1(y)}\gamma - e^{-i\theta_1(y)}\delta\right|^2\right].\end{aligned}$$

Since $(2n-2)\pi/(4n+2) < \theta(2) < (2n-1)\pi/(4n+2)$ we have $-2\pi + 3\pi/(2n+1) < \varphi_1(2) < 2\pi/(2n+1)$.

As $y \to 2^+$, ρ_1 approaches a reducible representation and so $L \to 1$, $\varphi_1(y) \to 0$. As $y \to (y^*)^-$, we have $x(y) \to 4$, $M \to -1$, $L = -M^{-4n}(M^{-1}\gamma - M\delta)/(M\gamma - M^{-1}\delta) \to -1$ and hence $\theta_1(y) \to \pi$, $\varphi_1(y) \to -(2n-1)\pi$. This implies that the image of $g_1(y) := -\varphi_1(y)/\theta_1(y)$ on the interval $(2, y^*)$ contains the interval $(0, 2n-1)$.

The rest of the proof of Proposition 3.1 for $C(2m+1, 2n)$ is similar to that for $C(2m, -2n)$.

Lastly, we consider the case $K = C(2m+1, -2n)$ and $n \geq 2$. Consider the continuous real function

$$x : [2, \infty) \to \left(4\cos^2\frac{(2n-1)\pi}{4n+2}, \infty\right)$$

in Lemma 2.4. Since $x(2) < 4\cos^2(2n-3)\pi/(4n+2)$ and $\lim_{y\to\infty} x(y) = \infty$, there exists $y^* > 2$ such that $x(y^*) = 4$ and $4\cos^2(2n-1)\pi/(4n+2) < x(y) < 4$ for all $y \in [2, y^*)$.

For each $y \in [2, y^*)$ we let $\theta(y) = \arccos(\sqrt{x(y)}/2)$. Then $\theta(2) > (2n-3)\pi/(4n+2)$, and for $y \in [2, y^*)$ we have $0 < \theta(y) < (2n-1)\pi/(4n+2)$ and $x(y) = 4\cos^2\theta(y)$. Since $R_K(x(y), y) = 0$ there exists a non-abelian representation $\rho : \pi_1(X) \to \mathrm{SL}_2(\mathbb{C})$ such that

$$\rho(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} M & 0 \\ 2-y & M^{-1} \end{bmatrix},$$

where $M = e^{i\theta(y)}$. Moreover, the image of the canonical longitude λ corresponding to the meridian $\mu = a$ has the form $\rho(\lambda) = \left[\begin{smallmatrix} L & * \\ 0 & L^{-1} \end{smallmatrix}\right]$, where

$$L = -M^{4n}\frac{M^{-1}\gamma - M\delta}{M\gamma - M^{-1}\delta}$$

and $\gamma = S_m(y)$, $\delta = S_{m-1}(y)$. Note that $\gamma > \delta > 0$, since $y > 2$.

As above, we write $L = e^{i\varphi(y)}$ where

$$\varphi(y) = -(2n-2)\pi + 4n\theta(y) + \arccos\left[\left(2\gamma\delta - (\gamma^2+\delta^2)\cos 2\theta(y)\right) / \left|e^{i\theta(y)}\gamma - e^{-i\theta(y)}\delta\right|^2\right].$$

Since $(2n-3)\pi/(4n+2) < \theta(2) < (2n-1)\pi/(4n+2)$ we have $-2\pi + 4\pi/(2n+1) < \varphi(2) < 2\pi - (2n-1)\pi/(2n+1)$.

As $y \to 2^+$, ρ approaches a reducible representation and so $L \to 1$, $\varphi(y) \to \varphi(2) = k2\pi$ for some integer k. Since $-2\pi + 4\pi/(2n+1) < \varphi(2) < 2\pi - (2n-1)\pi/(2n+1)$, we must have $\varphi(2) = 0$.

As $y \to (y^*)^-$, we have $x(y) \to 4$, $M \to 1$, $L = -M^{4n}(M^{-1}\gamma - M\delta)/(M\gamma - M^{-1}\delta) \to -1$ and hence $\theta(y) \to 0^+$, $\varphi(y) \to (2l-1)\pi$ for some integer l. Since

$$\begin{aligned}(2l-1)\pi = &\lim_{y\to(y^*)^-} -(2n-2)\pi + 4n\theta(y) \\ &\quad + \arccos\left[\left(2\gamma\delta - (\gamma^2+\delta^2)\cos 2\theta(y)\right) / \left|e^{i\theta(y)}\gamma - e^{-i\theta(y)}\delta\right|^2\right] \\ = &\lim_{y\to(y^*)^-} -(2n-2)\pi \\ &\quad + \arccos\left[\left(2\gamma\delta - (\gamma^2+\delta^2)\cos 2\theta(y)\right) / \left|e^{i\theta(y)}\gamma - e^{-i\theta(y)}\delta\right|^2\right],\end{aligned}$$

we have $-(2n-2)\pi \le (2l-1)\pi \le -(2n-3)\pi$. This implies that $2l-1 = -(2n-3)$ and $\varphi(y) \to -(2n-3)\pi$ as $y \to (y^*)^-$. Hence the image of $h(y) := -\varphi(y)/\theta(y)$ on the interval $(2, y^*)$ contains the interval $(0, \infty)$.

Similarly, with $\theta_1(y) = \pi - \theta(y)$ we have $x(y) = 4\cos^2(\theta_1(y))$ and hence for each $y \in [2, y^*)$ there exists a non-abelian representation $\rho_1 : \pi_1(X) \to \mathrm{SL}_2(\mathbb{C})$ such that

$$\rho_1(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho_1(b) = \begin{bmatrix} M & 0 \\ 2-y & M^{-1} \end{bmatrix},$$

where $M = e^{i\theta_1(y)}$. Moreover, the image of the canonical longitude λ corresponding to the meridian $\mu = a$ has the form $\rho_1(\lambda) = \left[\begin{smallmatrix} L & * \\ 0 & L^{-1} \end{smallmatrix}\right]$, where $L = e^{i\varphi_1(y)}$ and

$$\begin{aligned} \varphi_1(y) &= (2n-2)\pi - 4n\pi + 4n\theta_1(y) \\ &\qquad - \arccos\Big[\big(2\gamma\delta - (\gamma^2+\delta^2)\cos 2\theta_1(y)\big) \,/\, \big|e^{i\theta_1(y)}\gamma - e^{-i\theta_1(y)}\delta\big|^2\Big] \\ &= (2n-2)\pi - 4n\theta(y) \\ &\qquad - \arccos\Big[\big(2\gamma\delta - (\gamma^2+\delta^2)\cos 2\theta_1(y)\big) \,/\, \big|e^{i\theta_1(y)}\gamma - e^{-i\theta_1(y)}\delta\big|^2\Big]. \end{aligned}$$

Since $(2n-3)\pi/(4n+2) < \theta(2) < (2n-1)\pi/(4n+2)$ we have $-2\pi + (2n-1)\pi/(2n+1) < \varphi_1(2) < 2\pi - 4\pi/(2n+1)$.

As $y \to 2^+$, ρ_1 approaches a reducible representation and so $L \to 1$, $\varphi_1(y) \to \varphi_1(2) = 0$. As $y \to (y^*)^-$, we have $x(y) \to 4$, $M \to -1$, $L = -M^{4n}(M^{-1}\gamma - M\delta)/(M\gamma - M^{-1}\delta) \to -1$ and hence $\theta_1(y) \to \pi$, $\varphi_1(y) \to (2n-3)\pi$. This implies that the image of $h_1(y) := -\varphi_1(y)/\theta_1(y)$ on the interval $(2, y^*)$ contains the interval $(-(2n-3), 0)$.

The rest of the proof of Proposition 3.1 for $C(2m+1, -2n)$ is similar to that for $C(2m, -2n)$. □

We now finish the proof of Theorem 1. Suppose r is a rational number such that $r \in \mathrm{LO}_K$. If $r \neq 0$, by Proposition 3.1, there exists a representation $\rho : \pi_1(X_r) \to \mathrm{SL}_2(\mathbb{R})$ such that $\rho\big|_{\pi_1(\partial X)}$ is an elliptic representation. This representation lifts to a representation $\tilde{\rho} : \pi_1(X_r) \to \widetilde{\mathrm{SL}_2(\mathbb{R})}$, where $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ is the universal covering group of $\mathrm{SL}_2(\mathbb{R})$. See e.g. [**CD**, Section 3.5] and [**Va**, Section 2.2]. Note that X_r is an irreducible 3-manifold (by [**HTh**]) and $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ is a left orderable group (by [**Be**]). Hence, by [**BRW**], $\pi_1(X_r)$ is a left orderable group. Finally, 0-surgery along a knot always produces a prime manifold whose first Betti number is 1, and by [**BRW**] such manifold has left orderable fundamental group.

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