# Simpliciality of strongly convex problems

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**Abstract.** A multiobjective optimization problem is $C^r$ simplicial if the Pareto set and the Pareto front are $C^r$ diffeomorphic to a simplex and, under the $C^r$ diffeomorphisms, each face of the simplex corresponds to the Pareto set and the Pareto front of a subproblem, where $0 \leq r \leq \infty$. In the paper titled "Topology of Pareto sets of strongly convex problems", it has been shown that a strongly convex $C^r$ problem is $C^{r-1}$ simplicial under a mild assumption on the ranks of the differentials of the mapping for $2 \leq r \leq \infty$. On the other hand, in this paper, we show that a strongly convex $C^1$ problem is $C^0$ simplicial under the same assumption. Moreover, we establish a specialized transversality theorem on generic linear perturbations of a strongly convex $C^r$ mapping ($r \geq 2$). By the transversality theorem, we also give an application of singularity theory to a strongly convex $C^r$ problem for $2 \leq r \leq \infty$.

## 1. Introduction.

In this paper, $m$ and $n$ are positive integers, and we denote the index set $\{1, \ldots, m\}$ by $M$.

We consider the problem of optimizing several functions simultaneously. More precisely, let $f : X \to \mathbb{R}^m$ be a mapping, where $X$ is a given arbitrary set. A point $x \in X$ is called a *Pareto optimum* of $f$ if there does not exist another point $y \in X$ such that $f_i(y) \leq f_i(x)$ for all $i \in M$ and $f_j(y) < f_j(x)$ for at least one index $j \in M$. We denote the set consisting of all Pareto optimums of $f$ by $X^*(f)$, which is called the *Pareto set* of $f$. The set $f(X^*(f))$ is called the *Pareto front* of $f$. The problem of determining $X^*(f)$ is called the *problem of minimizing $f$*.

Let $f = (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ be a mapping, where $X$ is a given arbitrary set. For a non-empty subset $I = \{i_1, \ldots, i_k\}$ of $M$ such that $i_1 < \cdots < i_k$, set

$$f_I = (f_{i_1}, \ldots, f_{i_k}).$$

The problem of determining $X^*(f_I)$ is called a *subproblem* of the problem of minimizing $f$. Set

$$\Delta^{m-1} = \left\{ (w_1, \ldots, w_m) \in \mathbb{R}^m \,\middle|\, \sum_{i=1}^m w_i = 1, \ w_i \geq 0 \right\}.$$

We also denote a face of $\Delta^{m-1}$ for a non-empty subset $I$ of $M$ by

$$\Delta_I = \big\{ (w_1, \ldots, w_m) \in \Delta^{m-1} \,\big|\, w_i = 0 \ (i \notin I) \big\}.$$

For a $C^r$ manifold $N$ (possibly with corners) and a subset $V$ of $\mathbb{R}^\ell$, a mapping $g : N \to V$ is called a $C^r$ *mapping* (resp., a $C^r$ *diffeomorphism*) if $g : N \to \mathbb{R}^\ell$ is of class $C^r$ (resp., $g : N \to \mathbb{R}^\ell$ is a $C^r$ immersion and $g : N \to V$ is a homeomorphism), where $r \geq 1$. In this paper, $C^0$ mappings and $C^0$ diffeomorphisms are continuous mappings and homeomorphisms, respectively.

By referring to [**2**], we give the definition of (weakly) simplicial problems in this paper.

DEFINITION 1.　Let $f = (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ be a mapping, where $X$ is a subset of $\mathbb{R}^n$. The problem of minimizing $f$ is $C^r$ *simplicial* if there exists a $C^r$ mapping $\Phi : \Delta^{m-1} \to X^*(f)$ such that both the mappings $\Phi|_{\Delta_I} : \Delta_I \to X^*(f_I)$ and $f|_{X^*(f_I)} : X^*(f_I) \to f(X^*(f_I))$ are $C^r$ diffeomorphisms for any non-empty subset $I$ of $M$, where $0 \leq r \leq \infty$. The problem of minimizing $f$ is $C^r$ *weakly simplicial*[1] if there exists a $C^r$ mapping $\phi : \Delta^{m-1} \to X^*(f)$ such that $\phi(\Delta_I) = X^*(f_I)$ for any non-empty subset $I$ of $M$, where $0 \leq r \leq \infty$.

As described in [**2**], simpliciality is an important property, which can be seen in several practical problems ranging from facility location studied half a century ago [**7**] to sparse modeling actively developed today [**2**]. If a problem is simplicial, then we can efficiently compute a parametric-surface approximation of the entire Pareto set with few sample points [**6**].

A subset $X$ of $\mathbb{R}^n$ is *convex* if $tx + (1-t)y \in X$ for all $x, y \in X$ and all $t \in [0,1]$. Let $X$ be a convex set in $\mathbb{R}^n$. A function $f : X \to \mathbb{R}$ is *strongly convex* if there exists $\alpha > 0$ such that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{1}{2}\alpha t(1-t) \|x - y\|^2$$

for all $x, y \in X$ and all $t \in [0,1]$, where $\|z\|$ is the Euclidean norm of $z \in \mathbb{R}^n$. The constant $\alpha$ is called a *convexity parameter* of the function $f$. A mapping $f = (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ is *strongly convex* if $f_i$ is strongly convex for any $i \in M$. The problem of minimizing a strongly convex $C^r$ mapping is called the *strongly convex $C^r$ problem*.

In [**2**], we have the following result for the simpliciality of strongly convex $C^r$ problems, where $2 \leq r \leq \infty$.

THEOREM 1 ([**2**]).　*Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a strongly convex $C^r$ mapping, where $2 \leq r \leq \infty$. Then, the problem of minimizing $f$ is $C^{r-1}$ simplicial if the rank of the differential $df_x$ is equal to $m-1$ for any $x \in X^*(f)$.*

We give the following remark on Theorem 1.

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[1]In [**2**], the problem of minimizing $f : X \to \mathbb{R}^m$ is said to be $C^r$ *weakly simplicial* if there exists a $C^r$ mapping $\phi : \Delta^{m-1} \to f(X^*(f))$ satisfying $\phi(\Delta_I) = f(X^*(f_I))$ for any non-empty subset $I$ of $M$. On the other hand, a surjective mapping of $\Delta^{m-1}$ into $X^*(f)$ is important to describe $X^*(f)$. Hence, the definition of weak simpliciality in this paper is updated from that in [**2**].

REMARK 1. It is shown that if we remove the assumption on the rank of $df_x$ in Theorem 1, then the problem becomes $C^{r-1}$ weakly simplicial in the sense of [**2**] (for the definition of weak simpliciality in the sense of [**2**], see also Footnote 1 in this paper). In this paper, we show that the problem becomes $C^{r-1}$ weakly simplicial in the sense of Definition 1 (for the result, see Theorem 5 in Section 7.1).

As in [**2**], the assumption $r \geq 2$ is essentially used in the proof of Theorem 1. It is difficult to apply the same method as in the proof of Theorem 1 to strongly convex $C^1$ mappings. Hence, as the first purpose of this paper, we give a theorem in the case $r = 1$ as follows:

THEOREM 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a strongly convex $C^1$ mapping. Then, the problem of minimizing $f$ is $C^0$ weakly simplicial. Moreover, this problem is $C^0$ simplicial if the rank of the differential $df_x$ is equal to $m - 1$ for any $x \in X^*(f)$.*

In [**2**], as an application of singularity theory to a strongly convex problem, we have the following result (Theorem 3) on generic linear perturbations of a strongly convex $C^r$ mapping ($2 \leq r \leq \infty$). Here, note that strong convexity is preserved under linear perturbations (see Lemma 14 in Section 5). Let $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ be the space consisting of all linear mappings of $\mathbb{R}^n$ into $\mathbb{R}^m$. In what follows we will regard $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ as the Euclidean space $(\mathbb{R}^n)^m$ in the obvious way.

THEOREM 3 ([**2**]). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ ($n \geq m$) be a strongly convex $C^r$ mapping, where $2 \leq r \leq \infty$. If $n - 2m + 4 > 0$, then there exists a Lebesgue measure zero subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) - \Sigma$, the problem of minimizing $f + \pi : \mathbb{R}^n \to \mathbb{R}^m$ is $C^{r-1}$ simplicial.*

In Theorem 3, in order to make a given strongly convex $C^r$ problem simplicial, linear perturbations of all functions $f_1, \ldots, f_m$ are considered, where $f_1, \ldots, f_m$ are the components of $f$. On the other hand, as the second purpose of this paper, we show that it is sufficient to consider linear perturbations of only $m - 1$ functions (see Theorem 4).

Let $s$ be an arbitrary integer satisfying $1 \leq s \leq m$. Set

$$\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)_s = \big\{ (\pi_1, \ldots, \pi_m) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \,\big|\, \pi_s = 0 \big\}.$$

THEOREM 4. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ ($n \geq m$) be a strongly convex $C^r$ mapping, where $2 \leq r \leq \infty$. Let $s$ be an arbitrary integer satisfying $1 \leq s \leq m$. If $n - 2m + 4 > 0$, then there exists a Lebesgue measure zero subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)_s$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)_s - \Sigma$, the problem of minimizing $f + \pi : \mathbb{R}^n \to \mathbb{R}^m$ is $C^{r-1}$ simplicial.*

In this paper, in order to prove Theorem 4, we also give a specialized transversality theorem on generic linear perturbations of a strongly convex mapping (see Proposition 2 in Section 5). Hence, Theorem 4 is also an application of singularity theory to a strongly convex problem.

The remainder of this paper is organized as follows. In Section 2, some examples of (weakly) simplicial problems and remarks on Theorems 2 and 4 are presented. By lemmas prepared in Section 3, we prove Theorem 2 in Section 4. Moreover, in Section 5, preliminaries for the proof of Theorem 4 are given, where the specialized transversality

theorem (Proposition 2) is shown. By the transversality theorem, we show Theorem 4 in Section 6. Section 7 is an appendix for Remark 1 and Lemma 1 (for Lemma 1, see Section 2).

## 2.   Examples of (weakly) simplicial problems and remarks on Theorems 2 and 4.

First, we give some examples of (weakly) simplicial problems. In order to show given mappings are strongly convex, we prepare Lemma 1, which is a well-known result. For the sake of readers' convenience, the proof of Lemma 1 is given in Section 7.2.

Let $X$ be a convex subset of $\mathbb{R}^n$. A function $f : X \to \mathbb{R}$ is said to be *convex* if

$$f(tx + (1 - t)y) \le tf(x) + (1 - t)f(y)$$

for all $x, y \in X$ and all $t \in [0, 1]$.

LEMMA 1.    *Let $X$ be a convex subset of $\mathbb{R}^n$. Then, a function $f : X \to \mathbb{R}$ is strongly convex with a convexity parameter $\alpha > 0$ if and only if the function $g : X \to \mathbb{R}$ defined by $g(x) = f(x) - (\alpha/2)\|x\|^2$ is convex.*

EXAMPLE 1.    Let $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be the mapping defined by

$$f_1(x_1, x_2, x_3) = a(x_1 - 1)^2 + x_2^2 + x_3^2 \quad (a > 0),$$
$$f_2(x_1, x_2, x_3) = x_1^2 + (x_2 - 1)^2 + x_3^2,$$
$$f_3(x_1, x_2, x_3) = x_1^2 + x_2^2 + (x_3 - 1)^2.$$

First, we show that $f$ is strongly convex.

Let $\widetilde{f} : \mathbb{R}^3 \to \mathbb{R}$ be the mapping defined by $\widetilde{f}(x) = \sum_{i=1}^{3} c_i(x_i - p_i)^2$, where $c_i > 0$ for any $i = 1, 2, 3$, $x = (x_1, x_2, x_3)$ and $(p_1, p_2, p_3) \in \mathbb{R}^3$. Set $\alpha = \min\{c_1, c_2, c_3\}$ and $g(x) = \widetilde{f}(x) - (\alpha/2)\|x\|^2$. Then, we have

$$g(x) = \sum_{i=1}^{3} \left( \left( c_i - \frac{\alpha}{2} \right) x_i^2 - 2c_i p_i x_i + c_i p_i^2 \right).$$

Since $c_i - \alpha/2 > 0$ for all $i = 1, 2, 3$, the function $g$ is convex. Therefore, $\widetilde{f}$ is a strongly convex function with a convexity parameter $\alpha$ by Lemma 1.

Since $\widetilde{f}$ is strongly convex, $f$ is also strongly convex for all $a > 0$. Since rank $df_x \ge 2$ for any $x \in \mathbb{R}^3$ and $a > 0$, the problem of minimizing $f$ is $C^\infty$ simplicial for any $a > 0$ by Theorem 1 (see Figure 1). With the parameter $a$, the shapes of the Pareto set and the Pareto front change while the simpliciality is maintained. If $a = 1$, the Pareto set is a triangle as shown in Figure 1 (b). If $a = 4$ or $a = 1/4$, the Pareto set is a curved triangle as shown in Figures 1 (c) and 1 (d). For the precise description of $X^*(f)$, see Remark 7 in Section 4.
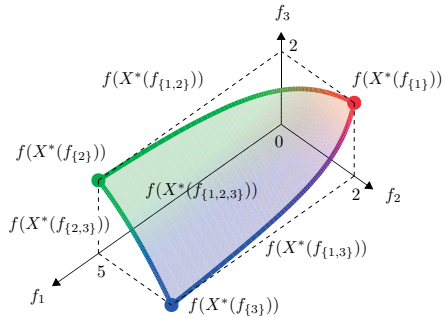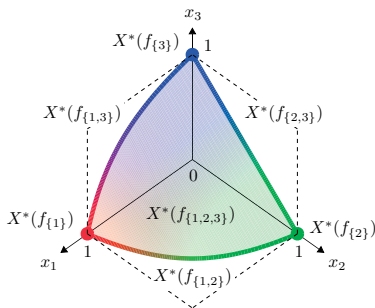
In Example 2, we give a simple example of a strongly convex $C^1$ mapping which is not of class $C^2$.
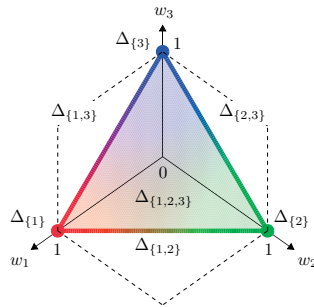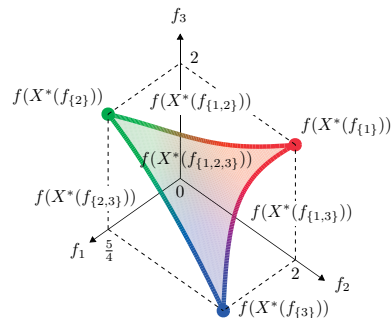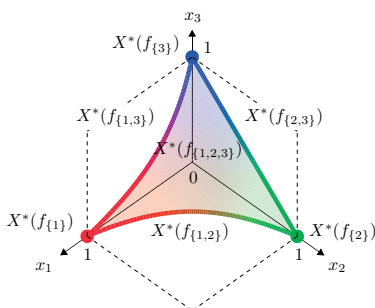
(a) Simplex $\Delta^2$.
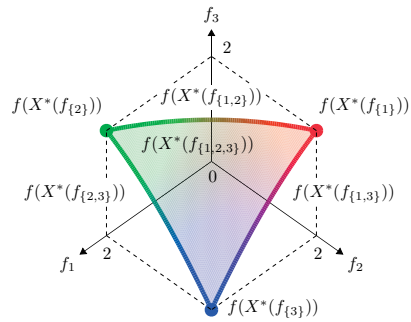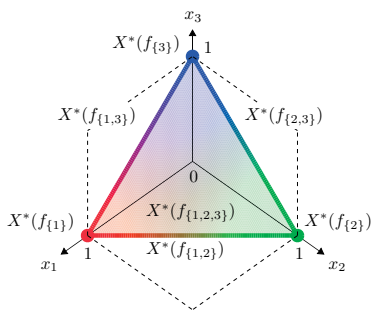


(b) Pareto set (left) and Pareto front (right) of $f$ with $a = 1$.



(c) Pareto set (left) and Pareto front (right) of $f$ with $a = 4$.



(d) Pareto set (left) and Pareto front (right) of $f$ with $a = 1/4$.

Figure 1. Example 1 with $a = 1, 4, 1/4$.

Example 2.   Let $f = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2$ be the mapping defined by

$$f_1(x) = (x - 2)^2,$$

$$f_2(x) = \begin{cases} x^2 & \text{if } x < 1, \\ x^2 + (x - 1)^2 & \text{if } x \geq 1. \end{cases}$$

Let $g_i : \mathbb{R} \to \mathbb{R}$ be the function defined by $g_i(x) = f_i(x) - (2/2)x^2$, where $i = 1, 2$. Since $g_1$ and $g_2$ are convex, $f_1$ and $f_2$ are strongly convex functions with a convexity parameter 2 by Lemma 1, respectively. Hence, $f$ is strongly convex. Since $f_2$ is not of class $C^2$, we cannot apply Theorem 1 to $f$. However, since $f$ is of class $C^1$, we can apply Theorem 2. Since rank $df_x = 1$ for any $x \in \mathbb{R}$, the problem of minimizing $f$ is $C^0$ simplicial by Theorem 2.

Remark 2.   We give the following remarks on Theorem 2.

1. Note that (strict) convexity of a mapping does not necessarily imply that the problem is $C^0$ simplicial. For example, the problem of minimizing $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$ does not have a Pareto optimum (i.e. a minimizer). Thus, it is not $C^0$ simplicial although $f$ is strictly convex.

2. We give an example such that Theorem 2 does not hold without the rank assumption. Let $f = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2$ be the mapping defined by $f(x) = (x^2, x^2)$. By Lemma 1, the mapping $f$ is strongly convex. Since $0 \in \mathbb{R}$ is a Pareto optimum and rank $df_0 = 0$, the mapping $f$ does not satisfy the rank assumption in Theorem 2. Since $X^*(f) = \{0\}$, the problem of minimizing $f$ is not $C^0$ simplicial.

Remark 3.   We give a remark on Theorem 4. Let $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be the mapping defined by $f_i(x) = \|x\|^2$ for any integer $i$ $(1 \leq i \leq 3)$. By Lemma 1, the mapping $f$ is strongly convex. In order to make the problem of minimizing $f$ simplicial by generic linear perturbations, it is necessary to perturb at least two components of $f$.

First, we consider the case without linear perturbations. Since $f_1$, $f_2$ and $f_3$ have the unique minimizer $0 \in \mathbb{R}^3$, we have $X^*(f) = \{0\}$. Hence, the problem of minimizing $f$ is not $C^0$ simplicial.

Next, we linearly perturb only one component $f_{s_1}$ of $f$, where $s_1$, $s_2$ and $s_3$ are three elements satisfying $\{s_1, s_2, s_3\} = \{1, 2, 3\}$. Set

$$\mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)_{(s_2, s_3)} = \left\{ (\pi_1, \pi_2, \pi_3) \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3) \,\middle|\, \pi_{s_2} = \pi_{s_3} = 0 \right\}.$$

Let $\pi = (\pi_1, \pi_2, \pi_3)$ be an arbitrary element of $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)_{(s_2, s_3)}$. Since

$$(f_{s_2} + \pi_{s_2})(x) = (f_{s_3} + \pi_{s_3})(x) = \|x\|^2,$$

the origin $0 \in \mathbb{R}^3$ is the unique minimizer of $f_{s_2} + \pi_{s_2}$ and $f_{s_3} + \pi_{s_3}$. Since $f_{s_1} + \pi_{s_1}$ is a distance-squared function, $f_{s_1} + \pi_{s_1}$ has a unique minimizer. Let $p \in \mathbb{R}^3$ be the unique minimizer. Then, it is not hard to see that

$$X^*(f + \pi) = \left\{ tp \in \mathbb{R}^3 \,\middle|\, t \in [0, 1] \right\}.$$

Therefore, the problem of minimizing $f + \pi$ is not $C^0$ simplicial.

Finally, we consider linear perturbations of two components of $f$. Let $s$ be an arbitrary integer satisfying $1 \leq s \leq 3$. By Theorem 4, there exists a Lebesgue measure zero subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)_s$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)_s - \Sigma$, the problem of minimizing $f + \pi : \mathbb{R}^3 \to \mathbb{R}^3$ is $C^\infty$ simplicial.

## 3. Preliminaries for the proof of Theorem 2.

In this section, we prepare some lemmas for the proof of Theorem 2.

Let $f : U \to \mathbb{R}^m$ be a $C^1$ mapping, where $U$ is a non-empty open subset of $\mathbb{R}^n$. A point $x \in U$ is called a *critical point* of $f$ if rank $df_x < m$. We denote the set consisting of all critical points of $f$ by $C(f)$. The following lemma gives a relationship between critical points and Pareto optimums.

LEMMA 2. *Let $f : U \to \mathbb{R}^m$ be a $C^1$ mapping, where $U$ is a non-empty open subset of $\mathbb{R}^n$. Then, $X^*(f) \subset C(f)$.*

PROOF OF LEMMA 2. In the case $n < m$, since $C(f) = U$, Lemma 2 clearly holds. Next, we consider the case $n \geq m$. Suppose that there exists $x \in X^*(f)$ such that $x \notin C(f)$. Since $x \notin C(f)$, there exists an open neighborhood $U_x$ of $x$ such that $f(U_x)$ is an open neighborhood of $f(x)$ by the implicit function theorem. This contradicts $x \in X^*(f)$. □

We give the following two lemmas (Lemmas 3 and 4) in [**8**].

LEMMA 3 ([**8**, Theorem 3.1.3 in Part II (p.79)]). *Let $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a (not necessarily continuous) mapping and let $(w_1, \ldots, w_m) \in \Delta^{m-1}$. If $x \in \mathbb{R}^n$ is the unique minimizer of the function $\sum_{i=1}^m w_i f_i$, then $x \in X^*(f)$.*

The following is a special case of the Karush–Kuhn–Tucker necessary condition for Pareto optimality.

LEMMA 4 ([**8**, Theorem 3.1.5 in Part I (p.39)]). *Let $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a $C^1$ mapping. If $x \in X^*(f)$, then there exists an element $(w_1, \ldots, w_m) \in \Delta^{m-1}$ satisfying $\sum_{i=1}^m w_i (df_i)_x = 0$.*

Now, we prepare the following four lemmas (Lemmas 5 to 8) on strongly convex mappings.

LEMMA 5 ([**9**, Theorem 2.2.6 (p.85)]). *A strongly convex $C^1$ function $f : \mathbb{R}^n \to \mathbb{R}$ has a unique minimizer.*

LEMMA 6 ([**9**, Theorem 2.1.9 (p.64)]). *A $C^1$ function $f : \mathbb{R}^n \to \mathbb{R}$ is strongly convex with a convexity parameter $\alpha > 0$ if and only if*

$$f(x) + df_x \cdot (y - x) + \frac{\alpha}{2} \|y - x\|^2 \leq f(y)$$

*for any $x, y \in \mathbb{R}^n$.*

Lemma 7 ([**9**, Lemma 2.1.4 (p.64)]). *Let $f_i : \mathbb{R}^n \to \mathbb{R}$ be a strongly convex $C^1$ function with a convexity parameter $\alpha_i > 0$, where $i$ is a positive integer $(1 \le i \le m)$. Then, for any $w = (w_1, \ldots, w_m) \in \Delta^{m-1}$, the function $\sum_{i=1}^m w_i f_i : \mathbb{R}^n \to \mathbb{R}$ is a strongly convex $C^1$ function with a convexity parameter $\sum_{i=1}^m w_i \alpha_i$.*

Lemma 8 ([**3**]). *Let $f : X \to \mathbb{R}^m$ be a strongly convex (not necessarily continuous) mapping, where $X$ is a convex subset of $\mathbb{R}^n$. Then, $f|_{X^*(f)} : X^*(f) \to \mathbb{R}^m$ is injective.*

In order to give the last lemma (Lemma 12) in this section, which is essentially used in the proof of Theorem 2, we prepare the following three lemmas (Lemmas 9 to 11).

Let $f : X \to \mathbb{R}^m$ be a mapping, where $X$ is a given arbitrary set. A point $x \in X$ is called a *weakly Pareto optimum* of $f$ if there does not exist another point $y \in X$ such that $f_i(y) < f_i(x)$ for all $i \in M$. Then, by $X^{\mathrm{w}}(f)$, we denote the set consisting of all weakly Pareto optimums of $f$.

Lemma 9 ([**3**]). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a strongly convex (not necessarily continuous) mapping. Then, we have $X^*(f) = X^{\mathrm{w}}(f)$.*

Lemma 10. *Let $f : X \to \mathbb{R}^m$ be a continuous mapping, where $X$ is a topological space. Then, $X^{\mathrm{w}}(f)$ is a closed set of $X$.*

Proof of Lemma 10. For the proof, it is sufficient to show that $X - X^{\mathrm{w}}(f)$ is open. Let $x_0 \in X - X^{\mathrm{w}}(f)$ be an arbitrary element. Then, there exists $\widetilde{x}_0 \in X$ such that $f_i(\widetilde{x}_0) < f_i(x_0)$ for any $i \in M$, where $f = (f_1, \ldots, f_m)$. Set

$$O = \big\{ (y_1, \ldots, y_m) \in \mathbb{R}^m \ \big| \ f_i(x_0) - \varepsilon_i < y_i \ \text{ for any } \ i \in M \big\},$$

where

$$\varepsilon_i = \frac{f_i(x_0) - f_i(\widetilde{x}_0)}{2}.$$

Since $f$ is continuous and $O$ is an open neighborhood of $f(x_0)$, the set $f^{-1}(O)$ is an open neighborhood of $x_0$. Since $f^{-1}(O) \subset X - X^{\mathrm{w}}(f)$, the set $X - X^{\mathrm{w}}(f)$ is open in $X$. $\quad\square$

Lemma 11. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a strongly convex $C^1$ mapping. Then, $X^*(f)$ is compact.*

Proof of Lemma 11. By Lemmas 9 and 10, it follows that $X^*(f)$ is closed. Thus, for the proof, it is sufficient to show that $X^*(f)$ is bounded. Let $\alpha_i > 0$ be a convexity parameter of $f_i$, where $f = (f_1, \ldots, f_m)$ and $i \in M$. By Lemma 5, the function $f_i$ has a unique minimizer for any $i \in M$. Let $x_i \in \mathbb{R}^n$ be the unique minimizer of $f_i$. Set

$$\Omega_i = \left\{ x \in \mathbb{R}^n \ \Big| \ f_i(x_i) + \frac{\alpha_i}{2} \|x - x_i\|^2 \le f_i(x_1) \right\}.$$

Since every $\Omega_i$ is compact, $\Omega = \bigcup_{i=1}^m \Omega_i$ is also compact. Hence, in order to show that $X^*(f)$ is bounded, it is sufficient to show that $X^*(f) \subset \Omega$. Suppose that there exists an element $x' \in X^*(f)$ such that $x' \notin \Omega$. Then, it follows that

$$f_i(x_i) + \frac{\alpha_i}{2} \|x' - x_i\|^2 > f_i(x_1) \tag{3.1}$$

for any $i \in M$. Since $(df_i)_{x_i} = 0$ for any $i \in M$, by Lemma 6, we have

$$f_i(x_i) + \frac{\alpha_i}{2} \|x' - x_i\|^2 \leq f_i(x'). \tag{3.2}$$

From (3.1) and (3.2), it follows that $f_i(x') > f_i(x_1)$ for any $i \in M$. This contradicts $x' \in X^*(f)$. $\qquad\square$

Now, we give a mapping from $\Delta^{m-1}$ into $X^*(f)$, which is introduced in [**2**].

Let $w = (w_1, \ldots, w_m) \in \Delta^{m-1}$. Since $\sum_{i=1}^m w_i f_i : \mathbb{R}^n \to \mathbb{R}$ is a strongly convex $C^1$ function by Lemma 7, the function $\sum_{i=1}^m w_i f_i$ has a unique minimizer by Lemma 5. By Lemma 3, this minimizer is contained in $X^*(f)$. Hence, we can define a mapping $x^* : \Delta^{m-1} \to X^*(f)$ as follows:

$$x^*(w) = \arg \min_{x \in \mathbb{R}^n} \left( \sum_{i=1}^m w_i f_i(x) \right), \tag{3.3}$$

where $\arg \min_{x \in \mathbb{R}^n} (\sum_{i=1}^m w_i f_i(x))$ is the minimizer of $\sum_{i=1}^m w_i f_i$.

LEMMA 12. *Let $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a strongly convex $C^1$ mapping. Let $\alpha_i > 0$ be a convexity parameter of $f_i$ and $K_i$ be the maximal value of $F_i : X^*(f) \times X^*(f) \to \mathbb{R}$ defined by $F_i(x, y) = |f_i(x) - f_i(y)|$ for any $i \in M$. Then, for any $w = (w_1, \ldots, w_m), \widetilde{w} = (\widetilde{w}_1, \ldots, \widetilde{w}_m) \in \Delta^{m-1}$, we have that*

$$\|x^*(w) - x^*(\widetilde{w})\| \leq \sqrt{\frac{K_0}{\alpha_0} \sum_{i=1}^m |w_i - \widetilde{w}_i|},$$

*where $\alpha_0 = \min\{\alpha_1, \ldots, \alpha_m\}$ and $K_0 = \max\{K_1, \ldots, K_m\}$.*

REMARK 4. In Lemma 12, the Pareto set $X^*(f)$ is compact by Lemma 11. Hence, for any $i \in M$, the function $F_i$ has the maximal value $K_i$.

PROOF OF LEMMA 12. Let $w, \widetilde{w} \in \Delta^{m-1}$ be arbitrary elements. By Lemma 7, the function $\sum_{i=1}^m w_i f_i : \mathbb{R}^n \to \mathbb{R}$ (resp., $\sum_{i=1}^m \widetilde{w}_i f_i : \mathbb{R}^n \to \mathbb{R}$) is a strongly convex function with a convexity parameter $\sum_{i=1}^m w_i \alpha_i$ (resp., $\sum_{i=1}^m \widetilde{w}_i \alpha_i$). Since $x^*(w)$ (resp., $x^*(\widetilde{w})$) is the minimizer of the function $\sum_{i=1}^m w_i f_i$ (resp., $\sum_{i=1}^m \widetilde{w}_i f_i$), we get $d(\sum_{i=1}^m w_i f_i)_{x^*(w)} = 0$ (resp., $d(\sum_{i=1}^m \widetilde{w}_i f_i)_{x^*(\widetilde{w})} = 0$). Thus, by Lemma 6, we obtain

$$\left( \sum_{i=1}^m w_i f_i \right)(x^*(w)) + \frac{\sum_{i=1}^m w_i \alpha_i}{2} \|x^*(\widetilde{w}) - x^*(w)\|^2 \leq \left( \sum_{i=1}^m w_i f_i \right)(x^*(\widetilde{w})), \tag{3.4}$$

$$\left( \sum_{i=1}^m \widetilde{w}_i f_i \right)(x^*(\widetilde{w})) + \frac{\sum_{i=1}^m \widetilde{w}_i \alpha_i}{2} \|x^*(w) - x^*(\widetilde{w})\|^2 \leq \left( \sum_{i=1}^m \widetilde{w}_i f_i \right)(x^*(w)). \tag{3.5}$$

By (3.4) and (3.5), we get

$$\frac{\sum_{i=1}^m w_i \alpha_i}{2} \|x^*(\widetilde{w}) - x^*(w)\|^2 \le \sum_{i=1}^m w_i \big(f_i(x^*(\widetilde{w})) - f_i(x^*(w))\big), \tag{3.6}$$

$$\frac{\sum_{i=1}^m \widetilde{w}_i \alpha_i}{2} \|x^*(\widetilde{w}) - x^*(w)\|^2 \le \sum_{i=1}^m \widetilde{w}_i \big(f_i(x^*(w)) - f_i(x^*(\widetilde{w}))\big), \tag{3.7}$$

respectively. By (3.6) and (3.7), we have

$$\frac{\sum_{i=1}^m (w_i + \widetilde{w}_i)\alpha_i}{2} \|x^*(\widetilde{w}) - x^*(w)\|^2 \le \sum_{i=1}^m (w_i - \widetilde{w}_i)\big(f_i(x^*(\widetilde{w})) - f_i(x^*(w))\big).$$

By the inequality above and $\sum_{i=1}^m (w_i + \widetilde{w}_i) = 2$, we obtain

$$\alpha_0 \|x^*(\widetilde{w}) - x^*(w)\|^2 \le \sum_{i=1}^m (w_i - \widetilde{w}_i)\big(f_i(x^*(\widetilde{w})) - f_i(x^*(w))\big). \tag{3.8}$$

We also have

$$\sum_{i=1}^m (w_i - \widetilde{w}_i)\big(f_i(x^*(\widetilde{w})) - f_i(x^*(w))\big) \le \sum_{i=1}^m |w_i - \widetilde{w}_i| \, \big|f_i(x^*(\widetilde{w})) - f_i(x^*(w))\big|$$

$$\le \sum_{i=1}^m |w_i - \widetilde{w}_i| K_i.$$

$$\le K_0 \sum_{i=1}^m |w_i - \widetilde{w}_i|.$$

By the inequality above and (3.8), we obtain

$$\alpha_0 \|x^*(w) - x^*(\widetilde{w})\|^2 \le K_0 \sum_{i=1}^m |w_i - \widetilde{w}_i|.$$

Hence, it follows that

$$\|x^*(w) - x^*(\widetilde{w})\| \le \sqrt{\frac{K_0}{\alpha_0} \sum_{i=1}^m |w_i - \widetilde{w}_i|}. \qquad \square$$

## 4. Proof of Theorem 2.

First, we give an essential result for the proof of Theorem 2 as follows (for the definition of $x^* : \Delta^{m-1} \to X^*(f)$ in Proposition 1, see (3.3)).

PROPOSITION 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a strongly convex $C^1$ mapping. Then, the following properties hold.*

(1) *The mapping $x^* : \Delta^{m-1} \to X^*(f)$ is surjective and continuous. Moreover, if $\operatorname{rank} df_x = m - 1$ for any $x \in X^*(f)$, then $x^*$ is a homeomorphism.*

(2) *The mapping $f|_{X^*(f)} : X^*(f) \to \mathbb{R}^m$ is a homeomorphism into the image.*

Thus, Theorem 2 follows from Proposition 1 as follows: let $I = \{i_1, \ldots, i_k\}$ $(i_1 < \cdots < i_k)$ be an arbitrary non-empty subset of $M$ as in Section 1. Since $f_I : \mathbb{R}^n \to \mathbb{R}^k$ is a strongly convex $C^1$ mapping, $x^*|_{\Delta_I} : \Delta_I \to X^*(f_I)$ is surjective and continuous by Proposition 1 (1). Hence, the problem of minimizing $f$ is $C^0$ weakly simplicial. Next, suppose that rank $df_x = m - 1$ for any $x \in X^*(f)$. Since

$$X^*(f_I) = x^*(\Delta_I) \subset x^*(\Delta^{m-1}) = X^*(f),$$

it follows that $\operatorname{rank}(df_I)_x \geq k - 1$ for any $x \in X^*(f_I)$. By Lemma 2, it follows that $\operatorname{rank}(df_I)_x = k - 1$ for any $x \in X^*(f_I)$. Therefore, by Proposition 1 (1), the mapping $x^*|_{\Delta_I} : \Delta_I \to X^*(f_I)$ is a homeomorphism. Since $X^*(f_I) \subset X^*(f)$, the mapping $f|_{X^*(f_I)} : X^*(f_I) \to \mathbb{R}^m$ is a homeomorphism into the image. Thus, the problem of minimizing $f$ is $C^0$ simplicial.

By the argument above, in order to complete the proof of Theorem 2, it is sufficient to show Proposition 1.

PROOF OF PROPOSITION 1 (1). Note that the bijectivity of $x^*$ is shown by the same method as in the proof of [**2**]. For the sake of readers' convenience, we give the proof in this paper.

First, we show that $x^*$ is surjective. Let $x \in X^*(f)$ be an arbitrary point. By Lemma 4, there exists $w = (w_1, \ldots, w_m) \in \Delta^{m-1}$ such that $\sum_{i=1}^m w_i(df_i)_x = 0$. Namely, we get $d(\sum_{i=1}^m w_i f_i)_x = 0$. Since the function $\sum_{i=1}^m w_i f_i$ is strongly convex, the point $x$ is the unique minimizer of $\sum_{i=1}^m w_i f_i$ by Lemma 6. This implies $x^*(w) = x$. Hence, $x^*$ is surjective.

Second, we show that $x^*$ is continuous. Let $\widetilde{w} = (\widetilde{w}_1, \ldots, \widetilde{w}_m) \in \Delta^{m-1}$ be an arbitrary element. For the proof, it is sufficient to show that $x^*$ is continuous at $\widetilde{w}$. Let $\varepsilon$ be an arbitrary positive real number. Then, there exists an open neighborhood $V$ of $\widetilde{w}$ in $\Delta^{m-1}$ satisfying

$$\sqrt{\frac{K_0}{\alpha_0} \sum_{i=1}^m |w_i - \widetilde{w}_i|} < \varepsilon$$

for any $w \in V$, where $K_0$ and $\alpha_0$ are defined in Lemma 12. From Lemma 12, it follows that

$$\|x^*(w) - x^*(\widetilde{w})\| < \varepsilon$$

for any $w \in V$.

Finally, we show that $x^*$ is a homeomorphism if rank $df_x = m - 1$ for any $x \in X^*(f)$. Since $x^*$ is surjective and continuous from a compact space $\Delta^{m-1}$ into a Hausdorff space, for this proof, it is sufficient to show that $x^*$ is injective.

Suppose that $x^*(w) = x^*(\widetilde{w})$, where $w = (w_1, \ldots, w_m)$ and $\widetilde{w} = (\widetilde{w}_1, \ldots, \widetilde{w}_m)$. Since $x^*(w) \in X^*(f)$ is the unique minimizer of $\sum_{i=1}^m w_i f_i$, we have $d(\sum_{i=1}^m w_i f_i)_{x^*(w)} = 0$. Namely, we get

$$(w_1, \ldots, w_m) df_{x^*(w)} = (0, \ldots, 0).$$

By the above argument, we also have $(\widetilde{w}_1, \ldots, \widetilde{w}_m) df_{x^*(\widetilde{w})} = (0, \ldots, 0)$. Since $x^*(w) = x^*(\widetilde{w})$, we obtain

$$(\widetilde{w}_1, \ldots, \widetilde{w}_m) df_{x^*(w)} = (0, \ldots, 0).$$

Since $m = \dim \operatorname{Ker} df_{x^*(w)} + \operatorname{rank} df_{x^*(w)}$ and $\operatorname{rank} df_{x^*(w)} = m - 1$, it follows that $\dim \operatorname{Ker} df_{x^*(w)} = 1$. Since $w, \widetilde{w} \in \operatorname{Ker} df_{x^*(w)} \cap \Delta^{m-1}$, we obtain $w = \widetilde{w}$. $\qquad \square$

PROOF OF PROPOSITION 1 (2). By Proposition 1 (1), $X^*(f) \ (= x^*(\Delta^{m-1}))$ is compact. By Lemma 8, $f|_{X^*(f)} : X^*(f) \to \mathbb{R}^m$ is injective. Since $f|_{X^*(f)} : X^*(f) \to f(X^*(f))$ is a bijective and continuous mapping from a compact space into a Hausdorff space, the mapping $f|_{X^*(f)}$ is a homeomorphism onto the image. $\qquad \square$

Finally, as supplements to this section, we give the following two remarks.

REMARK 5. In Proposition 1 (1), the assumption that $\operatorname{rank} df_x = m - 1$ for any $x \in X^*(f)$ yields $m - 1 \le n$. On the other hand, when $m - 1 > n$, it is impossible that $x^* : \Delta^{m-1} \to X^*(f)(\subset \mathbb{R}^n)$ is a homeomorphism by the invariance of domain theorem. For the invariance of domain theorem, see [4].

REMARK 6. The mapping $x^*$ in Proposition 1 (1) is not necessarily differentiable as follows. Let $f = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2$ be the mapping defined in Example 2 of Section 2. Let $\varphi : [0, 1] \to \Delta^1$ be the diffeomorphism defined by $\varphi(w_1) = (w_1, 1 - w_1)$. Since if $x^*(w_1, w_2) = x$ then $d(w_1 f_1 + w_2 f_2)_x = 0$, we can easily obtain the following:

$$x^* \circ \varphi(w_1) = \begin{cases} 2w_1 & \text{if } 0 \le w_1 < \dfrac{1}{2}, \\ \dfrac{w_1 + 1}{-w_1 + 2} & \text{if } \dfrac{1}{2} \le w_1 \le 1. \end{cases}$$

Since

$$\lim_{h \to +0} \frac{(x^* \circ \varphi)(1/2 + h) - (x^* \circ \varphi)(1/2)}{h} = \frac{4}{3},$$
$$\lim_{h \to -0} \frac{(x^* \circ \varphi)(1/2 + h) - (x^* \circ \varphi)(1/2)}{h} = 2,$$

the mapping $x^* \circ \varphi$ is not differentiable at $w_1 = 1/2$.

REMARK 7. The mapping $x^*$ in Proposition 1 (1) is useful for describing a Pareto set as follows.

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the mapping defined by Example 1. Let $w = (w_1, w_2, w_3) \in \Delta^2$. Since $x^*(w)$ is a minimizer of $\sum_{i=1}^{3} w_i f_i$ by the definition of $x^*$, we have $d(\sum_{i=1}^{3} w_i f_i)_{x^*(w)} = 0$. Thus, by simple calculations, $x^* : \Delta^2 \to X^*(f)$ can be described as follows:

$$x^*(w_1, w_2, w_3) = \left( \frac{aw_1}{aw_1 + (1 - w_1)}, w_2, w_3 \right).$$

Since $x^*(\Delta^2) = X^*(f)$, the Pareto set $X^*(f)$ can be described as follows:

$$X^*(f) = \left\{ \left( \frac{aw_1}{aw_1 + (1 - w_1)}, w_2, w_3 \right) \in \mathbb{R}^3 \ \middle| \ (w_1, w_2, w_3) \in \Delta^2 \right\}.$$

## 5. Preliminaries for the proof of Theorem 4.

In this section, unless otherwise stated, all manifolds are without boundary and assumed to have countable bases.

The purpose of this section is to establish the specialized transversality theorem (Proposition 2) for generically linearly perturbed strongly convex mappings, which is an essential tool for the proof of Theorem 4. First, we prepare the following two lemmas.

LEMMA 13 ([**9**, Theorem 2.1.11 (p.65)]). *Let $U$ be a convex open subset of $\mathbb{R}^n$ ($U \neq \emptyset$). A $C^2$ function $f : U \to \mathbb{R}$ is strongly convex with a convexity parameter $\alpha > 0$ if and only if $m(f)_x \geq \alpha$ for any $x \in U$, where $m(f)_x$ is the minimal eigenvalue of the Hessian matrix of $f$ at $x$.*

LEMMA 14 ([**2**]). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a strongly convex mapping. Then, for any $\pi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, the mapping $f + \pi : \mathbb{R}^n \to \mathbb{R}^m$ is also strongly convex.*

For the statement and the proof of Proposition 2, we prepare some definitions. Let $U$ be a non-empty open set of $\mathbb{R}^n$ and $J^1(U, \mathbb{R}^m)$ be the space of 1-jets of mappings of $U$ into $\mathbb{R}^m$. Then, note that $J^1(U, \mathbb{R}^m)$ is a $C^\infty$ manifold. For a given $C^r$ mapping $f : U \to \mathbb{R}^m$ ($r \geq 2$), the mapping $j^1 f : U \to J^1(U, \mathbb{R}^m)$ is defined by $x \mapsto j^1 f(x)$. Then, notice that $j^1 f : U \to J^1(U, \mathbb{R}^m)$ is of class $C^{r-1}$. Further, set

$$\Sigma^k = \left\{ j^1 f(0) \in J^1(n, m) \ \middle| \ \operatorname{corank} Jf(0) = k \right\},$$

where $J^1(n, m) = \{ j^1 f(0) \mid f : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0) \}$, $\operatorname{corank} Jf(0) = \min\{n, m\} - \operatorname{rank} Jf(0)$ and $k = 1, \ldots, \min\{n, m\}$. Set

$$\Sigma^k(U, \mathbb{R}^m) = U \times \mathbb{R}^m \times \Sigma^k.$$

Then, the set $\Sigma^k(U, \mathbb{R}^m)$ is a submanifold of $J^1(U, \mathbb{R}^m)$ satisfying

$$\operatorname{codim} \Sigma^k(U, \mathbb{R}^m) = \dim J^1(U, \mathbb{R}^m) - \dim \Sigma^k(U, \mathbb{R}^m)$$
$$= (n - v + k)(m - v + k),$$

where $v = \min\{n, m\}$. For details on $j^1 f : U \to J^1(U, \mathbb{R}^m)$, $\Sigma^k$ and $\Sigma^k(U, \mathbb{R}^m)$, see [**1**].

Now, we recall the definition of transversality.

DEFINITION 2. Let $X$ and $Y$ be $C^r$ manifolds, and $Z$ be a $C^r$ submanifold of $Y$ ($r \geq 1$). Let $f : X \to Y$ be a $C^1$ mapping.

1. We say that $f : X \to Y$ is *transverse* to $Z$ at $x \in X$ if $f(x) \notin Z$ or in the case $f(x) \in Z$, the following holds:

$$df_x(T_x X) + T_{f(x)} Z = T_{f(x)} Y.$$

2. We say that $f : X \to Y$ is *transverse* to $Z$ if for any $x \in X$, the mapping $f$ is transverse to $Z$ at $x$.

The following is the basic transversality result, which is a key lemma for the proof of Proposition 2.

LEMMA 15 ([**1**], [**5**]).   *Let $X$, $A$ and $Y$ be $C^r$ manifolds, $Z$ be a $C^r$ submanifold of $Y$ and $\Gamma : X \times A \to Y$ be a $C^r$ mapping. If $r > \max\{\dim X - \text{codim}\, Z, 0\}$ and $\Gamma$ is transverse to $Z$, then there exists a Lebesgue measure zero subset $\Sigma$ of $A$ such that for any $a \in A - \Sigma$, the $C^r$ mapping $\Gamma_a : X \to Y$ is transverse to $Z$, where $\text{codim}\, Z = \dim Y - \dim Z$ and $\Gamma_a(x) = \Gamma(x, a)$, $x \in X$.*

In [**1**], Lemma 15 is shown in the case that all manifolds and mappings are of class $C^\infty$. By the same method, Lemma 15 can be shown (cf. [**5**]).

PROPOSITION 2.   *Let $f : U \to \mathbb{R}^m$ be a strongly convex $C^r$ mapping, where $U$ is a convex open subset of $\mathbb{R}^n$ ($U \neq \emptyset$). Let $s$ be an arbitrary integer satisfying $1 \leq s \leq m$, and $k$ be an arbitrary integer satisfying $1 \leq k \leq \min\{n, m\}$. If*

$$r > \max\{n - \text{codim}\, \Sigma^k(U, \mathbb{R}^m), 0\} + 1,$$

*then there exists a Lebesgue measure zero subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)_s$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)_s - \Sigma$, the mapping $j^1(f + \pi) : U \to J^1(U, \mathbb{R}^m)$ is transverse to $\Sigma^k(U, \mathbb{R}^m)$.*

REMARK 8.   We give an example such that Proposition 2 does not hold without the hypothesis of strong convexity. Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined by $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = x_1^2 + x_2^2$. Note that $f_1$ is not strongly convex by Lemma 1. Let $\pi = (\pi_1, \pi_2) \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)_1$ be an arbitrary element. Then, it follows that $j^1(f + \pi)(p) \in \Sigma^2(\mathbb{R}^2, \mathbb{R}^2)$ and rank $d(j^1(f + \pi))_p \leq 2$, where $p$ is the unique minimizer of $f_2 + \pi_2$. Since $\text{codim}\, \Sigma^2(\mathbb{R}^2, \mathbb{R}^2) = 4$, the mapping $j^1(f + \pi)$ is not transverse to $\Sigma^2(\mathbb{R}^2, \mathbb{R}^2)$.

PROOF OF PROPOSITION 2.   In the case $m = 1$, Proposition 2 clearly holds by Lemma 13.

Hence, we will consider the case $m \geq 2$. For a positive integer $\ell$, we denote the $\ell \times \ell$ unit matrix by $E_\ell$. For simplicity, set

$$A = \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)_s.$$

In order to show Proposition 2, it is sufficient to give the proof in the case $s = 1$.

Let $\Gamma : U \times A \to J^1(U, \mathbb{R}^m)$ be the $C^{r-1}$ mapping defined by

$$\Gamma(x, \pi) = j^1(f + \pi)(x).$$

Note that $r - 1 > \max\{n - \text{codim}\, \Sigma^k(U, \mathbb{R}^m), 0\}$. If $\Gamma$ is transverse to $\Sigma^k(U, \mathbb{R}^m)$, then there exists a Lebesgue measure zero subset $\Sigma$ of $A$ such that for any $\pi \in A - \Sigma$, the mapping $\Gamma_\pi : U \to J^1(U, \mathbb{R}^m)$ is transverse to $\Sigma^k(U, \mathbb{R}^m)$ by Lemma 15, where $\Gamma_\pi(x) = \Gamma(x, \pi)$, $x \in U$. Thus, in order to finish the proof, it is sufficient to show that

$\Gamma$ is transverse to $\Sigma^k(U, \mathbb{R}^m)$. Let $(\widetilde{x}, \widetilde{\pi}) \in U \times A$ be an arbitrary element satisfying $\Gamma(\widetilde{x}, \widetilde{\pi}) \in \Sigma^k(U, \mathbb{R}^m)$. Then, it is sufficient to show that

$$\dim\left( d\Gamma_{(\widetilde{x}, \widetilde{\pi})}\left( T_{(\widetilde{x}, \widetilde{\pi})}(U \times A)\right) + T_{\Gamma(\widetilde{x}, \widetilde{\pi})}\Sigma^k(U, \mathbb{R}^m)\right) = n + m + nm. \qquad (5.1)$$

Let $(a_{ij})_{1 \le i \le m, 1 \le j \le n}$ be a representing matrix of a linear mapping $\pi \in A$. Since $s = 1$, note that $a_{1j} = 0$ for any $j$ $(1 \le j \le n)$. Thus, $f + \pi : U \to \mathbb{R}^m$ is given as follows:

$$(f + \pi)(x) = \left( f_1(x), f_2(x) + \sum_{j=1}^{n} a_{2j}x_j, \ldots, f_m(x) + \sum_{j=1}^{n} a_{mj}x_j \right),$$

where $f = (f_1, \ldots, f_m)$, $x = (x_1, \ldots, x_n)$ and $(a_{21}, \ldots, a_{2n}, \ldots, a_{m1}, \ldots, a_{mn}) \in (\mathbb{R}^n)^{m-1}$.

Hence, the mapping $\Gamma$ is given by

$$\Gamma(x, \pi)$$
$$= \left( x, (f + \pi)(x), \frac{\partial f_1}{\partial x_1}(x), \ldots, \frac{\partial f_1}{\partial x_n}(x), \right.$$
$$\left. \frac{\partial f_2}{\partial x_1}(x) + a_{21}, \ldots, \frac{\partial f_2}{\partial x_n}(x) + a_{2n}, \ldots\ldots, \frac{\partial f_m}{\partial x_1}(x) + a_{m1}, \ldots, \frac{\partial f_m}{\partial x_n}(x) + a_{mn} \right).$$

The Jacobian matrix of $\Gamma$ at $(\widetilde{x}, \widetilde{\pi})$ is as follows:

$$J\Gamma_{(\widetilde{x}, \widetilde{\pi})} = \left( \begin{array}{c|ccc} \begin{array}{c} E_n \\ * \\ H(f_1)_{\widetilde{x}} \end{array} & & 0 & \\ \hline * & E_n & & 0 \\ \vdots & & \ddots & \\ * & 0 & & E_n \end{array} \right),$$

where $H(f_1)_{\widetilde{x}}$ is the Hessian matrix of $f_1$ at $\widetilde{x}$. Notice that there are $m - 1$ copies of $E_n$ in the lower right partition of the above description of $J\Gamma_{(\widetilde{x}, \widetilde{\pi})}$. Since $\Sigma^k(U, \mathbb{R}^m)$ is a sub-bundle of $J^1(U, \mathbb{R}^m)$ with the fiber $\Sigma^k$, in order to show (5.1), it is sufficient to show that the matrix $R$ has rank $n + m + nm$:

$$R = \left( \begin{array}{c|c|ccc} E_{n+m} & * & & 0 & \\ \hline 0 & H(f_1)_{\widetilde{x}} & & 0 & \\ \hline & * & E_n & & 0 \\ 0 & \vdots & & \ddots & \\ & * & 0 & & E_n \end{array} \right).$$

Notice that there are $m - 1$ copies of $E_n$ in the above description of $R$. Note that for any $i$ $(1 \le i \le nm)$, the $(n + m + i)$-th column vector of $R$ coincides with the $i$-th column vector of $J\Gamma_{(\widetilde{x}, \widetilde{\pi})}$. Since $f_1$ is a strongly convex $C^2$ function, we have rank $H(f_1)_{\widetilde{x}} = n$ by Lemma 13. Hence, it follows that rank $R = n + m + nm$. Therefore, we obtain (5.1). $\quad \square$

## 6. Proof of Theorem 4.

Since Theorem 4 clearly holds by combining the following result (Corollary 1) and Theorem 1, in order to show Theorem 4, it is sufficient to prove Corollary 1.

Corollary 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ $(n \geq m)$ be a strongly convex $C^r$ mapping $(r \geq 2)$. Let $s$ be an arbitrary integer satisfying $1 \leq s \leq m$. If $n - 2m + 4 > 0$, then there exists a Lebesgue measure zero subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)_s$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)_s - \Sigma$ and any $x \in \mathbb{R}^n$, we have $\operatorname{rank} d(f + \pi)_x \geq m - 1$.*

Proof of Corollary 1. In the case $m = 1$, Corollary 1 clearly holds. Hence, we consider the case $m \geq 2$. Since $n \geq m$, we have

$$\operatorname{codim} \Sigma^2(\mathbb{R}^n, \mathbb{R}^m) = 2(n - m + 2).$$

Since $n - 2m + 4 > 0$, we also have $\operatorname{codim} \Sigma^2(\mathbb{R}^n, \mathbb{R}^m) > n$.

Let $k$ be an arbitrary integer satisfying $2 \leq k \leq m$. It follows that

$$n - \operatorname{codim} \Sigma^k(\mathbb{R}^n, \mathbb{R}^m) \leq n - \operatorname{codim} \Sigma^2(\mathbb{R}^n, \mathbb{R}^m) < 0. \qquad (6.1)$$

Furthermore, we have

$$r \geq 2 > \max\{n - \operatorname{codim} \Sigma^k(\mathbb{R}^n, \mathbb{R}^m), 0\} + 1.$$

By Proposition 2, there exists a Lebesgue measure zero subset $\Sigma_k$ of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)_s$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)_s - \Sigma_k$, the mapping $j^1(f + \pi)$ is transverse to $\Sigma^k(\mathbb{R}^n, \mathbb{R}^m)$. Set $\Sigma = \bigcup_{k=2}^m \Sigma_k$. Then, $\Sigma$ has Lebesgue measure zero in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)_s$.

Let $\pi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)_s - \Sigma$ and $x \in \mathbb{R}^n$ be arbitrary elements. Suppose $\operatorname{rank} d(f + \pi)_x \leq m - 2$. Then, there exists an integer $k$ $(2 \leq k \leq m)$ satisfying $j^1(f + \pi)(x) \in \Sigma^k(\mathbb{R}^n, \mathbb{R}^m)$. Since the mapping $j^1(f + \pi)$ is transverse to $\Sigma^k(\mathbb{R}^n, \mathbb{R}^m)$, we obtain

$$d(j^1(f + \pi))_x(T_x\mathbb{R}^n) + T_{j^1(f+\pi)(x)}\Sigma^k(\mathbb{R}^n, \mathbb{R}^m) = T_{j^1(f+\pi)(x)}J^1(\mathbb{R}^n, \mathbb{R}^m).$$

This equation implies that

$$\dim d(j^1(f + \pi))_x(T_x\mathbb{R}^n) \geq \operatorname{codim} \Sigma^k(\mathbb{R}^n, \mathbb{R}^m).$$

This contradicts (6.1). □

## 7. Appendix.

### 7.1. On Remark 1.

As described in Remark 1, we show that the problem of minimizing a strongly convex $C^r$ mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ $(2 \leq r \leq \infty)$ becomes $C^{r-1}$ weakly simplicial in the sense of Definition 1 as follows.

Theorem 5. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a strongly convex $C^r$ mapping, where $2 \leq r \leq \infty$. Then, the problem of minimizing $f$ is $C^{r-1}$ weakly simplicial.*

In order to show Theorem 5, we prepare the following result in [**2**].

PROPOSITION 3 ([**2**]). *Let $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a strongly convex $C^r$ mapping $(2 \leq r \leq \infty)$. Then, $x^* : \Delta^{m-1} \to X^*(f)$ is a surjective mapping of class $C^{r-1}$.*

PROOF OF THEOREM 5. Let $I = \{i_1, \ldots, i_k\}$ $(i_1 < \cdots < i_k)$ be an arbitrary non-empty subset of $M$ as in Section 1. Since $f_I : \mathbb{R}^n \to \mathbb{R}^k$ is a strongly convex $C^r$ mapping, $x^*|_{\Delta_I} : \Delta_I \to X^*(f_I)$ is a surjective mapping of class $C^{r-1}$ by Proposition 3, where $2 \leq r \leq \infty$. Hence, the problem of minimizing $f$ is $C^{r-1}$ weakly simplicial. $\square$

### 7.2. Proof of Lemma 1.
In order to show Lemma 1, we prepare the following lemma.

LEMMA 16. *For any $t \in \mathbb{R}$ and any $x, y \in \mathbb{R}^n$, we have*

$$t \|x\|^2 + (1-t) \|y\|^2 - \|tx + (1-t)y\|^2 = t(1-t) \|x-y\|^2 .$$

PROOF OF LEMMA 16. We have

$$
\begin{aligned}
& t \|x\|^2 + (1-t) \|y\|^2 - \|tx + (1-t)y\|^2 \\
&= t \sum_{i=1}^{n} x_i^2 + (1-t) \sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{n} (tx_i + (1-t)y_i)^2 \\
&= t(1-t) \sum_{i=1}^{n} (x_i^2 + y_i^2 - 2x_i y_i) \\
&= t(1-t) \|x-y\|^2 ,
\end{aligned}
$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. $\square$

Now, we will prove Lemma 1. A mapping $f : X \to \mathbb{R}$ is strongly convex with a convexity parameter $\alpha > 0$ if and only if for all $t \in [0, 1]$ and all $x, y \in X$, we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{1}{2}\alpha t(1-t) \|x-y\|^2 . \tag{7.1}$$

By Lemma 16, the inequality (7.1) holds for all $t \in [0, 1]$ and all $x, y \in X$ if and only if we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{1}{2}\alpha \left( t \|x\|^2 + (1-t) \|y\|^2 - \|tx + (1-t)y\|^2 \right), \tag{7.2}$$

for all $t \in [0, 1]$ and all $x, y \in X$. The inequality (7.2) holds for all $t \in [0, 1]$ and all $x, y \in X$ if and only if we have

$$f(tx + (1-t)y) - \frac{1}{2}\alpha \|tx + (1-t)y\|^2 \leq t \left( f(x) - \frac{1}{2}\alpha \|x\|^2 \right) + (1-t) \left( f(y) - \frac{1}{2}\alpha \|y\|^2 \right) \tag{7.3}$$

for all $t \in [0,1]$ and all $x, y \in X$. The inequality (7.3) holds for all $t \in [0,1]$ and all $x, y \in X$ if and only if the function $g : X \to \mathbb{R}$ defined by $g(x) = f(x) - (\alpha/2)\|x\|^2$ is convex. $\qquad\square$

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