

# Equivariant $K$ -theory of toric orbifolds

By Soumen SARKAR and Vikraman UMA

(Received Oct. 20, 2019)

**Abstract.** Toric orbifolds are a topological generalization of projective toric varieties associated to simplicial fans. We introduce some sufficient conditions on the combinatorial data associated to a toric orbifold to ensure the existence of an invariant cell structure on it and call such a toric orbifold *retractable*. In this paper, our main goal is to study equivariant cohomology theories of retractable toric orbifolds. Our results extend the corresponding results on *divisive* weighted projective spaces.

## 1. Introduction.

The notion of a *toric orbifold* was introduced by Davis and Januszkiewicz in [DJ91, Section 7]. It was later studied by Poddar and the first author in [PS10] who call it a “quasitoric orbifold”. Loosely speaking a toric orbifold  $X$  is an orbifold admitting an effective action by a compact torus  $T \cong (\mathbb{S}^1)^n$  with orbit space a simple convex polytope  $Q$ . A toric orbifold can alternately be constructed from  $Q$  and the data encoded by a  $\mathbb{Z}^n$ -valued function  $\lambda$  on the set of codimension-one faces of  $Q$  (see Definition 2.1). In particular, when  $\lambda$  satisfies the  $(*)$  condition in [DJ91, p.423], the toric orbifold is smooth and is called a *toric manifold*, also known as a “quasitoric manifold” (see [BP02, Section 5.2]). A toric manifold is known to admit a canonical  $T$ -invariant cell structure given by means of a height function on  $Q$  [DJ91, Theorem 3.1]. A torus manifold is an even-dimensional manifold acted on by a half-dimensional torus with non-empty fixed point set and some additional orientation data (see [HM03], [MP06]). The orbit space of a torus manifold has a rich combinatorial structure, e.g., it is a manifold with corners provided that the action is locally standard. These are generalizations of toric manifolds. But unlike a toric manifold, in general, a torus manifold does not come equipped with an invariant cellular structure, unless we impose some additional combinatorial conditions on it (see for example [VU06, Lemma 2.2]).

In this paper, we give some sufficient conditions for a possibly singular toric orbifold to admit a  $T$ -invariant cell structure. We do this by using the concept of retraction sequence of the simple polytope  $Q$  which was introduced in [BSS17]. We call a toric orbifold satisfying the above sufficiency condition a *retractable toric orbifold*. Our definition of a retractable toric orbifold was motivated by the invariant cell structure on a *divisive* weighted projective space described by Harada, Holm, Ray and Williams in [HHRW16, Proposition 2.7, Corollary 2.9]. Indeed, the divisive weighted projective

---

2010 *Mathematics Subject Classification.* Primary 57S12; Secondary 19L47, 55N91, 55N22.

*Key Words and Phrases.* toric orbifold, quasitoric orbifold, toric variety, projective toric variety, piecewise polynomial, piecewise Laurent polynomial.

space is a particular example of a retractable toric orbifold which has the simplex as the quotient polytope (see Example 2.9).

Let  $pt$  denote the 1-point space with the trivial  $T$ -action. The equivariant projection  $X \rightarrow pt$  induces the structure of a graded  $E_T^*(pt)$ -algebra structure on  $E_T^*(X)$ , where  $E_T^*$  is any generalized  $T$ -equivariant cohomology theory. We describe  $E_T^*(X)$  as an algebra over  $E_T^*(pt)$  when  $X$  is a retractable toric orbifold (see Corollary 3.3). In particular, we get a description of the  $T$ -equivariant  $K$ -theory ring of  $X$  over  $K_T^*(pt) \cong R(T)[z, z^{-1}]$ , where  $R(T) = K_T^0(pt)$  denotes the ring of finite dimensional complex representations of  $T$  and  $z$  denotes the Bott periodicity element in  $K^{-2}(pt)$ . This result generalizes the corresponding result on a divisive weighted projective space in [HHRW16, Proposition 3.10]. As in [HHRW16], our main tools here are the methods developed by Harada, Henriques and Holm in [HHH05]. For a topological group  $G$ , they prove a GKM-type theorem for the  $G$ -equivariant generalized cohomology theory of a  $G$ -space equipped with a  $G$ -invariant stratification satisfying some additional conditions [HHH05, Section 3]. In Proposition 3.2, we show that a retractable toric orbifold  $X$  has a  $T$ -invariant stratification satisfying these conditions.

In Section 4, we introduce the notion of *piecewise algebra* for a characteristic pair  $(Q, \lambda)$ , see Definition 4.1, in a dual manner to the concept of piecewise algebra associated to a fan (see [HHRW16, Section 4]). In our main result, Theorem 4.2, we prove that the generalized equivariant cohomology theory ring of a retractable toric orbifold is isomorphic as an  $E_T^*(pt)$ -algebra to the corresponding piecewise algebra for  $(Q, \lambda)$ . This result extends the corresponding result for a divisive weighted projective space in [HHRW16, Theorem 5.5] to the larger class of retractable toric orbifolds (see Example 2.10).

In particular, we show that the equivariant integral cohomology ring  $H_T^*(X)$  is isomorphic to the piecewise polynomial functions on  $(Q, \lambda)$ , the equivariant topological  $K$ -theory ring  $K_T^*(X)$  is isomorphic to the piecewise Laurent polynomial functions on  $(Q, \lambda)$ , and the equivariant complex cobordism ring  $MU_T^*(X)$  is isomorphic to piecewise cobordism forms on  $(Q, \lambda)$ .

Note that examples of toric orbifolds include simplicial projective toric varieties which correspond to simplicial polytopal fans. Thus Corollary 3.3 and Theorem 4.2 give generalizations of the corresponding results on the toric variety associated to a smooth polytopal fan in [HHRW16, Theorem 7.1] and [HHRW16, Corollary 7.2] respectively, to toric varieties associated to simplicial polytopal fans which have the underlying structure of a retractable toric orbifold under the action of the compact torus.

We refer to [May96] for the definitions and results on  $T$ -equivariant generalized cohomology theories  $E_T^*$ , [Seg68] for  $T$ -equivariant  $K$ -theory  $K_T^*$  and [tD70] and [Sin01] for  $T$ -equivariant complex cobordism theory  $MU_T^*$ .

The paper is organized as follows. We recall some basics of toric orbifolds, the concept of local groups, and the concept of retraction of simple polytopes in Sections 2.1–2.2, 2.3, and 2.4 respectively. We introduce the notion of “retractable toric orbifolds” in Section 2.5. We discuss GKM-theory of retractable toric orbifolds in Section 3 (see Proposition 3.2 and Corollary 3.3). The concept of “piecewise algebras” for a characteristic pair is introduced in Section 4. We prove our main result in Theorem 4.2.

## 2. Cell-structure of toric orbifolds.

In this section we briefly recall the concept of a *characteristic pair*  $(Q, \lambda)$  from [DJ91] and [PS10], and explain how it is used to construct a *toric orbifold*  $X := X(Q, \lambda)$ . The main goal of this section is to introduce some sufficient conditions on a toric orbifold  $X$  which is singular to have a cell structure. To complete this goal, we recall three additional concepts, namely the *characteristic subspaces* of  $X$ , the *local groups* corresponding to a face and a vertex of it, and the *retraction* of the simple polytope  $Q$ .

### 2.1. Definition of toric orbifolds by construction.

In this subsection, we briefly recall the constructive definition of (quasi)toric orbifolds following [DJ91] and [PS10]. Let  $M$  be a submodule of  $\mathbb{Z}^n \subset \mathbb{R}^n$  over  $\mathbb{Z}$ ,  $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$  and  $T_M := M_{\mathbb{R}}/M$ . Then we have the natural inclusions  $f: M_{\mathbb{R}} \rightarrow \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^n$  and  $f_*: T_M \rightarrow \mathbb{R}^n/M$ . Note that the inclusion  $i: M \rightarrow \mathbb{Z}^n$  induces a group homomorphism

$$i_*: (\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R})/M \rightarrow (\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R})/\mathbb{Z}^n,$$

defined by  $i_*(a + M) = a + \mathbb{Z}^n$ . Then  $\text{Ker}(i_*) \cong \mathbb{Z}^n/M$ . The range space is the  $n$ -dimensional standard torus. We denote this torus by  $T$ . Let  $f_M$  be the composition  $i_* \circ f_*: T_M \rightarrow T$ . If the rank of  $M$  is  $n$ , then the map  $f_M: T_M \rightarrow T$  is a surjective homomorphism with kernel  $G_M = \mathbb{Z}^n/M$ , a finite abelian group.

Let  $Q$  be an  $n$ -dimensional simple convex polytope in  $\mathbb{R}^n$  and

$$\mathcal{F}(Q) = \{F_i : i \in \{1, \dots, d\} = I\}$$

be the codimension-one faces (facets) of  $Q$ .

**DEFINITION 2.1.** A function  $\lambda: \mathcal{F}(Q) \rightarrow \mathbb{Z}^n$  is called a *characteristic function* on  $Q$  if  $\lambda(F_{i_1}), \dots, \lambda(F_{i_k})$  are linearly independent primitive vectors whenever the intersection  $F_{i_1} \cap \dots \cap F_{i_k}$  is nonempty. Then  $\lambda_i := \lambda(F_i)$  is called the *characteristic vector* corresponding to the facet  $F_i$ . The pair  $(Q, \lambda)$  is called a *characteristic pair*.

We remark here that in the above definition it suffices for  $\lambda$  to satisfy the linear independency at each vertex which is an intersection of  $n$  facets. An example of a characteristic function is given in Figure 1.

Let  $F$  be a codimension- $k$  face of  $Q$ . Since  $Q$  is a simple polytope,  $F$  is the unique intersection of  $k$  facets  $F_{i_1}, \dots, F_{i_k}$ . Let  $M(F)$  be the submodule of  $\mathbb{Z}^n$  generated by the characteristic vectors  $\{\lambda_{i_1}, \dots, \lambda_{i_k}\}$ . Then,  $T_{M(F)} = M(F)_{\mathbb{R}}/M(F)$  is a torus of dimension  $k$ . We shall adopt the convention that  $T_{M(Q)} = 1$ . Let

$$T_F = \text{Im}\{f_{M(F)}: T_{M(F)} \rightarrow T\}. \quad (2.1)$$

Define an equivalence relation  $\sim$  on the product  $T \times Q$  by

$$(t, x) \sim (s, y) \text{ if and only if } x = y \text{ and } s^{-1}t \in T_F \quad (2.2)$$

where  $F$  is the smallest face containing  $x$ . The quotient space

$$X(Q, \lambda) = (T \times Q) / \sim$$

has an orbifold structure with a natural  $T$ -action induced by the group operation, see Section 2 in [PS10]. Clearly, the orbit space of  $T$ -action on  $X(Q, \lambda)$  is  $Q$ . Let

$$\pi: X(Q, \lambda) \rightarrow Q \text{ defined by } \pi([p, t]_{\sim}) = p$$

be the orbit map. The space  $X(Q, \lambda)$  is called the *(quasi)toric orbifold* associated to the characteristic pair  $(Q, \lambda)$ .

We note that if, in addition,  $\lambda$  satisfies the Davis and Januszkiewicz's condition  $(*)$  in [DJ91, p.423], namely that the primitive vectors  $\lambda(F_{i_1}), \dots, \lambda(F_{i_k})$  can be extended to form a basis of  $\mathbb{Z}^n$  whenever the intersection  $F_{i_1} \cap \dots \cap F_{i_k}$  is nonempty, then  $X$  is a smooth manifold called a *(quasi)toric manifold*.

After analyzing the orbifold structure of  $X(Q, \lambda)$ , in [PS10, Section 2.2], Poddar and the first author also gave an axiomatic definition of (quasi)toric orbifolds, which generalizes the axiomatic definition of toric manifolds of [DJ91, Section 1]. In [PS10, Section 2], the authors give explicit orbifold charts (in the sense of [ALR07, Section 1.1]) of  $X(Q, \lambda)$ .

**REMARK 2.2.** One can alternately construct a toric orbifold as the quotient of a *moment angle complex* by the action of a torus determined by the characteristic function, see [BP02, Chapter 6] for the arguments.

## 2.2. Invariant and characteristic subspaces.

We now describe certain closed invariant subspaces of a toric orbifold  $X(Q, \lambda)$ . Let  $F$  be a face of  $Q$  of codimension  $k$ . Then, the pre-image  $\pi^{-1}(F)$  is a closed invariant subspace. Indeed with the subspace topology,  $\pi^{-1}(F)$  is a toric orbifold of dimension  $2n - 2k$ . The corresponding characteristic pair for  $\pi^{-1}(F)$  can be described as follows.

Let

$$M^*(F) = M(F)_{\mathbb{R}} \cap \mathbb{Z}^n \text{ and } G_F = M^*(F)/M(F). \quad (2.3)$$

Here,  $M(F) \subseteq M^*(F)$  and both are free  $\mathbb{Z}$ -modules of rank  $k$ , therefore  $G_F$  is a finite abelian group. Note that if  $F$  is a face of  $F'$ , then the natural inclusion of  $M(F')$  into  $M(F)$  induces a surjective homomorphism from  $G_{F'}$  to  $G_F$ . Moreover, since  $M^*(F)$  is a free  $\mathbb{Z}$ -module of rank  $k$ , one may identify  $M^*(F)$  with  $\mathbb{Z}^k$  by fixing a proper isomorphism.

Consider the following projection homomorphism:

$$\varrho_F: \mathbb{Z}^n \rightarrow \mathbb{Z}^n / M^*(F) \cong \mathbb{Z}^{n-k}. \quad (2.4)$$

Let  $\{H_1, \dots, H_\ell\}$  be the facets of  $F$ . Then for each  $j \in \{1, \dots, \ell\}$ , there is a unique facet  $F_{i_j}$  of  $Q$  such that  $H_j = F \cap F_{i_j}$ . We define a map

$$\begin{aligned} \lambda_F: \{H_1, \dots, H_\ell\} &\rightarrow \mathbb{Z}^{n-k} \text{ by} \\ \lambda_F(H_j) &= \text{prim}(\varrho_F \circ \lambda(F_{i_j})) \text{ for } j \in \{1, \dots, \ell\}, \end{aligned} \quad (2.5)$$

where  $\text{prim}(x)$  denotes the primitive vector of  $x$  in  $\mathbb{Z}^{n-k}$ . Then  $\lambda_F$  is a characteristic

function on  $F$ . Let  $X(F, \lambda_F)$  be the toric orbifold corresponding to the characteristic pair  $(F, \lambda_F)$ . Then

$$X(F, \lambda_F) = (T^{n-k} \times F) / \sim_{\lambda_F},$$

where the equivalence relation  $\sim_{\lambda_F}$  is defined similar to (2.2). Further, by [BSS17, Proposition 3.2], as topological spaces,  $\pi^{-1}(F)$  and  $X(F, \lambda_F)$  are homeomorphic.

In particular, if  $F_i$  is a facet of  $Q$ , the subspace  $\pi^{-1}(F_i)$  is called a *characteristic subspace* of  $X(Q, \lambda)$ .

### 2.3. Toric orbifolds and local groups.

In this subsection, we define the *local groups*  $G_F(v)$  of  $X(Q, \lambda)$  corresponding to a codimension- $k$  face  $F$  of  $Q$  containing a vertex  $v$ . There is a bijective correspondence between the fixed points of the  $T^{n-k}$ -action on the toric orbifold  $\pi^{-1}(F)$  and the vertices of  $F$ . Let  $v$  be a vertex of  $Q$ . Then  $v = F_{i_1} \cap \cdots \cap F_{i_n}$  for a unique collection of facets  $F_{i_1}, \dots, F_{i_n}$  of  $Q$ . Let  $M(v)$  be the submodule of  $\mathbb{Z}^n$  generated by  $\{\lambda(F_{i_1}), \dots, \lambda(F_{i_n})\}$  where  $\lambda$  is as in Definition 2.1. Also, we have  $v = H_{j_1} \cap \cdots \cap H_{j_{n-k}}$  for a unique collection of facets  $H_{j_1}, \dots, H_{j_{n-k}}$  of  $F$ . Let  $M_F(v)$  be the submodule of  $\mathbb{Z}^{n-k}$  generated by  $\{\lambda_F(H_{j_1}), \dots, \lambda_F(H_{j_{n-k}})\}$  where  $\lambda_F$  is defined in (2.5). We define

$$\begin{aligned} G_Q(v) &:= \mathbb{Z}^n / M(v), \\ G_F(v) &:= \mathbb{Z}^{n-k} / M_F(v). \end{aligned}$$

These are finite abelian groups. Notice that the orders  $|G_Q(v)|$  and  $|G_F(v)|$  of each group are obtained by computing the corresponding determinant. More precisely,

$$\begin{aligned} |G_Q(v)| &= |\det[\lambda(F_{i_1}) \mid \cdots \mid \lambda(F_{i_n})]|, \\ |G_F(v)| &= |\det[\lambda_F(H_{j_1}) \mid \cdots \mid \lambda_F(H_{j_{n-k}})]|. \end{aligned}$$

In particular, for a vertex  $v$  of  $Q$ ,  $G_v(v) = \{1\}$ .

### 2.4. Retraction sequences of simple polytopes.

In this subsection, we recall the concept of *retraction sequence* of a simple polytope which was introduced in [BSS17].

Let  $Q$  be an  $n$ -dimensional simple polytope with  $m$  vertices. We now construct a sequence of triples  $\{(B_\ell, P_\ell, v_\ell)\}$ . Let  $B_1 = Q = P_1$  and  $v_1 \in V(B_1)$ . Suppose  $(B_\ell, P_\ell, v_\ell)$  has been defined for  $1 \leq \ell \leq k-1$ . We define  $(B_k, P_k, v_k)$  inductively as follows:

- Let  $B_k$  denote the subset of  $B_{k-1}$  such that

$$B_k := \bigcup \{F \mid F \text{ is a face of } Q \text{ contained in } B_{k-1} \text{ and } v_{k-1} \notin V(F)\}.$$

- Let  $v_k \in V(B_k)$  be such that  $v_k$  has a neighbourhood  $U_k$  in  $B_k$  which is homeomorphic to  $\mathbb{R}_{\geq}^{s_k}$  as a manifold with corners for some  $0 \leq s_k \leq \dim(B_k)$ .
- Let  $P_k$  be the smallest face of  $B_k$  containing  $U_k$ . Hence  $U_k$  is obtained from  $P_k$  by deleting all its faces which do not contain  $v_k$ .

The sequence  $\{(B_\ell, P_\ell, v_\ell)\}$  stops if there is no vertex of  $B_\ell$  which has a neighbourhood in  $B_\ell$  that is homeomorphic to  $\mathbb{R}_{\geq}^{s_\ell}$  as a manifold with corners for some  $0 \leq s_\ell \leq \dim(B_\ell)$ . Proceeding in this way, if we get that  $B_m$  is the vertex  $v_m$ , we set  $P_m = \{v_m\} = B_m$ . At this point we introduce the following definition.

**DEFINITION 2.3.** Let  $Q$  be a simple polytope. If there exists a sequence  $\{(B_\ell, P_\ell, v_\ell)\}_{\ell=1}^m$  as constructed above such that  $B_m = P_m = v_m$  is a vertex, then we say that  $\{(B_\ell, P_\ell, v_\ell)\}_{\ell=1}^m$  is a *retraction sequence* of  $Q$  starting with the vertex  $v_1$  and ending at  $v_m$ .

In relation to the above, we also recall the following definition.

**DEFINITION 2.4.** A vertex  $v$  is called a *free vertex* of  $B_\ell$  if it has a neighbourhood in  $B_\ell$  which is homeomorphic to  $\mathbb{R}_{\geq}^s$  as manifold with corners for some  $0 \leq s \leq \dim(B_\ell)$ .

**REMARK 2.5.** (1) In the retraction sequence, a choice of a free vertex  $v_\ell$  in  $B_\ell$  determines  $B_{\ell+1}$ .

- (2) In a retraction sequence of a simple polytope  $Q$ , the number of retraction steps is  $m = |V(Q)|$ . In particular, it is independent of the choice of the retraction sequence.
- (3) By [BSS17, Proposition 2.3], the height function on a simple polytope  $Q$  gives a retraction sequence of it.
- (4) See Figure 2 for a retraction sequence of a 3-prism.

**REMARK 2.6.** Given a retraction sequence of simple polytope  $Q$  one can define a directed graph on the 1-skeleton of  $Q$  in the following way. Let  $\{(B_i, P_i, v_i)\}_{i=1}^m$  be a retraction sequence of  $Q$ . We order the vertices of  $Q$  as  $v_1 < v_2 < \dots < v_m$ . We assign a direction from  $v_s$  to  $v_r$  if there is an edge with end points  $v_s, v_r$  and  $v_s > v_r$ . This directed graph has the property that if  $s_i$  number of edges end at the vertex  $v_i$  then  $\dim P_i = s_i$ .

## 2.5. Torus invariant cell structures on toric orbifolds.

In this subsection, we first define *retractable* toric orbifolds, and then show the existence of a *genuine* invariant cell structure on a retractable toric orbifold justifying the terminology. (Here genuine is as opposed to the existence of a  $\mathbf{q}$ -cellular structure on any toric orbifold by [PS10, Section 4]). We adhere to the notation of the previous subsections.

**DEFINITION 2.7.** Let  $X$  be a toric orbifold over the simple polytope  $Q$ . Then  $X$  is called *retractable* if  $Q$  has a retraction  $\{(B_i, P_i, v_i)\}_{i=1}^m$  such that  $G_{P_i}(v_i)$  is the trivial group for  $i = 1, \dots, m-1$ .

**LEMMA 2.8.** If  $X$  is a retractable toric orbifold of dimension  $2n$ , then  $X$  has a cell structure with  $T$ -invariant cells.

PROOF. Let  $\{H_1, \dots, H_{s_i}\}$  be the facets of  $P_i$  such that  $H_1 \cap \dots \cap H_{s_i} = v_i$ . Let  $U_i \simeq \mathbb{R}_{\geq 0}^{s_i}$  be the open neighbourhood of  $v_i$  in  $P_i$  obtained by deleting all faces of  $P_i$  not containing the free vertex  $v_i$ . Since  $G_{P_i}(v_i)$  is trivial, the collection of vectors  $\{\lambda_{P_i}(H_1), \dots, \lambda_{P_i}(H_{s_i})\}$  form a basis of  $\mathbb{Z}^{s_i}$ . Now, following the arguments in [DJ91, Section 1.5 and Lemma 1.6], we have that

$$(T^{s_i} \times U_i) / \sim_{\lambda_{P_i}} = \pi_{P_i}^{-1}(U_i)$$

can be identified with  $\mathbb{C}^{s_i}$  having the standard  $T^{s_i}$  action. Alternately by [PS10, Section 4.2],  $\pi_{P_i}^{-1}(U_i)$  is equivariantly homeomorphic to the quotient of a  $2s_i$ -dimensional open disk in  $\mathbb{R}^{2s_i}$  by the group  $G_{P_i}(v_i)$  which is trivial by assumption, and hence equivariantly homeomorphic to  $\mathbb{C}^{s_i}$ . Now by [BSS17, Proposition 3.2],  $\pi_{P_i}^{-1}(U_i) \subset X(P_i, \lambda_{P_i})$  is equivariantly homeomorphic to  $\widehat{U}_i := \pi^{-1}(U_i) \subset X(Q, \lambda)$  for  $i = 1, \dots, m$ . Further, note that  $Q = \bigcup_{i=1}^m U_i$ . Therefore  $X(Q, \lambda) = \bigcup_{i=1}^m \widehat{U}_i$ . This proves the lemma.  $\square$

EXAMPLE 2.9. The *divisive* weighted projective spaces of [HHRW16, Definition 2.2] are examples of *retractable* toric orbifolds over an  $n$ -simplex  $Q$ . Let  $F_1, \dots, F_{n+1}$  denote the facets of  $Q$ . The characteristic function  $\lambda$  is given by  $\lambda(F_i) = e_i$  for  $1 \leq i \leq n$  and  $\lambda(F_{n+1}) = (-\chi_1, -\chi_2, \dots, -\chi_n)$  where  $\chi_1 = 1$  and  $\chi_i$  divides  $\chi_{i+1}$  for  $1 \leq i \leq n-1$  (divisive condition). Let  $v_0$  denote the intersection of the facets  $F_1, \dots, F_n$ . For  $1 \leq i \leq n$ , let  $v_i$  denote the vertex of  $Q$  which is the intersection of the facets  $F_j$  for  $1 \leq j \leq n+1$  such that  $j \neq i$ . The ordering of the vertices  $v_0 < v_1 < v_2 < \dots < v_n$  of  $Q$  by means of a generic height function gives a retraction sequence  $\{(B_i, P_i, v_i)\}_{i=0}^n$ . One can check using the divisive condition that the local groups satisfy  $G_{P_i}(v_i) = 1$  for every  $0 \leq i \leq n$ . For details of this computation see [Sar20].

EXAMPLE 2.10. We now give an example of a retractable toric orbifold which is not a divisive weighted projective space. Consider the characteristic function  $\lambda$  on a 3-prism  $Q$  as in Figure 1. One can compute that  $G_Q(v_1) = \{1\}$ ,  $G_Q(v_2) = \mathbb{Z}/3\mathbb{Z}$ ,  $G_Q(v_3) = \{1\}$ ,  $G_Q(v_4) = \{1\}$ ,  $G_Q(v_5) = \mathbb{Z}/21\mathbb{Z}$ ,  $G_Q(v_6) = \mathbb{Z}/7\mathbb{Z}$ , where, for example,  $G_Q(v_1)$  is the determinant of  $\{\lambda(F_1), \lambda(F_2), \lambda(F_3)\}$ , and similarly for the other groups.

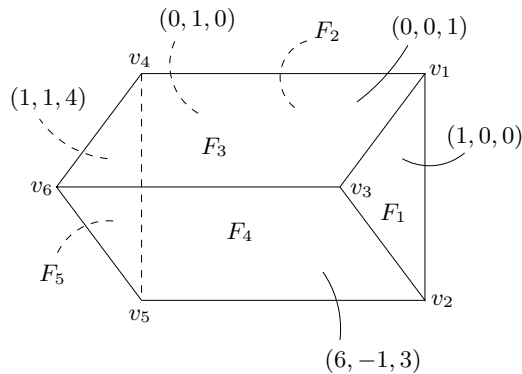


Figure 1. A characteristic function on 3-prism  $Q$ .

Now we consider the following retraction of  $Q$ , see Figure 2. The retraction sequence

is given by  $(B_1, B_1, v_1)$ ,  $(B_2, F_4, v_2)$ ,  $(B_3, F_3 \cap F_4, v_3)$ ,  $(B_4, F_5, v_4)$ ,  $(B_5, F_4 \cap F_5, v_5)$ ,  $(B_6, v_6, v_6)$ . We only compute the local group  $G_{F_4}(v_2)$  and the computation for other  $G_{P_i}(v_i)$  is similar. In this case  $P_i = F_4$  and  $v_i = v_2$ . So  $M(F_4) = \langle \lambda(F_4) \rangle = \langle (6, -1, 3) \rangle$ . Thus

$$M^*(F_4) = \langle \lambda(F_5) \rangle \cap \mathbb{Z} = \langle (6, -1, 3) \rangle = M(F_4) \cong \mathbb{Z}.$$

Consider the basis  $\{e_1 = (1, 0, 0), e_2 = (0, 0, 1), e_3 = (6, -1, 3)\}$  of  $\mathbb{Z}^3$ . Then one gets the projection

$$\rho: \mathbb{Z}^3 \rightarrow \mathbb{Z}^3 / M^*(F_4) = \langle e_1, e_2, e_3 \rangle / \langle e_3 \rangle \cong \mathbb{Z}^2.$$

The facets of  $F_4$  which intersects at  $v_2$  are  $F_1 \cap F_4$  and  $F_2 \cap F_4$ . Therefore we get  $\lambda_{F_4}$  on  $F_1 \cap F_4$  and  $F_2 \cap F_4$  which are given by

$$\lambda_{F_4}(F_1 \cap F_4) = \text{prim}(\rho(\lambda(F_1))) = (1, 0)$$

and

$$\lambda_{F_4}(F_2 \cap F_4) = \text{prim}(\rho(\lambda(F_2))) = \text{prim}(6, 3) = (2, 1).$$

Therefore

$$G_{F_4}(v_2) = \mathbb{Z}^2 / \langle \lambda_{F_4}(F_1 \cap F_4), \lambda_{F_4}(F_2 \cap F_4) \rangle = \mathbb{Z}^2 / \langle (1, 0), (2, 1) \rangle = \{1\}.$$

Similarly, one can compute that  $G_{F_3 \cap F_4}(v_3) = \{1\}$ ,  $G_{F_5}(v_4) = \{1\}$ ,  $G_{F_4 \cap F_5}(v_5) = \{1\}$ .

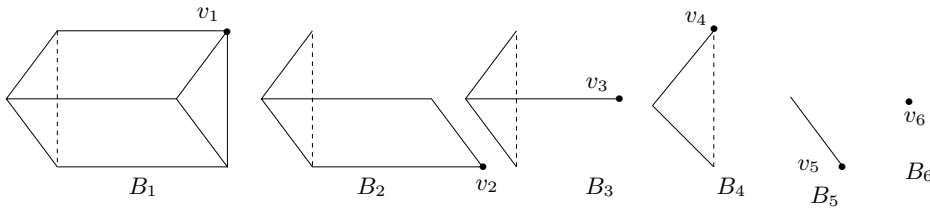


Figure 2. A retraction sequence of 3-prism  $Q$ .

In the above example if instead of  $\lambda(F_2) = (6, -1, 3)$  we let  $\lambda(F_2) = (2, -1, 3)$  then the local groups would be  $G_Q(v_1) = \{1\}$ ,  $G_Q(v_2) = \mathbb{Z}/3\mathbb{Z}$ ,  $G_Q(v_3) = \{1\}$ ,  $G_Q(v_4) = \{1\}$ ,  $G_Q(v_5) = \mathbb{Z}/5\mathbb{Z}$ ,  $G_Q(v_6) = \mathbb{Z}/3\mathbb{Z}$ .

For the retraction of  $Q$ , given in Figure 2 and the corresponding retraction sequence described above we check that the local group  $G_{F_4}(v_2)$  is not  $\{1\}$ . In this case we have  $M(F_4) = \langle \lambda(F_4) \rangle = \langle (2, -1, 3) \rangle$ . Thus

$$M^*(F_4) = \langle \lambda(F_5) \rangle \cap \mathbb{Z} = \langle (2, -1, 3) \rangle = M(F_4) \cong \mathbb{Z}.$$

Consider the basis  $\{e_1 = (1, 0, 0), e_2 = (0, 0, 1), e_3 = (2, -1, 3)\}$  of  $\mathbb{Z}^3$ . Then one gets



the projection

$$\rho: \mathbb{Z}^3 \rightarrow \mathbb{Z}^3/M^*(F_4) = \langle e_1, e_2, e_3 \rangle / \langle e_3 \rangle \cong \mathbb{Z}^2.$$

The facets of  $F_4$  which intersects at  $v_2$  are  $F_1 \cap F_4$  and  $F_2 \cap F_4$ . Therefore we get  $\lambda_{F_4}$  on  $F_1 \cap F_4$  and  $F_2 \cap F_4$  which are given by

$$\lambda_{F_4}(F_1 \cap F_4) = \text{prim}(\rho(\lambda(F_1))) = (1, 0)$$

and

$$\lambda_{F_4}(F_2 \cap F_4) = \text{prim}(\rho(\lambda(F_2))) = \text{prim}(2, 3) = (2, 3).$$

Therefore

$$G_{F_4}(v_2) = \mathbb{Z}^2 / \langle \lambda_{F_4}(F_1 \cap F_4), \lambda_{F_4}(F_2 \cap F_4) \rangle = \mathbb{Z}^2 / \langle (1, 0), (2, 3) \rangle = \mathbb{Z}/3\mathbb{Z}.$$

REMARK 2.11. Note that if one consider the retraction sequence  $\{(B_i, P_i, v_i)\}_{i=1}^6$  as in Figure 3, then it induces a  $T$ -invariant CW-structure on  $X(Q, \lambda)$ . Therefore [HHH05, Proposition 4.3] can be applied to get explicit generators of  $H_T^*(X(Q, \lambda))$  as a  $H_T^*(pt)$ -module. More generally, let  $(Q, \lambda)$  be a characteristic pair and let  $\{(B_i, P_i, v_i)\}_{i=1}^m$  be a retraction sequence of  $Q$  obtained from a height function on  $Q$  such that  $G_{P_i}(v_i) = 1$  for  $i = 1, \dots, m$ . Then it can be seen that the numbers  $s_i$ 's in the definition of the retraction sequence are non-decreasing. Thus the induced  $T$ -invariant retractable structure on the toric orbifold  $X(Q, \lambda)$  is in fact a  $T$ -invariant CW structure. Consequently, one can apply [HHH05, Proposition 4.3] to  $X(Q, \lambda)$ .

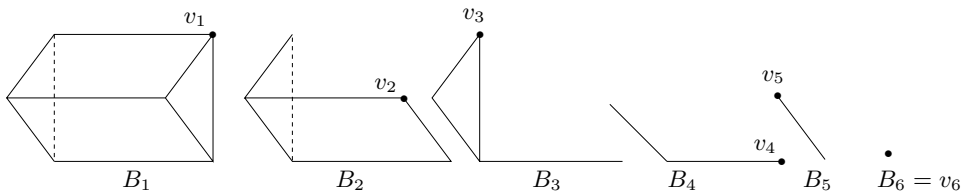


Figure 3. A retraction sequence of 3-prism  $Q$ .

### 3. GKM theory on retractable toric orbifolds.

We begin this section by recalling the GKM theory from [HHH05, Section 3]. We shall then verify that these results can be applied to a retractable toric orbifold and hence give a precise description of its equivariant generalized cohomology theory ring.

Let  $X$  be a  $G$ -space equipped with a  $G$ -invariant stratification

$$X_m \subseteq X_{m-1} \subseteq \dots \subseteq X_1 = X.$$

That is,  $X_i \setminus X_{i+1}$  has a  $G$ -invariant subspace  $Y_i$  having a  $G$ -stable neighbourhood which is homeomorphic to the total space  $V_i$  of a  $G$ -equivariant vector bundle  $\rho_i = (V_i, \varpi_i, Y_i)$

with projection  $\varpi_i: V_i \rightarrow Y_i$ . In particular, when  $Y_i = x_i$  is a  $G$ -fixed point then  $\rho_i = (V_i, \varpi_i, x_i)$  is a  $G$ -representation.

Let  $E_G^*$  be a generalized  $G$ -equivariant cohomology theory. We now make the following assumptions on  $X$ .

- (A1) Each subquotient  $X_i/X_{i+1}$  is homeomorphic to the Thom space  $Th(\rho_i)$  with corresponding attaching map  $\phi_i: S(\rho_i) \rightarrow X_{i+1}$ .
- (A2) Every  $\rho_i$  admits a  $G$ -equivariant direct sum decomposition  $\bigoplus_{j>i} \rho_{ij}$  into  $G$ -equivariant subbundles  $\rho_{ij} = (V_{ij}, \varpi_{ij}, Y_i)$ . We allow the case  $V_{ij} = 0$ .
- (A3) There exist  $G$ -equivariant maps  $f_{ij}: Y_i \rightarrow Y_j$  such that the restrictions  $f_{ij} \circ \varpi_{ij}|_{S(\rho_{ij})}$  and  $\phi_i|_{S(\rho_{ij})}$  agree for every  $j > i$ .
- (A4) The equivariant Euler classes  $e_G(\rho_{ij})$  for  $j > i$ , are not zero divisors and are pairwise relatively prime in  $E_G^*(Y_i)$ .

We now recall the precise description of the generalized  $G$ -equivariant cohomology ring of  $X$ .

**THEOREM 3.1** ([HHH05, Theorem 3.1]). *Let  $X$  be a  $G$ -space satisfying the four assumptions (A1) to (A4). Then the restriction map*

$$\iota^*: E_G^*(X) \rightarrow \prod_{i=1}^m E_G^*(Y_i)$$

*is monic and its image  $\Gamma_X$  can be described as*

$$\left\{ (a_i) \in \prod_{i=1}^m E_G^*(Y_i) : \text{for every } j > i, e_G(\rho_{ij}) \mid a_i - f_{ij}^*(a_j) \right\}.$$

We show below that the GKM theory of [HHH05] described above can be applied to a retractable toric orbifold  $X := X(Q, \lambda)$ . We further use this to give an explicit description of the  $T$ -equivariant  $K$ -theory ring  $K_T^*(X)$ , the  $T$ -equivariant complex cobordism ring  $MU_T^*(X)$  and the  $T$ -equivariant integral cohomology ring  $H_T^*(X)$  (see Corollary 3.3).

Let  $X$  be a retractable toric orbifold and  $\{(B_i, P_i, v_i)\}_{i=1}^m$  be the corresponding retraction sequence of the polytope  $Q$ . By Lemma 2.8 there is a  $T$ -invariant retractable structure on  $X$  associated to this retraction. Let

$$X_i := \pi^{-1}(B_i), \quad \text{and let } x_i := \pi^{-1}(v_i)$$

for  $1 \leq i \leq m$ . Then  $x_1, \dots, x_{m-1}$  and  $x_m$  are the  $T$ -fixed points of  $X$ . Let  $U_i \cong \mathbb{R}_{\geq}^{s_i}$  denote the open neighbourhood of  $v_i$  in  $P_i$  obtained by deleting all faces of  $P_i$  not containing the free vertex  $v_i$  and let  $\widehat{U}_i := \pi^{-1}(U_i)$ . Thus we have the following  $T$ -invariant stratification

$$\{x_m\} = X_m \subseteq X_{m-1} \subseteq \dots \subseteq X_1 = X \tag{3.1}$$

of  $X$  such that  $X_i \setminus X_{i+1} = \widehat{U}_i \cong \mathbb{C}^{s_i}$ , where  $s_i = \dim P_i$  for  $i = 1, \dots, m$  with  $X_{m+1} = \emptyset$ . Moreover,  $x_i \in \widehat{U}_i \subseteq X_i$ ,  $1 \leq i \leq m$ . Since  $\widehat{U}_i \cong \mathbb{C}^{s_i}$  are  $T$ -stable we have a  $T$ -representation  $\rho_i = (V_i, \varpi_i, x_i)$  for  $1 \leq i \leq m$ . Here  $T$  acts on  $V_i := \widehat{U}_i$  via its projection to  $T^{s_i}$  given by the characters  $u_{j_1}, \dots, u_{j_{s_i}}$  (described in the proof of Proposition 3.2) followed by the standard action of  $T^{s_i}$  on  $\mathbb{C}^{s_i}$ .

**PROPOSITION 3.2.** *A retractable toric orbifold  $X$  with the  $T$ -invariant stratification in (3.1) satisfies assumptions (A1) to (A4) listed above.*

**PROOF.** Checking for (A2): Let  $P_i = F_{i_1} \cap \dots \cap F_{i_{n-s_i}}$ . Consider the  $\mathbb{Z}$ -linear map

$$\psi: \mathbb{Z}^n \longrightarrow \mathbb{Z}^{n-s_i} \quad (3.2)$$

defined by the  $(n-s_i) \times n$ -matrix with rows  $\lambda(F_{i_1}), \dots, \lambda(F_{i_{n-s_i}})$ . We note that  $T$  acts on  $V_i$  via the characters  $u_{j_1}, \dots, u_{j_{s_i}}$  which form a  $\mathbb{Z}$ -basis of the kernel of  $\psi$ . In other words,  $V_i$  is a direct sum of the one-dimensional representations  $V_{ij_r} := \mathbb{C}_{u_{j_r}}$ ,  $1 \leq r \leq s_i$  of  $T$ . Recall that for  $1 \leq r \leq s_i$ , there is an edge  $e_{j_r}$  in the directed graph associated to the retraction sequence which points towards  $v_i$ . Then  $u_{j_r}$  is a  $\mathbb{Z}$ -basis of the kernel of the  $\mathbb{Z}$ -linear map

$$\psi_r: \mathbb{Z}^n \longrightarrow \mathbb{Z}^{n-1}$$

defined by an  $(n-1) \times n$ -matrix which has a row  $\lambda(F)$  corresponding to every facet  $F$  of  $Q$  containing  $e_{j_r}$ . Note that  $u_{j_r}$  is unique up to a sign. Since  $T$  acts on the invariant subspace  $\pi^{-1}(e_{j_r})$  via the character  $u_{j_r}$ , the sign of  $u_{j_r}$  for  $1 \leq r \leq s_i$  is determined by the  $T$ -action on  $X$ . It follows that  $V_{ij_r}$  corresponds to the directed edge  $e_{j_r}$  for  $1 \leq r \leq s_i$ . By putting  $V_{ik} = 0$  for  $k > i$  and  $k \notin \{j_1, \dots, j_{s_i}\}$  we see that (A2) follows. We denote by  $\rho_i$  the representation  $(V_i, \varpi_i, x_i)$  and by  $\rho_{ij_r} = (V_{ij_r}, \varpi_i|_{V_{ij_r}}, x_i)$  the one-dimensional sub-representation associated to the character  $u_{ij_r}$  for  $1 \leq r \leq s_i$ .

Checking for (A1): Since  $X_i \setminus X_{i+1} = \widehat{U}_i = V_i$  is a complex  $T$ -representation where  $T$  acts by characters  $u_{ij_r}$  for  $1 \leq r \leq s_i$ , it follows that  $X_i/X_{i+1}$  is the representation sphere  $S(\rho_i)$ . Here  $S(\rho_i)$  is the equivariant one point compactification of  $V_i$  with infinity viewed as a fixed point. This is, therefore, nothing but the Thom space  $Th(\rho_i)$  of the vector bundle  $V_i$  over the point  $x_i$ . We now describe in detail the attaching map

$$\phi_i: S(\rho_i) \longrightarrow X_{i+1}.$$

Consider the neighbourhood  $W_i = U_i \cap D$  of  $v_i$  in  $Q$  where  $U_i$  is the open neighbourhood of  $v_i$  in  $P_i$  obtained by deleting all faces of  $P_i$  not containing the free vertex  $v_i$  and  $D$  is a closed disc in  $\mathbb{R}^n$  with centre  $v_i$  such that  $D$  does not contain any other vertex of  $Q$ . Note that  $P_i$  is the smallest face of  $Q$  containing the edges  $e_{j_r}$  for  $1 \leq r \leq s_i$ . The  $\text{Link}(v_i)$  in  $P_i$  is  $P_i \setminus U_i$ . So  $P_i = \text{Star}(v_i) = \text{Link}(v_i) \star v_i$ . Thus it follows from polyhedral geometry that for every  $p \in (P_i \setminus U_i)$ , the line segment joining  $p$  and  $v_i$  meets  $W_i \cap \partial D$  at a unique point  $y_p$ . Moreover,  $y_p$  determines  $p$  uniquely and vice versa, see Figure 4 for an example.

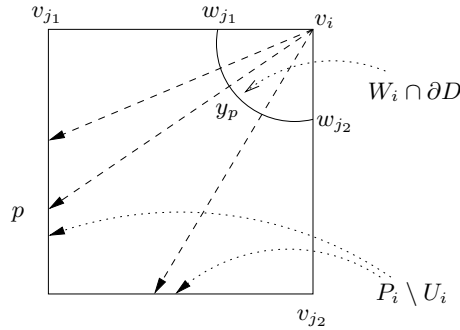


Figure 4. Correspondence of cell attaching.

This gives a bijective correspondence  $g_i: \partial D \cap W_i \rightarrow P_i \setminus U_i$ . Therefore we have following commutative diagram,

$$\begin{array}{ccc}
 T \times (\partial D \cap W_i) & \xrightarrow{\text{id} \times g_i} & T \times (P_i \setminus U_i) \\
 \downarrow & & \downarrow \\
 (T \times (\partial D \cap W_i)) / \sim & \xrightarrow{\phi_i} & (T \times (P_i \setminus U_i)) / \sim \xrightarrow{\subseteq} X_{i+1}.
 \end{array} \tag{3.3}$$

The map  $\phi_i$  sends  $[t, y_p]$  to  $[t, p]$ . This map is well defined because if  $y_p$  belongs to the relative interior of a face  $F$  of  $P_i$  then  $p$  also belongs to  $F$ . Moreover, under the identification of  $\widehat{U}_i$  with the complex representation  $\rho_i$ ,

$$\widehat{W}_i = \pi^{-1}(W_i) = (T \times W_i) / \sim \subseteq \widehat{U}_i$$

can be identified with the disc bundle  $D(\rho_i)$  and  $\partial(\widehat{W}_i) = \pi^{-1}(\partial D \cap W_i)$  with the sphere bundle  $S(\rho_i)$  of the representation space associated with  $\rho_i$ . The above identifications induce the following homeomorphisms

$$X_i / X_{i+1} \cong \widehat{W}_i / \partial(\widehat{W}_i) \cong D(\rho_i) / S(\rho_i) = Th(\rho_i),$$

where  $\widehat{U}_i \subseteq X_i$  maps homeomorphically onto the interior of  $\widehat{W}_i$ . This verifies (A1).

Checking for (A3): Let the initial vertex of  $e_{j_r}$  be  $v_{j_r}$  and let  $f_{ij_r}: x_i \rightarrow x_{j_r}$  denote the constant map for  $1 \leq r \leq s_i$ . Further,  $\pi^{-1}(e_{j_r} \setminus v_{j_r}) \subseteq \widehat{U}_i$  can be identified with the one dimensional sub-representation  $\rho_{ij_r}$  of  $\rho_i$  for  $r = 1, \dots, s_i$ . Let  $S(\rho_{ij_r})$  denote the circle bundle associated with  $\rho_{ij_r}$ . Let  $w_{j_r}$  be a point where  $e_{j_r}$  meets  $\partial D \cap W_i$ . Then the attaching map  $\phi_i|_{S(\rho_{ij_r})}$  (see (3.3)) sends  $\pi^{-1}(w_{j_r})$  in  $S(\rho_{ij_r})$  to  $x_{j_r}$ . Further,  $\pi^{-1}(w_{j_r}) \in \widehat{U}_i$  is mapped to  $x_i$  under the canonical projection in the radial direction of the representation  $\rho_{ij_r}$ . It follows that the restriction of the map  $\phi_i$  and the composition of the projection of  $\rho_{ij_r}$  with  $f_{ij_r}$  agree on  $S(\rho_{ij_r})$  for every  $1 \leq r \leq s_i$ . This verifies assumption (A3).

Checking for (A4): Recall that  $u_{j_r}$  for  $1 \leq r \leq s_i$  are a  $\mathbb{Z}$ -basis for the kernel of  $\psi$

(see 3.2). In particular, for  $1 \leq r \leq s_i$ ,  $u_{j_r}$  is a primitive non-zero vector. Hence the  $K$ -theoretic equivariant Euler class

$$e^T(\rho_{ij_r}) = 1 - e^{-u_{j_r}}$$

is a non-zero divisor in the integral domain  $K_T^0(x_i) = R(T)$ . Also,  $u_{j_r}$  for  $1 \leq r \leq s_i$ , are pairwise linearly independent. Thus  $1 - e^{-u_{j_r}}$  and  $1 - e^{-u_{j_{r'}}}$  are relatively prime in the unique factorization domain  $R(T)$  for  $r \neq r'$ . This can also be seen more generally for the equivariant Euler classes in  $MU_T^*$  and also for  $H_T^*(; \mathbb{Z})$  (see [HHH05, Lemma 5.2]).  $\square$

### 3.1. Equivariant generalized cohomology theory of retractable toric orbifolds.

In this subsection, we describe the  $T$ -equivariant generalized cohomology ring of a retractable toric orbifold  $X$  as an  $E_T^*(pt)$ -algebra.

The following corollary extends [HHRW16, Proposition 3.10] on a divisive weighted projective space and [HHRW16, Theorem 7.1] on the toric variety associated with a smooth polytopal fan to any retractable toric orbifold. Consider  $f_{i1}^*: E_T^*(x_m) \rightarrow E_T^*(x_i)$  induced by the constant map  $f_{i1}: x_i \rightarrow x_m$  for  $1 \leq i \leq m$ . This gives  $\prod_{i=1}^m E_T^*(x_i)$  a canonical  $E_T^*(x_m)$ -algebra structure via the inclusion defined by  $(f_{i1}^*(a))$  for  $a \in E_T^*(x_m)$ .

Let  $V_{ij_r}$  denote the 1-dimensional  $T$ -representation corresponding to the primitive character  $u_{j_r} \in \mathbb{Z}^n$ . When there is no edge between  $v_i$  and  $v_k$  for  $k > i$  then  $V_{ik}$  is trivial.

**COROLLARY 3.3.** *Let  $X = X(Q, \lambda)$  be a retractable toric orbifold. The  $T$ -equivariant generalized cohomology theory ring  $E_T^*(X)$ , for  $E = K, MU, H$  is isomorphic to the  $E_T^*(x_m)$ -subalgebra*

$$\Gamma_X = \{(a_i) : \forall k > i, e^T(V_{ik}) \text{ divides } a_i - f_{ik}^*(a_k) \in E_T^*(x_i)\} \subseteq \prod_{i=1}^m E_T^*(x_i). \quad (3.4)$$

(Here  $H$  denotes integral cohomology.)

**PROOF.** By Proposition 3.2 and Theorem 3.1 above it follows that  $E_T^*(X)$  is isomorphic to the subring  $\Gamma_X$  of  $\prod_{i=1}^m E_T^*(x_i)$ . Note that,  $\forall a \in E_T^*(x_1)$ ,  $(f_{i1}^*(a)) \in \Gamma_X$ . This is because for every  $k > i$ ,  $f_{i1}^*(a) - f_{ik}^*(f_{k1}^*(a)) = f_{i1}^*(a) - f_{i1}^*(a) = 0$  and is hence trivially divisible by  $e^T(V_{ik})$ . Thus  $\Gamma_X$  is an  $E_T^*(x_m)$ -subalgebra of  $\prod_{i=1}^m E_T^*(x_i)$ .  $\square$

## 4. Piecewise algebra and its applications.

In this section, we introduce the concept of piecewise algebra associated to the characteristic pair  $(Q, \lambda)$  in a dual manner to the notion of piecewise algebra associated to a fan (see [HHRW16, Section 4]). Consider the category  $\text{Face}(Q)$  whose objects are the faces  $F$  of  $Q$  and whose morphisms are their inclusions  $i_{F, F'}: F \hookrightarrow F'$ . Then  $\text{Face}(Q)$  is a small category in which  $Q$  is the final object.

We have a covariant functor  $\mathcal{E}_T^*$ -CGA from  $\text{Face}(Q)$  to the category of graded commutative  $E_T^*(pt)$ -algebras defined by sending a face  $F$  of  $Q$  to

$$\mathcal{E}_T^*\text{-CGA}(F) := E_T^*(T/T_F)$$

and the inclusion  $i_{F,F'}$  to the  $E_T^*(pt)$ -algebra morphism  $i_{F,F'}^*: E_T^*(T/T_F) \longrightarrow E_T^*(T/T_{F'})$  induced by the inclusion  $T_{F'} \hookrightarrow T_F$  (see [HHRW16, Definitions 4.3 and 4.4]).

DEFINITION 4.1. The limit  $\lim \mathcal{E}_T^*$ -CGA is called the  $E_T^*(pt)$ -algebra of piecewise  $E_T^*$ -coefficients for the characteristic pair  $(Q, \lambda)$  denoted by  $\mathcal{P}_E(Q, \lambda)$ .

We note that here  $\mathcal{P}_E(Q, \lambda)$  is an  $E_T^*(pt)$ -subalgebra of  $\prod_F E_T^*(T/T_F)$ , so every piecewise coefficient has one component  $f_F$  for every face  $F$  of  $Q$ . For  $1 \leq i \leq m$  we have

$$\mathcal{E}_T^*\text{-CGA}(v_i) = E_{T_{v_i}}^*(pt) = E_T^*(pt)$$

since  $T_{v_i} = T$ ; on the other hand

$$\mathcal{E}_T^*\text{-CGA}(Q) = E_{T_Q}^*(pt)$$

since  $T_Q = \{1\}$ . Moreover, if  $(f_F) \in \mathcal{P}_E(Q, \lambda)$  then  $i_{F,F'}^*(f_F) = f_{F'}$  whenever  $F \subseteq F'$  in  $Q$  which is called the *compatibility* condition. Sums and products of piecewise coefficients are taken facewise. We have a canonical diagonal inclusion  $E_T^*(pt) \subseteq \mathcal{P}_E(Q, \lambda)$  as  $(i_F^*(f))$ , for  $f \in E_T^*$  where  $i_F^*: E_T^*(pt) \longrightarrow E_T^*(T/T_F)$  is induced by the projection

$$T/T_F \longrightarrow T/T = pt$$

associated to the canonical inclusion  $T_F \subseteq T$ , which clearly satisfies the compatibility condition. The image of the diagonal inclusion is the subalgebra of *global coefficients*. Also, the constants (0) and (1) act as the zero and identity element in  $\mathcal{P}_E(Q, \lambda)$  respectively (see [HHRW16, Remark 4.8]).

Let  $F$  be a face of  $Q$  of codimension  $n - k$ . Let  $F$  be the intersection of the facets  $F_1, F_2, \dots, F_{n-k}$ . Then consider the  $\mathbb{Z}$ -linear map

$$\psi_F: \mathbb{Z}^n \longrightarrow \mathbb{Z}^{n-k},$$

defined by the  $(n - k) \times n$ -matrix with rows  $\lambda(F_1), \dots, \lambda(F_{n-k})$ . The kernel of  $\psi_F$  is a free  $\mathbb{Z}$ -module generated by the primitive vectors  $u_1, \dots, u_k$  in  $\mathbb{Z}^n$ . Since  $u_i$ 's are pairwise linearly independent,  $e^T(u_i), 1 \leq i \leq k$  are relatively prime in  $E_T^*(pt)$  for  $E = K, MU, H$ , see proof of (A4) in Proposition 3.2. (Here  $e^T(u_i) \in E_T^*(pt)$  denotes the  $T$ -equivariant Euler class of the 1-dimensional  $T$ -representation corresponding to  $u_i$ .) Thus as in [HHRW16, Example 4.12] we have the isomorphism

$$E_T^*(pt)/J_F \cong E_T^*(T/T_F), \quad (4.1)$$

where  $J_F$  is an ideal of  $E_T^*(pt)$  generated by  $e^T(u_i), 1 \leq i \leq k$ .

Further, if  $F \subseteq F'$  then  $\text{kernel}(\psi_F) \subseteq \text{kernel}(\psi_{F'})$ . This gives the inclusion of the ideals  $J_F \subseteq J_{F'}$  of  $E_T^*(pt)$  inducing the projection

$$r_{F,F'}^*: E_T^*(pt)/J_F \longrightarrow E_T^*(pt)/J_{F'},$$

which corresponds to  $i_{F,F'}^*$  under the identification (4.1).

The following theorem is an extension of [HHRW16, Theorem 5.5] for a divisive weighted projective space and [HHRW16, Corollary 7.2] for a toric variety corresponding to a smooth polytopal fan to any retractable toric orbifold. The proof follows closely that of [HHRW16, Theorem 5.5] suitably adapted to this setting.

**THEOREM 4.2.** *For a retractable toric orbifold  $X = X(Q, \lambda)$ ,  $E_T^*(X(Q, \lambda))$  is isomorphic as an  $E_T^*(pt)$ -algebra to  $\mathcal{P}_E(Q, \lambda)$  for each  $E = K, MU, H$ . In the case when  $E = H$ ,  $\mathcal{P}_E(Q, \lambda)$  is the ring of piecewise polynomial functions, when  $E = K$ , it is the ring of piecewise Laurent polynomial functions and when  $E = MU$ , it is the ring of piecewise cobordism forms on  $(Q, \lambda)$ .*

**PROOF.** It suffices to identify the algebra  $\Gamma_X$  defined by (3.4) with  $\lim \mathcal{E}_T^* \text{--CGA}$ . By the universal property of  $\lim \mathcal{E}_T^* \text{--CGA}$ , we first find compatible homomorphisms

$$h_F: \Gamma_X \longrightarrow \mathcal{E}_T^* \text{--CGA}(F)$$

for every face  $F$  of  $Q$ . If  $a = (a_i) \in \Gamma_X$ , on the vertex  $v_k$  of  $Q$  we define  $h_{v_k}(a) := a_k$  for each  $1 \leq k \leq m$ . On an edge  $e_{ij}$  joining  $v_i$  and  $v_j$  we let

$$h(a)_{e_{ij}} := a_i \pmod{J_{e_{ij}}} \in E_T^*(pt)/J_{e_{ij}}.$$

This is well defined since  $e^T(u_{ij}) \mid a_i - a_j$ , where  $u_{ij}$  generates  $\ker(\psi_{e_{ij}})$ , and  $J_{e_{ij}}$  contains  $e^T(u_{ij})$ . For any face  $F$  of  $Q$  having vertices  $v_{i_1}, \dots, v_{i_k}$  we let

$$h_F(a) := a_{i_1} \pmod{J_F} \in E_T^*(pt)/J_F.$$

Since  $u_{ij}$  generates  $\ker(\psi_{e_{ij}}) \subseteq \ker(\psi_F)$ ,  $J_F$  contains  $e^T(u_{ij})$  for each edge  $e_{ij} \subset F$  and hence contains  $a_i - a_j$ . Further, since the 1-skeleton of  $F$  is connected, any two vertices  $v_{i_j}$  and  $v_{i_r}$  are connected by a path of edges in  $F$ . It follows that  $a_{i_j} - a_{i_r}$  belongs to  $J_F$ , and therefore the map  $h_F$  is well defined for each face  $F$  of  $Q$ . Furthermore, since  $J_F$  is an ideal in  $E_T^*(pt)$ , it follows that  $h_F$  is a homomorphism of  $E_T^*(pt)$ -algebras.

Moreover,  $h_F$ 's are compatible over  $\text{Face}(Q)$ . This follows as  $F \subseteq F'$  implies  $J_{F'}$  is obtained from  $J_F$  by adjoining  $e^T(u)$  for  $u \in \ker(\psi_{F'}) \setminus \ker(\psi_F)$ . Thus the corresponding projection

$$r_{F, F'}^*: E_T^*(pt)/J_F \longrightarrow E_T^*(pt)/J_{F'}$$

satisfies  $h_{F'} = r_{F, F'}^* \circ h_F$  whenever  $F \subseteq F'$ . Therefore we have constructed a well defined homomorphism

$$h: \Gamma_X \longrightarrow \mathcal{P}_E(Q, \lambda)$$

of  $E_T^*(pt)$ -algebras.

We now conclude by showing that the map  $h$  is an isomorphism. Given  $a \neq a' \in \Gamma_X \subseteq \prod_{i=1}^m E_T^*(x_i)$ . There exists at least one  $v_i$  such that  $a_i \neq a'_i$ . Thus  $h_{v_i}(a) \neq h_{v_i}(a')$  in  $E_T^*(x_i)$ . Hence  $h$  is injective. Let  $(a_F)$  be an element in the limit  $\mathcal{P}_E(Q, \lambda)$  of the functor  $\mathcal{E}_T^*$ . Then  $(a_F)$  determines  $(a_i)$  in  $\prod_{i=1}^m E_T^*(x_i)$  by restricting to the vertices

$a_i := a_{v_i} \in E_T^*(pt)$  for  $1 \leq i \leq m$ . Whenever  $v_i$  and  $v_j$  are connected by an edge  $e_{ij}$  in  $Q$ , we have

$$a_{e_{ij}} = r_{v_i, e_{ij}}^*(a_{v_i}) \quad \text{and} \quad a_{e_{ij}} = r_{v_j, e_{ij}}^*(a_{v_j})$$

and hence  $a_i \equiv a_j \pmod{J_{e_{ij}}}$ . Since  $J_{e_{ij}}$  is generated by  $e^T(u_{ij})$  it follows that  $a_i - a_j$  is divisible by  $e^T(u_{ij})$ . This implies by (3.4) that  $(a_i) \in \Gamma_X$  proving the surjectivity of  $h$ .  $\square$

We now illustrate the above theorem by describing the equivariant generalized cohomology ring of the retractable toric manifold of Example 2.10, as a piecewise polytopal algebra.

EXAMPLE 4.3. The edges  $e_{ij}$  in the polytope  $Q$  given in Figure 1 are listed below:

$$\begin{aligned} e_{13} &= F_1 \cap F_3, & e_{12} &= F_1 \cap F_2, & e_{23} &= F_1 \cap F_4, \\ e_{14} &= F_3 \cap F_2, & e_{36} &= F_3 \cap F_4, & e_{45} &= F_2 \cap F_5, \\ e_{46} &= F_3 \cap F_5, & e_{56} &= F_4 \cap F_5, & e_{25} &= F_2 \cap F_4. \end{aligned}$$

We list below the corresponding  $\psi_{e_{ij}}$  and  $u_{ij}$ .

$$\begin{aligned} \psi_{e_{13}} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & \psi_{e_{12}} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, & \psi_{e_{23}} &= \begin{bmatrix} 1 & 0 & 0 \\ 6 & -1 & 3 \end{bmatrix}, \\ \psi_{e_{14}} &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, & \psi_{e_{36}} &= \begin{bmatrix} 0 & 0 & 1 \\ 6 & -1 & 3 \end{bmatrix}, & \psi_{e_{45}} &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 4 \end{bmatrix}, \\ \psi_{e_{46}} &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix}, & \psi_{e_{56}} &= \begin{bmatrix} 6 & -1 & 3 \\ 1 & 1 & 4 \end{bmatrix}, & \psi_{e_{25}} &= \begin{bmatrix} 0 & 1 & 0 \\ 6 & -1 & 3 \end{bmatrix}, \\ u_{13} &= (0, 1, 0), & u_{12} &= (0, 0, 1), & u_{23} &= (0, 3, 1), \\ u_{14} &= (1, 0, 0), & u_{36} &= (1, -6, 0), & u_{45} &= (-4, 0, 1), \\ u_{46} &= (-1, 1, 0), & u_{56} &= (1, 3, -1), & u_{25} &= (1, 0, -2). \end{aligned}$$

Then  $E_T^*(X)$  consists of  $(a_i) \in \prod_{i=1}^6 E_T^*(x_i)$  satisfying the following relations:

$$\begin{aligned} a_1 - a_3 &\equiv 0 \pmod{e^T(0, 1, 0)}, & a_4 - a_6 &\equiv 0 \pmod{e^T(-1, 1, 0)}, \\ a_1 - a_4 &\equiv 0 \pmod{e^T(1, 0, 0)}, & a_3 - a_6 &\equiv 0 \pmod{e^T(1, -6, 0)}, \\ a_1 - a_2 &\equiv 0 \pmod{e^T(0, 0, 1)}, & a_4 - a_5 &\equiv 0 \pmod{e^T(-4, 0, 1)}, \\ a_2 - a_3 &\equiv 0 \pmod{e^T(0, 3, 1)}, & a_5 - a_6 &\equiv 0 \pmod{e^T(1, 3, -1)}, \\ a_2 - a_5 &\equiv 0 \pmod{e^T(1, 0, -2)}. \end{aligned}$$

REMARK 4.4. (1) In Section 4.2 of [BNSS21], the authors constructed the characteristic pair corresponding to a polytopal simplicial complex. So one can define retractable toric variety using this characteristic pair following Definition 2.7. Therefore, as a consequence we can get similar description of the equivariant generalized cohomology theories for retractable toric varieties arising from polytopal simplicial complexes.



- (2) In [DJ91], toric manifolds were studied and an invariant CW-structure of a toric manifold was constructed. So in particular, toric manifolds are retractable toric orbifolds, and hence Theorem 4.2 holds for this class of manifolds. See [DKU19] for the description of the equivariant  $K$ -ring of toric manifolds as a Stanley–Reisner ring.
- (3) In Subsection 4.3 of [BNSS21], the local groups  $G_F(v)$  are computed for torus orbifolds which are generalizations of toric orbifolds. So Definition 2.7 can be introduced in this category. Thus one can get similar description of the equivariant generalized cohomology theories for retractable torus orbifolds.
- (4) In a recent related paper [HW19], the authors address the following question (see [HW19, Question 1.4]): for which fans the  $T$ -equivariant  $K$ -theory ring of the associated toric variety is isomorphic to the ring of piecewise Laurent polynomial functions on the fan. In particular in [HW19, Theorem 7.2] they show that fans with *distant singular cones* satisfy this property. Our result Theorem 4.2 shows in particular that simplicial polytopal fans whose associated toric varieties are retractable toric orbifolds with respect to the action of the compact torus satisfy this property.
- (5) In the paper [LT97] the authors study symplectic toric orbifolds. In particular, in [LT97, Section 9] they show that every symplectic toric orbifold has the structure of a toric variety associated to the fan dual to the corresponding moment polytope and is hence a simplicial projective toric variety. The symplectic toric orbifold is therefore retractable if the associated moment polytope admits a retraction sequence satisfying the conditions of Definition 2.7.

ACKNOWLEDGEMENTS. The authors would like to thank Anthony Bahri, Nigel Ray and Jongbaek Song for helpful conversations. The authors are grateful to the referee for valuable comments and suggestions for improvement of the manuscript.

## References

- [ALR07] A. Adem, J. Leida and Y. Ruan, Orbifolds and Stringy Topology, [Cambridge Tracts in Math.](#), **171**, Cambridge Univ. Press, Cambridge, 2007.
- [BNSS21] A. Bahri, D. Notbohm, S. Sarkar and J. Song, On integral cohomology of certain orbifolds, [Int. Math. Res. Not. IMRN](#), **2021** (2021), 4140–4168.
- [BP02] V. M. Buchstaber and T. E. Panov, Torus Actions and Their Applications in Topology and Combinatorics, [Univ. Lecture Ser.](#), **24**, Amer. Math. Soc., Providence, RI, 2002.
- [BSS17] A. Bahri, S. Sarkar and J. Song, On the integral cohomology ring of toric orbifolds and singular toric varieties, [Algebr. Geom. Topol.](#), **17** (2017), 3379–3810.
- [DKU19] J. Dasgupta, B. Khan and V. Uma, Equivariant  $K$ -theory of quasitoric manifolds, [Proc. Indian Acad. Sci. Math. Sci.](#), **129** (2019), no. 5, art no. 72.
- [DJ91] M. W. Davis and T. Januszkiewicz, Convex polytopes, Coxeter orbifolds and torus actions, [Duke Math. J.](#), **62** (1991), 417–451.
- [tD70] T. tom Dieck, Bordism of  $G$ -manifolds and integrality theorems, [Topology](#), **9** (1970), 345–358.
- [HHH05] M. Harada, A. Henriques and T. S. Holm, Computation of generalized equivariant cohomologies of Kac–Moody flag varieties, [Adv. Math.](#), **197** (2005), 198–221.

- [HHRW16] M. Harada, T. S. Holm, N. Ray and G. Williams, The equivariant  $K$ -theory and cobordism rings of divisive weighted projective spaces, *Tohoku Math. J. (2)*, **68** (2016), 487–513.
- [HM03] A. Hattori and M. Masuda, Theory of multi-fans, *Osaka J. Math.*, **40** (2003), 1–68.
- [HW19] T. S. Holm and G. Williams, Mayer–Vietoris sequences and equivariant  $K$ -theory rings of toric varieties, *Homology Homotopy Appl.*, **21** (2019), 375–401.
- [LT97] E. Lerman and S. Tolman, Hamiltonian torus actions on symplectic orbifolds and toric varieties, *Trans. Amer. Math. Soc.*, **349** (1997), 4201–4230.
- [MP06] M. Masuda and T. Panov, On the cohomology of torus manifolds, *Osaka J. Math.*, **43** (2006), 711–746.
- [May96] J. P. May, Equivariant Homotopy and Cohomology Theory, *CBMS Regional Conf. Ser. in Math.*, **91**, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the Amer. Math. Soc., Providence, RI, 1996, with contributions by M. Cole, G. Comezana, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou and S. Waner.
- [PS10] M. Poddar and S. Sarkar, On quasitoric orbifolds, *Osaka J. Math.*, **47** (2010), 1055–1076.
- [Sar20] S. Sarkar, Equivariant  $K$ -theory of divisive torus orbifolds, *Rocky Mountain J. Math.*, **50** (2020), 255–266.
- [Seg68] G. Segal, Equivariant  $K$ -theory, *Inst. Hautes Études Sci. Publ. Math.*, **34** (1968), 129–151.
- [Sin01] D. P. Sinha, Computations of complex equivariant bordism rings, *Amer. J. Math.*, **123** (2001), 577–605.
- [VU06] V. Uma,  $K$ -theory of torus manifolds, In: *Toric Topology. International Conference, May 28–June 3, 2006, Osaka City Univ., Osaka, Japan, Contemp. Math.*, **460**, Amer. Math. Soc., 2008, 385–389.
- [Zie95] G. M. Ziegler, Lectures on Polytopes, *Grad. Texts in Math.*, **152**, Springer-Verlag, New York, 1995.

Soumen SARKAR

Department of Mathematics  
 Indian Institute of Technology Madras  
 India  
 E-mail: soumensarkar20@gmail.com

Vikraman UMA

Department of Mathematics  
 Indian Institute of Technology Madras  
 India  
 E-mail: vuma@iitm.ac.in