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Effective Łojasiewicz gradient inequality for Nash functions with application to finite determinacy of germs

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Abstract. Let $X \subset \mathbb{R}^n$ be a compact semialgebraic set and let $f : X \to \mathbb{R}$ be a nonzero Nash function. We give a Solernó and D'Acunto–Kurdyka type estimation of the exponent $\varrho \in [0, 1)$ in the Łojasiewicz gradient inequality $|\nabla f(x)| \geq C|f(x)|^{\varrho}$ for $x \in X$, $|f(x)| < \varepsilon$ for some constants $C, \varepsilon > 0$, in terms of the degree of a polynomial P such that $P(x, f(x)) = 0$, $x \in X$. As a corollary we obtain an estimation of the degree of sufficiency of non-isolated Nash function singularities.

1. Introduction.

Łojasiewicz inequalities are important tools in various branches of mathematics: differential equations, singularity theory and optimization (for more detailed references, see for example [**16**], [**18**], [**19**], [**22**] and [**34**]). Quantitative aspects, like estimates (or exact computation), of these exponents are subject of intensive study in real and complex algebraic geometry (see for instance [**18**], [**19**], [**20**] and [**33**]). The main results of this paper are effective estimations of the exponent in the Łojasiewicz gradient inequality for Nash functions (Theorems 2.1 and 2.2). As a corollary we obtain an effective estimation of the degree of sufficiency of non-isolated Nash functions singularities and a sufficient condition for Nash function germs at zero to be isotopical and topologically trivial along $[0, 1]$ (Corollary 1.3).

Determinacy of jets of functions with isolated singularity at zero was investigated by many authors, including Kuiper [**14**], Kuo [**15**], Bochnak and Łojasiewicz [**2**] for real functions and Chang and Lu [**5**], Teissier [**40**] and Bochnak and Kucharz [**1**] for complex functions. Similar investigations were also carried out for functions in a neighbourhood of infinity by Cassou-Noguès and Vui [**4**] (see also [**35**], [**37**]). The case of real jets with non-isolated singularities was studied among others by Grandjean [**11**] and Xu [**41**], and for complex functions by Siersma [**36**] and Pellikaan [**30**]. In the case of nondegenerate analytic functions f, g, a condition for topological triviality of deformations $f + tg$, $t \in [0, 1]$ in terms of Newton polyhedra was obtained by Damon and Gaffney [**8**], and for blow analytic triviality by Fukui and Yoshinaga [**9**]. Some algebraic conditions for finite determinacy of a smooth function jet were obtained by Kushner [**21**].

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1.1. Łojasiewicz gradient inequality.

Let $U \subset \mathbb{R}^n$ be an open semialgebraic set and let $a \in U$. Let $f, F : U \to \mathbb{R}$ be continuous semialgebraic functions such that $a \in F^{-1}(0) \subset f^{-1}(0) \subset U$. Then the following *Łojasiewicz inequality* holds:

$$|F(x)| \geq C|f(x)|^{\eta} \text{ in a neighbourhood of } a \in \mathbb{R}^n \text{ for some constant } C > 0. \tag{1.1}$$

The lower bound of the exponents η in (1.1) is called the *Łojasiewicz exponent* of the pair (F, f) *at* a and is denoted by $\mathcal{L}_a(F, f)$. It is known that $\mathcal{L}_a(F, f)$ is a rational number (see [**3**]) and the inequality (1.1) holds actually with $\eta = \mathcal{L}_a(F, f)$ on some neighbourhood of the point a for some positive constant C (see for instance [**39**]). An asymptotic estimate for $\mathcal{L}_a(F, f)$ was obtained by Solernó [**38**]:

$$\mathcal{L}_a(F, f) \leq D^{M^{c\ell}}, \tag{S}$$

where D is a bound for the degrees of the polynomials involved in a description of F, f and U; M is the number of variables in these formulas; ℓ is the maximum number of alternating blocks of quantifiers in these formulas; and c is an unspecified universal constant.

In this paper, we consider the case when F is equal to the gradient $\nabla f := (\partial f/\partial x_1, \ldots, \partial f/\partial x_n) : U \to \mathbb{R}^n$ of a Nash function f in $x = (x_1, \ldots, x_n)$. Recall that semialgebraic and analytic functions are called *Nash functions*.

Our main goal is to obtain an effective estimate for the exponent $\varrho \in [0, 1)$ in the following *Łojasiewicz gradient inequality* (see [**23**] or [**24**], cf. [**40**]):

$$|\nabla f(x)| \geq C|f(x)|^{\varrho} \text{ in a neighbourhood of } a \in \mathbb{R}^n \text{ for some constant } C > 0 \tag{Ł}$$

for an arbitrary Nash function $f : U \to \mathbb{R}$, where $f(a) = 0$, in terms of the degree of a polynomial $P \in \mathbb{R}[x, y]$ describing the graph of f. We denote by $|\nabla f(x)|$ the Euclidean norm of $\nabla f(x)$, i.e. $|\nabla f(x)|^2 = ((\partial f/\partial x_1)(x))^2 + \cdots + ((\partial f/\partial x_n)(x))^2$.

The smallest exponent ϱ in (Ł), denoted by $\varrho_a(f)$, is called the *Łojasiewicz exponent in the gradient inequality* at a. It is known that (Ł) holds with $\varrho = \varrho_a(f)$.

In the case of a polynomial function $f : \mathbb{R}^n \to \mathbb{R}$ of degree $d > 0$ such that 0 is an isolated point of $f^{-1}(0)$, Gwoździewicz [**12**] (cf. [**13**]) proved that

$$\varrho_0(f) \leq 1 - \frac{1}{(d-1)^n + 1}, \tag{G2}$$

and in the general case of an arbitrary polynomial f, D'Acunto and Kurdyka [**6**] (cf. [**7**], [**10**] and [**31**]) showed that

$$\varrho_0(f) \leq 1 - \frac{1}{d(3d-3)^{n-1}}, \quad \text{provided} \quad d \geq 2. \tag{DK}$$

If f is a rational function of the form $f = p/q$, where p, $q \in \mathbb{R}[x]$, $p(0) = 0$ and $q(0) \neq 0$, then $\varrho_0(f) = \varrho_0(p)$, so (G2) and (DK) hold with $d = \deg p$.

The aim of this paper is to show generalizations of the above estimates for Nash functions. The main results are Theorems 2.1 and 2.2 in Section 2. More precisely, let $U \subset \mathbb{R}^n$ be a neighbourhood of $a \in \mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a nonzero Nash function. We give a Solernó and D'Acunto–Kurdyka type estimation of the exponent $\varrho \in [0, 1)$ in the Łojasiewicz gradient inequality (Ł) in terms of the degree d of a nonzero polynomial P such that $P(x, f(x)) = 0$, $x \in U$. Namely, in Theorem 2.2 we obtain

$$\varrho_a(f) \le 1 - \frac{1}{2(2d-1)^{3n+1}}.$$

If additionally $n \ge 2$ and $(\partial P/\partial y)(a, f(a)) \ne 0$, then in Theorem 2.1 we obtain

$$\varrho_a(f) \le 1 - \frac{1}{d(3d-2)^n + 1}, \quad \text{provided} \quad d \ge 2.$$

The above estimates are comparable with the Solernó estimate (S), but our estimates are explicit.

As a corollary, we obtain the following inequality (see Corollary 3.6):

$$|\nabla f(x)| \ge C \operatorname{dist}(x, f^{-1}(0))^{2(2d-1)^{3n+1}-1} \quad \text{in a neighbourhood of } a. \tag{1.2}$$

If additionally $n \ge 2$ and $(\partial P/\partial y)(a, f(a)) \ne 0$, then

$$|\nabla f(x)| \ge C \operatorname{dist}(x, f^{-1}(0))^{d(3d-2)^n} \quad \text{in a neighbourhood of } a. \tag{1.3}$$

The inequalities (1.2), (1.3) are essential points in the effective estimate of the degree of sufficiency of non-isolated Nash function singularities given in the next section. The proof of these inequalities is based on Theorem 2.2 and estimates of the length of trajectories of the vector field ∇f in $U \setminus f^{-1}(0)$ (see Theorem 3.4).

1.2. Sufficiency of non-isolated Nash function singularities.

Let $\mathscr{C}_a^k(n)$ denote the set of $\mathscr{C}^k$ real functions defined in neighbourhoods of $a \in \mathbb{R}^n$.

By a *k-jet* at $a \in \mathbb{R}^n$ in the class $\mathscr{C}^\ell$ we mean a family of functions $w \subset \mathscr{C}_a^\ell(n)$, called *$\mathscr{C}^\ell$-realizations* of this jet, possessing the same Taylor polynomial of degree k at a. We also say that f *determines* a k-jet at a in $\mathscr{C}^\ell$ if f is a $\mathscr{C}^\ell$-realization of this jet. For a function $f \in \mathscr{C}_a^k(n)$, we denote by $j^k f(a)$ the k-jet at a (in $\mathscr{C}^k$) determined by f.

Let $Z \subset \mathbb{R}^n$ be a set such that $0 \in Z$ and let $k \in \mathbb{Z}$, $k > 0$. By a *k-Z-jet in the class $\mathscr{C}^k$*, or briefly a *k-Z-jet*, we mean an equivalence class $w \subset \mathscr{C}_0^k(n)$ of the following equivalence relation: $f \sim g$ iff for some neighbourhood $U \subset \mathbb{R}^n$ of the origin, $j^k f(a) = j^k g(a)$ for $a \in Z \cap U$ (cf. [**27**], [**41**]). The functions $f \in w$ are called *$\mathscr{C}^k$-Z-realizations* of the jet w and we write $w = j_Z^k f$. The set of all jets $j_Z^k f$ is denoted by $J_Z^k(n)$.

The k-Z-jet $w \in J_Z^k(n)$ is said to be *$\mathscr{C}^r$-Z-sufficient* (resp. *Z-v-sufficient*) in the class $\mathscr{C}^k$ if for every pair of its $\mathscr{C}^k$-Z-realizations f and g there exist sufficiently small neighbourhoods $U_1, U_2 \subset \mathbb{R}^n$ of 0, and a $\mathscr{C}^r$ diffeomorphism $\varphi : U_1 \to U_2$, such that $f \circ \varphi = g$ in U_1 (resp. there exists a homeomorphism $\varphi : [f^{-1}(0) \cup Z] \cap U_1 \to [g^{-1}(0) \cup Z] \cap U_2$ with $\varphi(0) = 0$ and $\varphi(Z \cap U_1) = Z \cap U_2$).

The classical and significant result on sufficiency of jets is the following:

Theorem 1.1 (Kuiper, Kuo, Bochnak–Łojasiewicz). *Let w be a k-jet at $0 \in \mathbb{R}^n$ and let f be its $\mathscr{C}^k$-realization. If $f(0) = 0$ then the following conditions are equivalent*:

(a) *w is $\mathscr{C}^0$-sufficient in $\mathscr{C}^k$*,

(b) *w is v-sufficient in $\mathscr{C}^k$*,

(c) *$|\nabla f(x)| \geq C|x|^{k-1}$ in a neighbourhood of the origin for some $C > 0$.*

The implication (c)⇒(a) was proved by Kuiper [**14**] and Kuo [**15**], (b)⇒(c) by Bochnak and Łojasiewicz [**2**], and (a)⇒(b) is obvious (cf. [**29**]).

Let us recall the notions of isotopy and topological triviality. Let $\Omega \subset \mathbb{R}^n$ be a neighbourhood of $0 \in \mathbb{R}^n$ and let $Z \subset \mathbb{R}^n$ with $0 \in Z$.

A continuous mapping $H : \Omega \times [0,1] \to \mathbb{R}^n$ is called an *isotopy near Z at zero* if:
(a) $H_0(x) = x$ for $x \in \Omega$ and $H_t(x) = x$ for $t \in [0,1]$ and $x \in \Omega \cap Z$,
(b) for any t the mapping $H_t : \Omega \to \mathbb{R}^n$ is a homeomorphism onto $H_t(\Omega)$,
where $H_t(x) = H(x,t)$ for $x \in \Omega$, $t \in [0,1]$.

Functions $f : \Omega_1 \to \mathbb{R}$, $g : \Omega_2 \to \mathbb{R}$, where $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are neighbourhoods of $0 \in \mathbb{R}^n$, are called *isotopical near Z at zero* if there exists an isotopy near Z at zero, $H : \Omega \times [0,1] \to \mathbb{R}^n$, with $\Omega \subset \Omega_1 \cap \Omega_2$, such that $f(H_1(x)) = g(x)$, $x \in \Omega$.

A deformation $f + tg$ is called *topologically trivial near Z* along $[0,1]$ if there exists an isotopy near Z at zero, $H : \Omega \times [0,1] \to \mathbb{R}^n$, with $\Omega \subset \Omega_1 \cap \Omega_2$, such that $f(H(t,x)) + tg(H(t,x))$ does not depend on t.

Theorem 1.1 concerns the case of an isolated singularity of f at 0, i.e. 0 is an isolated zero of ∇f. In the case of a non-isolated singularity of f at 0, from [**27**, Theorems 1.3 and 1.4] (cf. [**41**]) we have the following criterion for sufficiency of jets.

Theorem 1.2. *Let $f \in \mathscr{C}_0^k(n)$ be a $\mathscr{C}^k$-Z-realization of a k-Z-jet $w \in J_Z^k(n)$, where $k > 1$ and $Z = f^{-1}(0)$, $0 \in Z$, and suppose $(\nabla f)^{-1}(0) \subset Z$. Then the following conditions are equivalent*:

(a) *The k-Z-jet w is $\mathscr{C}^0$-Z-sufficient in $\mathscr{C}^k$.*

(b) *For any $\mathscr{C}^k$-Z-realizations f_1, f_2 of w, the deformation $f_1 + t(f_2 - f_1)$, $t \in \mathbb{R}$, is topologically trivial along $[0,1]$.*

(c) *Any two $\mathscr{C}^k$-Z-realizations of w are isotopical at zero.*

(d) *The k-Z-jet w is Z-v-sufficient in $\mathscr{C}^k$.*

(e) *There exists a positive constant C such that*

$$|\nabla f(x)| \geq C \operatorname{dist}(x,Z)^{k-1} \quad \textit{in a neighbourhood of the origin.}$$

Let $f : U \to \mathbb{R}$ be a Nash function, where $U \subset \mathbb{R}^n$ is a neighbourhood of the origin, let $Z = f^{-1}(0)$, and suppose $0 \in Z$.

As a consequence of Theorem 1.2 and inequality (1.2) we obtain

Corollary 1.3. *Let $k = 2(2d-1)^{3n+1}$, where $d = \deg_0 f$, and let $w \in J_Z^k(n)$ be the k-Z-jet for which f is a $\mathscr{C}^k$-Z-realization. Then the following conditions hold*:

(a) *The k-Z-jet w is $\mathscr{C}^0$-Z-sufficient in $\mathscr{C}^k$.*

(b) *For any $\mathscr{C}^k$-Z-realizations f_1, f_2 of w, the deformation $f_1 + t(f_2 - f_1)$, $t \in \mathbb{R}$, is topologically trivial along* $[0, 1]$.

(c) *Any two $\mathscr{C}^k$-Z-realizations of w are isotopical at zero.*

(d) *The k-Z-jet w is Z-v-sufficient in $\mathscr{C}^k$.*

Under additional assumption on f, from Theorem 1.2 and inequality (1.3), we obtain

COROLLARY 1.4. *Assume that there exists a nonzero polynomial $P \in \mathbb{R}[x, y]$ such that $P(x, f(x)) = 0$ for $x \in U$ and $(\partial P/\partial y)(a, f(a)) \neq 0$. Then the assertion of Corollary* 1.3 *holds with $k = d(3d-2)^n + 1$, where $d = \deg P$.*

REMARK 1.5. If f is a polynomial of degree $d > 1$ or a rational function $f = p/q$, where $p(0) = 0$, $q(0) \neq 0$ and $d = \deg p$, then from Theorem 1.2 and by (DK), the assertion of Corollary 1.3 holds with $k = d(3d-3)^{n-1}$. If additionally the origin is an isolated zero of f, then by (G2) the assertion of Corollary 1.3 holds with $k = (d-1)^n + 1$.

2. Łojasiewicz gradient inequality.

Let $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$ is a connected neighbourhood of $a \in \mathbb{R}^n$, be a Nash function. Let $P \in \mathbb{R}[x, y]$ be the unique irreducible real polynomial such that

$$P(x, f(x)) = 0 \quad \text{for } x \in U, \tag{2.1}$$

and let

$$d = \deg P.$$

We will call this number d the *degree of the Nash function f at a* and denote it by $\deg_a f$. Obviously $d = \deg_a f > 0$ is uniquely determined. For $d = 1$, the function f is linear and (Ł) holds with $\varrho = 0$, so we will assume that $d > 1$. We will also assume that $\nabla f(a) = 0$, because in the opposite case (Ł) holds with $\varrho = 0$.

Put

$$\mathcal{R}(n, d) = \max\{2d(2d-1), d(3d-2)^n\} + 1.$$

The main result of this section is the following theorem.

THEOREM 2.1. *Let $f : U \to \mathbb{R}$ be a nonzero Nash function such that $f(a) = 0$ and $\nabla f(a) = 0$. Assume that for the unique polynomial P satisfying* (2.1) *we have*

$$\frac{\partial P}{\partial y}(a, f(a)) \neq 0. \tag{2.2}$$

Then $\varrho_a(f) \leq 1 - 1/\mathcal{R}(n, d)$. Moreover, for $\varrho = 1 - 1/\mathcal{R}(n, d)$ and some constants $C, \varepsilon > 0$,

$$|\nabla f(x)| \geq C|f(x)|^{\varrho} \quad \textit{for} \quad |x-a|<\varepsilon, \quad |f(x)|<\varepsilon. \tag{2.3}$$

Without the assumption (2.2), we have a somewhat weaker estimation of the exponent $\varrho_a(f)$ than that in Theorem 2.1. Namely, let

$$\mathcal{S}(n,d) = 2(2d-1)^{3n+1}.$$

Theorem 2.2. *Let $f : U \to \mathbb{R}$ be a nonzero Nash function such that $f(a) = 0$ and $\nabla f(a) = 0$ and let P be the unique polynomial satisfying* (2.1). *Then $\varrho_a(f) \leq 1 - 1/\mathcal{S}(n,d)$. Moreover,* (2.3) *holds actually with $\varrho = 1 - 1/\mathcal{S}(n,d)$.*

Theorems 2.1 and 2.2 are generalizations for Nash functions of the above mentioned results by Gwoździewicz, D'Acunto and Kurdyka in the polynomial function case. They are also comparable with Solernó's estimate (S), but our estimates are explicit. In the case of Nash functions with isolated singularity at zero, a similar result was obtained in [**17**].

We give the proofs of Theorems 2.1 and 2.2 in Section 5.

3. Łojasiewicz inequality.

Let $X \subset \mathbb{R}^n$ be a compact semialgebraic set and let $f : X \to \mathbb{R}$ be a Nash function. Then f is defined in a neighbourhood of X. So, there exists a compact semialgebraic set $Y \subset \mathbb{R}^n$ such that $X \subset \operatorname{Int} Y$ and f is defined on Y.

The *degree* of f is defined to be $\sup\{\deg_a f : a \in X\}$ and is denoted by $\deg_X f$. In fact, $\deg_X f = \max\{\deg_a f : a \in X\}$. Moreover, one can assume that Y was chosen in such a manner that $\deg_X f = \deg_Y f$.

Let $\operatorname{dist}(x,V)$ denote the distance of a point $x \in \mathbb{R}^n$ to a set $V \subset \mathbb{R}^n$ in the Euclidean norm (with $\operatorname{dist}(x,V) = 1$ if $V = \emptyset$).

3.1. Global Łojasiewicz gradient inequality.

Theorems 2.1 and 2.2 have a local character. From these theorems we obtain a *global Łojasiewicz gradient inequality.*

Corollary 3.1. *Let $d = \deg_X f$. If $(\nabla f)^{-1}(0) \subset f^{-1}(0)$ then for some positive constant C,*

$$|\nabla f(x)| \geq C|f(x)|^{\varrho} \quad \textit{for} \quad x \in X \tag{3.1}$$

with $\varrho = 1 - 1/\mathcal{S}(n,d)$. If additionally there exists a polynomial $P \in \mathbb{R}[x,y]$ such that $P(x,f(x)) = 0$ and $(\partial P/\partial y)(x,f(x)) \neq 0$ for $x \in X$ and $d_1 = \deg P$, then (3.1) *holds with $\varrho = 1 - 1/\mathcal{R}(n,d_1)$.*

Denote by $\varrho_X(f)$ the smallest exponent ϱ for which (3.1) holds. We call it the *Łojasiewicz exponent in the gradient inequality* on X. It is known that the inequality (3.1) holds with $\varrho = \varrho_X(f)$. So, from Corollary 3.1 we obtain

Corollary 3.2. $\varrho_X(f) \leq 1 - 1/S(n,d)$.

3.2. Length of trajectory.

Let $f : X \to \mathbb{R}$ be a nonzero Nash function such that $(\nabla f)^{-1}(0) \subset f^{-1}(0)$, let $\varrho \in (0,1)$ and $C > 0$ be such that the global inequality (3.1) in Corollary 3.1 holds in X, and let $V = f^{-1}(0)$. Then $\nabla f(x) \neq 0$ for $x \in X \setminus V$.

Let $\varphi(t) = |t|^{1-\varrho}$ for $t \in \mathbb{R}$. By the same argument as in the proof of [**18**, Proposition 1] we obtain (cf. [**16**])

PROPOSITION 3.3 (Kurdyka–Łojasiewicz inequality). *Under the above notations,*

$$|\nabla(\varphi \circ f)(x)| \geq (1-\varrho)C \quad \textit{for } x \in X \setminus V.$$

We will also assume that $\overline{\operatorname{Int} X \setminus V} = X$. Let

$$U_{X,f} = \left\{ x \in \operatorname{Int} X : \frac{1}{C(1-\varrho)} |f(x)|^{1-\varrho} < \operatorname{dist}(x, \mathbb{R}^n \setminus X) \right\}.$$

Then $U_{X,f} \subset X$ is a neighbourhood of $(\operatorname{Int} X) \cap V$.

Take a global trajectory $\gamma : [0, s) \to U_{X,f} \setminus V$ of the vector field

$$H(x) = -\operatorname{sign} f(x) \frac{\nabla f(x)}{|\nabla f(x)|} \quad \text{for } x \in U_{X,f} \setminus V.$$

Then the function $f \circ \gamma$ is monotonic, so the limit $\lim_{t \to s} f \circ \gamma(t)$ exists.

Let $\operatorname{length} \gamma$ denote the length of γ. Since $|\gamma'(t)| = 1$, we have $\operatorname{length} \gamma = s$.

The following generalization of [**18**, Theorem 1] has a similar proof.

THEOREM 3.4. *The limit* $\lim_{t \to s} \gamma(t)$ *exists and belongs to* V. *Moreover,*

$$\operatorname{dist}(\gamma(0), V) \leq \operatorname{length} \gamma \leq \frac{1}{(1-\varrho)C} |f(\gamma(0))|^{1-\varrho}.$$

From Theorem 3.4 we have

COROLLARY 3.5. *Under the assumptions and notations of Theorem* 3.4,

$$|f(x)| \geq (C(1-\varrho))^{1/(1-\varrho)} \operatorname{dist}(x, V)^{1/(1-\varrho)}, \quad x \in U_{X,f},$$

and

$$|\nabla f(x)| \geq (C(1-\varrho))^{\varrho/(1-\varrho)} \operatorname{dist}(x, V)^{\varrho/(1-\varrho)}, \quad x \in U_{X,f}.$$

Similarly to [**18**], we obtain a version of the above corollary in the complex case with the same formulation.

From Corollaries 3.1, 3.5 and Theorem 2.2, we immediately obtain

COROLLARY 3.6. *Let* $d = \deg_X f$. *Then there exists a positive constant* C *such that*

$$|f(x)| \geq C \operatorname{dist}(x, V)^{2(2d-1)^{3n+1}}, \quad x \in X,$$

and

$$|\nabla f(x)| \geq C \operatorname{dist}(x, V)^{2(2d-1)^{3n+1}-1}, \quad x \in X.$$

If additionally $n \geq 2$ and there exists a polynomial $P \in \mathbb{R}[x, y]$ such that $P(x, f(x)) = 0$ and $(\partial P/\partial y)(x, f(x)) \neq 0$ for $x \in X$, and $d = \deg P$, then

$$|f(x)| \geq C \operatorname{dist}(x, V)^{d(3d-2)^n+1}, \quad x \in X,$$

and

$$|\nabla f(x)| \geq C \operatorname{dist}(x, V)^{d(3d-2)^n}, \quad x \in X.$$

3.3. Łojasiewicz exponent.

Corollary 3.5 implies the known fact that the exponents $\alpha > 0$ in the inequality

$$|f(x)| \geq C \operatorname{dist}(x, V)^{\alpha}, \quad x \in X, \tag{3.2}$$

for some positive constant C, are bounded below. The inequality (3.2) is called the *Łojasiewicz inequality for f on X* and the lower bound of the exponents $\alpha > 0$ is the *Łojasiewicz exponent* of f on X, denoted by $\mathcal{L}_X(f)$. It is known that (3.2) holds with $\alpha = \mathcal{L}_X(f)$ and some positive constant C.

From Theorem 3.4 we obtain

Corollary 3.7. $\mathcal{L}_X(f) \leq \dfrac{1}{1 - \varrho_X(f)}$.

Corollary 3.5 implies

Corollary 3.8. *If $d = \deg_X f$, then $\mathcal{L}_X(f) \leq 2(2d-1)^{3n+1}$.*

For $n \geq 4$ the above estimate is sharper than the one given in [**20**] for continuous semialgebraic functions: $\mathcal{L}_X(f) \leq d(6d-3)^{n+r-1}$, where $r \leq n(n+1)/2$ is the degree of complexity of f, equal to the number of inequalities necessary to define the graph of f, and d is the maximal degree of polynomials describing the graph of f. Consequently, this gives the estimate $\mathcal{L}_X(f) \leq d(6d-3)^{n+n(n+1)/2-1}$ in terms of the degree only. So, the estimate in Corollary 3.8 is more exact than the one above for $n \geq 4$.

4. Total degree of algebraic sets.

Let $\mathbb{C}[x]$ denote the ring of complex polynomials in $x = (x_1, \dots, x_n)$.

Let $f = (f_1, \dots, f_r) : \mathbb{C}^n \to \mathbb{C}^r$ be a polynomial mapping with $\deg f_i > 0$ for $i = 1, \dots, r$. Let $V = f^{-1}(0) \subset \mathbb{C}^n$.

The *total degree* of V is the number

$$\delta(V) = \deg V_1 + \cdots + \deg V_s,$$

where $V = V_1 \cup \cdots \cup V_s$ is the decomposition into irreducible components (see [**25**]).

We have the following useful fact (see [**25**]).

Fact 4.1. *If $V, W \subset \mathbb{C}^n$ are algebraic sets, then*

$$\delta(V \cap W) \leq \delta(V)\delta(W).$$

From Fact 4.1 and the definition of total degree of algebraic sets we have the following two facts (cf. [**25**]).

Fact 4.2. $\delta(V) \leq \deg f_1 \cdots \deg f_r$. *In particular, for any irreducible component V_j of V we have*

$$\deg V_j \leq \deg f_1 \cdots \deg f_r.$$

Fact 4.3. *Let $L : \mathbb{C}^n \to \mathbb{C}^k$ be a linear mapping. Then*

$$\delta(\overline{L(V)}) \leq \delta(V).$$

We will need the following lemma (see [**17**, Lemma 3.20]).

Lemma 4.4. *Let V_j be an irreducible component of the set V, and suppose* $\dim V_j \geq 1$. *Then for a generic linear mapping $L = (L_1, \ldots, L_{n-1}) : \mathbb{C}^r \to \mathbb{C}^{n-1}$ the set V_j is an irreducible component of the set of common zeros of the system of equations*

$$L_i \circ f = 0, \quad i = 1, \ldots, n-1.$$

In particular,

$$\deg V_j \leq \deg(L_1 \circ f) \cdots \deg(L_{n-1} \circ f).$$

Moreover, we can take $L_1(y_1, \ldots, y_r) = y_1$.

5. Proofs of Theorems 2.1 and 2.2.

The idea of the proofs is similar to that in [**17**, Proof of Theorem 1.2].

Without loss of generality, we may assume that $a = 0$. Let $f : U \to \mathbb{R}$ be a nonzero Nash function defined in an open neighbourhood $U \subset \mathbb{R}^n$ of the origin such that $f(0) = 0$ and $\nabla f(0) = 0$. Let $P \in \mathbb{R}[x, y]$ be the unique irreducible polynomial satisfying (2.1) and let $d = \deg P$.

Since the set of critical values of a differentiable semialgebraic function is finite, we have

Fact 5.1. *There exists $\varepsilon > 0$ such that f has no critical values in the interval $(-\varepsilon, \varepsilon)$ except* 0.

Let $\varepsilon > 0$ be as in Fact 5.1. Take $r > 0$. Denote by Ω the closed ball

$$\Omega := \{x \in \mathbb{R}^n : |x| \leq r\}$$

and by $\partial\Omega$ the sphere $\{x \in \mathbb{R}^n : |x| = r\}$. Suppose that $\Omega \subset U$. Define a semialgebraic set $\Gamma \subset \Omega$ by

$$\Gamma := \{x \in \Omega : \forall_{\zeta\in\Omega}\ f(x) = f(\zeta) \ \Rightarrow\ |\nabla f(x)| \le |\nabla f(\zeta)|\}.$$

Then by the definition of Γ we have

Fact 5.2. *Let $\varrho \in \mathbb{R}$ and let $C > 0$. If $|\nabla f(x)| \ge C|f(x)|^{\varrho}$ for $x \in \Gamma$ such that $|f(x)| < \varepsilon$, then $|\nabla f(x)| \ge C|f(x)|^{\varrho}$ for $x \in \Omega$, $|f(x)| < \varepsilon$.*

Let $\varrho_0 = \varrho_0(f)$. Then, decreasing r if necessary, we can assume that

$$|\nabla f(x)| \ge C|f(x)|^{\varrho_0} \quad \text{for } x \in \Omega \text{ and some constant } C > 0. \tag{5.1}$$

Let us fix such an r.

Consider the case $n = 1$. Denote by $\operatorname{ord}_0 f$ the order of f at zero. Then f has an isolated zero and singularity at zero, $\operatorname{ord}_0 f > 0$ and the inequality (2.3) holds with

$$\varrho_0(f) = \frac{\operatorname{ord}_0 f - 1}{\operatorname{ord}_0 f} = 1 - \frac{1}{\operatorname{ord}_0 f}. \tag{5.2}$$

Let the polynomial P be of the form $P(x_1, y) = p_0(x_1)y^d + p_1(x_1)y^{d-1} + \cdots + p_d(x_1)$, where $p_0, \ldots, p_d \in \mathbb{R}[x_1]$. As P is irreducible, $p_d \ne 0$ and $\operatorname{ord}_0 p_d \le d$. Since

$$-p_d(x_1) = f(x_1)\big(p_0(x_1)(f(x_1))^{d-1} + p_1(x_1)(f(x_1))^{d-2} + \cdots + p_{d-1}(x_1)\big),$$

we have $\operatorname{ord}_0 f \le \operatorname{ord}_0 p_d \le d$. Together with (5.2) this gives (2.3) with $\varrho_0(f) = 1 - 1/d$ and the assertions of Theorems 2.1 and 2.2 in the case $n = 1$.

In the remainder of this article we will assume that $n > 1$.

By (5.1) and the curve selection lemma, there exists an analytic curve $\varphi : [0, 1) \to \Omega$ for which $f(\varphi(0)) = 0$, $f(\varphi(\xi)) \ne 0$ for $\xi \in (0, 1)$ and for some constant $C_1 > 0$,

$$C|f(\varphi(\xi))|^{\varrho_0} \le |\nabla f(\varphi(\xi))| \le C_1|f(\varphi(\xi))|^{\varrho_0}, \quad \xi \in [0, 1) \tag{5.3}$$

(cf. [**39**]). By Fact 5.2 we may assume that $\varphi([0, 1)) \subset \Gamma$. Then we have two cases:

I. $\varphi\big((0, 1)\big) \subset \operatorname{Int}\Omega$,

II. $\varphi\big([0, 1)\big) \subset \partial\Omega$.

We will use the Lagrange multipliers theorem to describe the relation between the values $y = f(x)$ and $u = |\nabla f(x)|^2$ for $x \in \Gamma$, so we put

$$\Gamma_I = \{x \in \Omega : \exists_{\lambda\in\mathbb{R}}\ \nabla|\nabla f(x)|^2 - \lambda\nabla f(x) = 0\},$$
$$\Gamma_{II} = \{x \in \partial\Omega : |f(x)| < \varepsilon \ \wedge\ \exists_{\lambda_1,\lambda_2\in\mathbb{R}}\ \nabla|\nabla f(x)|^2 - \lambda_1\nabla f(x) - 2\lambda_2 x = 0\}.$$

To fulfill the assumptions of the Lagrange theorem we will need

Lemma 5.3. *There exists $\varepsilon > 0$ such that for every $x \in \partial\Omega$ and every $y \in \mathbb{R}$ such that $0 < |y| < \varepsilon$ and $y = f(x)$, the vectors $\nabla\big(|x|^2 - r^2\big)$ and $\nabla f(x)$ (that is, $2x$ and $\nabla f(x)$) are linearly independent.*

Proof. If $f|_{\partial\Omega}$ is a constant function then the assertion is obvious. Assume that f is not constant on $\partial\Omega$. Then, by Fact 5.1, there exists $\varepsilon > 0$ such that $\nabla f(x) \ne 0$ for $x \in \partial\Omega$, $0 < |f(x)| < \varepsilon$.

Suppose to the contrary that for any $\varepsilon > 0$ there exist $x \in \partial\Omega$ and $y_\varepsilon \in \mathbb{R}$ with $0 < |y_\varepsilon| < \varepsilon$ such that $y_\varepsilon = f(x)$ and $\nabla f(x) = \xi \cdot 2x$ for some $\xi \in \mathbb{R} \setminus \{0\}$. Then by the curve selection lemma there exist analytic curves $\gamma : [0,1) \to \partial\Omega$ with $\gamma((0,1)) \subset \Omega \setminus f^{-1}(0)$ and $f(\gamma(0)) = 0$, and $\alpha : [0,1) \to \mathbb{R}$, such that for $t \in (0,1)$,

$$\nabla f\big(\gamma(t)\big) = \alpha(t) \cdot 2\gamma(t).$$

Then

$$(f \circ \gamma)'(t) = \langle \nabla f(\gamma(t)), \gamma'(t)\rangle = \alpha(t)\langle \gamma(t), \gamma'(t)\rangle = 0,$$

since $|\gamma(t)|^2 = 1$, and consequently $f \circ \gamma$ is a constant function equal to 0. This contradicts the choice of γ and ends the proof. □

By the Lagrange multipliers theorem, Fact 5.1 and Lemma 5.3 we obtain

FACT 5.4. *Let $\varepsilon > 0$ fulfill Fact 5.1 and Lemma 5.3. Take a point $x_0 \in \Omega$ such that $0 < |f(x_0)| < \varepsilon$.*

(a) *If $x_0 \in \Gamma \cap \operatorname{Int}\Omega$ then x_0 is a critical point of the function $\Omega \ni x \mapsto |\nabla f(x)|^2 \in \mathbb{R}$ on the set $f^{-1}(f(x_0)) \cap \Omega$. In particular, $\Gamma \cap \operatorname{Int}\Omega \subset \Gamma_I$.*

(b) *If $n \geq 3$, $x_0 \in \Gamma \cap \partial\Omega$ then x_0 is a critical point of the function $\partial\Omega \ni x \mapsto |\nabla f(x)|^2 \in \mathbb{R}$ on the set $f^{-1}(f(x_0)) \cap \partial\Omega$. In particular, $\Gamma \cap \partial\Omega \subset \Gamma_{II}$.*

Let $\mathbb{M} = \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n$, and let $\mathbb{X} \subset \mathbb{M}$ be the Zariski closure of the set

$$\Big\{\big(x, f(x), |\nabla f(x)|^2, \nabla f(x), \nabla|\nabla f(x)|^2\big) \in \mathbb{M} : x \in \Omega\Big\}.$$

We will determine polynomials describing a certain algebraic set $\mathbb{Y} \subset \mathbb{M}$ containing $\mathbb{X}$ as an irreducible component. Let $G \in \mathbb{C}[x,y,u]$, where u is a variable, be the polynomial defined by

$$G(x,y,u) = \sum_{i=1}^{n} \left(\frac{\partial P}{\partial x_i}(x,y)\right)^2 - \left(\frac{\partial P}{\partial y}(x,y)\right)^2 \cdot u. \tag{5.4}$$

It is easy to observe that $G(x, f(x), |\nabla f(x)|^2) = 0$ for $x \in \Omega$. In particular, the polynomial G vanishes on $\mathbb{X}$.

Take systems of variables $t = (t_1, \ldots, t_n)$, $z = (z_1, \ldots, z_n)$, and let $G_1, G_{2,i}, G_{3,i} \in \mathbb{C}[x,y,u,t,z]$ be defined by

$$\begin{aligned}
G_1(u,t) &= u - t_1^2 - \cdots - t_n^2, \\
G_{2,i}(x,y,t) &= \frac{\partial P}{\partial x_i}(x,y) + \frac{\partial P}{\partial y}(x,y)t_i, && 1 \leq i \leq n, \\
G_{3,i}(x,y,u,t,z) &= \frac{\partial G}{\partial x_i}(x,y,u) + \frac{\partial G}{\partial y}(x,y,u)\, t_i - \left(\frac{\partial P}{\partial y}(x,y)\right)^2 \cdot z_i, && 1 \leq i \leq n.
\end{aligned}$$

Let $\mathbb{Y} \subset \mathbb{M}$ be the closure of the constructible set

$$\mathbb{Y}^0 = \Big\{ w = (x,y,u,t,z) \in \mathbb{M} : P(x,y) = 0,\ \frac{\partial P}{\partial y}(x,y) \neq 0,\ G_1(x,y,u) = 0,\\ G_{2,i}(x,y,t) = 0,\ G_{3,i}(w) = 0,\ 1 \le i \le n \Big\}.$$

Obviously $\mathbb{X} \subset \mathbb{Y}$, and locally $\mathbb{Y}^0$ is the graph of a complex Nash mapping (i.e. a holomorphic mapping whose graph is contained in a complex algebraic set of the same dimension). Moreover, we have

Lemma 5.5. *The set $\mathbb{X}$ is an irreducible component of $\mathbb{Y}$. Moreover, $\mathbb{Y}^0$ is a Zariski open and dense subset of $\mathbb{Y}$, and any point $w = (x_0, y_0, u_0, t_0, z_0) \in \mathbb{Y}^0$ has a neighbourhood $B \subset \mathbb{M}$ such that $\mathbb{Y} \cap B = \mathbb{Y}^0 \cap B$ and*

$$\mathbb{Y}^0 \cap B = \big\{ w = (x, g(x), h(x), \nabla g(x), \nabla h(x)) \in \mathbb{M} : x \in \Delta \big\}$$

for some holomorphic function $g : \Delta \to \mathbb{C}$, where $\Delta \subset \mathbb{C}^n$ is a neighbourhood of x_0, and $h(x) = ((\partial g/\partial x_1)(x))^2 + \cdots + ((\partial g/\partial x_n)(x))^2$.

Proof. Since P is an irreducible polynomial, $\partial P/\partial y$ does not vanish on $\mathbb{X}$. So, by the implicit function theorem, $\{w = (x,y,u,t,z) \in \mathbb{X} : (\partial P/\partial y)(x,y) \neq 0\}$ is an open and dense subset of $\mathbb{X}$, and moreover it is a smooth and connected submanifold of $\mathbb{Y}^0$. Consequently, $\mathbb{X}$ is an irreducible component of $\mathbb{Y}$. The "moreover" part of the assertion follows immediately from the implicit function theorem. □

Define $G_0, G_{4,i,j}, G_{4,i,j,k} \in \mathbb{C}[x,y,u,t,z]$ by

$$G_0(x) = x_1^2 + \cdots + x_n^2 - r^2,$$

$$G_{4,i,j}(t,z) = \det \begin{bmatrix} t_i & z_i \\ t_j & z_j \end{bmatrix}, \qquad 1 \le i < j \le n,$$

$$G_{4,i,j,k}(x,t,z) = \det \begin{bmatrix} t_i & z_i & x_i \\ t_j & z_j & x_j \\ t_k & z_k & x_k \end{bmatrix}, \qquad 1 \le i < j < k \le n,$$

where the polynomials $G_{4,i,j,k}$ are defined if $n \ge 3$. Put

$$\mathbb{X}_I = \big\{ w = (x,y,u,t,z) \in \mathbb{X} : G_{4,i,j}(t,z) = 0,\ 1 \le i < j \le n \big\},$$
$$\mathbb{X}_{II} = \big\{ w = (x,y,u,t,z) \in \mathbb{X} : G_0(x) = 0,\ G_{4,i,j,k}(x,t,z) = 0,\ 1 \le i < j < k \le n \big\},$$
$$\mathbb{L}_I = \big\{ (w,\lambda) = (x,y,u,t,z,\lambda) \in \mathbb{X} \times \mathbb{C} : z = \lambda t \big\},$$
$$\mathbb{L}_{II} = \big\{ (w,\lambda_1,\lambda_2) = (x,y,u,t,z,\lambda_1,\lambda_2) \in \mathbb{X} \times \mathbb{C} \times \mathbb{C} : G_0(x) = 0,\ z = \lambda_1 t + \lambda_2 x \big\},$$
$$\mathbb{Y}_I = \big\{ w = (x,y,u,t,z) \in \mathbb{Y} : G_{4,i,j}(t,z) = 0,\ 1 \le i < j \le n \big\},$$
$$\mathbb{Y}_{II} = \big\{ w = (x,y,u,t,z) \in \mathbb{Y} : G_0(x) = 0,\ G_{4,i,j,k}(x,t,z) = 0,\ 1 \le i < j < k \le n \big\},$$
$$\mathscr{Z}_I = \big\{ w = (x,y,u,t,z) \in \mathbb{X} : x \in \Gamma_I \big\},$$
$$\mathscr{Z}_{II} = \big\{ w = (x,y,u,t,z) \in \mathbb{X} : x \in \Gamma_{II} \big\},$$
$$\mathscr{F} = \big\{ w = (x,y,u,t,z) \in \mathbb{X} : x \in \varphi((0,1)) \big\},$$

where the sets $\mathbb{X}_{II}$, $\mathbb{L}_{II}$ and $\mathbb{Y}_{II}$ are defined for $n \geq 3$.

Obviously $\mathbb{X}_I \subset \mathbb{Y}_I$ and $\mathbb{X}_{II} \subset \mathbb{Y}_{II}$. Moreover, any irreducible component of $\mathbb{X}_I$ is an irreducible component of $\mathbb{Y}_I$. The same holds for $\mathbb{X}_{II}$ and $\mathbb{Y}_{II}$. Additionally, by the Lagrange multiplier theorem and Facts 5.1, 5.4 we immediately obtain

FACT 5.6. (a) *Let*

$$A_I = \{w \in \mathbb{X} : \exists_{\lambda \in \mathbb{C}}\, (w, \lambda) \in \mathbb{L}_I\}.$$

If $\varphi((0,1)) \subset \operatorname{Int}\Omega$ *then* $\mathscr{F} \subset \mathcal{Z}_I \subset A_I \subset \mathbb{X}_I \subset \mathbb{Y}_I$ *and there exists an irreducible component* $\mathbb{X}_{I,*}$ *of* $\overline{A_I}$ *which contains* $\mathscr{F}$ *and is an irreducible component of* $\mathbb{X}_I$.

(b) *Let*

$$A_{II} = \{w \in \mathbb{X} : \exists_{\lambda_1, \lambda_2 \in \mathbb{C}}\, (w, \lambda_1, \lambda_2) \in \mathbb{L}_{II}\}.$$

If $\varphi((0,1)) \subset \partial\Omega$ *then* $\mathscr{F} \subset \mathcal{Z}_{II} \subset A_{II} \subset \mathbb{X}_{II} \subset \mathbb{Y}_{II}$ *and there exists an irreducible component* $\mathbb{X}_{II,*}$ *of* $\overline{A_{II}}$ *which contains* $\mathscr{F}$ *and is an irreducible component of* $\mathbb{X}_{II}$.

PROOF. From Fact 5.4(a) we have $\mathscr{F} \subset \{(x,y,u,t,z) \in \mathbb{X} : x \in \Gamma_I\} \subset A_I$. Since all the polynomials $G_{4,i,j}$ vanish on $\mathbb{X}_I$, the vectors t, z are linearly dependent provided $(x,y,u,t,z) \in \mathbb{X}_I$ for some x, y, u. So $\mathbb{X}_I = \mathcal{X}_I \cup A_I$, where

$$\mathcal{X}_I = \{w = (x,y,u,t,z) \in \mathbb{X}_I : t = 0\}.$$

Obviously, the set $\mathcal{X}_I$ is contained in the hyperplane H defined by $t = 0$, and by Fact 5.1 we have $\mathscr{F} \setminus H \neq \emptyset$, so $\overline{A_I}$ has an irreducible component containing $\mathscr{F}$ which is an irreducible component of $\mathbb{X}_I$. This gives assertion (a).

Analogously, from Fact 5.4(b) we obtain $\mathscr{F} \subset A_{II}$. Moreover, the vectors x, t, z are linearly dependent provided $(x,y,u,t,z) \in \mathbb{X}_{II}$ for some y, u, so $\mathbb{X}_{II} = \mathcal{X}_{II} \cup A_{II}$, where

$$\mathcal{X}_{II} = \{w = (x,y,u,t,z) \in \mathbb{X}_I : G_0(x) = 0,\ G_{4,i,j}(x,t) = 0,\ 1 \leq i < j \leq n\}.$$

Obviously, $\mathcal{X}_{II}$ is contained in the set W defined by $G_{4,i,j}(x,t) = 0$, $1 \leq i < j \leq n$. By Lemma 5.3 we have $\mathscr{F} \setminus W \neq \emptyset$, so as above, the set A_{II} has an irreducible component satisfying (b). □

From Fact 5.6 and Lemmas 4.4 and 5.5 and the definition of $\mathbb{Y}$ we have

FACT 5.7. $\delta(\mathbb{X}_{I,*}) \leq \delta(\mathbb{Y}_I) \leq 2(2d-1)^{3n+1}$ *and* $\delta(\mathbb{X}_{II,*}) \leq \delta(\mathbb{Y}_{II}) \leq 2(2d-1)^{3n+1}$.

The proofs of Theorems 2.1 and 2.2 consist in showing that the projections of the sets $\mathbb{X}_{I,*}$ and $\mathbb{X}_{II,*}$ onto the space of $(y,u) \in \mathbb{C}^2$ are proper algebraic subsets of $\mathbb{C}^2$, since we have

LEMMA 5.8. *If* $Q \in \mathbb{C}[y,u]$ *is a nonzero polynomial of degree* D *such that*

$$Q\big(f(\varphi(t)), |\nabla f(\varphi(t))|^2\big) = 0 \quad \text{for } t \in [0,1),$$

where φ *is the curve fulfilling* (5.3), *then*

(a) $\varrho_0(f) \leq 1 - \dfrac{1}{D}$ *if* D *is even,*

(b) $\varrho_0(f) \leq 1 - \dfrac{1}{D+1}$ *if* D *is odd.*

Proof. Let $\operatorname{ord}_0(f \circ \varphi) = M$ and $\operatorname{ord}_0 |\nabla f \circ \varphi|^2 = K$. Then $M, K > 0$ and

$$\operatorname{ord}_0 (f \circ \varphi)^K = \operatorname{ord}_0 |\nabla f \circ \varphi|^{2M},$$

i.e. $|f \circ \varphi|^{K/2M} \sim |\nabla f \circ \varphi|$ near zero[1], so by (5.3) we have

$$\varrho_0(f) = \frac{K}{2M}. \tag{5.5}$$

Then, by definitions of M and K there exists a pair of different monomials $\alpha u^N y^S$ and $\beta u^{N_1} y^{S_1}$ of the polynomial Q such that

$$N + S \leq D \quad \text{and} \quad N_1 + S_1 \leq D,$$

and

$$NK + SM = N_1 K + S_1 M.$$

Hence $N - N_1 \neq 0$, $S_1 - S \neq 0$, and

$$\frac{K}{2M} = \frac{S_1 - S}{2(N - N_1)}.$$

Since $M > 0$, we have $\operatorname{ord}_0 |\nabla f \circ \varphi| \leq M - 1$, and so $K \leq 2M - 2$, and $K/2M < 1$. On the other hand, $|S_1 - S|, |N - N_1| \in \{1, \dots, D\}$, so by (5.5), $\varrho_0(f)$ is estimated from above by the maximal possible rational number less than 1 with numerator from the set $\{1, \dots, D\}$ and denominator from $\{2, 4, \dots, 2D\}$. Consequently, we obtain the assertion. □

5.1. Proof of Theorem 2.1 in case I when $\varphi((0,1)) \subset \operatorname{Int} \Omega$.
By the assumption (2.2), in the definition of $\mathbb{Y}$ one can take the polynomials

$$\begin{aligned} K_{3,i}(x,y,u,z) = {} & \frac{\partial G}{\partial x_i}(x,y,u)\frac{\partial P}{\partial y}(x,y) - \frac{\partial G}{\partial y}(x,y,u)\frac{\partial P}{\partial x_i}(x,y) \\ & - \left(\frac{\partial P}{\partial y}(x,y)\right)^3 \cdot z_i \end{aligned} \tag{5.6}$$

instead of $G_{3,i}$, $1 \leq i \leq n$; also in the definitions of $\mathbb{X}_I$ and $\mathbb{Y}_I$ one can take

$$K_{4,i,j}(x,y,u) = \frac{\partial P}{\partial x_i}(x,y)\frac{\partial G}{\partial x_j}(x,y,u) - \frac{\partial P}{\partial x_j}(x,y)\frac{\partial G}{\partial x_i}(x,y,u)$$

instead of $G_{4,i,j}$, $1 \leq i < j \leq n$.

[1] That is, there are $C_1, C_2 > 0$ such that $C_1 |f \circ \varphi|^{K/2M} \leq |\nabla f \circ \varphi| \leq C_2 |f \circ \varphi|^{K/2M}$ near zero.

From the above and Fact 5.6 we obtain the following fact.

FACT 5.9. *For $x \in \Gamma_I$ and $v = (x, y, u) = (x, f(x), |\nabla f(x)|^2)$ we have*

$$P(v) = 0, \tag{5.7}$$

$$G(v) = 0, \tag{5.8}$$

$$K_{4,i,j}(v) = 0, \qquad 1 \le i < j \le n. \tag{5.9}$$

Let $\boldsymbol{Y}_{I,0} \subset \boldsymbol{M}$, where $\boldsymbol{M} = \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}$, be an algebraic set defined by the system of equations (5.7)–(5.9), and let

$$\mathbb{Y}_I^0 = \left\{ (x, y, u, t, z) \in \mathbb{Y}_I : \frac{\partial P}{\partial y}(x, y) \neq 0 \right\},$$

$$\boldsymbol{Y}_I^0 = \left\{ (x, y, u) \in \boldsymbol{Y}_{I,0} : \frac{\partial P}{\partial y}(x, y) \neq 0 \right\},$$

$$\boldsymbol{Y}_I = \overline{\boldsymbol{Y}_I^0}.$$

We have the following fact (cf. [**17**, Fact 2.11]).

FACT 5.10. *The mapping*

$$\mathbb{Y}_I^0 \ni (x, y, u, t, z) \mapsto (x, y, u) \in \boldsymbol{Y}_I^0$$

is a bijection.

PROOF. Taking any $(x, y, u, t, z) \in \mathbb{Y}_I^0$ (respectively $(x, y, u) \in \boldsymbol{Y}_I^0$), by the implicit function theorem there are a neighbourhood $\Delta \subset \mathbb{C}^n$ of x, a holomorphic function $g : \Delta \to \mathbb{C}$ and neighbourhoods $U_1 \subset \mathbb{C} \times \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n$ and $U_2 \subset \mathbb{C} \times \mathbb{C}$ of (y, u, t, z) and (y, u) respectively such that

$$\mathbb{Y}_I^0 \cap (\Delta \times U_1) = \big\{ (\zeta, g(\zeta), h(\zeta), \nabla g(\zeta), \nabla h(\zeta)) \in \mathbb{M} : \zeta \in \Delta \cap V \big\},$$

$$\boldsymbol{Y}_I^0 \cap (\Delta \times U_2) = \big\{ (\zeta, g(\zeta), h(\zeta)) \in \boldsymbol{M} : \zeta \in \Delta \cap V \big\},$$

where $h(\zeta) = ((\partial g/\partial x_1)(\zeta))^2 + \cdots + ((\partial g/\partial x_n)(\zeta))^2$, and

$$V = \big\{ \zeta \in \Delta : K_{4,i,j}(\zeta, g(\zeta), h(\zeta)) = 0,\ 1 \le i < j \le n \big\}.$$

In particular, $g(x) = y$, $u = h(x)$, $t = \nabla g(x)$ and $z = \nabla h(x)$. Thus, we obtain the assertion. □

Let $\boldsymbol{L}_I \subset \boldsymbol{M} \times \mathbb{C}$ be the Zariski closure of the set

$$\boldsymbol{L}_{I,0} = \big\{ (x, y, u, \lambda) \in \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} : y = f(x),\ u = |\nabla f(x)|^2, \nabla |\nabla f(x)|^2 = \lambda \nabla f(x) \big\}.$$

From Fact 5.6(a) we obtain

Fact 5.11. *There exists an irreducible component $\boldsymbol{L}_{I,*}$ of $\boldsymbol{L}_I$ which contains a Zariski open and dense subset $\mathcal{U}$ such that for any $(x, y, u, \lambda) \in \mathcal{U}$ there exist $t, z \in \mathbb{C}^n$ such that $(x, y, u, t, z) \in \mathbb{X}_{I,*}$ and in particular $z = \lambda t$.*

Proof. The set $\boldsymbol{L}_I$ is the projection of the union of some irreducible components of $\mathbb{L}_I$ onto $(x, y, u, \lambda) \in \boldsymbol{M} \times \mathbb{C}$. So by Fact 5.6(a) we obtain the assertion. □

Let

$$\pi : \boldsymbol{M} \times \mathbb{C} \ni (x, y, u, \lambda) \mapsto (x, y, u) \in \boldsymbol{M},$$

let $\boldsymbol{L}_{I,*}$ be an irreducible component of $\boldsymbol{L}_I$ as in Fact 5.11 and let

$$\boldsymbol{X}_I := \overline{\pi(\boldsymbol{L}_{I,*})}.$$

Lemma 5.12. *The set $\boldsymbol{X}_I$ is an irreducible component of the algebraic set $\boldsymbol{Y}_I$. Moreover, $\boldsymbol{X}_I$ contains a Zariski open and dense subset $\mathcal{U}_I$ such that $\mathcal{U}_I \subset \boldsymbol{Y}_I^0 \cap \pi(\boldsymbol{L}_{I,*})$, and any point $(x_0, y_0, u_0) \in \mathcal{U}_I$ has a neighbourhood $B \subset \boldsymbol{M}$ such that $\boldsymbol{Y}_I \cap B = \mathcal{U}_I \cap B$ and*

$$\mathcal{U}_I \cap B = \left\{ \left(x, g(x), \left(\frac{\partial g}{\partial x_1}(x) \right)^2 + \cdots + \left(\frac{\partial g}{\partial x_n}(x) \right)^2 \right) : x \in \Delta \cap V \right\} \tag{5.10}$$

for some analytic set $V \subset \Delta$ with $x_0 \in V$ and a holomorphic function $g : \Delta \to \mathbb{C}$, where $\Delta \subset \mathbb{C}^n$ is a neighbourhood of x_0.

Proof. By Facts 5.6, 5.10 and 5.11 we have $\pi(\boldsymbol{L}_{I,0}) \subset \boldsymbol{Y}_I$, so $\boldsymbol{X}_I \subset \boldsymbol{Y}_I$ and $\boldsymbol{X}_I$ is an algebraic subset of $\boldsymbol{Y}_I$. Since any irreducible component of $\mathbb{X}_I$ is an irreducible component of $\mathbb{Y}_I$, the same holds for $\pi(\boldsymbol{L}_I)$ and $\boldsymbol{Y}_I$, because these sets are projections onto the space $\boldsymbol{M}$ of some collections of irreducible components of $\mathbb{X}_I$ and $\mathbb{Y}_I$, respectively. In particular, this holds for $\boldsymbol{X}_I$ and $\boldsymbol{Y}_I$. This gives the first part of the assertion. We prove the "moreover" part analogously to Fact 5.10. □

Let

$$\pi_y : \boldsymbol{X}_I \ni v = (x, y, u) \mapsto y \in \mathbb{C},$$
$$\pi_u : \boldsymbol{X}_I \ni v = (x, y, u) \mapsto u \in \mathbb{C}.$$

We have the following lemma (cf. [**17**, Lemma 2.12 and Lemma 2.14]):

Lemma 5.13. *For generic $y_0 \in \mathbb{C}$, i.e. for any $y_0 \in \mathbb{C}$ off a finite set, the function π_u is constant on each connected component of $(\pi_y)^{-1}(y_0)$.*

Proof. If $\dim \boldsymbol{X}_I = 0$ or $\dim(\pi_y)^{-1}(y) \leq 0$ for generic $y \in \mathbb{C}$, then the assertion holds. Assume that $\dim \boldsymbol{X}_I > 0$ and $\dim(\pi_y)^{-1}(y) > 0$ for generic $y \in \mathbb{C}$. Then by Lemma 5.12, and under the notations of this lemma, we have $\overline{\pi_y(\mathcal{U}_I)} = \overline{\pi_y(\boldsymbol{X}_I)} = \mathbb{C}$ and $(\pi_y)^{-1}(y) \cap \mathcal{U}_I \neq \emptyset$ for generic $y \in \mathbb{C}$.

Take any $y_0 \in \mathbb{C}$ such that $(\pi_y)^{-1}(y_0) \cap \mathcal{U}_I \neq \emptyset$. Take any $x_0 \in \mathbb{C}^n$ and $u_0 \in \mathbb{C}$ such that $(x_0, y_0, u_0) \in \mathcal{U}_I$. By Lemma 5.12 there exist a neighbourhood $B \subset \boldsymbol{M}$ of (x_0, y_0, u_0) and a holomorphic function $g : \Delta \to \mathbb{C}$, where $\Delta \subset \mathbb{C}^n$ is a neighbourhood of x_0, such that (5.10) holds for some analytic set $V \subset \Delta$.

Take any smooth curve $\gamma : [0,1] \to \Delta \cap V$ such that $g(\gamma(t)) = y_0$ for $t \in [0,1]$. Let $h(x) = ((\partial g/\partial x_1)(x))^2 + \cdots + ((\partial g/\partial x_n)(x))^2$ for $x \in \Delta$ and take a function $u : [0,1] \to \mathbb{C}$ defined by

$$u(t) = h \circ \gamma(t).$$

Observe that the function u is constant. Indeed, by definition of $\mathcal{U}_I$ we see that for any $x \in \Delta \cap V$ there exists $\lambda_x \in \mathbb{C}$ such that

$$\nabla h(x) = \lambda_x \nabla g(x).$$

So,

$$u'(t) = \lambda_{\gamma(t)} \langle \nabla g(\gamma(t)), \overline{\gamma'(t)} \rangle \quad \text{for } t \in [0,1],$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian product in $\mathbb{C}^n$. Since $g(\gamma(t)) = y_0$ for $t \in [0,1]$, we have $\langle \nabla g(\gamma(t)), \overline{\gamma'(t)} \rangle = 0$, and consequently $u'(t) = 0$ for $t \in [0,1]$ and u is constant. Summing up, the function π_u is constant on each connected component of $(\pi_y)^{-1}(y_0) \cap \mathcal{U}_I$.

Since $\mathcal{U}_I$ is a Zariski open and dense subset of $\boldsymbol{X}_I$, any irreducible component of $\boldsymbol{X}_I \setminus \mathcal{U}_I$ has dimension smaller than the dimension of $\boldsymbol{X}_I$, and for generic $y \in \mathbb{C}$ any irreducible component A of the fibre $\pi_y^{-1}(y)$ has a dense subset of the form $A \cap \mathcal{U}_I$ (see [**28**, Chapter 3]). Then by the above we obtain the assertion. □

Since Γ is an infinite set, it follows that $\dim \boldsymbol{L}_{I,0} \geq 1$, so by Fact 5.10, $\dim \boldsymbol{L}_I \geq 1$, and since $d = \deg P \geq 2$, Lemma 4.4 and the definition of $\boldsymbol{Y}_I$ yield $\delta(\boldsymbol{X}_I) \leq d(3d-2)^n$, where $\delta(\boldsymbol{X}_I)$ is the total degree of $\boldsymbol{X}_I$. So, from Lemma 5.13, the closure of the projection of $\boldsymbol{X}_I$, $W = \overline{\{(y,u) \in \mathbb{C}^2 : \exists_{x \in \mathbb{C}^n}\ (x,y,u) \in \boldsymbol{X}_I\}}$, is a proper algebraic subset of $\mathbb{C}^2$ and by Fact 4.3, $\delta(W) \leq \delta(\boldsymbol{X}_I)$. Then there exists a nonzero polynomial $Q \in \mathbb{C}[y,u]$ such that

$$\deg Q \leq d(3d-2)^n \leq \mathcal{R}(n,d) - 1$$

and $Q(y,u) = 0$ for $(x,y,u) \in \boldsymbol{X}_I$. In particular, $Q(f(\varphi(t)), |\nabla f(\varphi(t))|^2) = 0$ for $t \in [0,1)$. Since $D = d(3d-2)^n$ may be odd, by Lemma 5.8(b) we obtain the assertion of Theorem 2.1 in case I.

5.2. Proof of Theorem 2.1 in case II when $\varphi([0,1)) \subset \partial\Omega$.

For any $x \in \partial\Omega \setminus f^{-1}(0)$ sufficiently close to $f^{-1}(0)$ the tangent spaces to $\partial\Omega$ and $f^{-1}(f(x))$ are transversal, as shown in Lemma 5.3.

We will prove Theorem 2.1 in two dimensions and in the multidimensional case separately.

Proof of Theorem 2.1 in case II for $n = 2$. Take a polynomial $G \in \mathbb{C}[x, y, u]$, where $x = (x_1, x_2)$ and y, u are single variables, defined by (5.4), i.e. $G(x, y, u) = \sum_{i=1}^{2}((\partial P/\partial x_i)(x, y))^2 - ((\partial P/\partial y)(x, y))^2 \cdot u$. Let

$$\begin{aligned}
\boldsymbol{Y}_{II,0} &= \big\{(x, y, u) \in \mathbb{C}^2 \times \mathbb{C} \times \mathbb{C} : P(x, y) = 0,\ G_0(x) = 0,\ G(x, y, u) = 0\big\},\\
\boldsymbol{Y}_{II}^0 &= \left\{(x, y, u) \in \boldsymbol{Y}_{II,0} : \frac{\partial P}{\partial y}(x, y) \neq 0\right\},\\
\boldsymbol{Y}_{II} &= \overline{\boldsymbol{Y}_{II}^0}.
\end{aligned}$$

Then for any $x \in \Gamma \cap \partial\Omega$ we have $(x, f(x), |\nabla f(x)|^2) \in \boldsymbol{Y}_{II}$. Consequently,

$$\big(\varphi(t), f(\varphi(t)), |\nabla f(\varphi(t))|^2\big) \in \boldsymbol{Y}_{II} \quad \text{for } t \in [0, 1).$$

In particular, $\dim \boldsymbol{Y}_{II} \geq 1$ and by Fact 4.2 we have $\delta(\boldsymbol{Y}_{II}) \leq 2d(2d - 1)$.

Since P is an irreducible polynomial of positive degree with respect to y, for any $y \in \mathbb{C} \setminus \{0\}$ sufficiently close to 0 the set $\{x \in \mathbb{C}^2 : P(x, y) = 0, G_0(x) = 0\}$ is finite, so the set $\{(x, u) \in \mathbb{C}^2 \times \mathbb{C} : (x, y, u) \in \boldsymbol{Y}_{II}\}$ is also finite. Then the projection

$$W = \big\{(y, u) \in \mathbb{C}^2 : \exists_{x \in \mathbb{C}^2} (x, y, u) \in \boldsymbol{Y}_{II}\big\}$$

is contained in a proper algebraic subset of $\mathbb{C}^2$. By Fact 4.3,

$$\delta(\overline{W}) \leq 2d(2d - 1) \leq \mathcal{R}(n, d).$$

Then there exists a nonzero polynomial $Q \in \mathbb{C}[y, u]$ of degree $\deg Q \leq \delta(\overline{W}) \leq \mathcal{R}(n, d)$ which vanishes on W. Since $2d(2d-1)$ is even, by Lemma 5.8(a) we obtain the assertion of Theorem 2.1 in case II for $n = 2$. □

Let us consider the case $n \geq 3$. Let $\varepsilon > 0$ be as in Lemma 5.3.

By the assumption (2.2), in the definition of the set $\mathbb{Y}$ one can take the polynomials $K_{3,i}$ of the form (5.6) instead of $G_{3,i}$; also, in the definitions of $\mathbb{X}_{II}$ and $\mathbb{Y}_{II}$, one can take the polynomials

$$K_{4,i,j,k}(x, y, u) = \begin{vmatrix} \dfrac{\partial P}{\partial x_i}(x, y) & \dfrac{\partial G}{\partial x_i}(x, y, u) & x_i \\ \dfrac{\partial P}{\partial x_j}(x, y) & \dfrac{\partial G}{\partial x_j}(x, y, u) & x_j \\ \dfrac{\partial P}{\partial x_k}(x, y) & \dfrac{\partial G}{\partial x_k}(x, y, u) & x_k \end{vmatrix},$$

instead of $G_{4,i,j,k}$ for $1 \leq i < j < k \leq n$, where G is defined in (5.4). Then

$$\begin{aligned}
\mathbb{X}_{II} &= \big\{w = (x, y, u, t, z) \in \mathbb{X} : G_0(x) = 0,\ K_{4,i,j,k}(x, y, u) = 0,\ 1 \leq i < j < k \leq n\big\},\\
\mathbb{Y}_{II} &= \big\{w = (x, y, u, t, z) \in \mathbb{Y} : G_0(x) = 0,\ K_{4,i,j,k}(x, y, u) = 0,\ 1 \leq i < j < k \leq n\big\}.
\end{aligned}$$

Let $\boldsymbol{Y}_{II,0} \subset \boldsymbol{M}$, where $\boldsymbol{M} = \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}$, be the algebraic set defined by

$$\boldsymbol{Y}_{II,0} = \{(x,y,u) \in \boldsymbol{M} : P(x,y) = 0,\ G_0(x) = 0,\ G(x,y,u) = 0,\\ K_{4,i,j,k}(x,y,u) = 0,\ 1 \le i < j < k \le n\}$$

and let

$$\mathbb{Y}_{II}^0 = \left\{(x,y,u,t,z) \in \mathbb{Y}_{II} : \frac{\partial P}{\partial y}(x,y) \neq 0\right\},$$
$$\boldsymbol{Y}_{II}^0 = \left\{(x,y,u) \in \mathbb{V}_{II,0} : \frac{\partial P}{\partial y}(x,y) \neq 0\right\},$$
$$\boldsymbol{Y}_{II} = \overline{\boldsymbol{Y}_{II}^0}.$$

By an analogous argument to the proof of Fact 5.10 we obtain

Fact 5.14. *The mapping*

$$\mathbb{Y}_{II}^0 \ni (x,y,u,t,z) \mapsto (x,y,u) \in \boldsymbol{Y}_{II}^0$$

is a bijection.

Let $\boldsymbol{L}_{II} \subset \boldsymbol{M} \times \mathbb{C}^2$ be the Zariski closure of the set

$$\boldsymbol{L}_{II,0} = \{(x,y,u,(\lambda_1,\lambda_2)) \in \partial\Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 : y = f(x),\ u = |\nabla f(x)|^2,\\ \nabla|\nabla f(x)|^2 = \lambda_1 \nabla f(x) + \lambda_2 x\}.$$

By a similar argument to the proof of Fact 5.11, from Fact 5.6(b) we obtain

Fact 5.15. *There exists an irreducible component $\boldsymbol{L}_{II,*}$ of $\boldsymbol{L}_{II}$ which contains a Zariski open, dense subset $\mathcal{U}$ such that for any $(x,y,u,\lambda_1,\lambda_2) \in \mathcal{U}$ there exist $t,z \in \mathbb{C}^n$ such that $(x,y,u,t,z) \in \mathbb{X}_{II,*}$ and in particular $z = \lambda_1 t + \lambda_2 x$.*

Let

$$\pi' : \boldsymbol{M} \times \mathbb{C}^2 \ni (x,y,u,(\lambda_1,\lambda_2)) \mapsto (x,y,u) \in \boldsymbol{M},$$

and let

$$\boldsymbol{X}_{II} = \overline{\pi'(\boldsymbol{L}_{II,*})}.$$

By an analogous argument to the proof of Lemma 5.12 we obtain

Lemma 5.16. *The set $\boldsymbol{X}_{II}$ is an irreducible component of the algebraic set $\boldsymbol{Y}_{II}$. Moreover, $\boldsymbol{X}_{II}$ contains a Zariski open and dense subset $\mathcal{U}_{II}$ such that $\mathcal{U}_{II} \subset \boldsymbol{Y}_{II}^0 \cap \pi'(\boldsymbol{L}_{II,*})$ and any point $(x_0,y_0,u_0) \in \mathcal{U}_{II}$ has a neighbourhood $B \subset \boldsymbol{M}$ such that $\boldsymbol{Y}_{II} \cap B = \mathcal{U}_{II} \cap B$ and*

$$\mathcal{U}_{II} \cap B = \left\{\left(x, g(x), \left(\frac{\partial g}{\partial x_1}(x)\right)^2 + \cdots + \left(\frac{\partial g}{\partial x_n}(x)\right)^2\right) : x \in \Delta \cap V\right\} \tag{5.11}$$

for some analytic set $V \subset \Delta$, where $x_0 \in V$ and G_0 vanishes on V, and a holomorphic function $g : \Delta \to \mathbb{C}$, where $\Delta \subset \mathbb{C}^n$ is a neighbourhood of x_0.

Let

$$\pi_y : \boldsymbol{X}_{II} \ni v = (x, y, u) \mapsto y \in \mathbb{C}, \qquad \pi_u : \boldsymbol{X}_{II} \ni v = (x, y, u) \mapsto u \in \mathbb{C}.$$

We have the following lemma (cf. Lemma 5.13 and [**17**, Lemmas 2.12, 2.14]).

Lemma 5.17. *For generic $y_0 \in \mathbb{C}$ the function π_u is constant on each connected component of $(\pi_y)^{-1}(y_0)$.*

Proof. As in the proof of Lemma 5.13, we may assume that $\dim \boldsymbol{X}_{II} > 0$ and $\dim (\pi_y)^{-1}(y) > 0$ for generic $y \in \mathbb{C}$. Then by Lemma 5.16, and under the notations of that lemma, $\overline{\pi_y(\mathcal{U}_{II})} = \overline{\pi_y(\boldsymbol{X}_{II})} = \mathbb{C}$ and $(\pi_y)^{-1}(y) \cap \mathcal{U}_{II} \neq \emptyset$ for generic $y \in \mathbb{C}$.

Take any $y_0 \in \mathbb{C}$ such that $(\pi_y)^{-1}(y_0) \cap \mathcal{U}_{II} \neq \emptyset$. Take any $x_0 \in \mathbb{C}^n$ and $u_0 \in \mathbb{C}$ such that $(x_0, y_0, u_0) \in \mathcal{U}_{II}$. By Lemma 5.16 there exist a neighbourhood $B \subset \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}$ of (x_0, y_0, u_0) and a holomorphic function $g : \Delta \to \mathbb{C}$, where $\Delta \subset \mathbb{C}^n$ is a neighbourhood of x_0, such that (5.11) holds for some analytic set $V \subset \Delta$ such that G_0 vanishes on V.

Take a smooth curve $\gamma = (\gamma_1, \dots, \gamma_n) : [0,1] \to \Delta \cap V$ such that $g(\gamma(t)) = y_0$. Then

$$G_0(\gamma(t)) = 0 \quad \text{for } t \in [0,1]. \tag{5.12}$$

Let $h(x) = ((\partial g/\partial x_1)(x))^2 + \dots + ((\partial g/\partial x_n)(x))^2$, $x \in \Delta$. Take a function $u : [0,1] \to \mathbb{C}$ defined by

$$u(t) = h \circ \gamma(t), \quad t \in [0,1].$$

Observe that the function u is constant. Indeed, by definition of $\mathcal{U}_{II}$, for any $x \in \Delta \cap V$ there exist $\lambda_{1,x}, \lambda_{2,x} \in \mathbb{C}$ such that

$$\nabla h(x) = \lambda_{1,x} \nabla g(x) + \lambda_{2,x} x.$$

So

$$u'(t) = \lambda_{1,\gamma(t)} \langle \nabla g(\gamma(t)), \overline{\gamma'(t)} \rangle + \lambda_{2,\gamma(t)} \langle \gamma(t), \overline{\gamma'(t)} \rangle \quad \text{for } t \in [0,1].$$

Since $g(\gamma(t)) = y_0$, we have $\langle \nabla g(\gamma(t)), \overline{\gamma'(t)} \rangle = 0$ for $t \in [0,1]$. Moreover, by (5.12) we have $\langle \gamma(t), \overline{\gamma'(t)} \rangle = 0$ for $t \in [0,1]$. Consequently, $u'(t) = 0$ for $t \in [0,1]$ and u is constant. Summing up, the function π_u is constant on each connected component of $(\pi_y)^{-1}(y_0) \cap \mathcal{U}_{II}$. Since $\mathcal{U}_{II}$ is a dense subset of $\boldsymbol{X}_{II}$, we obtain the assertion. ☐

Since Γ is an infinite set, we have $\dim \boldsymbol{L}_{II,0} \geq 1$, so by Fact 5.14, $\dim \boldsymbol{L}_{II} \geq 1$, and since $d = \deg P \geq 2$, Lemma 4.4 and the definition of $\boldsymbol{Y}_{II}$ yield $\delta(\boldsymbol{X}_{II}) \leq d(3d-2)^n$. So, from Lemma 5.17, the closure of the projection of $\boldsymbol{X}_{II}$,

$$W = \overline{\{(y, u) \in \mathbb{C}^2 : \exists_{x \in \mathbb{C}^n} \ (x, y, u) \in \boldsymbol{X}_{II}\}},$$

is a proper algebraic subset of $\mathbb{C}^2$ and $\delta(W) \le \delta(\boldsymbol{X}_{II})$. Then there exists a nonzero polynomial $Q \in \mathbb{C}[y,u]$ such that $\deg Q \le 2(3d-2)^n \le \mathcal{R}(n,d) - 1$ and $Q(y,u) = 0$ for $(x,y,u) \in \boldsymbol{X}_{II}$. Since $D = 2(3d-2)^n$ is an even number, by Lemma 5.8(a) we obtain the assertion of Theorem 2.1 in case II.

5.3. Proof of Theorem 2.2.

Analogously to the proof of Lemma 5.13, we prove that the set

$$W = \overline{\left\{(y,u) \in \mathbb{C}^2 : \exists_{x\in\mathbb{C}^n}\ \exists_{t\in\mathbb{C}^n}\ \exists_{z\in\mathbb{C}^n}\ (x,y,u,t,z) \in \mathbb{Y}_I\right\}}$$

is a proper algebraic subset of $\mathbb{C}^2$. Moreover, by Fact 5.7 we have $\delta(W) \le \delta(\mathbb{Y}_I) \le 2d(2d-1)$ if $n=1$ and $\delta(W) \le \delta(\mathbb{Y}_I) \le 2(2d-1)^{3n+1}$ for $n \ge 2$. Then by Lemma 5.8(a) we obtain the assertion of Theorem 2.2 in case I. An analogous argument gives the assertion in case II.

References

[1] J. Bochnak and W. Kucharz, Sur les germes d'applications différentiables à singularités isolées, Trans. Amer. Math. Soc., **252** (1979), 115–131.

[2] J. Bochnak and S. Łojasiewicz, A converse of the Kuiper–Kuo theorem, In: Proceedings of Liverpool Singularities—Symposium I (1969/70), Lecture Notes in Math., **192**, Springer, Berlin, 1971, 254–261.

[3] J. Bochnak and J. J. Risler, Sur les exposants de Łojasiewicz, Comment. Math. Helv., **50** (1975), 493–507.

[4] P. Cassou-Noguès and H. H. Vui, Théorème de Kuiper–Kuo–Bochnak–Łojasiewicz à l'infini, Ann. Fac. Sci. Toulouse Math. (6), **5** (1996), 387–406.

[5] S. H. Chang and Y. C. Lu, On C^0-sufficiency of complex jets, Canad. J. Math., **25** (1973), 874–880.

[6] D. D'Acunto and K. Kurdyka, Explicit bounds for the Łojasiewicz exponent in the gradient inequality for polynomials, Ann. Polon. Math., **87** (2005), 51–61.

[7] D. D'Acunto and K. Kurdyka, Bounds for gradient trajectories and geodesic diameter of real algebraic sets, Bull. London Math. Soc., **38** (2006), 951–965.

[8] J. Damon and T. Gaffney, Topological triviality of deformations of functions and Newton filtrations, Invent. Math., **72** (1983), 335–358.

[9] T. Fukui and E. Yoshinaga, The modified analytic trivialization of family of real analytic functions, Invent. Math., **82** (1985), 467–477.

[10] A. Gabrielov, Multiplicities of Pfaffian intersections, and the Łojasiewicz inequality, Selecta Math. (N.S.), **1** (1995), 113–127.

[11] V. Grandjean, Finite determinacy relative to closed and finitely generated ideals, Manuscripta Math., **103** (2000), 313–328.

[12] J. Gwoździewicz, The Łojasiewicz exponent of an analytic function at an isolated zero, Comment. Math. Helv., **74** (1999), 364–375.

[13] J. Kollár, An effective Łojasiewicz inequality for real polynomials, Period. Math. Hungar., **38** (1999), 213–221.

[14] N. H. Kuiper, C^1-equivalence of functions near isolated critical points, In: Proceedings of Symposium on Infinite-Dimensional Topology, Baton Rouge, 1967, Ann. of Math. Studies, **69**, Princeton Univ. Press, Princeton, NJ, 1972, 199–218.

[15] T.-C. Kuo, On C^0-sufficiency of jets of potential functions, Topology, **8** (1969), 167–171.

[16] K. Kurdyka, T. Mostowski and A. Parusiński, Proof of the gradient conjecture of R. Thom, Ann. of Math. (2), **152** (2000), 763–792.

[17] K. Kurdyka, B. Osińska-Ulrych, G. Skalski and S. Spodzieja, Effective Łojasiewicz gradient inequality for generic Nash functions with isolated singularity, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform., **66** (2016), 47–64.

[18] K. Kurdyka and S. Spodzieja, Separation of real algebraic sets and the Łojasiewicz exponent, Proc. Amer. Math. Soc., **142** (2014), 3089–3102.

[19] K. Kurdyka and S. Spodzieja, Convexifying positive polynomials and sums of squares approximation, SIAM J. Optim., **25** (2015), 2512–2536.

[20] K. Kurdyka, S. Spodzieja and A. Szlachcińska, Metric properties of semialgebraic mappings, Discrete Comput. Geom., **55** (2016), 786–800.

[21] L. Kushner, Finite determination on algebraic sets, Trans. Amer. Math. Soc., **331** (1992), 553–561.

[22] M. Lejeune-Jalabert and B. Teissier, Clôture Intégrale des Idéaux et Équisingularité, Centre de Mathématiques Ecole Polytechnique, Palaiseau, 1974.

[23] S. Łojasiewicz, Ensembles semi-analytiques, preprint IHES, (1965).

[24] S. Łojasiewicz, Sur les trajectoires du gradient d'une fonction analytique, In: Geometry Seminars, 1982–1983, Univ. Stud. Bologna, Bologna, 1984, 115–117.

[25] S. Łojasiewicz, Introduction to Complex Analytic Geometry, Birkhäuser Verlag, Basel, 1991.

[26] M. Merle, Invariants polaires des courbes planes, Invent. Math., **41** (1977), 103–111.

[27] P. Migus, T. Rodak and S. Spodzieja, Finite determinacy of non-isolated singularities, Ann. Polon. Math., **117** (2016), 197–206.

[28] D. Mumford, Algebraic Geometry I. Complex Projective Varieties, Springer, Berlin, 1976.

[29] B. Osińska-Ulrych, G. Skalski and S. Spodzieja, On C^0-sufficiency of jets, In: Analytic and Algebraic Geometry, Faculty of Math. and Computer Science, Univ. of Łódź, Łódź, 2013, 95–113.

[30] R. Pellikaan, Finite determinacy of functions with non-isolated singularities, Proc. London Math. Soc. (3), **57** (1988), 357–382.

[31] T. S. Pham, An explicit bound for the Łojasiewicz exponent of real polynomials, Kodai Math. J., **35** (2012), 311–319.

[32] P. J. Rabier, Ehresmann fibrations and Palais–Smale conditions for morphisms of Finsler manifolds, Ann. of Math. (2), **146** (1997), 647–691.

[33] T. Rodak and S. Spodzieja, Effective formulas for the local Łojasiewicz exponent, Math. Z., **268** (2011), 37–44.

[34] T. Rodak and S. Spodzieja, Łojasiewicz exponent near the fibre of a mapping, Proc. Amer. Math. Soc., **139** (2011), 1201–1213.

[35] T. Rodak and S. Spodzieja, Equivalence of mappings at infinity, Bull. Sci. Math., **136** (2012), 679–686.

[36] D. Siersma, Singularities with critical locus a 1-dimensional complete intersection and transversal type A_1, Topology Appl., **27** (1987), 51–73.

[37] G. Skalski, On analytic equivalence of functions at infinity, Bull. Sci. Math., **135** (2011), 517–530.

[38] P. Solernó, Effective Łojasiewicz inequalities in semialgebraic geometry, Appl. Algebra Engrg. Comm. Comput., **2** (1991), 1–14.

[39] S. Spodzieja, The Łojasiewicz exponent of subanalytic sets, Ann. Polon. Math., **87** (2005), 247–263.

[40] B. Teissier, Variétés polaires. I. Invariants polaires des singularités d'hypersurfaces, Invent. Math., **40** (1977), 267–292.

[41] X. Xu, C^0-sufficiency, Kuiper–Kuo and Thom conditions for non-isolated singularity, Acta Math. Sin. (Engl. Ser.), **23** (2007), 1251–1256.

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