

Functional calculus of Laplace transform type on non-doubling parabolic manifolds with ends

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Abstract. Let M be a non-doubling parabolic manifold with ends and L a non-negative self-adjoint operator on $L^2(M)$ which satisfies a suitable heat kernel upper bound named the upper bound of Gaussian type. These operators include the Schrödinger operators $L = \Delta + V$ where Δ is the Laplace–Beltrami operator and V is an arbitrary non-negative potential. This paper will investigate the behaviour of the Poisson semi-group kernels of L together with its time derivatives and then apply them to obtain the weak type $(1, 1)$ estimate of the functional calculus of Laplace transform type of \sqrt{L} which is defined by $\mathfrak{M}(\sqrt{L})f(x) := \int_0^\infty [\sqrt{L}e^{-t\sqrt{L}}f(x)]m(t)dt$ where $m(t)$ is a bounded function on $[0, \infty)$. In the setting of our study, both doubling condition of the measure on M and the smoothness of the operators’ kernels are missing. The purely imaginary power L^{is} , $s \in \mathbb{R}$, is a special case of our result and an example of weak type $(1, 1)$ estimates of a singular integral with non-smooth kernels on non-doubling spaces.

1. Introduction.

Let (\mathcal{X}, d, μ) be a metric space equipped with a metric d and a measure μ . We assume that T is a bounded linear operator on $L^2(\mathcal{X})$ with an associated kernel $k(x, y)$ in the sense

$$Tf(x) = \int_{\mathcal{X}} k(x, y)f(y)d\mu(y) \quad (1.1)$$

for all continuous functions f with compact support and for almost all x not in the support of f .

In the standard Calderón–Zygmund theory, sufficient conditions named doubling condition and Hörmander condition were established to get the L^p boundedness of singular integral T on $L^p(\mathcal{X})$ for $p \neq 2$. Let us recall that

- A measure μ on the space \mathcal{X} is said to be doubling if there exists some positive constant C such that for all balls $B(x, r) = \{y \in \mathcal{X} : d(x, y) < r\}$,

$$0 < \mu(B(x, 2r)) < C\mu(B(x, r)) < \infty$$

for all $x \in \mathcal{X}$ and $r > 0$.

- The associated kernel $k(x, y)$ satisfies the Hörmander condition if there exist positive constants c and C such that

$$\int_{d(x, y_1) \geq cd(y_1, y_2)} |k(x, y_1) - k(x, y_2)| d\mu(x) \leq C$$

uniformly of y_1, y_2 .

Under these conditions, we can show that the operator T is of weak type $(1, 1)$. Then by Marcinkiewicz interpolation and duality, T is bounded on $L^p(\mathcal{X})$ for all $1 < p < \infty$.

Although the Calderón–Zygmund theory is well established, there are many problems in which the assumptions of this theory are not fully satisfied. For more than twenty years, a lot of research has been done to develop this theory. Those studies can be classified into two main directions.

In the first direction, ones have studied singular integrals on non-homogeneous spaces (i.e., spaces do not satisfy the doubling condition). Authors that have made significant contributions to this direction are Nazarov, Treil, Volberg, Tolsa and others. However, to obtain the boundedness of singular integrals, a certain strong regularity on the associated kernels is needed as a compensation for the lack of the doubling condition, i.e., the Hölder continuity on the space variables of the kernels (see [16], [17], [18], [20]).

In the other direction, many scholars have focused on singular integrals with non-smooth kernels (i.e., kernels do not satisfy the Hörmander condition). Substantial progress of this direction has been made by Auscher, Coulhon, Duong, Martell, McIntosh and others. Their studies showed that the weak type $(1, 1)$ estimates can be obtained when the Hörmander condition is replaced by a weaker one (see [1], [3], [12], [11]). However, the achievements in this direction are mostly obtained for operators acting on doubling spaces.

Recently, a great number of researchers have investigated singular integrals with non-smooth kernels acting on non-doubling spaces. The interesting and challenging thing of this approach is that both key conditions of the standard Calderón–Zygmund theory are missing. Notable improvements are of [2], [5], [10], [15] and others. Those authors achieved weak type $(1, 1)$ estimates and then the L^p boundedness for some certain singular integrals with non-smooth kernels on non-doubling spaces.

In [14], Grigor’yan, Ishiwata and Saloff-Coste achieved the sharp estimates for heat kernels of the semi-group $e^{-t\Delta}$ where Δ is the Laplace–Beltrami operator on parabolic manifolds with ends. Motivated by the work in [2] where authors obtained weak type $(1, 1)$ estimate for the holomorphic functional calculus of Laplace transform type on non-doubling Riemannian manifolds with ends $\mathcal{R}^m \sharp \mathcal{R}^n$ for $m > n \geq 3$, we aim to study the boundedness of such operator on a new setting space, i.e., the parabolic manifolds with ends (see [14]). Although the method used in [2] is powerful, it cannot be directly applied to our setting. The difficulty originates from the heavy dependence on the position of x, y on manifolds and the occurrence of log terms in the estimates of the heat kernels $g_t(x, y)$ of the heat semi-group $e^{-t\Delta}$. For example (see [14] for details)

$$g_t(x, y) \approx \begin{cases} \frac{1}{t} e^{-b(d^2(x,y)/t)} & \text{if } |x| > \sqrt{t}, \\ \frac{1}{t} \left(1 + \frac{|x|}{\sqrt{t}} \log \frac{\sqrt{t}}{|y|} \right) & \text{if } |x|, |y| \leq \sqrt{t}, \\ \frac{1}{t} \log \frac{\sqrt{t}}{|y|} e^{-b(d^2(x,y)/t)} & \text{if } |x| > \sqrt{t} \geq |y|, \end{cases}$$

for $t > 1, x \in \mathcal{R}^1$ and $y \in \mathcal{R}^2$. Therefore, we have to carry out a number of subtle decomposition and employ particular treatments for log terms.

We first recall some basic facts about the parabolic manifolds with ends studied in [14]. Let M be a complete non-compact Riemannian manifold and $K \subset M$ be a connected compact subset of M with non-empty interior and smooth boundary such that $M \setminus K$ has k non-compact connected components E_1, E_2, \dots, E_k . We refer to each E_i ($i = 1, \dots, k$) as an end of M and K as its central part. We also assume that each E_i is isometric to the exterior of a compact set in another manifold M_i ; therefore, M can be written as follows:

$$M = M_1 \# M_2 \# \dots \# M_k.$$

For a fixed integer $N > 0$, take an arbitrary integer $m \in [1, N]$. The manifold \mathcal{R}^m is defined by

$$\mathcal{R}^1 = \mathbb{R}_+ \times S^{N-1} \quad \text{and} \quad \mathcal{R}^m = \mathbb{R}^m \times S^{N-m} \quad \text{for all } m \geq 2.$$

Thus, we can construct a finite connected sum of the \mathcal{R}^m 's:

$$M = \mathcal{R}^{m_1} \# \mathcal{R}^{m_2} \# \dots \# \mathcal{R}^{m_k},$$

where $m_1, m_2, \dots, m_k \in [1, N]$.

We note that a manifold is called parabolic if any positive super-harmonic function on M is constant. This implies that $m_1, m_2, \dots, m_k \leq 2$. We refer the readers to [13, p.164] and [14, p.5] for more details.

For the sake of simplicity, we restrict ourselves by setting $M = \mathcal{R}^1 \# \mathcal{R}^2$. The assumption of our main result is the so-called upper bound of Gaussian type which is originated in [2].

DEFINITION 1.1. Let Δ be the Laplace–Beltrami operator and L a non-negative self-adjoint operator on $L^2(\mathcal{R}^1 \# \mathcal{R}^2)$. We say that the heat kernel $\mathbf{g}_t(x, y)$ of the operator e^{-tL} has an upper bound of Gaussian type if $\mathbf{g}_t(x, y)$ satisfies the following estimate

$$|\mathbf{g}_t(x, y)| \leq C g_{\alpha t}(x, y), \tag{1.2}$$

for some positive constants C and α , where $g_{\alpha t}(x, y)$ is the kernel of the heat semi-group $e^{-\alpha t \Delta}$.

We notice that operators whose heat kernels satisfying Definition 1.1 include the Schrödinger operators $L = \Delta + V$ where V is a non-negative potential. Moreover, by using the inequality (1.2), we can deduce the upper bounds of the heat kernels of the

operator e^{-tL} from the upper bounds of the heat kernels of the operator $e^{-t\Delta}$.

From now on, we always assume that L is a non-negative self-adjoint operator whose kernel has an upper bound of Gaussian type. The following theorem is our main result.

THEOREM 1.1. *Let L be a non-negative self-adjoint operator on $L^2(\mathcal{R}^1 \sharp \mathcal{R}^2)$ whose heat kernels have upper bounds of Gaussian type. Let $\mathfrak{M}(\sqrt{L})$ be the holomorphic functional calculus of Laplace transform type of \sqrt{L} defined by*

$$\mathfrak{M}(\sqrt{L})f(x) := \int_0^\infty \left[\sqrt{L}e^{-t\sqrt{L}}f(x) \right] m(t)dt \tag{1.3}$$

where $m(t)$ is a bounded function on $[0, \infty)$.

Then the operator $\mathfrak{M}(\sqrt{L})$ is of weak type $(1, 1)$ and is bounded on $L^p(\mathcal{R}^1 \sharp \mathcal{R}^2)$ for all $1 < p < \infty$.

REMARK 1.1.

- (i) Theorem 1.1 covers the purely imaginary power L^{is} , $s \in \mathbb{R}$, which is an example of singular integrals acting on non-doubling spaces whose kernels do not satisfy the Hörmander condition.
- (ii) The subordination formula was employed to overcome the lack of estimates of the time derivatives of the Poisson kernel based on the estimates of the heat kernel of the operator $e^{-t\Delta}$ obtained by [14]. We then use this result to estimate the holomorphic functional calculus of Laplace transform type $\mathfrak{M}(\sqrt{L})$.
- (iii) The approach of our study can be adapted to other manifolds, e.g., $M = \mathcal{R}^1 \sharp \mathcal{R}^1 \sharp \mathcal{R}^2$, to obtain the weak type $(1, 1)$ estimate of $\mathfrak{M}(\sqrt{L})$.

The paper is organized as follows. In Section 2, we investigate the estimates of the time derivatives of the Poisson semi-group kernels based on the upper bounds of the heat kernel obtained by Grigor’yan, Ishiwata and Saloff-Coste (see [14]). In the last section, we then use these estimates to achieve the boundedness of the holomorphic functional calculus of Laplace transform type $\mathfrak{M}(\sqrt{L})$.

2. Time derivatives of Poisson semi-group kernels.

We recall here the parabolic manifold with two ends $\mathcal{R}^1 \sharp \mathcal{R}^2$ as well as the behaviour of the heat kernel $g_t(x, y)$ of the operator $e^{-t\Delta}$ obtained by Grigor’yan, Ishiwata and Saloff-Coste (see [14]) where Δ is a Laplace–Beltrami operator on $\mathcal{R}^1 \sharp \mathcal{R}^2$.

For each $x \in \mathcal{R}^1 \sharp \mathcal{R}^2$, the modulus of x is defined by $|x| := d(x, K) + 1$ where d is the geodesic distance on $\mathcal{R}^1 \sharp \mathcal{R}^2$ and $d(x, K) = \inf_{y \in K} d(x, y)$. The geodesic ball with center x and radius $r > 0$ is defined by $B(x, r) := \{y \in \mathcal{R}^1 \sharp \mathcal{R}^2 : d(x, y) < r\}$.

We denote by $V(x, r)$ the measure of the ball $B(x, r)$ on $\mathcal{R}^1 \sharp \mathcal{R}^2$. Then we have

- (i) $V(x, r) \approx r^2$ for all $x \in \mathcal{R}^1 \sharp \mathcal{R}^2$, when $r \leq 1$;
- (ii) $V(x, r) \approx r$ for $B(x, r) \subset \mathcal{R}^1$, when $r > 1$; and
- (iii) $V(x, r) \approx r^2$ for $x \in \mathcal{R}^1 \setminus K$, $r > 2|x|$ or $x \in \mathcal{R}^2$, $r > 1$.

Based on these properties, it can be verified that the doubling property does not hold for the manifold $\mathcal{R}^1 \sharp \mathcal{R}^2$.

The following theorem is the result obtained in [14].

THEOREM 2.1 ([14]). *Let Δ be the Laplace–Beltrami operator acting on the manifold $\mathcal{R}^1 \sharp \mathcal{R}^2$. The heat kernel $g_t(x, y)$ associated with the heat semi-group $e^{-t\Delta}$ satisfies the following estimates:*

1. *If $t \leq 1$, then*

$$g_t(x, y) \approx \frac{C}{V(x, \sqrt{t})} e^{-b(d^2(x,y)/t)}$$

for all $x, y \in \mathcal{R}^1 \sharp \mathcal{R}^2$;

2. *If $t > 1$, then*

(i) *For $x, y \in K$*

$$g_t(x, y) \approx \frac{C}{t} e^{-b(d^2(x,y)/t)};$$

(ii) *For $x \in \mathcal{R}^2 \setminus K$ and $y \in K$*

(ii₁) $|x| > \sqrt{t}$,

$$g_t(x, y) \approx \frac{C}{t} e^{-b(d^2(x,y)/t)};$$

(ii₂) $|x| \leq \sqrt{t}$,

$$g_t(x, y) \approx \frac{C}{t};$$

(iii) *For $x \in \mathcal{R}^1 \setminus K$ and $y \in K$*

(iii₁) $|x| > \sqrt{t}$,

$$g_t(x, y) \approx C \frac{\log t}{t} e^{-b(d^2(x,y)/t)};$$

(iii₂) $|x| \leq \sqrt{t}$,

$$g_t(x, y) \approx \frac{C}{t} \left(1 + \frac{|x|}{\sqrt{t}} \log t \right) e^{-b(d^2(x,y)/t)};$$

(iv) *For $x \in \mathcal{R}^1 \setminus K$ and $y \in \mathcal{R}^2 \setminus K$*

(iv₁) $|y| > \sqrt{t}$,

$$g_t(x, y) \approx \frac{C}{t} e^{-b(d^2(x,y)/t)};$$

(iv₂) $|x|, |y| \leq \sqrt{t}$,

$$g_t(x, y) \approx \frac{C}{t} \left(1 + \frac{|x|}{\sqrt{t}} \log \frac{\sqrt{t}}{|y|} \right);$$

(iv₃) $|x| > \sqrt{t} \geq |y|,$

$$g_t(x, y) \approx \frac{C}{t} \log \frac{\sqrt{t}}{|y|} e^{-b(d^2(x,y)/t)};$$

(v) For $x, y \in \mathcal{R}^1 \setminus K$

(v₁) $|x|, |y| \leq \sqrt{t},$

$$g_t(x, y) \approx \frac{C}{t} \left(1 + \frac{|x||y|}{\sqrt{t}} + \frac{|x| + |y|}{\sqrt{t}} \log t \right) e^{-b(d^2(x,y)/t)};$$

(v₂) $|x| > \sqrt{t} \geq |y|,$

$$g_t(x, y) \approx \frac{C}{t} (|y| + \log t) e^{-b(d^2(x,y)/t)};$$

(v₃) $|y| > \sqrt{t} \geq |x|,$

$$g_t(x, y) \approx \frac{C}{t} (|x| + \log t) e^{-b(d^2(x,y)/t)};$$

(v₄) $|x|, |y| > \sqrt{t},$

$$g_t(x, y) \approx \frac{C}{\sqrt{t}} e^{-b(d^2(x,y)/t)};$$

(vi) For $x, y \in \mathcal{R}^2 \setminus K$

$$g_t(x, y) \approx \frac{C}{t} e^{-b(d^2(x,y)/t)}.$$

Following is a result in [9].

THEOREM 2.2 ([9]). *Let \mathcal{M}_Δ be the maximal operator defined by $\mathcal{M}_\Delta f(x) := \sup_{t>0} |e^{-t\Delta} f(x)|$. Then \mathcal{M}_Δ is of weak type $(1, 1)$ and is bounded on $L^p(\mathcal{R}^1 \# \mathcal{R}^2)$ for all $1 < p \leq \infty$, i.e.,*

$$\|\mathcal{M}_\Delta\|_{L^p(\mathcal{R}^1 \# \mathcal{R}^2)} \leq C \|f\|_{L^p(\mathcal{R}^1 \# \mathcal{R}^2)}.$$

REMARK 2.1. Due to the inequality (1.2) in Definition 1.1,

- (i) Theorem 2.2 still holds if Δ is replaced by L . More specifically, the maximal function $\mathcal{M}_L f(x) := \sup_{t>0} |e^{-tL} f(x)|$ where $f \in L^p(\mathcal{R}^1 \# \mathcal{R}^2)$ is of weak type $(1, 1)$ and is bounded on $L^p(\mathcal{R}^1 \# \mathcal{R}^2)$ for all $1 < p \leq \infty$.
- (ii) The upper bounds of the heat kernel $\mathfrak{g}_t(x, y)$ of the semi-group e^{-tL} have the same form as those of the heat kernel $g_t(x, y)$ of the heat semi-group $e^{-t\Delta}$.

Next, we will study the behaviours of the kernel of the Poisson semi-group e^{-tL} and its time derivatives. Let $k \in \mathbb{N}$, we denote by $\mathfrak{p}_{t,k}(x, y)$ the kernel of the operator $(t\sqrt{L})^k e^{-t\sqrt{L}}$ which includes the Poisson semi-group $e^{-t\sqrt{L}}$. To simplify notation, we write $\mathfrak{p}_t(x, y)$ instead of $\mathfrak{p}_{t,0}(x, y)$. The following is the estimates of $\mathfrak{p}_{t,k}(x, y)$ on the manifold $\mathcal{R}^1 \# \mathcal{R}^2$.

THEOREM 2.3. For $k \in \mathbb{N}$, we set $k \vee 1 = \max\{k, 1\}$. Then the kernel $\mathfrak{p}_{t,k}(x, y)$ of the operator $(t\sqrt{L})^k e^{-t\sqrt{L}}$ acting on the manifold $\mathcal{R}^1 \sharp \mathcal{R}^2$ satisfies the following estimates:

1. For $x, y \in \mathcal{R}^2$,

$$|\mathfrak{p}_{t,k}(x, y)| \leq \frac{C}{t^2} \left[\frac{t}{d(x, y) + t} \right]^{k \vee 1 + 2};$$

2. For $x \in \mathcal{R}^1 \setminus K, y \in K$ or $x, y \in \mathcal{R}^1 \setminus K$,

$$|\mathfrak{p}_{t,k}(x, y)| \leq \frac{C_1}{t^2} \left[\frac{t}{d(x, y) + t} \right]^{k \vee 1 + 2} + \frac{C_2}{t} \left[\frac{t}{d(x, y) + t} \right]^{k \vee 1 + 1};$$

3. For $x \in \mathcal{R}^1 \setminus K, y \in \mathcal{R}^2 \setminus K$,

$$|\mathfrak{p}_{t,k}(x, y)| \leq \frac{C_1}{t^2} \left[\frac{t}{d(x, y) + t} \right]^{k \vee 1 + 2} + \frac{C_2}{t|y|} \left[\frac{t}{d(x, y) + t} \right]^{k \vee 1 + 1}.$$

LEMMA 2.1. For $k \in \mathbb{N}$ and for all $x, y \in \mathcal{R}^1 \sharp \mathcal{R}^2$,

(i) $\int_0^\infty \left(\frac{t}{\sqrt{s}} \right)^k \frac{1}{s} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \leq \frac{C}{t^2} \left[\frac{t}{d(x, y) + t} \right]^{k+2};$

(ii) $\int_0^\infty \left(\frac{t}{\sqrt{s}} \right)^k \frac{1}{\sqrt{s}} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \leq \frac{C}{t} \left[\frac{t}{d(x, y) + t} \right]^{k+1}.$

PROOF. (i) One has,

$$\begin{aligned} & \int_0^\infty \left(\frac{t}{\sqrt{s}} \right)^k \frac{1}{s} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \\ &= \left(\int_0^{d^2(x,y)+t^2} + \int_{d^2(x,y)+t^2}^\infty \right) \left(\frac{t}{\sqrt{s}} \right)^k \frac{1}{s} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \\ &=: E + F. \end{aligned}$$

Noting that $e^{-b((d^2(x,y)+t^2)/s)} \leq C_\alpha [s/(d^2(x, y) + t^2)]^\alpha$ for all $\alpha > 0$; therefore, by choosing some suitable $\alpha > 0$,

$$E \leq C \frac{t^k}{[d^2(x, y) + t^2]^\alpha} \int_0^{d^2(x,y)+t^2} \frac{1}{s^{k/2+1-\alpha}} \frac{ds}{s} \leq \frac{C}{t^2} \left[\frac{t}{d(x, y) + t} \right]^{k+2},$$

and

$$F \leq C \frac{t^k}{[d^2(x, y) + t^2]^\alpha} \int_{d^2(x,y)+t^2}^\infty \frac{1}{s^{k/2+1-\alpha}} \frac{ds}{s} \leq \frac{C}{t^2} \left[\frac{t}{d(x, y) + t} \right]^{k+2}.$$

This follows (i).

(ii) Using the same arguments as those in (i),

$$\begin{aligned} \int_0^\infty \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{\sqrt{s}} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} &\leq C \int_0^\infty \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{\sqrt{s}} \left[\frac{s}{d^2(x,y)+t^2}\right]^\alpha \frac{ds}{s} \\ &\leq C \frac{t^k}{[d^2(x,y)+t^2]^\alpha} \int_0^\infty \frac{1}{s^{(k+1)/2-\alpha}} \frac{ds}{s} \\ &\leq \frac{C}{t} \left[\frac{t}{d(x,y)+t}\right]^{k+1}, \end{aligned}$$

for some suitable $\alpha > 0$. □

PROOF OF THEOREM 2.3. Though the kernel’s behaviours of the heat semigroup were obtained in [14], there is no further information about the estimates of the time derivatives of the Poisson semigroup $e^{-t\sqrt{L}}$. To achieve the estimates of the kernel upper bounds of the operator $(t\sqrt{L})^k e^{-t\sqrt{L}}$ ($k \in \mathbb{N}$), we employed the technique in [2, Theorem 2.2]. The main idea is to use the subordination formula to obtain those estimates via the known heat kernel upper bounds in [14]. However, unlike those in [2] where the heat kernel upper bounds are sharper than the Gaussian upper bounds, the heat kernel in our setting does not satisfy the Gaussian upper bounds due to the occurrence of logarithmic terms (e.g., the terms $(1/s) \log(\sqrt{s}/|y|) e^{-b(d^2(x,y)/s)}$ and $(1/s)(1 + (|x|/\sqrt{s}) \log(\sqrt{s}/|y|))$ in the heat kernel upper bounds).

Therefore, treatments for the heat kernel which does not satisfy the Gaussian upper bounds need to be modified to obtain good estimates which ensure the weak type $(1, 1)$ property of the operator $\mathfrak{M}(\sqrt{L})$ in Theorem 1.1.

The first part of our proof is quite standard, see [2], [4]; however, for the sake of completeness, we provide it here.

Note that, by the subordination formula,

$$e^{-t\sqrt{L}} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{te^{-t^2/4s}}{\sqrt{s}} e^{-sL} \frac{ds}{s}.$$

Then, taking its k^{th} derivative, we get that

$$\begin{aligned} (t\sqrt{L})^k e^{-t\sqrt{L}} &= (-1)^k \frac{t^k}{2\sqrt{\pi}} \int_0^\infty \partial_t^k (te^{-t^2/4s}) e^{-sL} \frac{ds}{s\sqrt{s}} \\ &= (-1)^{k+1} \frac{t^k}{\sqrt{\pi}} \int_0^\infty \partial_t^{k+1} (e^{-t^2/4s}) e^{-sL} \frac{ds}{\sqrt{s}}. \end{aligned}$$

This implies that

$$\mathfrak{p}_{t,k}(x, y) = (-1)^{k+1} \frac{t^k}{\sqrt{\pi}} \int_0^\infty \partial_t^{k+1} (e^{-t^2/4s}) \mathfrak{g}_s(x, y) \frac{ds}{\sqrt{s}}, \tag{2.1}$$

where $\mathfrak{g}_s(x, y)$ is the kernel of e^{-sL} .

Let $\nu > 0$ and $k \in \mathbb{N}$. By Faà di Bruno’s formula, we can write

$$\partial_t^{k+1} (e^{-t^2/\nu}) = \sum \frac{(-1)^{m_1+m_2} (k+1)! 2^{m_1}}{m_1! m_2!} e^{-t^2/\nu} \left(\frac{t}{\nu}\right)^{m_1} \left(\frac{1}{\nu}\right)^{m_2},$$

where the sum is taken over all pairs (m_1, m_2) of non-negative integers satisfying $m_1 + 2m_2 = k + 1$. For such a pair (m_1, m_2) , there exists a constant $C > 0$ such that

$$e^{-t^2/\nu} \left(\frac{t}{\nu}\right)^{m_1} \left(\frac{1}{\nu}\right)^{m_2} = e^{-t^2/\nu} \left(\frac{t}{\sqrt{\nu}}\right)^{m_1} \left(\frac{1}{\sqrt{\nu}}\right)^{m_1+2m_2} \leq C e^{-t^2/\nu} \left(\frac{1}{\sqrt{\nu}}\right)^{k+1} \max \left\{ 1, \left(\frac{t}{\sqrt{\nu}}\right)^{k+1} \right\}.$$

Combining all the above estimates, we get

$$\left| \partial_t^{k+1} (e^{-t^2/\nu}) \right| \leq C e^{-t^2/2\nu} \left(\frac{1}{\sqrt{\nu}}\right)^{k+1}. \tag{2.2}$$

From (2.1) and (2.2) we deduce that

$$|\mathfrak{p}_{t,k}(x, y)| \leq \frac{C}{4\sqrt{\pi}} \int_0^\infty e^{-t^2/8s} \left(\frac{t}{\sqrt{s}}\right)^k |\mathfrak{g}_s(x, y)| \frac{ds}{s}.$$

Note that the upper bounds of the heat kernel $\mathfrak{g}_s(x, y)$ are analogous as those in Theorem 2.1 due to Remark 2.1(ii). To get the estimates of $\mathfrak{p}_{t,k}(x, y)$, we split the above integral into two parts:

$$\mathcal{I}(x, y) := \frac{C}{4\sqrt{\pi}} \int_0^1 e^{-t^2/8s} \left(\frac{t}{\sqrt{s}}\right)^k |\mathfrak{g}_s(x, y)| \frac{ds}{s}$$

and

$$\mathcal{J}(x, y) := \frac{C}{4\sqrt{\pi}} \int_1^\infty e^{-t^2/8s} \left(\frac{t}{\sqrt{s}}\right)^k |\mathfrak{g}_s(x, y)| \frac{ds}{s}.$$

Applying Theorem 2.1(1), and then Lemma 2.1(i), we have

$$\begin{aligned} \mathcal{I}(x, y) &\leq \frac{C}{4\sqrt{\pi}} \int_0^1 e^{-t^2/8s} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{s} e^{-b(d^2(x,y)/s)} \frac{ds}{s} \\ &\leq C \int_0^\infty \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{s} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \\ &\leq \frac{C}{t^2} \left[\frac{t}{d(x, y) + t} \right]^{k+2}. \end{aligned}$$

For the latter part, $\mathcal{J}(x, y)$, we have some observations as follows:

- (a) If $x, y \in \mathcal{R}^2$, by Theorem 2.1(2), the heat kernel $\mathfrak{g}_s(x, y)$ has the Gaussian upper bound; hence, by Lemma 2.1(i),

$$\mathcal{J}(x, y) \leq \frac{C}{t^2} \left[\frac{t}{d(x, y) + t} \right]^{k+2}.$$

- (b) In other cases, the heat kernel upper bounds vary depending on the positions of the variables x, y on the manifolds and the modulus of these variables against the value of the time scaling s . Further, most of them no longer satisfy the Gaussian upper bound due to the influence of logarithmic terms. Therefore, we will apply particular analyses for each case.

Case 1: $x \in \mathcal{R}^1 \setminus K, y \in K$.

$$\begin{aligned} \mathcal{J}(x, y) &= \frac{C}{4\sqrt{\pi}} \left(\int_1^{|x|^2} + \int_{|x|^2}^\infty \right) e^{-t^2/8s} \left(\frac{t}{\sqrt{s}} \right)^k |\mathfrak{g}_s(x, y)| \frac{ds}{s} \\ &=: \mathcal{J}_1 + \mathcal{J}_2. \end{aligned}$$

As for \mathcal{J}_1 , by Theorem 2.1,2.(ii₁), we have

$$\begin{aligned} \mathcal{J}_1 &\leq C \int_1^{|x|^2} e^{-t^2/8s} \left(\frac{t}{\sqrt{s}} \right)^k \frac{\log s}{s} e^{-b(d^2(x,y)/s)} \frac{ds}{s} \\ &\leq C \int_1^\infty \left(\frac{t}{\sqrt{s}} \right)^k \frac{\log s}{s} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \\ &\leq C \int_1^\infty \left(\frac{t}{\sqrt{s}} \right)^k \frac{1}{\sqrt{s}} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s}, \quad \text{since } \frac{\log s}{\sqrt{s}} \leq C. \end{aligned}$$

Then, by Lemma 2.1(ii),

$$J_1 \leq \frac{C}{t} \left[\frac{t}{d(x, y) + t} \right]^{k+1}.$$

As for \mathcal{J}_2 , by Theorem 2.1,2.(ii₂), we have

$$\begin{aligned} \mathcal{J}_2 &\leq C \int_{|x|^2}^\infty e^{-t^2/8s} \left(\frac{t}{\sqrt{s}} \right)^k \frac{1}{s} \left(1 + \frac{|x|}{\sqrt{s}} \log s \right) e^{-b(d^2(x,y)/s)} \frac{ds}{s} \\ &\leq C \int_{|x|^2}^\infty \left(\frac{t}{\sqrt{s}} \right)^k \frac{1}{s} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \\ &\quad + C \int_{|x|^2}^\infty \left(\frac{t}{\sqrt{s}} \right)^k \frac{1}{s} \frac{|x| \log s}{\sqrt{s}} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \\ &=: \mathcal{J}_{21} + \mathcal{J}_{22}. \end{aligned}$$

Note that, by Lemma 2.1(i),

$$\mathcal{J}_{21} \leq \frac{C}{t^2} \left[\frac{t}{d(x, y) + t} \right]^{k+2}.$$

As for \mathcal{J}_{22} ,

$$\mathcal{J}_{22} = C \int_{|x|^2}^\infty \left(\frac{t}{\sqrt{s}} \right)^k \frac{1}{s} \frac{|x| \log s}{\sqrt{s}} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s}$$

$$\leq C \int_{|x|^2}^{\infty} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{\sqrt{s}} e^{-b((d^2(x,y)+t^2)/v)} \frac{ds}{s},$$

since $|x| \leq \sqrt{s}$ and $\log s/\sqrt{s} \leq C$.

Then, by Lemma 2.1(ii),

$$\mathcal{J}_{22} \leq \frac{C}{t} \left[\frac{t}{d(x,y)+t} \right]^{k+1}.$$

Hence,

$$\mathcal{J}(x,y) \leq \frac{C_1}{t^2} \left[\frac{t}{d(x,y)+t} \right]^{k+2} + \frac{C_2}{t} \left[\frac{t}{d(x,y)+t} \right]^{k+1}.$$

Case 2: $x \in \mathcal{R}^1 \setminus K, y \in \mathcal{R}^2 \setminus K$.

$$\begin{aligned} \mathcal{J}(x,y) &= \frac{C}{4\sqrt{\pi}} \left(\int_1^{|y|^2} + \int_{|y|^2}^{\infty} \right) e^{-t^2/8s} \left(\frac{t}{\sqrt{s}}\right)^k |\mathfrak{g}_s(x,y)| \frac{ds}{s} \\ &=: \mathcal{J}_3 + \mathcal{J}_4. \end{aligned}$$

The estimate of \mathcal{J}_3 is omitted since the heat kernel $\mathfrak{g}_s(x,y)$ has the Gaussian upper bound.

As for \mathcal{J}_4 , we consider the two following cases.

(iv₁) If $|x| \leq |y|$ then, by Theorem 2.1,2.(iv₂), we have

$$|\mathfrak{g}_s(x,y)| \leq \frac{C}{s} \left(1 + \frac{|x|}{\sqrt{s}} \log \frac{\sqrt{s}}{|y|} \right).$$

Therefore,

$$\begin{aligned} \mathcal{J}_4 &\leq C \int_{|y|^2}^{\infty} e^{-t^2/8s} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{s} \left(1 + \frac{|x|}{\sqrt{s}} \log \frac{\sqrt{s}}{|y|} \right) \frac{ds}{s} \\ &\leq C \int_{|y|^2}^{\infty} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{s} \left(1 + \frac{|x|}{\sqrt{s}} \log \frac{\sqrt{s}}{|y|} \right) e^{-t^2/8s} \frac{ds}{s} \\ &\leq C \int_{|y|^2}^{\infty} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{s} e^{-t^2/8s} \frac{ds}{s} \\ &\quad + C \int_{|y|^2}^{\infty} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{s} \frac{|x|}{\sqrt{s}} \log \frac{\sqrt{s}}{|y|} e^{-t^2/8s} \frac{ds}{s} \\ &=: \mathcal{J}_{41} + \mathcal{J}_{42}. \end{aligned}$$

It should be noted that $|x|, |y| \leq \sqrt{s}$ and $d(x,y) \approx |x| + |y|$; therefore,

$$e^{-t^2/8s} \leq C e^{-b((d^2(x,y)+t^2)/s)},$$

where C is a positive constant.

So,

$$\begin{aligned} \mathcal{J}_{41} &\leq C \int_{|y|^2}^{\infty} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{s} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \\ &\leq \frac{C}{t^2} \left[\frac{t}{d(x,y)+t}\right]^{k+2}. \end{aligned}$$

As for \mathcal{J}_{42} , we have

$$\begin{aligned} \mathcal{J}_{42} &\leq C \int_{|y|^2}^{\infty} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{s} \frac{\sqrt{s}}{|y|} e^{-t^2/8s} \frac{ds}{s} \\ &\leq \frac{C}{|y|} \int_{|y|^2}^{\infty} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{\sqrt{s}} e^{-t^2/8s} \frac{ds}{s}. \end{aligned}$$

By using similar arguments as those for \mathcal{J}_{41} ,

$$\begin{aligned} \mathcal{J}_{42} &\leq \frac{C}{|y|} \int_{|y|^2}^{\infty} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{\sqrt{s}} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \\ &\leq \frac{C}{t|y|} \left[\frac{t}{d(x,y)+t}\right]^{k+1}. \end{aligned}$$

(iv₂) If $|x| > |y|$ then we have

$$\begin{aligned} \mathcal{J}_4 &= \frac{C}{4\sqrt{\pi}} \left(\int_{|y|^2}^{|x|^2} + \int_{|x|^2}^{\infty} \right) e^{-t^2/8s} \left(\frac{t}{\sqrt{s}}\right)^k |\mathfrak{g}_s(x,y)| \frac{ds}{s} \\ &= \mathcal{J}_{43} + \mathcal{J}_{44}. \end{aligned}$$

As for \mathcal{J}_{43} , by Theorem 2.1,2.(iv₃), we have

$$|\mathfrak{g}_s(x,y)| \leq \frac{C}{s} \log \frac{\sqrt{s}}{|y|} e^{-b(d^2(x,y)/s)}.$$

Hence,

$$\begin{aligned} \mathcal{J}_{43} &\leq C \int_{|y|^2}^{|x|^2} e^{-t^2/8s} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{s} \log \frac{\sqrt{s}}{|y|} e^{-b(d^2(x,y)/s)} \frac{ds}{s} \\ &\leq C \int_{|y|^2}^{|x|^2} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{s} \frac{\sqrt{s}}{|y|} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \\ &\leq \frac{C}{|y|} \int_1^{\infty} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{\sqrt{s}} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \\ &\leq \frac{C}{t|y|} \left[\frac{t}{d(x,y)+t}\right]^{k+1}. \end{aligned}$$

As for \mathcal{J}_{44} , by Theorem 2.1,2.(iv₂), we have

$$\mathcal{J}_{44} \leq C \int_{|x|^2}^{\infty} e^{-t^2/8s} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{s} \left(1 + \frac{|x|}{\sqrt{s}} \log \frac{\sqrt{s}}{|y|}\right) \frac{ds}{s}.$$

Arguing similarly to the estimate of \mathcal{J}_4 in the case (iv₁), we have

$$\mathcal{J}_{44} \leq \frac{C_1}{t^2} \left[\frac{t}{d(x,y)+t}\right]^{k+2} + \frac{C_2}{t|y|} \left[\frac{t}{d(x,y)+t}\right]^{k+1}.$$

Case 3: $x, y \in \mathcal{R}^1 \setminus K$.

Due to the symmetry of the heat kernel upper bounds in this case, we only need to consider the case $|x| \leq |y|$. We now decompose

$$\begin{aligned} \mathcal{J}(x, y) &= \frac{C}{4\sqrt{\pi}} \left(\int_1^{|x|^2} + \int_{|x|^2}^{|y|^2} + \int_{|y|^2}^{\infty} \right) e^{-t^2/8s} \left(\frac{t}{\sqrt{s}}\right)^k |\mathfrak{g}_s(x, y)| \frac{ds}{s} \\ &=: \mathcal{J}_5 + \mathcal{J}_6 + \mathcal{J}_7. \end{aligned}$$

As for \mathcal{J}_5 , by Theorem 2.1,2.(v₄), we have

$$\begin{aligned} \mathcal{J}_5 &\leq C \int_1^{|x|^2} e^{-t^2/8s} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{\sqrt{s}} e^{-b(d^2(x,y)/s)} \frac{ds}{s} \\ &\leq C \int_1^{\infty} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{\sqrt{s}} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \\ &\leq \frac{C}{t} \left[\frac{t}{d(x,y)+t}\right]^{k+1}. \end{aligned}$$

As for \mathcal{J}_6 , by Theorem 2.1,2.(v₃), we have

$$|\mathfrak{g}_s(x, y)| \leq \frac{C}{s} (|x| + \log s) e^{-b(d^2(x,y)/s)}.$$

Then,

$$\begin{aligned} \mathcal{J}_6 &\leq C \int_{|x|^2}^{|y|^2} e^{-t^2/8s} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{s} (|x| + \log s) e^{-b(d^2(x,y)/s)} \frac{ds}{s} \\ &\leq C \int_{|x|^2}^{|y|^2} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{s} (|x| + \log s) e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \\ &\leq C \int_{|x|^2}^{|y|^2} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{\sqrt{s}} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \\ &\quad + C \int_{|x|^2}^{|y|^2} \left(\frac{t}{\sqrt{s}}\right)^k \frac{\log s}{s} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s}, \text{ since } |x| \leq \sqrt{s} \\ &\leq C \int_1^{\infty} \left(\frac{t}{\sqrt{s}}\right)^k \frac{1}{\sqrt{s}} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \\ &\quad + C \int_1^{\infty} \left(\frac{t}{\sqrt{s}}\right)^k \frac{\log s}{s} e^{-b((d^2(x,y)+t^2)/s)} \frac{ds}{s} \end{aligned}$$

$$\leq \frac{C}{t} \left[\frac{t}{d(x, y) + t} \right]^{k+1}.$$

As for \mathcal{J}_7 , by Theorem 2.1,2.(v₁), we have

$$|\mathfrak{g}_s(x, y)| \leq \frac{C}{s} \left(1 + \frac{|x||y|}{\sqrt{s}} + \frac{|x| + |y|}{\sqrt{s}} \log s \right) e^{-b(d^2(x, y)/s)}.$$

Hence,

$$\begin{aligned} \mathcal{J}_7 &\leq C \int_{|y|^2}^\infty e^{-t^2/8s} \left(\frac{t}{\sqrt{s}} \right)^k \frac{1}{s} \left(1 + \frac{|x||y|}{\sqrt{s}} + \frac{|x| + |y|}{\sqrt{s}} \log s \right) e^{-b(d^2(x, y)/s)} \frac{ds}{s} \\ &\leq C \int_{|y|^2}^\infty \left(\frac{t}{\sqrt{s}} \right)^k \frac{1}{s} \left(1 + \frac{|x||y|}{\sqrt{s}} + \frac{|x| + |y|}{\sqrt{s}} \log s \right) e^{-b((d^2(x, y)+t^2)/s)} \frac{ds}{s} \\ &\leq C \int_{|y|^2}^\infty \left(\frac{t}{\sqrt{s}} \right)^k \left(\frac{1}{s} + \frac{1}{\sqrt{s}} \right) e^{-b((d^2(x, y)+t^2)/s)} \frac{ds}{s}, \end{aligned}$$

since $|x|, |y| \leq \sqrt{s}$ and $\log s/\sqrt{s} \leq C$

By Lemma 2.1(i) and (ii), we have

$$\mathcal{J}_7 \leq \frac{C_1}{t^2} \left[\frac{t}{d(x, y) + t} \right]^{k+2} + \frac{C_2}{t} \left[\frac{t}{d(x, y) + t} \right]^{k+1}.$$

The proof is complete. □

3. The Holomorphic functional calculus.

We now present the proof of our main result, Theorem 1.1. The method of proof here is based on the Calderón–Zygmund decomposition and the method of Duong and McIntosh [11]. We make use of the technique in [2] in order to deal with the blowing up of non-doubling volumes of balls on the non-doubling manifolds with ends to get the weak type (1, 1) estimate.

PROOF. We first note that the L^p boundedness of $\mathfrak{M}(\sqrt{L})$ for $1 < p < \infty$ can be obtained by the Littlewood–Paley theory [19] or transference method [7]. Therefore, we just need to prove that the operator $\mathfrak{M}(\sqrt{L})$ is of weak type (1, 1), i.e., there is a positive constant C such that for all functions $f \in L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)$ and for every $\lambda > 0$,

$$\left| \left\{ x \in \mathcal{R}^1 \sharp \mathcal{R}^2 : \left| \mathfrak{M}(\sqrt{L})f(x) \right| > \lambda \right\} \right| \leq \frac{C}{\lambda} \|f\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)}. \tag{3.1}$$

Then, by the Marcinkiewicz interpolation and duality, $\mathfrak{M}(\sqrt{L})$ is bounded on $L^p(\mathcal{R}^1 \sharp \mathcal{R}^2)$ for all $1 < p < \infty$.

Let us consider the following inequalities:

$$\left| \left\{ x \in \mathcal{R}^2 \setminus K : \left| \mathfrak{M}(\sqrt{L})f(x) \right| > \lambda \right\} \right| \leq \frac{C}{\lambda} \|f\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)}, \tag{3.2a}$$

$$\left| \left\{ x \in \mathcal{R}^1 \setminus K : \left| \mathfrak{M}(\sqrt{L})f(x) \right| > \lambda \right\} \right| \leq \frac{C}{\lambda} \|f\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}, \tag{3.2b}$$

$$\left| \left\{ x \in K : \left| \mathfrak{M}(\sqrt{L})f(x) \right| > \lambda \right\} \right| \leq \frac{C}{\lambda} \|f\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}. \tag{3.2c}$$

We now set $f_1(x) = f(x)\chi_{\mathcal{R}^2 \setminus K}$, $f_2(x) = f(x)\chi_{\mathcal{R}^1 \setminus K}$ and $f_3(x) = f(x)\chi_K$. Then we can write

$$f = f_1 + f_2 + f_3.$$

Since $\mathfrak{M}(\sqrt{L})$ is a linear operator, the estimates of (3.2a), (3.2b) and (3.2c) can be obtained by exploring the three estimates below:

$$\begin{aligned} \left| \left\{ x \in \mathcal{R}^2 \setminus K : \left| \mathfrak{M}(\sqrt{L})f(x) \right| > \lambda \right\} \right| &\leq \left| \left\{ x \in \mathcal{R}^2 \setminus K : \left| \mathfrak{M}(\sqrt{L})f_1(x) \right| > \frac{\lambda}{3} \right\} \right| \\ &\quad + \left| \left\{ x \in \mathcal{R}^2 \setminus K : \left| \mathfrak{M}(\sqrt{L})f_2(x) \right| > \frac{\lambda}{3} \right\} \right| \\ &\quad + \left| \left\{ x \in \mathcal{R}^2 \setminus K : \left| \mathfrak{M}(\sqrt{L})f_3(x) \right| > \frac{\lambda}{3} \right\} \right| \\ &=: I_1 + I_2 + I_3, \end{aligned}$$

$$\begin{aligned} \left| \left\{ x \in \mathcal{R}^1 \setminus K : \left| \mathfrak{M}(\sqrt{L})f(x) \right| > \lambda \right\} \right| &\leq \left| \left\{ x \in \mathcal{R}^1 \setminus K : \left| \mathfrak{M}(\sqrt{L})f_1(x) \right| > \frac{\lambda}{3} \right\} \right| \\ &\quad + \left| \left\{ x \in \mathcal{R}^1 \setminus K : \left| \mathfrak{M}(\sqrt{L})f_2(x) \right| > \frac{\lambda}{3} \right\} \right| \\ &\quad + \left| \left\{ x \in \mathcal{R}^1 \setminus K : \left| \mathfrak{M}(\sqrt{L})f_3(x) \right| > \frac{\lambda}{3} \right\} \right| \\ &=: II_1 + II_2 + II_3, \end{aligned}$$

and

$$\begin{aligned} \left| \left\{ x \in K : \left| \mathfrak{M}(\sqrt{L})f(x) \right| > \lambda \right\} \right| &\leq \left| \left\{ x \in K : \left| \mathfrak{M}(\sqrt{L})f_1(x) \right| > \frac{\lambda}{3} \right\} \right| \\ &\quad + \left| \left\{ x \in K : \left| \mathfrak{M}(\sqrt{L})f_2(x) \right| > \frac{\lambda}{3} \right\} \right| \\ &\quad + \left| \left\{ x \in K : \left| \mathfrak{M}(\sqrt{L})f_3(x) \right| > \frac{\lambda}{3} \right\} \right| \\ &=: III_1 + III_2 + III_3. \end{aligned}$$

To get the desired estimate (3.1), we aim to prove that each of the terms above is dominated by $(C/\lambda)\|f\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}$. To do this, we will investigate the following terms: I_1, I_2, II_1 and II_2 . The rest can be handled by an analogous approach with some adjustments.

3.1. Estimate of I_1 .

Observing that, in this case, $x \in \mathcal{R}^2 \setminus K$ and function f_1 is supported on $\mathcal{R}^2 \setminus K$ which is a homogeneous space (in the sense of Coifman and Weiss, [6]); therefore, we can construct a sequence of disjoint dyadic cubes $\{Q_{1,i}\}$ on \mathbb{R}^2 (for reference, see [2]) and then employ the Calderón–Zygmund decomposition to decompose function

$$f_1(x) = g_1(x) + b_1(x) = g_1(x) + \sum_i b_{1,i}(x),$$

such that

- (a) $|g_1(x)| \leq C\lambda$ for almost all $x \in \mathcal{R}^2 \setminus K$;
- (b) the support of each function $b_{1,i}$ is contained in $Q_{1,i}$ and

$$\int_{Q_{1,i}} |b_{1,i}(x)| dx \leq C\lambda|Q_{1,i}|;$$

- (c) $\sum_i |Q_{1,i}| \leq \frac{C}{\lambda} \int_{\mathcal{R}^2 \setminus K} |f_1(x)| dx$;
- (d) $\sum_i \chi_{Q_{1,i}} \leq C$,

where $\chi_{Q_{1,i}}$ is the characteristic function of $Q_{1,i}$.

Hence, we have

$$\begin{aligned} I_1 &\leq \left| \left\{ x \in \mathcal{R}^2 \setminus K : \left| \mathfrak{M}(\sqrt{L})g_1(x) \right| > \frac{\lambda}{6} \right\} \right| \\ &\quad + \left| \left\{ x \in (\mathcal{R}^2 \setminus K) \setminus \bigcup_i 8Q_{1,i} : \left| \mathfrak{M}(\sqrt{L}) \sum_i b_{1,i}(x) \right| > \frac{\lambda}{6} \right\} \right| + \left| \bigcup_i 8Q_{1,i} \right| \\ &=: I_{11} + I_{12} + I_{13}. \end{aligned}$$

Using the facts that $\mathfrak{M}(\sqrt{L})$ is bounded on $L^2(\mathbb{R}^2)$ and $|g_1(x)| \leq C\lambda$, it is easy to verify that

$$I_{11} \leq \frac{C}{\lambda^2} \|\mathfrak{M}(\sqrt{L})g_1\|_{L^2(\mathcal{R}^2 \setminus K)}^2 \leq \frac{C}{\lambda^2} \|g_1\|_{L^2(\mathcal{R}^2 \setminus K)}^2 \leq \frac{C}{\lambda} \|f\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}.$$

Since (c) and the doubling property of \mathbb{R}^2 , we have

$$I_{13} \leq C \sum_i |Q_{1,i}| \leq \frac{C}{\lambda} \|f_1\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}.$$

The estimate of the bad part I_{12} was based on the method of Duong and McIntosh (see [11]). In addition, we employ a particular refined classification of the dyadic cubes $\{Q_{1,i}\}$ in [2] to deal with the lack of power of the time scaling t in the kernels' upper bounds, i.e., $(C/t|y|)[t/(d(x, y) + t)]^2$.

Hence, we split all the cubes $Q_{1,i}$'s into two groups:

$$\mathcal{J}_1 := \{i : \text{none of the corners of } Q_{1,i} \text{ is the origin}\},$$

and

$$\mathcal{J}_2 := \{i : \text{one of the corners of } Q_{1,i} \text{ is the origin}\}.$$

We then write

$$\mathfrak{M}(\sqrt{L}) \sum_i b_{1,i}(x) = \sum_{i \in \mathcal{J}_1} \mathfrak{M}(\sqrt{L}) b_{1,i}(x) + \sum_{i \in \mathcal{J}_2} \mathfrak{M}(\sqrt{L}) b_{1,i}(x).$$

For each $i \in \mathcal{J}_1$, we further decompose

$$\mathfrak{M}(\sqrt{L}) b_{1,i}(x) = \mathfrak{M}(\sqrt{L}) e^{-t_i \sqrt{L}} b_{1,i}(x) + \mathfrak{M}(\sqrt{L}) \left(I - e^{-t_i \sqrt{L}} \right) b_{1,i}(x),$$

where $\{e^{-t\sqrt{L}}\}_{t>0}$ is the Poisson semi-group of L studied in Section 2, and for each i , t_i is the size length of the cube $Q_{1,i}$.

Hence, we have

$$\begin{aligned} I_{12} &\leq \left| \left\{ x \in (\mathcal{R}^2 \setminus K) \setminus \bigcup_i 8Q_{1,i} : \left| \mathfrak{M}(\sqrt{L}) \left(\sum_{i \in \mathcal{J}_1} e^{-t_i \sqrt{L}} b_{1,i} \right) (x) \right| > \frac{\lambda}{18} \right\} \right| \\ &\quad + \left| \left\{ x \in (\mathcal{R}^2 \setminus K) \setminus \bigcup_i 8Q_{1,i} : \left| \mathfrak{M}(\sqrt{L}) \left[\sum_{i \in \mathcal{J}_1} \left(I - e^{-t_i \sqrt{L}} \right) b_{1,i} \right] (x) \right| > \frac{\lambda}{18} \right\} \right| \\ &\quad + \left| \left\{ x \in (\mathcal{R}^2 \setminus K) \setminus \bigcup_i 8Q_{1,i} : \left| \mathfrak{M}(\sqrt{L}) \left(\sum_{i \in \mathcal{J}_2} b_{1,i} \right) (x) \right| > \frac{\lambda}{18} \right\} \right| \\ &=: I_{121} + I_{122} + I_{123}. \end{aligned}$$

It can be verified that we can get the estimate

$$I_{121} \leq \frac{C}{\lambda} \|f_1\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)},$$

if the following estimates

$$\left\| \sum_{i \in \mathcal{J}_1} e^{-t_i \sqrt{L}} b_{1,i} \right\|_{L^2(\mathcal{R}^2 \setminus K)} \leq C \lambda^{1/2} \|f_1\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}^{1/2}, \tag{3.3}$$

$$\left\| \sum_{i \in \mathcal{J}_1} e^{-t_i \sqrt{L}} b_{1,i} \right\|_{L^2(\mathcal{R}^1 \setminus K)} \leq C \lambda^{1/2} \|f_1\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}^{1/2}, \tag{3.4}$$

and

$$\left\| \sum_{i \in \mathcal{J}_1} e^{-t_i \sqrt{L}} b_{1,i} \right\|_{L^2(K)} \leq C \lambda^{1/2} \|f_1\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}^{1/2} \tag{3.5}$$

are held.

To estimate (3.3), let us consider the function $e^{-t_i\sqrt{L}}b_{1,i}(x)$ for $x \in \mathcal{R}^2 \setminus K$. Since

$$e^{-t_i\sqrt{L}}b_{1,i}(x) = \int_{\mathcal{R}^2 \setminus K} \mathbf{p}_{t_i}(x, y)b_{1,i}(y)dy,$$

applying Theorem 2.3(1), we obtain that

$$\begin{aligned} |e^{-t_i\sqrt{L}}b_{1,i}(x)| &\leq \int_{\mathcal{R}^2 \setminus K} |\mathbf{p}_{t_i}(x, y)| |b_{1,i}(y)| dy \\ &\leq C \int_{\mathcal{R}^2 \setminus K} \frac{1}{t_i^2} \left[\frac{t_i}{d(x, y) + t_i} \right]^3 |b_{1,i}(y)| dy =: F_i. \end{aligned}$$

It should be noted that in this case $x \in \mathcal{R}^2 \setminus K$ and $Q_{1,i} \subset \mathcal{R}^2 \setminus K$ is the dyadic cube with none of its corners being the origin; therefore,

$$\sup_{z \in Q_{1,i}} \frac{t_i}{[d(x, z) + t_i]^3} \leq C \inf_{z \in Q_{1,i}} \frac{t_i}{[d(x, z) + t_i]^3}.$$

Hence,

$$\begin{aligned} F_i &\leq C \sup_{z \in Q_{1,i}} \frac{t_i}{[d(x, z) + t_i]^3} \int_{\mathcal{R}^2 \setminus K} |b_{1,i}(y)| dy \\ &\leq C \inf_{z \in Q_{1,i}} \frac{t_i}{[d(x, z) + t_i]^3} \lambda|Q_{1,i}| \\ &\leq C\lambda \int_{\mathcal{R}^2 \setminus K} \frac{t_i}{[d(x, z) + t_i]^3} \chi_{Q_{1,i}}(z) dz, \end{aligned}$$

where $\chi_{Q_{1,i}}$ is the characteristic function of $Q_{1,i}$.

For any function $h \in L^2(\mathcal{R}^2 \setminus K)$ with $\|h\|_{L^2(\mathcal{R}^2 \setminus K)} = 1$, we get that

$$\begin{aligned} |\langle F_i, h \rangle| &= C\lambda \int_{\mathcal{R}^2 \setminus K} \int_{\mathcal{R}^2 \setminus K} \frac{t_i}{[d(x, z) + t_i]^3} |h(x)| dx \chi_{Q_{1,i}}(z) dz \\ &\leq C\lambda \langle \mathcal{M}_2(h), \chi_{Q_{1,i}} \rangle \end{aligned}$$

where \mathcal{M}_2 is the Hardy–Littlewood maximal function on $\mathcal{R}^2 \setminus K$.

As a consequence, we obtain that

$$\left| \left\langle \sum_{i \in \mathcal{J}_1} F_i, h \right\rangle \right| \leq C\lambda \left\langle \mathcal{M}_2(h), \sum_{i \in \mathcal{J}_1} \chi_{Q_{1,i}} \right\rangle,$$

which yields

$$\left\| \sum_{i \in \mathcal{J}_1} F_i \right\|_{L^2(\mathcal{R}^2 \setminus K)} \leq C\lambda \left\| \sum_{i \in \mathcal{J}_1} \chi_{Q_{1,i}} \right\|_{\mathcal{R}^2 \setminus K}$$

$$\begin{aligned} &\leq C\lambda \left(\sum_{i \in \mathcal{J}_1} |Q_{1,i}| \right)^{1/2} \\ &\leq C\lambda \frac{\|f_1\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}^{1/2}}{\lambda^{1/2}} \\ &\leq C\lambda^{1/2} \|f_1\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}^{1/2}. \end{aligned}$$

So

$$\left\| \sum_{i \in \mathcal{J}_1} e^{-t_i \sqrt{L}} b_{1,i} \right\|_{L^2(\mathcal{R}^2 \setminus K)} \leq \left\| \sum_{i \in \mathcal{J}_1} F_i \right\|_{L^2(\mathcal{R}^2 \setminus K)} \leq C\lambda^{1/2} \|f_1\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}^{1/2}.$$

To estimate (3.4), we consider the function $e^{-t_i \sqrt{L}} b_{1,i}(x)$ for $x \in \mathcal{R}^1 \setminus K$. Since

$$e^{-t_i \sqrt{L}} b_{1,i}(x) = \int_{\mathcal{R}^2 \setminus K} \mathfrak{p}_{t_i}(x, y) b_{1,i}(y) dy,$$

applying Theorem 2.3(3), we obtain that

$$\begin{aligned} \left| e^{-t_i \sqrt{L}} b_{1,i}(x) \right| &\leq \int_{\mathcal{R}^2 \setminus K} |\mathfrak{p}_{t_i}(x, y)| |b_{1,i}(y)| dy \\ &\leq \int_{\mathcal{R}^2 \setminus K} \left(\frac{C_1}{t_i^2} \left[\frac{t_i}{d(x, y) + t_i} \right]^3 + \frac{C_2}{t_i |y|} \left[\frac{t_i}{d(x, y) + t_i} \right]^2 \right) |b_{1,i}(y)| dy \\ &\leq \frac{C_1}{t_i^2} \int_{\mathcal{R}^2 \setminus K} \left[\frac{t_i}{d(x, y) + t_i} \right]^3 |b_{1,i}(y)| dy \\ &\quad + \frac{C_2}{t_i |y|} \int_{\mathcal{R}^2 \setminus K} \left[\frac{t_i}{d(x, y) + t_i} \right]^2 |b_{1,i}(y)| dy \\ &=: G_{1,i} + G_{2,i}. \end{aligned}$$

The first term $G_{1,i}$ could be estimated similarly to F_i . To handle the second term $G_{2,i}$, we observe that the distance of $Q_{1,i}$ to the central part K is comparable to the side length of $Q_{1,i}$ for all $i \in \mathcal{J}_1$, since none of the corners of $Q_{1,i}$ is the origin. Therefore,

$$\sup_{z \in Q_{1,i}} |z| \approx \inf_{z \in Q_{1,i}} |z|. \tag{3.6}$$

Taking (3.6) into account, we then have

$$\begin{aligned} G_{2,i} &\leq C \int_{\mathcal{R}^2 \setminus K} \frac{1}{|y|} \frac{t_i}{[d(x, y) + t_i]^2} |b_{1,i}(y)| dy \\ &\leq C \sup_{z \in Q_{1,i}} \frac{1}{|z|} \frac{t_i}{[d(x, z) + t_i]^2} \int_{\mathcal{R}^2 \setminus K} |b_{1,i}(y)| dy \\ &\leq C \inf_{z \in Q_{1,i}} \frac{1}{|z|} \frac{t_i}{[d(x, z) + t_i]^2} \int_{\mathcal{R}^2 \setminus K} |b_{1,i}(y)| dy \end{aligned}$$

$$\leq C\lambda \int_{\mathcal{R}^2 \setminus K} \frac{1}{|z|} \frac{t_i}{[d(x, z) + t_i]^2} \chi_{Q_{1,i}}(z) dz.$$

So for any $h \in L^2(\mathcal{R}^1 \setminus K)$ with $\|h\|_{L^2(\mathcal{R}^1 \setminus K)} = 1$,

$$\begin{aligned} |\langle G_{2,i}, h \rangle| &\leq C\lambda \int_{\mathcal{R}^2 \setminus K} \int_{\mathcal{R}^1 \setminus K} \frac{1}{|z|} \frac{t_j}{[d(x, z) + t_i]^2} |h(x)| dx \chi_{Q_{1,i}}(z) dz \\ &\leq C\lambda \int_{\mathcal{R}^2 \setminus K} \mathcal{T}(h)(z) \chi_{Q_{1,i}}(z) dz, \end{aligned}$$

where the operator \mathcal{T} is defined as follows

$$\mathcal{T}(h)(z) := \int_{\mathcal{R}^1 \setminus K} \frac{1}{|z|} \frac{t_i}{[d(x, z) + t_i]^2} |h(x)| dx.$$

We will show that \mathcal{T} is a bounded operator on $L^2(\mathcal{R}^2 \setminus K)$. In fact, since $d(x, z) \approx |x| + |z|$ for all $x \in \mathcal{R}^1 \setminus K$ and $z \in \mathcal{R}^2 \setminus K$, we get that

$$\begin{aligned} \|\mathcal{T}(h)\|_{L^2(\mathcal{R}^2 \setminus K)}^2 &\leq \|h\|_{L^2(\mathcal{R}^1 \setminus K)}^2 \int_{\mathcal{R}^2 \setminus K} \int_{\mathcal{R}^1 \setminus K} \frac{1}{|z|^2} \frac{t_i^2}{(d(x, z) + t_i)^4} dx dz \\ &\leq C \|h\|_{L^2(\mathcal{R}^1 \setminus K)}^2 \int_{\mathcal{R}^2 \setminus K} \int_{\mathcal{R}^1 \setminus K} \frac{1}{|z|^2} \frac{t_i^2}{(|x| + |z| + t_i)^4} dx dz \\ &\leq C \|h\|_{L^2(\mathcal{R}^1 \setminus K)}^2 \int_{\mathcal{R}^2 \setminus K} \int_{\mathcal{R}^1 \setminus K} \frac{1}{|z|^2} \frac{t_i^2}{(|x| + t_i)^3} \frac{1}{|z|} dx dz \\ &\leq C \|h\|_{L^1(\mathcal{R}^1 \setminus K)}^2 \int_{\mathcal{R}^2 \setminus K} \int_{\mathcal{R}^1 \setminus K} \frac{1}{t_i} \left(\frac{t_i}{|x| + t_i} \right)^3 dx \frac{1}{|z|^3} dz \\ &\leq C \|h\|_{L^1(\mathcal{R}^1 \setminus K)}^2. \end{aligned}$$

As a consequence, we obtain that

$$\begin{aligned} \left\| \sum_{i \in \mathcal{J}_1} G_{2,i} \right\|_{L^2(\mathcal{R}^1 \setminus K)} &\leq C\lambda \left\| \sum_{i \in \mathcal{J}_1} \chi_{Q_{1,i}} \right\|_{L^2(\mathcal{R}^1 \setminus K)} \leq C\lambda \left(\sum_{i \in \mathcal{J}_1} |Q_{1,i}| \right)^{1/2} \\ &\leq C\lambda^{1/2} \|f_1\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}^{1/2}. \end{aligned}$$

Combining the estimates of $G_{1,i}$ and $G_{2,i}$, we get the estimate (3.4).

The last estimate (3.5) can be done similarly to F_i since the upper bound of the operator $e^{-t_i \sqrt{L}} b_{1,i}(x)$ has the same form as that in the estimate (3.3).

Next, we observe that

$$\begin{aligned} I_{122} &\leq \left\| \left\{ x \in (\mathcal{R}^2 \setminus K) \setminus \bigcup_i 8Q_{1,i} : \left| \mathfrak{M}(\sqrt{L}) \sum_{i \in \mathcal{J}_1} (I - e^{-t_i \sqrt{L}}) b_{1,i}(x) \right| > \frac{\lambda}{12} \right\} \right\| \\ &\leq \frac{C}{\lambda} \sum_i \int_{(8Q_{1,i})^c} \left| \mathfrak{M}(\sqrt{L}) (I - e^{-t_i \sqrt{L}}) b_{1,i}(x) \right| dx. \end{aligned}$$

For each i , we get that

$$\begin{aligned}
 H &:= \int_{(8Q_{1,i})^c} \left| \mathfrak{M}(\sqrt{L}) \left(I - e^{-t_i \sqrt{L}} \right) b_{1,i}(x) \right| dx \\
 &\leq \int_{(8Q_{1,i})^c} \int_{Q_{1,i}} |k_{t_i}(x, y)| |b_{1,i}(y)| dy dx \\
 &= \int_{Q_{1,i}} \int_{(8Q_{1,i})^c} |k_{t_i}(x, y)| dx |b_{1,i}(y)| dy,
 \end{aligned} \tag{3.7}$$

where $k_{t_i}(x, y)$ is the kernel of the operator $\mathfrak{M}(\sqrt{L})(I - e^{-t_i \sqrt{L}})$.

By definition of the holomorphic functional calculus $\mathfrak{M}(\sqrt{L})$, we have

$$\begin{aligned}
 \mathfrak{M}(\sqrt{L}) \left(I - e^{-t_i \sqrt{L}} \right) &= \int_0^\infty \sqrt{L} e^{-s \sqrt{L}} m(s) ds \int_0^{t_i} -\frac{d}{dt} e^{-t \sqrt{L}} dt \\
 &= \int_0^\infty \sqrt{L} e^{-s \sqrt{L}} m(s) ds \int_0^{t_i} \sqrt{L} e^{-t \sqrt{L}} dt \\
 &= \int_0^{t_i} \int_0^\infty (\sqrt{L})^2 e^{-(s+t) \sqrt{L}} m(s) ds dt \\
 &= \int_0^{t_i} \int_0^\infty (s+t)^2 (\sqrt{L})^2 e^{-(s+t) \sqrt{L}} \frac{m(s)}{(s+t)^2} ds dt.
 \end{aligned}$$

Hence, we obtain that

$$k_{t_i}(x, y) = \int_0^{t_i} \int_0^\infty \mathfrak{p}_{s+t,2}(x, y) \frac{m(s)}{(s+t)^2} ds dt.$$

We will show that there exists a positive constant C such that

$$\int_{(8Q_{1,i})^c} |k_{t_i}(x, y)| dx \leq C. \tag{3.8}$$

To justify it, applying the kernel expression of $k_{t_i}(x, y)$ and Theorem 2.3(1), we will estimate the above integral as follows.

$$\begin{aligned}
 \int_{(8Q_{1,i})^c} |k_{t_i}(x, y)| dx &\leq \int_{(8Q_{1,i})^c} \int_0^{t_i} \int_0^\infty |\mathfrak{p}_{s+t,2}(x, y)| \frac{|m(s)|}{(s+t)^2} ds dt dx \\
 &\leq C \int_{(8Q_{1,i})^c} \int_0^{t_i} \int_0^\infty \frac{1}{(s+t)^2} \left[\frac{s+t}{d(x, y) + s+t} \right]^4 \frac{1}{(s+t)^2} ds dt dx \\
 &=: E.
 \end{aligned}$$

Note that

$$E \leq C \int_0^{t_i} \int_0^\infty \int_{d(x,y) \geq 2t_i} \left[\frac{1}{d(x, y) + s+t} \right]^4 dx ds dt$$

$$\begin{aligned}
 &\leq C \int_0^{t_i} \int_0^{t_i} \int_{d(x,y) \geq 2t_i} \left[\frac{1}{d(x,y) + s + t} \right]^4 dx ds dt \\
 &\quad + C \int_0^{t_i} \int_{t_i}^\infty \int_{d(x,y) \geq 2t_i} \frac{1}{(s+t)^2} \left[\frac{s+t}{d(x,y) + s + t} \right]^4 dx \frac{1}{(s+t)^2} ds dt \\
 &\leq C \int_0^{t_i} \int_0^{t_i} \int_{t_i}^\infty \frac{1}{r^4} r dr ds dt + C \int_0^{t_i} \int_{t_i}^\infty \frac{1}{(s+t)^2} ds dt \\
 &\leq C,
 \end{aligned}$$

where in the last inequality, we used polar coordinates for estimating the first term and the following fact

$$\int_{d(x,y) \geq 2t_i} \frac{1}{(s+t)^2} \left[\frac{s+t}{d(x,y) + s + t} \right]^4 dx \leq C,$$

for the second term.

Hence, (3.8) holds. As a consequence, from (3.7), we obtain that for each i ,

$$\int_{(8Q_{1,i})^c} \left| \mathfrak{M}(\sqrt{L}) \left(I - e^{-t_i \sqrt{L}} \right) b_{1,i}(x) \right| dx \leq C \int_{Q_{1,i}} |b_{1,i}(y)| dy \leq C \lambda |Q_i|,$$

which implies that

$$I_{122} \leq \frac{C}{\lambda} \sum_i \int_{(8Q_{1,i})^c} \left| \mathfrak{M}(\sqrt{L}) \left(I - e^{-t_i \sqrt{L}} \right) b_{1,i}(x) \right| dx \leq C \sum_i |Q_{1,i}| \leq \frac{C}{\lambda} \|f_1\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)}.$$

We now consider the term I_{123} . Note that for each $i \in \mathcal{I}_2$ we have $t_i \geq 1/2$. Fix $i \in \mathcal{I}_2$. Denote by $k_{\mathfrak{M}(\sqrt{L})}(x, y)$ the associated kernel of $\mathfrak{M}(\sqrt{L})$. For $x \in (\mathcal{R}^2 \setminus K) \setminus 8Q_{1,i}$ and any $y \in Q_{1,i}$, by Theorem 2.3(1), we have

$$\left| k_{\mathfrak{M}(\sqrt{L})}(x, y) \right| \leq C \int_0^\infty |p_{t,1}(x, y)| \frac{dt}{t} \leq C \int_0^\infty \frac{1}{t^2} \left[\frac{t}{d(x, y) + t} \right]^3 \frac{dt}{t} =: K(x, y).$$

Since $d(x, y) \approx d(x, x_{Q_{1,i}})$, we have

$$\begin{aligned}
 K(x, y) &\leq C \int_0^\infty \frac{1}{t^3} \left[\frac{t}{d(x, x_{Q_{1,i}}) + t} \right]^3 dt \\
 &\leq C \int_0^{d(x, x_{Q_{1,i}})} \frac{1}{d(x, x_{Q_{1,i}})^3} dt + C \int_{d(x, x_{Q_{1,i}})}^\infty \frac{1}{t^3} dt \\
 &\leq \frac{C}{d(x, x_{Q_{1,i}})^2}.
 \end{aligned}$$

From the estimate of $K(x, y)$ for each $i \in \mathcal{I}_2$ and $x \in (\mathcal{R}^2 \setminus K) \setminus 8Q_{1,i}$ we have

$$\sup_{y \in Q_{1,i}} \left| k_{\mathfrak{M}(\sqrt{L})}(x, y) \right| \leq \frac{C}{d(x, x_{Q_{1,i}})^2}.$$

Moreover, observing that since $i \in \mathcal{I}_2$ and $x \in (\mathcal{R}^2 \setminus K) \setminus 8Q_{1,i}$, we have

$$\frac{1}{d(x, x_{Q_{1,i}})^2} \approx \frac{1}{|x|^2}.$$

As a consequence, for each $i \in \mathcal{I}_2$ and $x \in (\mathcal{R}^2 \setminus K) \setminus 8Q_{1,i}$ we have

$$\sup_{y \in Q_{1,i}} \left| k_{\mathfrak{M}(\sqrt{L})}(x, y) \right| \leq \frac{C}{|x|^2}.$$

This implies that for each $x \in (\mathcal{R}^2 \setminus K) \setminus 8Q_{1,i}$,

$$\left| \sum_{i \in \mathcal{I}_2} \mathfrak{M}(\sqrt{L})b_{1,i}(x) \right| \leq \frac{C}{|x|^2} \left(\sum_{i \in \mathcal{I}_2} \|b_{1,i}\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)} \right).$$

Hence,

$$\begin{aligned} I_{123} &\leq \left| \left\{ x \in (\mathcal{R}^2 \setminus K) \setminus \bigcup_i 8Q_{1,i} : \left| \sum_{i \in \mathcal{I}_2} \mathfrak{M}(\sqrt{L})b_{1,i}(x) \right| > \frac{\lambda}{18} \right\} \right| \\ &\leq \left| \left\{ x \in (\mathcal{R}^2 \setminus K) \setminus \bigcup_i 8Q_{1,i} : \frac{C}{|x|^2} \left(\sum_{i \in \mathcal{I}_2} \|b_{1,i}\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)} \right) > \frac{\lambda}{18} \right\} \right| \\ &\leq \frac{C}{\lambda} \sum_{i \in \mathcal{I}_2} \|b_{1,i}\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)} \\ &\leq \frac{C}{\lambda} \|f_1\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)}. \end{aligned}$$

Combining all cases of $I_{11}, I_{13}, I_{121}, I_{122}$ and I_{123} we get the desired estimate

$$I_1 \leq \frac{C}{\lambda} \|f_1\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)}.$$

3.2. Estimate of I_2 .

The difficulties in this case come from the non-doubling of the measure and the lack of information about the exact location of Calderón–Zygmund cubes on the manifold. This implies that the standard Calderón–Zygmund decomposition on non-homogeneous space such as in [16] and [20] is not applicable. To overcome those issues, a Whitney type decomposition combined with a clever use of the Poisson kernel upper bound is employed to achieve the weak type $(1, 1)$ estimate. The idea of this approach is an adaptation of the techniques used in [2] and [16].

First, we split $\mathcal{R}^1 \setminus K$ into two parts according to f_2 . Define

$$\mathcal{F} := \{x \in \mathcal{R}^1 \setminus K : \mathcal{M}_1 f_2(x) \leq \lambda\}$$

and

$$\Omega := \{x \in \mathcal{R}^1 \setminus K : \mathcal{M}_1 f_2(x) > \lambda\},$$

where \mathcal{M}_1 is the Hardy–Littlewood maximal function defined on $\mathcal{R}^1 \setminus K$.

Then we define

$$f_{2,\lambda}(x) := f_2(x)\chi_{\mathcal{F}}(x) \text{ and } f_2^\lambda(x) := f_2(x)\chi_\Omega(x).$$

So we have

$$\begin{aligned} I_2 &\leq \left| \left\{ x \in \mathcal{R}^2 \setminus K : \left| \mathfrak{M}(\sqrt{L})f_{2,\lambda}(x) \right| > \frac{\lambda}{6} \right\} \right| \\ &\quad + \left| \left\{ x \in \mathcal{R}^2 \setminus K : \left| \mathfrak{M}(\sqrt{L})f_2^\lambda(x) \right| > \frac{\lambda}{6} \right\} \right| \\ &=: I_{21} + I_{22}. \end{aligned}$$

As for I_{21} , by using the L^2 boundedness of $\mathfrak{M}(\sqrt{L})$ combined with Chebycheff’s inequality,

$$I_{21} = \left| \left\{ x \in \mathcal{R}^2 \setminus K : \left| \mathfrak{M}(\sqrt{L})f_{2,\lambda}(x) \right| > \frac{\lambda}{6} \right\} \right| \leq \frac{C}{\lambda^2} \|f_{2,\lambda}\|_{L^2(\mathcal{R}^1 \setminus K)}^2 \leq \frac{C}{\lambda} \|f\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)},$$

where we use the fact that $|f_{2,\lambda}(x)| = |f_2(x)|\chi_{\mathcal{F}}(x) \leq |\mathcal{M}_1 f_2(x)|\chi_{\mathcal{F}}(x) \leq \lambda$.

As for I_{22} , we consider the function f_2^λ . We now apply a covering lemma in [6] (see also [8], Lemma 5.5) for the set Ω in the homogeneous space \mathcal{R}^1 to obtain a collection of balls $\{Q_i := B(x_i, r_i) : x_i \in \Omega, r_i = d(x_i, \Omega^c)/2, i = 1, \dots\}$ so that

- (i) $\Omega = \bigcup_i Q_i$;
- (ii) $\{B(x_i, r_i/5)\}_{i=1}^\infty$ are disjoint;
- (iii) there exists a universal constant C so that $\sum_k \chi_{Q_k}(x) \leq C$ for all $x \in \Omega$.

Hence, we can further decompose

$$f_2^\lambda(x) = \sum_i f_{2,i}^\lambda(x),$$

where $f_{2,i}^\lambda = (\chi_{Q_i}(x) / \sum_k \chi_{Q_k}(x)) f_2^\lambda(x)$.

Next, note that for $x \in \mathcal{R}^2 \setminus K$,

$$\begin{aligned} \left| \mathfrak{M}(\sqrt{L})f_{2,i}^\lambda(x) \right| &= \left| \int_0^\infty t\sqrt{L}e^{-t\sqrt{L}} f_{2,i}^\lambda(x) m(t) \frac{dt}{t} \right| \\ &\leq \int_0^\infty \int_{Q_i} |\mathfrak{p}_{t,1}(x, y)| |m(t)| |f_{2,i}^\lambda(y)| dy \frac{dt}{t}. \end{aligned}$$

Applying Theorem 2.3(3), we obtain that

$$\left| \mathfrak{M}(\sqrt{L})f_{2,i}^\lambda(x) \right| \leq C \int_0^\infty \int_{Q_i} \left[\frac{t}{(d(x, y) + t)^3} + \frac{1}{|x|} \frac{t}{(d(x, y) + t)^2} \right] |f_{2,i}^\lambda(y)| dy \frac{dt}{t}. \tag{3.9}$$

Note that $d(x, y) \approx |x| + |y|$ for all $x \in \mathcal{R}^2 \setminus K$ and $y \in \mathcal{R}^1 \setminus K$,

$$\int_0^\infty \left[\frac{t}{(d(x, y) + t)^3} + \frac{1}{|x|} \frac{t}{(d(x, y) + t)^2} \right] \frac{dt}{t} \leq C \int_0^\infty \left[\frac{t}{(|x| + t)^3} + \frac{1}{|x|} \frac{t}{(|x| + t)^2} \right] \frac{dt}{t} =: U.$$

Splitting U into two integrals as below, we have

$$\begin{aligned} U &\leq C \left(\int_0^{|x|} + \int_{|x|}^\infty \right) \left[\frac{t}{(|x| + t)^3} + \frac{1}{|x|} \frac{t}{(|x| + t)^2} \right] \frac{dt}{t} \\ &\leq C \int_0^{|x|} \frac{1}{|x|^3} dt + \int_{|x|}^\infty \frac{1}{|x|} \frac{1}{t^2} dt \\ &\leq \frac{C}{|x|^2}. \end{aligned}$$

Substituting the above estimate into (3.9), we have

$$\left| \mathfrak{M}(\sqrt{L})f_{2,i}^\lambda(x) \right| \leq \frac{C}{|x|^2} \int_{Q_i} |f_{2,i}^\lambda(y)| dy,$$

which implies that

$$\left| \mathfrak{M}(\sqrt{L})f_2^\lambda(x) \right| \leq \sum_i \left| \mathfrak{M}(\sqrt{L})f_{2,i}^\lambda(x) \right| \leq \frac{C}{|x|^2} \sum_i \int_{Q_i} |f_{2,i}^\lambda(y)| dy \leq \frac{C}{|x|^2} \|f_2\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}.$$

Thus,

$$\begin{aligned} I_{22} &= \left| \left\{ x \in \mathcal{R}^2 \setminus K : \left| \mathfrak{M}(\sqrt{L})f_2^\lambda(x) \right| > \frac{\lambda}{6} \right\} \right| \\ &\leq \left| \left\{ x \in \mathcal{R}^2 \setminus K : \frac{C}{|x|^2} \|f_2\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)} > \frac{\lambda}{6} \right\} \right| \\ &\leq \left| \left\{ x \in \mathcal{R}^2 \setminus K : |x|^2 < \frac{C}{\lambda} \|f_2\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)} \right\} \right| \\ &\leq \frac{C}{\lambda} \|f_2\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)} \\ &\leq \frac{C}{\lambda} \|f\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}. \end{aligned}$$

3.3. Estimate of II_1 .

We point out that this case can be handled by using the same way as those in the estimate of I_2 with some adjustments since x and y are at different ends of the manifold. We sketch the proof as follows.

Define

$$\mathcal{F} := \{x \in \mathcal{R}^2 \setminus K : \mathcal{M}_2 f_1(x) \leq \lambda\} \quad \text{and} \quad \Omega := \{x \in \mathcal{R}^2 \setminus K : \mathcal{M}_2 f_1(x) > \lambda\},$$

where \mathcal{M}_2 is the Hardy–Littlewood maximal function defined on $\mathcal{R}^2 \setminus K$. Then let

$$f_{1,\lambda}(x) := f_1(x)\chi_{\mathcal{F}}(x) \quad \text{and} \quad f_1^\lambda(x) := f_1(x)\chi_{\Omega}(x).$$

Then we have

$$\begin{aligned} II_1 &\leq \left| \left\{ x \in \mathcal{R}^1 \setminus K : \left| \mathfrak{M}(\sqrt{L})f_{1,\lambda}(x) \right| > \frac{\lambda}{6} \right\} \right| \\ &\quad + \left| \left\{ x \in \mathcal{R}^1 \setminus K : \left| \mathfrak{M}(\sqrt{L})f_1^\lambda(x) \right| > \frac{\lambda}{6} \right\} \right| \\ &=: II_{11} + II_{12}. \end{aligned}$$

Similarly to I_{21} , we have

$$II_{11} \leq \frac{C}{\lambda} \|f\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}.$$

As for II_{12} , by using the Whitney decomposition as those in Subsection 3.2, we obtain $\Omega = \bigcup_i Q_i$ such that $\sum_i |Q_i| = |\Omega|$, which gives

$$f_1^\lambda(x) = \sum_i f_{1,i}^\lambda(x),$$

where $f_{1,i}^\lambda = (\chi_{Q_i}(x) / \sum_k \chi_{Q_k}(x)) f_1^\lambda(x)$.

Next, for $x \in \mathcal{R}^1 \setminus K$,

$$\left| \mathfrak{M}(\sqrt{L})f_{1,i}^\lambda(x) \right| \leq \int_0^\infty \int_{Q_i} |\mathfrak{p}_{t,1}(x, y)| |m(t)| |f_{1,i}^\lambda(y)| dy \frac{dt}{t}.$$

Applying Theorem 2.3(3), we have

$$\left| \mathfrak{M}(\sqrt{L})f_{1,i}^\lambda(x) \right| \leq C \int_0^\infty \int_{Q_i} \left[\frac{t}{(d(x, y) + t)^3} + \frac{1}{|y|} \frac{t}{(d(x, y) + t)^2} \right] |f_{1,i}^\lambda(y)| dy \frac{dt}{t}. \tag{3.10}$$

Similarly to the estimate of U in Subsection 3.2, we get

$$\left| \mathfrak{M}(\sqrt{L})f_{1,i}^\lambda(x) \right| \leq \frac{C}{|x|} \int_{Q_i} |f_{1,i}^\lambda(y)| dy.$$

This implies

$$\left| \mathfrak{M}(\sqrt{L})f_1^\lambda(x) \right| \leq \sum_i \left| \mathfrak{M}(\sqrt{L})f_{1,i}^\lambda(x) \right| \leq \frac{C}{|x|} \sum_i \int_{Q_i} |f_{1,i}^\lambda(y)| dy \leq \frac{C}{|x|} \|f_1\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}.$$

Thus

$$\begin{aligned} II_{12} &\leq \left| \left\{ x \in \mathcal{R}^1 \setminus K : \left| \mathfrak{M}(\sqrt{L})f_1^\lambda(x) \right| > \frac{\lambda}{6} \right\} \right| \\ &\leq \left| \left\{ x \in \mathcal{R}^1 \setminus K : \frac{C}{|x|} \|f_1\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)} > \frac{\lambda}{6} \right\} \right| \\ &\leq \left| \left\{ x \in \mathcal{R}^1 \setminus K : |x| < \frac{C}{\lambda} \|f_1\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)} \right\} \right| \\ &\leq \frac{C}{\lambda} \|f\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}. \end{aligned}$$

3.4. Estimate of II_2 .

Since f_2 is supported on $\mathcal{R}^1 \setminus K$ which is a homogeneous space, we again use the Calderón–Zygmund decomposition as in the estimate of I_1 (see [2] and [11] for references) to get the weak type $(1, 1)$ estimate of II_2 .

We now have a sequence of disjoint dyadic cubes $\{Q_{2,i}\}$ on \mathbb{R}^1 and a decomposition

$$f_2(x) = g_2(x) + \sum_i b_{2,i}(x)$$

such that

- (a') $|g_2(x)| \leq C\lambda$ for almost all $x \in \mathcal{R}^1 \setminus K$;
- (b') the support of each function $b_{2,i}$ is contained in $Q_{2,i}$ and

$$\int_{Q_{2,i}} |b_{2,i}(x)| dx \leq C\lambda |Q_{2,i}|;$$

(c') $\sum_i |Q_{2,i}| \leq \frac{C}{\lambda} \int_{\mathcal{R}^2 \setminus K} |f_2(x)| dx;$

(d') $\sum_i \chi_{Q_{2,i}} \leq C,$

where $\chi_{Q_{2,i}}$ is the characteristic function of $Q_{2,i}$.

We then get

$$\begin{aligned} II_2 &\leq \left| \left\{ x \in \mathcal{R}^1 \setminus K : \left| \mathfrak{M}(\sqrt{L})g_2(x) \right| > \frac{\lambda}{6} \right\} \right| \\ &\quad + \left| \left\{ x \in (\mathcal{R}^1 \setminus K) \setminus \bigcup_i 8Q_{2,i} : \left| \mathfrak{M}(\sqrt{L}) \sum_i b_{2,i}(x) \right| > \frac{\lambda}{6} \right\} \right| + \left| \bigcup_i 8Q_{2,i} \right| \\ &=: II_{21} + II_{22} + II_{23}. \end{aligned}$$

Similarly to the estimates of I_{11} and I_{13} , we get that $II_{21} \leq (C/\lambda)\|f\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}$ and $II_{23} \leq (C/\lambda)\|f\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}$.

For the term II_{22} , we have

$$\begin{aligned} II_{22} &\leq \left| \left\{ x \in (\mathcal{R}^1 \setminus K) \setminus \bigcup_i 8Q_{2,i} : \left| \mathfrak{M}(\sqrt{L}) \sum_i e^{-t_i \sqrt{L}} b_{2,i}(x) \right| > \frac{\lambda}{12} \right\} \right| \\ &\quad + \left| \left\{ x \in (\mathcal{R}^1 \setminus K) \setminus \bigcup_i 8Q_{2,i} : \left| \mathfrak{M}(\sqrt{L}) \sum_i (I - e^{-t_i \sqrt{L}}) b_{2,i}(x) \right| > \frac{\lambda}{12} \right\} \right| \\ &=: II_{221} + II_{222} \end{aligned}$$

where t_i is the side length of the cube $Q_{2,i}$ for each i .

As for the term II_{222} we have

$$\begin{aligned}
 II_{222} &\leq \left| \left\{ x \in (\mathcal{R}^1 \setminus K) \setminus \bigcup_i 8Q_{2,i} : \left| \mathfrak{M}(\sqrt{L}) \sum_i (I - e^{-t_i \sqrt{L}}) b_{2,i}(x) \right| > \frac{\lambda}{12} \right\} \right| \\
 &\leq \frac{C}{\lambda} \sum_i \int_{(8Q_{2,i})^c} \left| \mathfrak{M}(\sqrt{L}) (I - e^{-t_i \sqrt{L}}) b_{2,i}(x) \right| dx.
 \end{aligned}$$

For each i , let

$$H := \int_{(8Q_{2,i})^c} \left| \mathfrak{M}(\sqrt{L}) (I - e^{-t_i \sqrt{L}}) b_{2,i}(x) \right| dx.$$

We then get

$$H \leq \int_{Q_{2,i}} \int_{(8Q_{2,i})^c} |k_{t_i}(x, y)| dx |b_{2,i}(y)| dy,$$

where $k_{t_i}(x, y)$ is the kernel of the operator $\mathfrak{M}(\sqrt{L})(I - e^{-t_i \sqrt{L}})$.

Using the definition of $\mathfrak{M}(\sqrt{L})$, we can get the expression of the kernel $k_{t_i}(x, y)$ as follows

$$k_{t_i}(x, y) = \int_0^{t_i} \int_0^\infty \mathfrak{p}_{s+t,2}(x, y) \frac{m(s)}{(s+t)^2} ds dt.$$

Applying Theorem 2.3(2), we have

$$\begin{aligned}
 \int_{(8Q_i)^c} |k_{t_i}(x, y)| dx &\leq \int_{(8Q_i)^c} \int_0^{t_i} \int_0^\infty |\mathfrak{p}_{s+t,2}(x, y)| \frac{|m(s)|}{(s+t)^2} ds dt dx \\
 &\leq C \int_{(8Q_i)^c} \int_0^{t_i} \int_0^\infty \frac{1}{(s+t)^2} \left[\frac{s+t}{d(x, y) + s+t} \right]^4 \frac{1}{(s+t)^2} ds dt dx \\
 &\quad + C \int_{(8Q_i)^c} \int_0^{t_i} \int_0^\infty \frac{1}{s+t} \left[\frac{s+t}{d(x, y) + s+t} \right]^3 \frac{1}{(s+t)^2} ds dt dx \\
 &=: E_1 + E_2.
 \end{aligned}$$

It should be noted that the technique used in the estimate of I_{122} can be applied for E_1 and E_2 ; therefore, we omit details.

As for II_{221} , we now split all the $Q_{2,i}$'s into two groups:

$$\mathcal{J}_1 := \{i : \text{none of the corners of } Q_{2,i} \text{ is the origin}\}$$

and

$$\mathcal{J}_2 := \{i : \text{one of the corners of } Q_{2,i} \text{ is the origin}\}.$$

Similarly to I_{12} , we need to verify the estimate

$$\left\| \sum_{i \in \mathcal{J}_1} e^{-t_i \sqrt{L}} b_{2,i} \right\|_{L^2(M)} \leq C \lambda^{1/2} \|f_2\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)}^{1/2}. \tag{3.11}$$

To see this claim, it suffices to show the following cases:

$$\left\| \sum_{i \in \mathcal{J}_1} e^{-t_i \sqrt{L}} b_{2,i} \right\|_{L^2(\mathcal{R}^2 \setminus K)} \leq C \lambda^{1/2} \|f_2\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}^{1/2}, \tag{3.12}$$

$$\left\| \sum_{i \in \mathcal{J}_1} e^{-t_i \sqrt{L}} b_{2,i} \right\|_{L^2(\mathcal{R}^1 \setminus K)} \leq C \lambda^{1/2} \|f_2\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}^{1/2}, \tag{3.13}$$

and

$$\left\| \sum_{i \in \mathcal{J}_1} e^{-t_i \sqrt{L}} b_{2,i} \right\|_{L^2(K)} \leq C \lambda^{1/2} \|f_2\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}^{1/2}. \tag{3.14}$$

We now point out that (3.12) and (3.14) can be obtained by using a similar approach to those for (3.3) and (3.5), respectively.

As for (3.13), applying Theorem 2.3(2) for $\mathfrak{p}_{t_i}(x, y)$, we obtain that

$$\begin{aligned} |e^{-t_i \sqrt{L}} b_{2,i}(x)| &\leq \int_{\mathcal{R}^1 \setminus K} |\mathfrak{p}_{t_i}(x, y)| |b_{2,i}(y)| dy \\ &\leq C \int_{\mathcal{R}^1 \setminus K} \left[\frac{t_i}{(d(x, y) + t_i)^3} + \frac{t_i}{(d(x, y) + t_i)^2} \right] |b_{2,i}(y)| dy \\ &=: \mathcal{F}_{1,i} + \mathcal{F}_{2,i}. \end{aligned}$$

For the term $\mathcal{F}_{2,i}$, by using the sup-inf technique similarly to those for F_i in Subsection 3.1,

$$\left\langle \sum_i \mathcal{F}_{2,i}, h \right\rangle \leq C \lambda \left\langle \mathcal{M}_1(h), \sum_i \chi_{Q_{2,i}} \right\rangle$$

for any function h with $\|h\|_{L^2(\mathcal{R}^1 \setminus K)} = 1$, which yields that

$$\left\| \sum_{i \in \mathcal{J}_1} \mathcal{F}_{2,i} \right\|_{L^2(\mathcal{R}^2 \setminus K)} \leq C \lambda^{1/2} \|f_2\|_{L^1(\mathcal{R}^1 \# \mathcal{R}^2)}^{1/2}.$$

For the term $\mathcal{F}_{1,i}$, we consider the position of $Q_{2,i}$, the support of $b_{2,i}$, as follows: if one of the corners of $Q_{2,i}$ is origin, then $t_i \geq 1/2$, since otherwise the function f_2 on $Q_{2,i}$ is zero, which yields that this $Q_{2,i}$ cannot be chosen from the Calderón–Zygmund decomposition; if none of the corners of $Q_{2,i}$ is origin, then if $t_i < 1, d(0, Q_{2,i}) \geq 1/2$. Combining all these cases, we get

$$\frac{t_i}{(d(x, y) + t_i)^3} \leq C \frac{t_i}{(d(x, y) + t_i)^2},$$

which shows that

$$\left\| \sum_{i \in \mathcal{J}_1} \mathcal{F}_{1,i} \right\|_{L^2(\mathcal{R}^2 \setminus K)} \leq C \left\| \sum_{i \in \mathcal{J}_1} \mathcal{F}_{2,i} \right\|_{L^2(\mathcal{R}^2 \setminus K)} \leq C \lambda^{1/2} \|f_2\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)}^{1/2}. \quad \square$$

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