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Estimates of the renewal measure

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Abstract. We prove sharp estimates for the renewal measure of a strongly nonlattice probability measure on the real line. In particular we consider the case where the measure has finite moments between 1 and 2. The proof uses Fourier analysis of tempered distributions.

1. Introduction.

Let μ be a nonlattice probability measure on $\mathbb{R}$. The renewal measure ν is defined by

$$\nu = \sum_{n=0}^{\infty} \mu^{n*},$$

where μ^{n*} is n-fold convolution of μ with itself and $\mu^{0*} = \delta$ is the Dirac measure at 0.

This paper considers the asymptotic behavior of ν at infinity. This study has a long history. Our starting point is Blackwell's renewal theorem from the middle of the twentieth century, see [**1**], [**2**].

Theorem B. *If μ is a nonlattice probability measure on the real line with a positive first moment $\mu_1 > 0$, then*

$$\nu(x + I) - \frac{|I|}{\mu_1} \to 0, \ x \to \infty. \tag{1.1}$$

Here I is a fixed interval and $|I|$ its length.

Since then many authors have studied the rate of this convergence using different techniques, see for instance [**10**], [**5**], [**9**] and [**8**]. The inspiration for this paper comes mostly from [**10**] and [**8**] that uses Fourier methods. [**5**] uses Banach algebra methods and [**9**] coupling methods.

We will use Fourier transforms of tempered distributions to prove the following theorem.

Theorem 1.1. *Assume that μ is a strongly nonlattice probability measure on the real line with finite moments of order $\alpha > 1$ and positive first moment μ_1. Then, its renewal measure ν satisfies*

(a) *if $\alpha \geq 2$,*

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$$\nu(x+I) = |I|\left(\frac{H(x)}{\mu_1} + \frac{1}{\mu_1^2}R_1(x)\right) + o\left(\frac{\log|x|}{|x|^\alpha}\right),\ |x| \to +\infty, \tag{1.2}$$

and

(b) *if* $1 < \alpha < 2$,

$$\nu(x+I) = |I|\left(\frac{H(x)}{\mu_1} + \frac{1}{\mu_1^2}R_1(x) + \frac{1}{\mu_1^3}R_2(x) + \cdots + \frac{1}{\mu_1^{m+1}}R_m(x)\right) + o\left(\frac{\log|x|}{|x|^\alpha}\right),\ |x| \to +\infty, \tag{1.3}$$

where $m = [1/(\alpha-1)] + 1$.

As in Theorem B I is a fixed interval and $|I|$ its length. H is the Heaviside function. That μ has finite moments of order α means that $\int_{\mathbb{R}} |x|^\alpha d\mu(x) < \infty$ and $\mu_1 = \int_{\mathbb{R}} x d\mu(x)$ is its first moment. μ is a strongly nonlattice measure if $\liminf_{|\xi|\to\infty} |1-f(\xi)| > 0$, where (as throughout this paper) f denotes the Fourier transform of μ, i.e. $f(\xi) = \int_{\mathbb{R}} e^{-ix\xi} d\mu(x)$.

The functions R_n (or rather their Fourier transforms) appear naturally in our proof. For their definition see (3.2) and (3.3) in Section 3. For now we only state the following lemma that is proved in Section 4.

LEMMA 1.2. *Assume that* μ *has finite moments of order* $\alpha > 1$. *Then the functions* R_n *are bounded.*

Also,

$$R_1(x) = o\left(\frac{1}{|x|^{\alpha-1}}\right),\ |x| \to \infty, \tag{1.4}$$

and if $n \geq 2$,

$$R_n(x) = \begin{cases} o\left(\dfrac{1}{|x|^\alpha}\right), & |x| \to \infty,\ \text{if}\ \ \alpha \geq 2, \\ o\left(\dfrac{1}{|x|^{2(\alpha-1)}}\right), & |x| \to \infty,\ \text{if}\ \ 1 < \alpha < 2. \end{cases} \tag{1.5}$$

Note that if $\alpha \geq 2$ and $n \geq 2$, then $R_n(x)$ decays more rapidly than the remainder in Theorem 1.1, and hence, although they simplify the proof, they do not appear in (1.2). Also note that when $1 < \alpha < 2$, $2(\alpha-1) < \alpha$ and the estimate of $R_n(x)$ is weaker than the remainder and thus R_n are needed in (1.3).

We are mostly interested in the case $1 < \alpha < 2$ in Theorem 1.1. The case when $\alpha \geq 2$ is an integer was proved already by Stone, see [**10**, Theorem 3]. We include a proof of (1.2), since the techniques developed to prove (1.3) give a simple proof also of this case.

The function R_2 was introduced by Isozaki [**8**] and he proved a version of (1.3) when $3/2 < \alpha < 2$ (and also suggested generalizations to all $1 < \alpha < 2$). However he was interested in the density of the renewal measure and assumed a strong regularity assumption of the measure μ, namely that its Fourier transform is in some $L^p(\mathbb{R})$, $p \in [1, \infty)$.

We are interested in less regular measures and only need the weak regularity condition "strongly non-lattice". This condition guarantees that $1/(1-f(\xi))$ is bounded at infinity. To see that some regularity condition of μ is needed, we refer to [**4**].

1.1. Notation.

Our proof uses Fourier transforms of tempered distributions. For the necessary background and the standard notation of distributions we refer to Hörmander's book [**7**].

In order not to have to write too many absolute value signs, we let $A(\xi) \lesssim B(\xi)$ mean that $|A(\xi)| \le C|B(\xi)|$, also when $A(\xi)$ and $B(\xi)$ are complex valued.

$A(\xi) \sim B(\xi)$ means that $A(\xi) \lesssim B(\xi)$ and $B(\xi) \lesssim A(\xi)$.

$f(\xi)$ is always the Fourier transform of the measure μ.

2. Sketch of proof.

Assume that m is a positive integer. The general idea of our proof is that in order to prove that

$$x^m r(x) \to 0, \ |x| \to \infty, \tag{2.1}$$

it is enough to show that

$$\widehat{r}^{(m)}(\xi) \in L^1(\mathbb{R}). \tag{2.2}$$

This follows from the Riemann–Lebesgue lemma, and that multiplying a function by x corresponds to differentiation of its Fourier transform.

We also have to generalize this to fractional powers α, see Section 5.

We start by discussing the Fourier transform of ν. It is not à priori clear that ν exists but it is of course well-known. ν is also a tempered distribution and hence has a well-defined Fourier transform. Formally,

$$\widehat{\nu}(\xi) = \sum_{n=0}^{\infty} f^n(\xi) = \frac{1}{1-f(\xi)}.$$

This formula is not correct due to the singularity at the origin. But we have

Lemma 2.1. *Assume that μ has finite moments of order $\alpha > 1$. Then*

$$\left(\nu - \frac{1}{\mu_1} H\right)^{\wedge}(\xi) = \frac{1}{1-f(\xi)} - \frac{1}{i\mu_1\xi} \quad \textit{in} \quad \mathscr{S}'(\mathbb{R}).$$

For a discussion of this see Section 3.

If $\xi \neq 0$, we have

$$\frac{1}{1-f(\xi)} - \frac{1}{i\mu_1\xi} = \frac{f(\xi)-1+i\mu_1\xi}{i\mu_1\xi}\frac{1}{1-f(\xi)},$$

or

$$\frac{1}{1-f(\xi)} = \frac{1}{i\mu_1\xi} + \frac{f(\xi)-1+i\mu_1\xi}{i\mu_1\xi}\frac{1}{1-f(\xi)}. \tag{2.3}$$

By the moment assumption, we have

$$f(\xi) - 1 + i\mu_1\xi = \int_{\mathbb{R}} \left(e^{-ix\xi} - 1 + ix\xi\right) d\mu(x) \lesssim |\xi|^{\min(\alpha,2)}, \; \xi \to 0.$$

Hence, for some $\gamma > 0$,

$$\frac{f(\xi) - 1 + i\mu_1\xi}{i\mu_1\xi} \lesssim |\xi|^{\min(\alpha,2)-1} = |\xi|^{\gamma}, \; \xi \to 0. \tag{2.4}$$

and

$$\frac{f(\xi) - 1 + i\mu_1\xi}{i\mu_1\xi}\frac{1}{1 - f(\xi)} \lesssim |\xi|^{\gamma-1}, \; \xi \to 0. \tag{2.5}$$

From this we make two observations.

One is that $1/(1 - f(\xi)) - 1/i\mu_1\xi$ is locally integrable and hence a well-defined distribution.

The other is that $(f(\xi) - 1 + i\mu_1\xi)/i\mu_1\xi$ is an improving factor; $((f(\xi) - 1 + i\mu_1\xi)/i\mu_1\xi)(1/(1 - f(\xi)))$ is less singular than $1/(1 - f(\xi))$ near the origin.

To be able to differentiate, we want to improve the estimate (2.5). This can be obtained by the following bootstrapping argument. By replacing the last occurence of $1/(1 - f(\xi))$ in (2.3) with the whole formula we get

$$\begin{aligned}
\frac{1}{1 - f(\xi)} &= \frac{1}{i\mu_1\xi} + \frac{f(\xi) - 1 + i\mu_1\xi}{i\mu_1\xi}\frac{1}{1 - f(\xi)} \\
&= \frac{1}{i\mu_1\xi} + \frac{f(\xi) - 1 + i\mu_1\xi}{i\mu_1\xi} \times \left(\frac{1}{i\mu_1\xi} + \frac{f(\xi) - 1 + i\mu_1\xi}{i\mu_1\xi}\frac{1}{1 - f(\xi)}\right) \\
&= \frac{1}{i\mu_1\xi} + \frac{f(\xi) - 1 + i\mu_1\xi}{(i\mu_1\xi)^2} + \left(\frac{f(\xi) - 1 + i\mu_1\xi}{i\mu_1\xi}\right)^2 \frac{1}{1 - f(\xi)}.
\end{aligned}$$

Thus we gain one more factor $(f(\xi) - 1 + i\mu_1\xi)/i\mu_1\xi$.

This can be repeated any number of times and we get

$$\begin{aligned}
&\frac{1}{1 - f(\xi)} - \frac{1}{i\mu_1\xi} \\
&\quad = \frac{f(\xi) - 1 + i\mu_1\xi}{(i\mu_1\xi)^2} + \frac{\left(f(\xi) - 1 + i\mu_1\xi\right)^2}{(i\mu_1\xi)^3} + \cdots + \frac{\left(f(\xi) - 1 + i\mu_1\xi\right)^m}{(i\mu_1\xi)^{m+1}} \\
&\qquad + \left(\frac{f(\xi) - 1 + i\mu_1\xi}{i\mu_1\xi}\right)^{m+1} \frac{1}{1 - f(\xi)}.
\end{aligned} \tag{2.6}$$

This is a pointwise identity for $\xi \neq 0$. But both sides of the equality are locally integrable functions and (2.6) is also an identity between tempered distributions.

By Fourier inversion we obtain

$$\nu = \omega_m + r_m(x),$$

where the main term ω_m is

$$\omega_m(x) = \frac{H(x)}{\mu_1} + \frac{1}{\mu_1^2} R_1(x) + \frac{1}{\mu_1^3} R_2(x) + \cdots + \frac{1}{\mu_1^{m+1}} R_m(x), \tag{2.7}$$

with

$$\widehat{R_n}(\xi) = \frac{\big(f(\xi) - 1 + i\mu_1\xi\big)^n}{(i\xi)^{n+1}}, \ n = 1, 2, 3, \ldots, \tag{2.8}$$

and the remainder r_m satisfies

$$\widehat{r_m}(\xi) = \left(\frac{f(\xi) - 1 + i\mu_1\xi}{i\mu_1\xi}\right)^{m+1} \frac{1}{1 - f(\xi)}. \tag{2.9}$$

Let us prove that $\widehat{r_m}^{(m)}(\xi) \in L^1_{\text{loc}}(\mathbb{R})$ if μ has finite moment of integer order $m \geq 2$. (This is the main difficulty in proving (1.2) for integer α.)

By the moment condition $f^{(m)}$ exists and is continuous. Since $\gamma = 1$, we have

$$\widehat{r_m}(\xi) = \left(\frac{f(\xi) - 1 + i\mu_1\xi}{i\mu_1\xi}\right)^{m+1} \frac{1}{1 - f(\xi)} \lesssim |\xi|^m, \ \xi \to 0.$$

Also, as

$$\big(f(\xi) - 1 + i\mu_1\xi\big)' = \int_{\mathbb{R}} -ix\big(e^{-ix\xi} - 1\big) d\mu(x) \lesssim |\xi|, \ \xi \to 0,$$

successive derivatives of $\widehat{r_m}$ introduces at worst a multiplicative factor of size $1/\xi$. Thus

$$\widehat{r_m}^{(m)}(\xi) \lesssim |\xi|^{m-m} = 1, \ \xi \to 0,$$

so $\widehat{r_m}^{(m)}$ is even bounded.

This simple argument can be compared with the delicate estimates needed to prove that $\widehat{r_1}^{(m)} \in L^1_{\text{loc}}(\mathbb{R})$ in Stone [**10**].

2.1. A truncation argument.

To prove Theorem 1.1 we want $\widehat{r_m}^{(m)}$ (and a generalization of this to non-integer α) to be integrable. As $\widehat{r_m}^{(m)}(\xi)$ do not decay at infinity, this is clearly not true. However there is a standard truncation argument to remedy this problem.

Let ϕ_ϵ be a smooth approximative identity, i.e. $\phi_\epsilon(x) = (1/\epsilon)\phi(x/\epsilon)$, where $0 \leq \phi \in C_0^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \phi(x)dx = 1$ and (say) $\operatorname{supp} \phi \subset [-1/2, 1/2]$ and $0 < \epsilon \leq 1/2$. We also let χ_s denote the characteristic function of $I_s = [-s, s]$. Then $\phi_\epsilon * \chi_{s-\epsilon} \leq \chi_s \leq \phi_\epsilon * \chi_{s+\epsilon}$.

Since ν is a positive measure we get

$$\phi_\epsilon * \chi_{s-\epsilon} * \nu(x) \leq \chi_s * \nu(x) = \nu(I_s + x) \leq \phi_\epsilon * \chi_{s+\epsilon} * \nu(x).$$

By Lemma 1.2, ω_m is bounded. Hence

$$\big|\omega_m(I_s + x) - \phi_\epsilon * \chi_{s\pm\epsilon} * \omega_m(x)\big| \lesssim \epsilon.$$

Then we show that

$$\mathcal{F}(\phi_\epsilon * \chi_s * (\nu - \omega_m))(\xi) = \widehat{\phi}(\epsilon\xi)\frac{\sin s\xi}{\xi}\widehat{r_m}(\xi)$$

has m derivatives in $L^1(\mathbb{R})$ with norm $\lesssim \log 1/\epsilon$. Using a uniform version of the Riemann–Lebesgue lemma we obtain

$$\left(\phi_\epsilon * \chi_s * (\nu - \omega_m)\right)(x) = o\left(\frac{1}{|x|^m}\right)\log\frac{1}{\epsilon},\ \ |x| \to +\infty,$$

and

$$\begin{aligned}(\nu - \omega_m)(I_s + x) &\lesssim \left(\phi_\epsilon * \chi_{s\pm\epsilon} * (\nu - \omega_m)\right)(x) + \epsilon \\ &\lesssim o\left(\frac{1}{|x|^m}\right)\log\frac{1}{\epsilon} + \epsilon,\ \ |x| \to +\infty,\end{aligned}$$

where $o(1/|x|^m)$ is independent of ϵ. Letting $\epsilon = 1/|x|^m$, we obtain Theorem 1.1 for integer m.

The details of this argument for general α are presented in Section 6.

3. Calculation of Fourier and inverse Fourier transforms.

Lemma 2.1 was proved in [**3**] under the assumption of a finite second moment. However it holds for measures with finite moments of order $\alpha > 1$. The moment condition was used to prove that

$$\lim_{N\to\infty} f^N(\xi)\left(\frac{1}{1-f(\xi)} - \frac{1}{i\mu_1\xi}\right) = 0 \ \text{ in } \ \mathscr{D}'(\mathbb{R}),$$

but that argument works provided $1/(1-f(\xi)) - 1/i\mu_1\xi \in L^1_{\text{loc}}(\mathbb{R})$. By (2.5) this holds also under our weaker moment condition.

Remark 3.1. As is well-known $\widehat{H}(\xi) = \text{pv}(1/i\xi) + \pi\delta$, and by Lemma 2.1 we get

$$\widehat{\nu} = \text{pv}\frac{1}{1-f(\xi)} + \frac{\pi}{\mu_1}\delta.$$

Next we compute R_n, the inverse Fourier transform of $\widehat{R_n}$. We write (2.8) as

$$\widehat{R_n}(\xi) = \frac{f(\xi) - 1 + i\mu_1\xi}{(i\xi)^2}\left(\frac{f(\xi) - 1 + i\mu_1\xi}{i\xi}\right)^{n-1} = \widehat{R_1}(\xi)\widehat{R_0}^{\,n-1}(\xi),$$

where

$$\widehat{R_0}(\xi) = \frac{f(\xi) - 1 + i\mu_1\xi}{i\xi} = \frac{f(\xi) - 1}{i\xi} + \mu_1.$$

Thus we have $R_n = R_1 * R_0^{(n-1)*}$.

If $F(x) = \int_{-\infty}^x d\mu(y)$ is the distribution function of μ, then $(F - H)' = \mu - \delta$ in $\mathscr{D}'(\mathbb{R})$. Thus

$$i\xi(F-H)^\wedge(\xi) = f(\xi) - 1 \;\text{ and }\; (F-H)^\wedge(\xi) = \frac{f(\xi)-1}{i\xi} + C\delta$$

for some constant C. But

$$\frac{f(\xi)-1}{i\xi} \in L^1(\mathbb{R}) + L^2(\mathbb{R}) \;\text{ and }\; \lim_{x\to-\infty}(F-H)(x) = 0.$$

Thus $C = 0$.

This implies that $(F-H)^\wedge(\xi) = (f(\xi)-1)/i\xi$ and

$$R_0 = F - H + \mu_1\delta.$$

For later use we note that

$$(F-H)(x) = \begin{cases} -\displaystyle\int_x^{+\infty} d\mu(y), \; x > 0, \\ \displaystyle\int_{-\infty}^x d\mu(y), \; x < 0. \end{cases}$$

Remark 3.2. When we consider integrals, such as $F(x) = \int_{-\infty}^x d\mu(y)$ with respect to measures that may have pointmasses, there is a choice to include the point x or not. The standard choice is to include x, i.e. $F(x) = \int_{(-\infty,x]} d\mu(y)$ (and thus $1 - F(x) = \int_{(x,\infty)} d\mu(y)$). However the choice does not matter as $\int_{(-\infty,x]} d\mu(y)$ and $\int_{(-\infty,x]} d\mu(y)$ are equal almost everywhere and thus equal as distributions.

Remark 3.3. The Fourier transform of $F - H$ can be computed with classical means (the Fubini theorem) as $F - H \in L^1(\mathbb{R})$ when μ has a finite first moment. Our proof is valid if some positive moment of μ is finite since this makes $(f(\xi)-1)/i\xi$ locally integrable.

Next we consider R_1. Let $\mathcal{R}_1$ be that primitive of $R_0 = F - H + \mu_1\delta$ that is given by

$$\mathcal{R}_1(x) = \int_{-\infty}^x R_0(y)dy. \tag{3.1}$$

Remark 3.4. Here and in the sequel we use the convention $\int_{-\infty}^x \delta\, dy = H(x)$ so that

$$\mathcal{R}_1(x) = \int_{-\infty}^x (F-H)(y)dy + \mu_1 H(x).$$

Note also that by Fubini's theorem, $\int_{\mathbb{R}}(F-H)(x)dx = -\mu_1$, and hence

$$\mathcal{R}_1(x) = \int_x^{+\infty} 1 - F(y)dy$$

if $x > 0$.

We have $i\xi\widehat{\mathcal{R}_1}(\xi) = \widehat{R_0}(\xi)$. By divison, this implies

$$\widehat{\mathcal{R}_1}(\xi) = \frac{1}{i\xi}\widehat{R_0}(\xi) + C\delta = \frac{f(\xi) - 1 + i\mu_1\xi}{(i\xi)^2} + C\delta.$$

Note that since μ has a finite moment greater than 1, $(f(\xi) - 1 + i\mu_1\xi)/(i\xi)^2 \in L^1_{\mathrm{loc}}(\mathbb{R})$, and is a well-defined distribution. Arguing as above we get that $C = 0$ and $R_1 = \mathcal{R}_1$.

To sum up,

$$R_0 = F - H + \mu_1\delta = \mu_1\delta + \begin{cases} -\displaystyle\int_x^{+\infty} d\mu(y), \ x > 0, \\ \displaystyle\int_{-\infty}^x d\mu(y), \ x < 0, \end{cases}$$

$$R_1(x) = \begin{cases} \displaystyle\int_x^{+\infty} 1 - F(y)\, dy, \ x \geq 0, \\ \displaystyle\int_{-\infty}^x F(y)\, dy, \ x < 0, \end{cases} \tag{3.2}$$

and if $n \geq 2$,

$$R_n = R_1 * R_0^{(n-1)*}. \tag{3.3}$$

4. Estimates of R_n.

Proof of Lemma 1.2. We only consider the case $x \to +\infty$, the case $x \to -\infty$ is similar.

The proof of (1.4) is simple. By Fubini's theorem we have

$$\begin{aligned} x^{\alpha-1}R_1(x) = x^{\alpha-1}\int_x^\infty (1 - F(y))dy &\leq \int_x^\infty y^{\alpha-1}\left(\int_y^\infty d\mu(t)\right) dy \\ &= \int_x^\infty \left(\int_x^t y^{\alpha-1}dy\right) d\mu(t) \lesssim \int_x^\infty t^\alpha d\mu(t) \to 0, \ x \to +\infty. \end{aligned}$$

With $\alpha = 1$, the argument shows that R_1 is bounded.

We will prove (1.5) by induction using that, by (3.3), we have

$$R_{n+1} = R_1 * R_0^{n*} = R_1 * R_0^{(n-1)*} * R_0 = R_n * R_0.$$

We first observe that if $Q = F - H$, then $R_0 = Q + \mu_1\delta$ and

$$x^\alpha Q(x) = -x^\alpha \int_x^\infty d\mu(y) \lesssim \int_x^\infty y^\alpha d\mu(y) \to 0, \ x \to +\infty,$$

i.e.

$$Q(x) = o\left(\frac{1}{x^\alpha}\right), \ x \to +\infty. \tag{4.1}$$

In particular $Q \in L^1(\mathbb{R})$ since $\alpha > 1$. As $R_{n+1} = R_n * Q + \mu_1 R_n$, the boundedness of R_n follows by induction.

Also, since $\int_{\mathbb{R}} Q(x)dx = -\mu_1$, we have

$$R_{n+1}(x) = \int_{\mathbb{R}} (R_n(x-y) - R_n(x))\, Q(y)dy. \tag{4.2}$$

We first prove (1.5) when $\alpha > 2$. This case is somewhat simpler as then $R_1 \in L^1(\mathbb{R})$ (and by induction also $R_n \in L^1(\mathbb{R})$) and $xQ(x) \in L^1(\mathbb{R})$.

For R_2, we have

$$R_2(x) = \int_{\mathbb{R}} (R_1(x-y) - R_1(x))\, Q(y)dy.$$

We divide the range of integration into two parts, $y \le x/2$ and $y > x/2$.

By (3.1) (recall that $R_1 = \mathcal{R}_1$), we have $R_1(x) = \int_{-\infty}^{x} R_0(t)dt$. Thus if $y \le x/2$, and hence $x - y \ge x/2$,

$$R_1(x-y) - R_1(x) = \int_x^{x-y} Q(t)dt \lesssim \int_x^{x-y} o\left(\frac{1}{t^\alpha}\right) dt \lesssim o\left(\frac{1}{x^\alpha}\right) y. \tag{4.3}$$

Thus,

$$\begin{aligned}&\int_{y\le x/2} (R_1(x-y) - R_1(x))Q(y)dy\\ &\quad\lesssim o\left(\frac{1}{x^\alpha}\right) \int_{\mathbb{R}} |yQ(y)|dy = o\left(\frac{1}{x^\alpha}\right),\ x \to +\infty.\end{aligned}$$

For the part where $y > x/2$ we have

$$\begin{aligned}&\int_{y>x/2} (R_1(x-y) - R_1(x))Q(y)dy\\ &\quad\lesssim o\left(\frac{1}{x^\alpha}\right) \int_{y>x/2} |R_1(x-y)|dy + o\left(\frac{1}{x^{\alpha-1}}\right) \int_{y>x/2} |Q(y)|dy\\ &\quad= o\left(\frac{1}{x^\alpha}\right) + o\left(\frac{1}{x^{2(\alpha-1)}}\right) = o\left(\frac{1}{x^\alpha}\right),\ x \to +\infty,\end{aligned}$$

as $\alpha > 2$.

For $n \ge 2$, we have $R_{n+1} = R_n * Q + \mu_1 R_n$, and since by induction

$$\begin{aligned}R_n * Q(x) = \int_{\mathbb{R}} R_n(x-y)Q(y)dy &\lesssim o\left(\frac{1}{x^\alpha}\right) \int_{y\le x/2} |Q(y)|dy\\ &+ o\left(\frac{1}{x^\alpha}\right) \int_{y>x/2} |R_n(x-y)|dy = o\left(\frac{1}{x^\alpha}\right), x \to +\infty,\end{aligned}$$

(1.5) follows.

When $\alpha = 2$, the same argument basically holds. In the proof above, we used that $R_1 \in L^1(\mathbb{R})$ and $xQ(x) \in L^1(\mathbb{R})$. This do not follow from our decay estimates (4.1) and $Q(x) = o(1/x^\alpha), x \to +\infty$. However, by Fubini's theorem it is not hard to see that $R_1 \in L^1(\mathbb{R})$ (and by induction also $R_n \in L^1(\mathbb{R})$), and $xQ(x) \in L^1(\mathbb{R})$ also when $\alpha = 2$. We omit the details.

When $1 < \alpha < 2$, we let $\alpha = 1+\beta$ and first consider R_2. Now we divide the integral in (4.2) into four parts.

When $y < -x/2$, $x - y \geq x$, and hence

$$\int_{-\infty}^{-x/2} (R_1(x-y) - R_1(x))Q(y)dy$$
$$= o\left(\frac{1}{x^\beta}\right) \int_{-\infty}^{-x/2} |Q(y)|dy = o\left(\frac{1}{x^{2\beta}}\right), \ x \to +\infty.$$

If $-x/2 \leq y < x/2$, then by (4.3),

$$\int_{-x/2}^{x/2} (R_1(x-y) - R_1(x))Q(y)dy$$
$$= o\left(\frac{1}{x^\alpha}\right) \int_{-x/2}^{x/2} |yQ(y)|dy = o\left(\frac{1}{x^{2\beta}}\right), \ x \to +\infty.$$

When $x/2 \leq y < 3x/2$, we have

$$\int_{x/2}^{3x/2} (R_1(x-y) - R_1(x))Q(y)dy$$
$$= o\left(\frac{1}{x^\alpha}\right) \int_{x/2}^{3x/2} |R_1(x-y)|dy + o\left(\frac{1}{x^\beta}\right) \int_{x/2}^{3x/2} |Q(y)|dy$$
$$= o\left(\frac{1}{x^\alpha}\right) \int_{-x/2}^{x/2} |R_1(t)|dt + o\left(\frac{1}{x^\beta}\right) \frac{x}{x^\alpha} = o\left(\frac{1}{x^{2\beta}}\right), \ x \to +\infty.$$

For the last part, we have

$$\int_{3x/2}^{\infty} (R_1(x-y) - R_1(x))Q(y)dy$$
$$= o\left(\frac{1}{x^\beta}\right) \int_{3x/2}^{\infty} |Q(y)|dy = o\left(\frac{1}{x^{2\beta}}\right), \ x \to +\infty.$$

For the induction step it is enough to show that for $n \geq 2$,

$$\int_{\mathbb{R}} R_n(x-y)Q(y)dy \lesssim o\left(\frac{1}{x^{2\beta}}\right), \ x \to +\infty.$$

We have

$$\int_{-\infty}^{x/2} R_n(x-y)Q(y)dy$$
$$= o\left(\frac{1}{x^{2\beta}}\right) \int_{-\infty}^{x/2} |Q(y)|dy = o\left(\frac{1}{x^{2\beta}}\right), \ x \to +\infty.$$

Next,

$$\int_{x/2}^{3x/2} R_n(x-y)Q(y)dy = o\left(\frac{1}{x^\alpha}\right)\int_{x/2}^{3x/2} |R_n(x-y)|dy$$
$$= o\left(\frac{1}{x^\alpha}\right)\int_{-x/2}^{x/2} |R_n(y)|dy = o\left(\frac{1}{x^{2\beta}}\right),\ x \to +\infty.$$

(Note that $R_n(y) = o(1/y^\beta), y \to \infty$, is enough for the last estimate.)

Finally,

$$\int_{3x/2}^{\infty} R_n(x-y)Q(y)dy = o\left(\frac{1}{x^{2\beta}}\right)\int_{3x/2}^{\infty} |Q(y)|dy = o\left(\frac{1}{x^{2\beta}}\right),\ x \to +\infty. \qquad \square$$

We conclude this section with the following simple lemma.

Lemma 4.1.

$$\int_{x+I} R_n(y)dy = R_n(x)|I| + o\left(\frac{1}{|x|^\alpha}\right),\ |x| \to \infty.$$

Proof. When $n = 1$ this follows directly from (4.3). If $y \in x + I$, we get

$$R_1(y) - R_1(x) \lesssim |I|\, o\left(\frac{1}{x^\alpha}\right) = o\left(\frac{1}{x^\alpha}\right),\ x \to +\infty.$$

For $n > 1$, the argument is the same after we have shown that

$$R_n(x) = \int_{-\infty}^{x} R_0^{n*}(y)dy. \tag{4.4}$$

Since by (3.1), $R_1' = R_0$, we have $R_n' = \left(R_1 * R_0^{(n-1)*}\right)' = R_0^{n*}$ in $\mathscr{D}'(\mathbb{R})$. Also the distributional derivative of $\int_{-\infty}^{x} R_0^{n*}(y)dy$ is R_0^{n*} and thus the integral differ from R_n by at most a constant. But both $R_n(x)$ and $\int_{-\infty}^{x} R_0^{n*}(y)dy$ tends to zero as $x \to -\infty$ and the constant vanishes. (In fact the value of the constant is irrelevant.) $\square$

5. Fractional derivatives.

In this section we generalise (2.1) and (2.2) to non integers. We first define the fractional derivative D^β, when $0 < \beta < 1$. Then if $1 < \alpha = m + \beta$, with m an integer and $0 < \beta < 1$, we let $D^\alpha g = D^\beta(g^{(m)})$ where $g^{(m)}$ is the classical derivative of order m.

The fact that multiplying a function by x corresponds to differentiation of its Fourier transform is a consequence of that the Fourier transform of x^m is a constant times $\delta^{(m)}$. Here $\delta^{(m)}$ is the m order derivative of the Dirac measure. Also $\delta^{(m)} * \widehat{g} = \widehat{g}^{(m)}$. So multiplication with x^m corresponds to convolution with $\delta^{(m)}$.

When $0 < \beta < 1$ is natural to replace x^m with $|x|^\beta$ in this argument. It is well-known, see for instance [**6**, p.173], that if $0 < \beta < 1$, $c_\beta \widehat{|x|^\beta} = \mathrm{fp}(1/|\xi|^{1+\beta})$, for some (non-vanishing) constant c_β. Thus the Fourier transform of $c_\beta |x|^\beta g(x)$ is $\mathrm{fp}(1/|\xi|^{1+\beta}) * \widehat{g}$ and we can think of convolution with $\mathrm{fp}(1/|\xi|^{1+\beta})$ as a fractional derivative. The distribution $\mathrm{fp}(1/|\xi|^{1+\beta})$ is defined by

$$\left\langle \mathrm{fp}\frac{1}{|\xi|^{1+\beta}}, \varphi \right\rangle = \int_{\mathbb{R}} \frac{\varphi(\xi)-\varphi(0)}{|\xi|^{1+\beta}} d\xi,\ \varphi \in \mathscr{D}(\mathbb{R}).$$

For technical reasons we replace the integral over $\mathbb{R}$ with that over $[-1,1]$ and put

$$\langle u_\beta, \varphi\rangle = \int_{-1}^{1} \frac{\varphi(\xi)-\varphi(0)}{|\xi|^{1+\beta}} d\xi.$$

The difference between u_β and $\mathrm{fp}(1/|\xi|^{1+\beta})$ is a distribution defined by an integrable function and a constant times the Dirac measure. The inverse Fourier transform of this difference is bounded. Thus, if U_β is the inverse Fourier transform of u_β, $U_\beta(x)$ differ from $c_\beta|x|^\beta$ by a bounded function. So $U_\beta(x) \sim |x|^\beta$, $|x| \to +\infty$.

We now define the fractional derivative as convolution with u_β.

DEFINITION 5.1. If $0<\beta<1$, the fractional derivative $D^\beta g$ is

$$D^\beta g = u_\beta * g, g \in \mathscr{D}'(\mathbb{R}).$$

Thus the inverse Fourier transform of $D^\beta \widehat{g}$ is $U_\beta(x)g(x)$ and $U_\beta(x) \sim |x|^\beta, |x| \to \infty$.

As u_β has compact support, $D^\beta g$ is a well-defined distribution for any $g \in \mathscr{D}'(\mathbb{R})$. However, we want a more concrete representation for $D^\beta g$ for certain g.

DEFINITION 5.2. Assume that $0<\beta<1$ and that g is a measurable function. If

$$|D|^\beta g(\xi) = \int_{-1}^{1} \frac{|g(\xi-t)-g(\xi)|}{|t|^{1+\beta}} dt < \infty,$$

we say that g has a finite (fractional) derivative of order β at ξ.

LEMMA 5.3. *Assume that $0<\beta<1$ and g is a locally integrable function such that $|D|^\beta g(\xi)$ is finite almost everywhere and locally integrable. Then*

$$D^\beta g(\xi) = u_\beta * g(\xi) = \int_{-1}^{1} \frac{g(\xi-t)-g(\xi)}{|t|^{1+\beta}} dt. \tag{5.1}$$

PROOF. If $g \in \mathscr{D}(\mathbb{R})$, (5.1) follows from the definition of convolution. As u_β has compact support it holds also for $\varphi \in C^\infty(\mathbb{R})$, i.e.

$$D^\beta \varphi(\xi) = \int_{-1}^{1} \frac{\varphi(\xi-t)-\varphi(\xi)}{|t|^{1+\beta}} dt.$$

The convolution $D^\beta g = u_\beta * g$ is characterized by associativity,

$$D^\beta g * \varphi = u_\beta * (g * \varphi),\ \varphi \in \mathscr{D}(\mathbb{R}).$$

Now $g*\varphi \in C^\infty(\mathbb{R})$, and we get

$$D^\beta g * \varphi(\xi) = \int_{-1}^{1} \frac{g * \varphi(\xi - t) - g * \varphi(\xi)}{|t|^{1+\beta}} dt.$$

Using $\langle u_\beta, \varphi \rangle = u_\beta * \overset{\vee}{\varphi}(0)$, where $\overset{\vee}{\varphi}(\xi) = \varphi(-\xi)$, we obtain

$$\begin{aligned} \left\langle D^\beta g, \varphi \right\rangle &= (u_\beta * g) * \overset{\vee}{\varphi}(0) = u_\beta * \left(g * \overset{\vee}{\varphi} \right)(0) = u_\beta * \left(\overset{\vee}{g} * \varphi \right)^{\vee}(0) \\ &= \left\langle u_\beta, \overset{\vee}{g} * \varphi \right\rangle = \int_{-1}^{1} \frac{\overset{\vee}{g} * \varphi(t) - \overset{\vee}{g} * \varphi(0)}{|t|^{1+\beta}} dt. \end{aligned}$$

But $\overset{\vee}{g} * \varphi(t) = \int_{\mathbb{R}} g(\xi - t)\varphi(\xi) d\xi$, and by Fubini's theorem we obtain

$$\begin{aligned} \left\langle D^\beta g, \varphi \right\rangle &= \int_{-1}^{1} \left(\int_{\mathbb{R}} (g(\xi - t) - g(\xi))\varphi(\xi) d\xi \right) \frac{1}{|t|^{1+\beta}} dt \\ &= \int_{\mathbb{R}} \left(\int_{-1}^{1} \frac{g(\xi - t) - g(\xi)}{|t|^{1+\beta}} dt \right) \varphi(\xi) d\xi \end{aligned}$$

as desired.

We may change the order of integration since $|D|^\beta g \in L^1_{\text{loc}}(\mathbb{R})$. □

We need a couple of results about fractional derivatives. The first is

LEMMA 5.4. *If μ has finite moments of order β, $0 < \beta < 1$, then $D^\beta f$ is a bounded uniformly continuous function.*

PROOF. We first show that

$$|D|^\beta f(\xi) = \int_{-1}^{1} \frac{|f(\xi - t) - f(\xi)|}{|t|^{1+\beta}} dt \in L^\infty(\mathbb{R}).$$

Since

$$f(\xi - t) - f(\xi) = \int_{\mathbb{R}} e^{-ix\xi}(e^{ixt} - 1) d\mu(x) \lesssim \int_{\mathbb{R}} \min(1, |xt|) d\mu(x), \tag{5.2}$$

we get, changing the order of integration,

$$|D|^\beta f(\xi) \lesssim \int_{\mathbb{R}} \left(\int_{-1}^{1} \frac{\min(1, |xt|)}{|t|^{1+\beta}} dt \right) d\mu(x). \tag{5.3}$$

The inner integral satisfies

$$\int_{-1}^{1} \frac{\min(1, |xt|)}{|t|^{1+\beta}} dt \le \int_{|t| \le 1/|x|} \frac{|x|}{|t|^\beta} dt + \int_{|t| > 1/|x|} \frac{1}{|t|^{1+\beta}} dt \lesssim |x|^\beta,$$

and altogether we have

$$|D|^\beta f(\xi) \lesssim \int_{\mathbb{R}} |x|^\beta du(x) < \infty,$$

as desired.

To prove the uniform continuity, assume that $|\xi-\eta|<\delta$. Since

$$(f(\xi-t)-f(\xi))-(f(\eta-t)-f(\eta))=\int_{\mathbb{R}}\big(e^{-ix\xi}-e^{-ix\eta}\big)\big(e^{-ixt}-1\big)d\mu(x),$$

and $e^{-ix\xi}-e^{-ix\eta}\lesssim|x\delta|$, (5.2) can be sharpened to

$$(f(\xi-t)-f(\xi))-(f(\eta-t)-f(\eta))\lesssim\int_{\mathbb{R}}\min(1,|xt|,|x\delta|)d\mu(x),$$

and (5.3) to

$$D^\beta f(\xi)-D^\beta f(\eta)\lesssim\int_{-1}^{1}\left(\int_{\mathbb{R}}\frac{\min(1,|xt|,|x\delta|)}{|t|^{1+\beta}}d\mu(x)\right)dt.$$

As $\min(1,|xt|,|x\delta|)/|t|^{1+\beta}\lesssim\min(1,|xt|)/|t|^{1+\beta}\in L^1(d\mu(x)dt)$, we get by dominated convergence that

$$\lim_{\delta\to0}\big(D^\beta f(\xi)-D^\beta f(\eta)\big)=0,$$

uniformly in ξ and η. □

LEMMA 5.5. *If g is bounded and g and h have finite derivatives of order β, $0<\beta<1$, then so has gh, and*

$$|D|^\beta(gh)(\xi)\le\|g\|_{I_\xi}|D|^\beta h(\xi)+|h(\xi)|\,|D|^\beta g(\xi),$$

where $I_\xi=[\xi-1,\xi+1]$.

PROOF. Since

$$gh(\xi-t)-gh(\xi)=g(\xi-t)(h(\xi-t)-h(\xi))+h(\xi)(g(\xi-t)-g(\xi)),$$

we have

$$\begin{aligned}|D|^\beta(gh)(\xi)&=\int_{-1}^{1}\frac{\big|g(\xi-t)(h(\xi-t)-h(\xi))+h(\xi)(g(\xi-t)-g(\xi))\big|}{|t|^{1+\beta}}dt\\&\le\|g\|_{I_\xi}|D|^\beta h(\xi)+|h(\xi)|\,|D|^\beta g(\xi).\end{aligned}$$

□

LEMMA 5.6. *If g is differentiable with a bounded derivative, then g has a finite (fractional) derivative of order β, $0<\beta<1$, and*

$$|D|^\beta g(\xi)\lesssim\|g'\|_{I_\xi}.$$

Furthermore, $D^\beta g$ is Hölder continuous of degree $1-\beta$, in particular $D^\beta g$ is uniformly continuous.

PROOF. First, by the mean value theorem,

$$|D|^\beta g(\xi) = \int_{-1}^{1} \frac{|g(\xi - t) - g(\xi)|}{|t|^{1+\beta}} dt = \int_{-1}^{1} |g'(\eta_t)| \frac{dt}{|t|^\beta} \lesssim \|g'\|_{I_\xi}.$$

Secondly, if $|\xi - \eta| < 1$,

$$\begin{aligned} D^\beta g(\xi) - D^\beta g(\eta) &= \int_{|t| \le |\xi - \eta|} \frac{(g(\xi - t) - g(\xi)) - (g(\eta - t) - g(\eta))}{|t|^{1+\beta}} dt \\ &\quad + \int_{|\xi - \eta| < |t| \le 1} \frac{(g(\xi - t) - g(\eta - t)) - (g(\xi) - g(\eta))}{|t|^{1+\beta}} dt \\ &\lesssim \int_{|t| \le |\xi - \eta|} \frac{\|g'\|_\infty}{|t|^\beta} dt + \int_{|\xi - \eta| < |t| \le 1} \frac{\|g'\|_\infty |\xi - \eta|}{|t|^{1+\beta}} dt \lesssim \|g'\|_\infty |\xi - \eta|^{1-\beta}. \end{aligned}$$

□

We remark that $\|g'\|_\infty$ can be sharpened to $\|g'\|_{[\xi-2,\xi+2]}$.
We also need the following sharpening of Lemma 5.6.

Lemma 5.7. *Assume that* $-1 < a < 1$ *and that*

$$g(\xi) \lesssim \frac{1}{|\xi|^a} \quad \textit{and} \quad g'(\xi) \lesssim \frac{1}{|\xi|^{a+1}}.$$

Then, if $, 0 < \beta < 1$, *we have*

$$|D|^\beta g(\xi) \lesssim \frac{1}{|\xi|^{a+\beta}}.$$

Before the proof we note that $1/|\xi|^a \in L^1_{\text{loc}}(\mathbb{R})$, and that $1/|\xi|^{a+1} \to \infty$, $\xi \to 0$. Also, if $\beta < 1 - a$, then $|D|^\beta g$ is locally integrable and $D^\beta g$ is well-defined by (5.1).

Proof. For notational convenience we assume that $\xi > 0$. To estimate $|D|^\beta g(\xi)$, we first consider $|t| \le \xi/2$. By the mean value theorem $g(\xi - t) - g(\xi) = g'(\eta)t$. Since $\eta \sim \xi$, we get $|g(\xi - t) - g(\xi)| \lesssim t/|\xi|^{a+1}$. Thus

$$\int_{-\xi/2}^{\xi/2} \frac{|g(\xi - t) - g(\xi)|}{|t|^{1+\beta}} dt \lesssim \frac{1}{|\xi|^{a+1}} \int_{-\xi/2}^{\xi/2} \frac{dt}{|t|^\beta} \lesssim \frac{1}{|\xi|^{a+\beta}}.$$

For the remaining region of integration we estimate $g(\xi - t)$ and $g(\xi)$ separately.
If $\xi/2 \le t \le 3\xi/2$, we have

$$\begin{aligned} \int_{\xi/2}^{3\xi/2} \frac{|g(\xi - t)|}{|t|^{1+\beta}} dt &\lesssim \frac{1}{|\xi|^{1+\beta}} \int_{\xi/2}^{3\xi/2} |g(\xi - t)| dt = \frac{1}{|\xi|^{1+\beta}} \int_{-\xi/2}^{\xi/2} |g(s)| ds \\ &\lesssim \frac{1}{|\xi|^{1+\beta}} \int_{-\xi/2}^{\xi/2} \frac{1}{|s|^a} ds \lesssim \frac{1}{|\xi|^{a+\beta}}. \end{aligned}$$

If $t \notin [\xi/2, 3\xi/2]$, $\xi - t \gtrsim \xi$ and both $g(\xi - t)$ and $g(\xi)$ are bounded by $1/|\xi|^a$. Hence, their contribution to the integral can be estimated by

$$\frac{1}{|\xi|^a}\int_{|t|>\xi/2}\frac{dt}{|t|^{1+\beta}}\lesssim\frac{1}{|\xi|^{a+\beta}}.\qquad\square$$

We conclude this section with

LEMMA 5.8. *Assume that* $0<\beta<1$ *and that* g, $|D|^\beta g$, h *and* $|D|^\beta h$ *are bounded and* g, $D^\beta g$, h *and* $D^\beta h$ *are uniformly continuous. Then* $D^\beta gh$ *is uniformly continuous.*

PROOF. The proof is elementary but somewhat tedious. Writing $gh(x-t)-gh(x)=h(x-t)(g(x-t)-g(x))+g(x)(h(x-t)-h(x))$, we get

$$\begin{aligned}D^\beta gh(x)=&\int_{-1}^{1}\frac{h(x-t)(g(x-t)-g(x))}{|t|^{1+\beta}}dt\\&+g(x)\int_{-1}^{1}\frac{h(x-t)-h(x)}{|t|^{1+\beta}}dt=\mathscr{I}(x)+g(x)D^\beta h(x).\end{aligned}$$

Since g and $D^\beta h$ are bounded and uniformly continuous so is $gD^\beta h$.

It remains to consider $\mathscr{I}(x)$. Let $\epsilon>0$. By assumption we can choose δ so that if $|x-y|<\delta$, then $|g(x)-g(y)|<\epsilon$, $|D^\beta g(x)-D^\beta g(y)|<\epsilon$, $|h(x)-h(y)|<\epsilon$ and $|D^\beta h(x)-D^\beta h(y)|<\epsilon$. We have

$$\mathscr{I}(x)-\mathscr{I}(y)=\int_{-1}^{1}\frac{h(x-t)(g(x-t)-g(x))-h(y-t)(g(y-t)-g(y))}{|t|^{1+\beta}}dt.$$

Writing

$$g(y-t)-g(y)=\big(g(x-t)-g(x)\big)+\Big(\big(g(y-t)-g(x-t)\big)-\big(g(y)-g(x)\big)\Big),$$

we get

$$\begin{aligned}h(x-t)&(g(x-t)-g(x))-h(y-t)(g(y-t)-g(y))\\&=\big(h(x-t)-h(y-t)\big)\big(g(x-t)-g(x)\big)\\&\quad-h(y-t)\Big(\big(g(y-t)-g(x-t)\big)-\big(g(y)-g(x)\big)\Big)\\&=\mathscr{A}(x,y)-\mathscr{B}(x,y).\end{aligned}$$

As h is uniformly continuous, we get $|\mathscr{A}(x,y)|\le\epsilon|g(x-t)-g(x)|$ and

$$\int_{-1}^{1}\frac{|\mathscr{A}(x,y)|}{|t|^{1+\beta}}dt\lesssim\epsilon\int_{-1}^{1}\frac{|g(x-t)-g(x)|}{|t|^{1+\beta}}dt\le\epsilon\big\||D|^\beta g\big\|_\infty\lesssim\epsilon$$

if $|x-y|<\delta$.

Finally, to deal with $\mathscr{B}$, we write

$$\begin{aligned}\mathscr{B}(x,y)=h(y)&\Big(\big(g(y-t)-g(y)\big)-\big(g(x-t)-g(x)\big)\Big)\\&+\big(h(y-t)-h(y)\big)\Big(\big(g(y-t)-g(x-t)\big)-\big(g(y)-g(x)\big)\Big).\end{aligned}$$

If $|x-y|<\delta$ this implies, as g is uniformly continuous,

$$\int_{-1}^{1}\frac{\mathscr{B}(x,y)}{|t|^{1+\beta}}dt \lesssim h(y)\big(D^\beta g(y)-D^\beta g(x)\big)+\epsilon|D|^\beta h(y)\lesssim \epsilon.$$

In last estimate the boundedness of h and $|D|^\beta h$, and the uniform continuity of $D^\beta g$ is used. □

6. The proof.

To prove Theorem 1.1 it is enough to prove that

$$\int_{\mathbb{R}} e^{ix\xi}D^\alpha\left(\widehat{\phi}(\epsilon\xi)\frac{\sin s\xi}{\xi}\widehat{r_m}(\xi)\right)d\xi = O(1)+o\,(1)\log 1/\epsilon,\ x\to\infty, \tag{6.1}$$

where $O(1)$ and $o(1)$ do not depend on ϵ and s in bounded sets. Here $m=[1/(\alpha-1)]+1$ if $1<\alpha<2$ and m is sufficiently large if $\alpha\geq 2$ ($m=[\alpha]+1$ is enough).

By the truncation argument in Section 2.1, (6.1) implies

$$(\nu-\omega_m)(I_s+x)\lesssim \frac{O(1)}{|x|^\alpha}+\frac{o(1)}{|x|^\alpha}\log\frac{1}{\epsilon}+\epsilon,\ |x|\to+\infty.$$

Letting $\epsilon=1/|x|^\alpha$, we obtain

$$\nu(I_s+x)=\omega_m(I_s+x)+o\left(\frac{\log|x|}{|x|^\alpha}\right),\ |x|\to+\infty,$$

and Theorem 1.1 follows by Lemma 4.1.

To prove (6.1), we divide the integral into three parts, $|\xi|\leq 10$, $10<|\xi|\leq 1/\epsilon$ and $|\xi|>1/\epsilon$.

By the discussion in the previous section it follows that the function $D^\alpha\big(\widehat{\phi}(\epsilon\xi)(\sin s\xi/\xi)\widehat{r_m}(\xi)\big)$ is bounded and uniformly continuous when $|\xi|>10$.

Furthermore the strongly non-lattice condition and the rapid decrease of $\widehat{\phi}(\epsilon\xi)$ and its derivatives implies

$$D^\alpha\left(\widehat{\phi}(\epsilon\xi)\frac{\sin s\xi}{\xi}\widehat{r_m}(\xi)\right)\lesssim\frac{\psi(\epsilon\xi)}{|\xi|},$$

for some $\psi\in\mathscr{S}(\mathbb{R})$. Thus

$$\int_{|\xi|>1/\epsilon} e^{ix\xi}D^\alpha\left(\widehat{\phi}(\epsilon\xi)\frac{\sin s\xi}{\xi}\widehat{r_m}(\xi)\right)d\xi\lesssim\int_{|\xi|>1/\epsilon}\psi(\epsilon\xi)\frac{d\xi}{|\xi|}=\int_{|\xi|>1}\psi(\xi)\frac{d\xi}{|\xi|}\lesssim 1.$$

When $10<|\xi|\leq 1/\epsilon$, we have $D^\alpha\big(\widehat{\phi}(\epsilon\xi)(\sin s\xi/\xi)\widehat{r_m}(\xi)\big)\lesssim 1/|\xi|$ and hence its L^1-norm is bounded by $\log 1/\epsilon$. By the following uniform Riemann–Lebesgue lemma, we get

$$\int_{10<|\xi|\leq 1/\epsilon} e^{ix\xi}D^\alpha\left(\widehat{\phi}(\epsilon\xi)\frac{\sin s\xi}{\xi}\widehat{r_m}(\xi)\right)d\xi=o\,(1)\log\frac{1}{\epsilon},\ x\to\infty.$$

Lemma 6.1. *Assume that $\psi_\lambda(\xi)$ are uniformly bounded and uniformly equicontinuous functions on $\mathbb{R}$. Then*

$$\int_{10\le|\xi|\le N} e^{ix\xi}\frac{\psi_\lambda(\xi)}{\xi}d\xi = o\,(1)\log N,\ \ |x|\to\infty,$$

where $o\,(1)$ is uniform in N and λ.

Sketch of proof. Let $\epsilon>0$. Let ϕ_δ, $0<\delta<1$ be a C^∞ approximation of the identity and $\psi_\lambda^\delta=\phi_\delta*\psi_\lambda$. Note that, since $(\psi_\lambda^\delta)'=(\phi_\delta)'*\psi_\lambda=(1/\delta)(\phi')_\delta*\psi_\lambda$, we have $\|(\psi_\lambda^\delta)'\|_\infty\lesssim 1/\delta$. Also, if δ is small enough, $\|\psi_\lambda-\psi_\lambda^\delta\|_\infty<\epsilon$. Write

$$\int_{10\le|\xi|\le N} e^{ix\xi}\frac{\psi_\lambda(\xi)}{\xi}d\xi=\int_{10\le|\xi|\le N} e^{ix\xi}\frac{\psi_\lambda(\xi)-\psi_\lambda^\delta(\xi)}{\xi}d\xi+\int_{10\le|\xi|\le N} e^{ix\xi}\frac{\psi_\lambda^\delta(\xi)}{\xi}d\xi.$$

The first integrand is bounded by $\epsilon/|\xi|$, and thus the integral by $\epsilon\log N$. An integration by parts shows that the second integral is bounded by $(1/\delta|x|)\log N$. So if $|x|>1/\delta\epsilon$,

$$\int_{10\le|\xi|\le N} e^{ix\xi}\frac{\psi_\lambda(\xi)}{\xi}d\xi\lesssim\epsilon\log N. \qquad \square$$

The hardest part to estimate is when $|\xi|\le 10$. This is the content of the next proposition.

Proposition 6.2. *Assume that μ has a finite moment of order α. Then $D^\alpha\big(\widehat{\phi}(\epsilon\xi)(\sin s\xi/\xi)\widehat{r_m}(\xi)\big)$ is locally integrable.*

Proof. We will only prove the case $1<\alpha<2$.

The case $\alpha\ge 2$ is easier as we may take m sufficiently large and prove that $D^\alpha\widehat{\phi}(\epsilon\xi)(\sin s\xi/\xi)\widehat{r_m}(\xi)$ is bounded. By (1.4), we may then discard the terms containing R_k with $k\ge 2$.

So assume that $\alpha<2$ and consider first $3/2<\alpha<2$. Let $\beta=\alpha-1$ and $\delta=\alpha-3/2$. When $3/2<\alpha<2$, we have $m=2$ and by (2.9) and (2.4)

$$\widehat{r_2}(\xi)=\left(\frac{f(\xi)-1+i\mu_1\xi}{i\mu_1\xi}\right)^3\frac{1}{1-f(\xi)}\lesssim|\xi|^{3\beta-1}=|\xi|^{3\delta+1/2},\ \ |\xi|\to 0.$$

Let $\rho(\xi)=\widehat{\phi}(\epsilon\xi)(\sin s\xi/\xi)\widehat{r_2}(\xi)$, and consider ρ'. By Leibniz formula it consists of several terms, two of which contain f'. One of these appears when we differentiate $1/(1-f(\xi))$ and one when we differentiate $(f(\xi)-1+i\mu_1\xi)^3$. We write

$$\rho'(\xi)=\rho_1(\xi)+\rho_2(\xi)f'(\xi)+\rho_3(\xi)h(\xi),$$

where $h(\xi)=f'(\xi)+i\mu_1$. The functions ρ_1, ρ_2 and ρ_3 has one more classical derivative if $\xi\ne 0$. Since a derivative of ρ, not acting on $(f(\xi)-1+i\mu_1\xi)^3$, introduces a singularity at most $1/\xi$, we get

$$\rho_j(\xi)\lesssim|\xi|^{3\delta-1/2}\ \ \text{and}\ \ \rho_j'(\xi)\lesssim|\xi|^{3\delta-3/2},\ j=1,2.$$

By Lemma 5.7, this implies

$$|D|^\beta \rho_i(\xi) \lesssim |\xi|^{3\delta-1/2-\beta} = |\xi|^{2\delta-1},\ i=1,2.$$

In particular $D^\beta \rho_1 \in L^1_{\mathrm{loc}}(\mathbb{R})$.

By Lemma 5.4, $|D|^\beta f'$ is bounded. Hence, by Lemma 5.5,

$$\begin{aligned}|D|^\beta(\rho_2 f')(\xi) &\le |\rho_2(\xi)|\,|D|^\beta f'(\xi) + \|f'\|_{I_\xi}\,|D|^\beta \rho_2(\xi) \\ &\lesssim |\xi|^{3\delta-1/2} + |\xi|^{2\delta-1} \lesssim |\xi|^{2\delta-1}.\end{aligned}$$

Thus also $D^\beta(\rho_2 f') \in L^1_{\mathrm{loc}}(\mathbb{R})$.

The last term is slightly different as ρ_3 is more singular (ρ_3' is not locally integrable). By Lemma 5.7,

$$\rho_3(\xi) \lesssim |\xi|^{2\delta-1},\ \rho_3'(\xi) \lesssim |\xi|^{2\delta-2} \ \text{ and } \ |D|^\beta \rho_3(\xi) \lesssim |\xi|^{2\delta-1-\beta}.$$

On the other hand,

$$h(\xi) = f'(\xi) + i\mu_1 = \int_{\mathbb{R}} -ix\big(e^{-ix\xi}-1\big)d\mu(x) \lesssim |\xi|^\beta$$

is small at the origin.

Arguing as in the proof of Lemma 5.5, we have

$$|D|^\beta(\rho_3 h)(\xi) \le \int_{-1}^{1} |\rho_3(\xi-t)|\frac{|h(\xi-t)-h(\xi)|}{|t|^{1+\beta}}dt + |h(\xi)|\,|D|^\beta \rho_3(\xi).$$

As

$$|h(\xi)|\,|D|^\beta \rho_3(\xi) \lesssim |\xi|^{\beta+2\delta-1-\beta} = |\xi|^{2\delta-1} \in L^1_{\mathrm{loc}}(\mathbb{R}),$$

it remains to estimate the integral. The argument is similar to the proof of Lemma 5.7. We have

$$\begin{aligned}\int_{|t|\le 1, |\xi-t|\le \xi/2} |\rho_3(\xi-t)|\frac{|h(\xi-t)-h(\xi)|}{|t|^{1+\beta}}dt &\lesssim |\xi|^\beta \frac{1}{|\xi|^{1+\beta}} \int_{|\xi-t|\le\xi/2} |\rho_3(\xi-t)|dt \\ = |\xi|^{-1}\int_{|t|\le\xi/2}|\rho_3(t)|dt &\lesssim |\xi|^{-1}\int_{|t|\le\xi/2}|t|^{2\delta-1}dt \lesssim |\xi|^{2\delta-1} \in L^1_{\mathrm{loc}}(\mathbb{R}).\end{aligned}$$

When $|\xi-t| > \xi/2$, we get

$$\begin{aligned}&\int_{|t|\le 1, |\xi-t|>\xi/2} |\rho_3(\xi-t)|\frac{|h(\xi-t)-h(\xi)|}{|t|^{1+\beta}}dt \\ &\lesssim \int_{|t|\le 1, |\xi-t|>\xi/2} |\xi-t|^{2\delta-1}\frac{|h(\xi-t)-h(\xi)|}{|t|^{1+\beta}}dt \\ &\lesssim |\xi|^{2\delta-1}|D|^\beta h(\xi) \lesssim |\xi|^{2\delta-1} \in L^1_{\mathrm{loc}}(\mathbb{R}).\end{aligned}$$

The proof for arbitrary α, $1 < \alpha < 2$ is basically the same. Assume that $1 + 1/m < \alpha \le 1 + 1/(m-1)$, and consider for instance the term $\rho_2 f'$ above, but now with $\widehat{r_2}$ replaced with

$$\widehat{r_m}(\xi) = \left(\frac{f(\xi) - 1 + i\mu_1\xi}{i\mu_1\xi}\right)^{m+1} \frac{1}{1 - f(\xi)}.$$

If $\alpha = 1 + \beta$ and $\beta = 1/m + \delta$, ρ_2 satisfies

$$\rho_2(\xi) \lesssim |\xi|^{(m+1)(1/m+\delta)-2} = |\xi|^{1/m+\delta+m\delta-1}, \ \rho_2'(\xi) \lesssim |\xi|^{1/m+\delta+m\delta-2}$$

and

$$|D|^\beta \rho_2(\xi) \lesssim |\xi|^{1/m+\delta+m\delta-1-\beta} = |\xi|^{m\delta-1}.$$

Thus

$$|D|^\beta(\rho_2 f')(\xi) \le |\rho_2(\xi)|\,|D|^\beta f'(\xi) + \|f'\|_{I_\xi}\,|D|^\beta \rho_2(\xi) \lesssim |\xi|^{m\delta-1},$$

which is locally integrable. □

7. Concluding remarks.

A slightly different way to prove Theorem 1.1, also when $1 < \alpha < 2$, is to include more terms in (1.3). If M is (much) larger than m, $\widehat{r_M}$ satisfies a better estimate than $\widehat{r_m}$ and it is easier to prove Proposition 6.2 for this M. Then one needs to prove that $R_k(x)$ decays rapidly when $k > m$ so that they can be discarded in (1.3).

Since $\widehat{R_{m+1}}$ and $\widehat{r_m}$ have the same singularity at the origin, using the Fourier methods in Section 6, we can prove this. Namely, we have $R_n(x) = o(1/|x|^\alpha)$, $|x| \to \infty$ when $n > [1/(\alpha-1)] + 1$. The logarithmic factor is not needed due to the decay of $\widehat{R_n}$ at infinity. However the proof of these estimate of R_n is almost as hard as that of r_n, so altogether this approach is not a simplification of the proof of Theorem 1.1.

If on the other hand, by sharpening the arguments in Section 4, we could prove the necessary estimates of R_n directly, this could simplify the proof. But I have not been able to do this.

We can also prove $R_n(x) = o(1/|x|^{n\beta})$, $|x| \to \infty$, when $n \le [1/(\alpha-1)] + 1$ except possibly when $\beta = 1/k$ and $n = k+1$, $k = 2, 3, 4, \dots$. Note that $n\beta = \alpha$ in this case.

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