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Uniqueness of the solution of nonlinear singular first order partial differential equations

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Abstract. This paper deals with nonlinear singular partial differential equations of the form $t\partial u/\partial t = F(t, x, u, \partial u/\partial x)$ with independent variables $(t, x) \in \mathbb{R} \times \mathbb{C}$, where $F(t, x, u, v)$ is a function continuous in t and holomorphic in the other variables. Under a very weak assumption we show the uniqueness of the solution of this equation. The results are applied to the problem of analytic continuation of local holomorphic solutions of equations of this type.

1. Introduction.

To investigate the uniqueness of the solution is one of the most important problems in the theory of partial differential equations, and there are many references in various situations. In this paper, we consider the case of first order nonlinear singular partial differential equations (1.1) given below, and show uniqueness results by a method quite similar to Cauchy's characteristic method.

Let $t \in \mathbb{R}$, $x \in \mathbb{C}$, $u \in \mathbb{C}$ and $v \in \mathbb{C}$ be the variables. For $r > 0$ we write $D_r = \{z \in \mathbb{C}\,;\, |z| < r\}$ where z represents x, u or v. Let $T_0 > 0$, $R_0 > 0$, $\rho_0 > 0$, and set $\Omega = \{(t, x, u, v) \in [0, T_0] \times D_{R_0} \times D_{\rho_0} \times D_{\rho_0}\}$.

Let $F(t, x, u, v)$ be a function on Ω. In this paper, we consider the equation

$$t\frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right) \tag{1.1}$$

under the following assumptions:

A$_1$) $F(t, x, u, v)$ is a continuous function on Ω which is holomorphic in the variable $(x, u, v) \in D_{R_0} \times D_{\rho_0} \times D_{\rho_0}$ for any fixed t.

A$_2$) There is a weight function $\mu(t)$ on $(0, T_0]$ satisfying the following:

$$\sup_{x \in D_{R_0}} |F(t, x, 0, 0)| = O(\mu(t)) \quad (\text{as } t \longrightarrow +0),$$
$$\left|\frac{\partial F}{\partial v}(t, 0, 0, 0)\right| = O(\mu(t)) \quad (\text{as } t \longrightarrow +0).$$

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Here, a weight function $\mu(t)$ on $(0, T_0]$ means that $\mu(t)$ is a positive real valued continuous function on $(0, T_0]$ which is increasing in t and satisfies

$$\int_0^{T_0} \frac{\mu(s)}{s} ds < \infty.$$

By this condition, we have $\mu(t) \longrightarrow 0$ (as $t \longrightarrow +0$).

By A_2) we can express $(\partial F/\partial v)(t, x, 0, 0)$ in the form

$$\frac{\partial F}{\partial v}(t, x, 0, 0) = b(t) + x^{p+1} c(t, x)$$

where $b(t)$ is a continuous function on $[0, T_0]$ satisfying $b(t) = O(\mu(t))$ (as $t \longrightarrow +0$), $c(t, x)$ is a continuous function on $[0, T_0] \times D_{R_0}$ that is holomorphic in x, and $p \in \{0, 1, 2, \ldots\}$. Then, we can divide our situation into the following three cases:

Case 1. $c(t, x) \equiv 0$ on $[0, T_0] \times D_{R_0}$,
Case 2. $p = 0$ and $c(t, 0) \not\equiv 0$ on $[0, T_0]$,
Case 3. $p \geq 1$ and $c(t, 0) \not\equiv 0$ on $[0, T_0]$.

In Case 1, equation (1.1) is a generalization of Briot–Bouquet's ordinary differential equations (in Briot–Bouquet [**4**]) to partial differential equations, and this type of equations was studied by Baouendi–Goulaouic [**3**], Gérard–Tahara [**8**], Yamazawa [**15**], Koike [**10**] and Lope–Roque–Tahara [**11**]. In Case 2, equation (1.1) has a regular singularity at $x = 0$, and this type of equations was studied by Chen–Tahara [**5**] and Bacani–Tahara [**1**]. In Case 3, equation (1.1) has an irregular singularity at $x = 0$, and this type of equations was studied by Chen–Luo–Zhang [**6**], Luo–Chen–Zhang [**12**] and Bacani–Tahara [**2**]. In these papers, mainly the solvability (or the unique solvability) of equation (1.1) is discussed.

As to the uniqueness of the solution, we know some results: in Case 1 we have a result in Tahara [**13**] under the assumption: $u(t, x) = O(\mu(t)^\epsilon)$ (as $t \longrightarrow +0$) for some $\epsilon > 0$, and in Case 2 we have a result in Tahara [**14**] under the assumption: $u(t, x) = O(|t|^\epsilon)$ (as $t \longrightarrow +0$) for some $\epsilon > 0$.

In this paper, we will show the uniqueness of the solution in each case under a much weaker assumption like

$$\overline{\lim_{R \to +0}} \left[\lim_{\sigma \to +0} \left(\frac{1}{R^2} \sup_{(0,\sigma) \times D_R} |u(t, x)| \right) \right] = 0.$$

2. Analysis in Case 1.

Let us consider Case 1 in a little bit general setting. We consider equation (1.1) under the following assumptions:

$$\sup_{x \in D_{R_0}} |F(t, x, 0, 0)| = O(\mu(t)) \quad (\text{as } t \longrightarrow +0), \tag{2.1}$$

$$\sup_{x \in D_{R_0}} \left| \frac{\partial F}{\partial v}(t, x, 0, 0) \right| = O(\mu(t)) \quad (\text{as } t \longrightarrow +0). \tag{2.2}$$

As to the existence of a solution, we know a unique solvability result in a certain function space. To state the existence result, let us prepare some notations. We set

$$\varphi(t) = \int_0^t \frac{\mu(s)}{s} ds, \quad 0 < t \le T_0.$$

This is also an increasing function on $(0, T_0]$ and we have $\varphi(t) \longrightarrow 0$ (as $t \longrightarrow +0$). For $T > 0$, $R > 0$ and $r > 0$ we set

$$W_{T,R,r} = \{(t,x) \in [0,T] \times \mathbb{C}\,;\, \varphi(t)/r + |x| < R\}.$$

For $W = W_{T,R,r}$, we denote by $\mathscr{X}_0(W)$ the set of all functions in $C^0(W)$ that are holomorphic in x for any fixed t, and by $\mathscr{X}_1(W)$ the set of all functions in $C^1(W \cap \{t > 0\}) \cap C^0(W)$ that are also holomorphic in x for any fixed t. We set

$$\lambda(t,x) = \frac{\partial F}{\partial u}(t,x,0,0).$$

By Theorem 1.1 (with $\alpha = 1$) in Lope–Roque–Tahara [**11**] we have

THEOREM 2.1. *Suppose the conditions* (2.1) *and* (2.2). *If* $\mathrm{Re}\lambda(0,0) < 0$ *holds, there are* $T > 0$, $R > 0$, $r > 0$ *and* $M > 0$ *such that equation* (1.1) *has a unique solution* $u_0(t,x) \in \mathscr{X}_1(W_{T,R,r})$ *satisfying*

$$|u_0(t,x)| \le M\mu(t) \quad \text{and} \quad \left|\frac{\partial u_0}{\partial x}(t,x)\right| \le M\mu(t)$$

on $W_{T,R,r}$.

2.1. Uniqueness result in Case 1.

For $T > 0$ and $R > 0$ we denote by $\mathscr{X}_1((0,T) \times D_R)$ the set of all functions in $C^1((0,T) \times D_R)$ that are holomorphic in the variable $x \in D_R$ for any fixed t.

The following theorem is the main result of this section.

THEOREM 2.2. *Suppose the conditions* (2.1), (2.2) *and* $\mathrm{Re}\lambda(0,0) < 0$. *Let* $u(t,x) \in \mathscr{X}_1((0,T) \times D_R)$ *be a solution of* (1.1) *with* $T > 0$ *and* $R > 0$. *If* $u(t,x)$ *satisfies*

$$\overline{\lim_{R \to +0}} \left[\lim_{\sigma \to +0} \left(\frac{1}{R^2} \sup_{(0,\sigma) \times D_R} |u(t,x)| \right) \right] = 0, \tag{2.3}$$

we have $u(t,x) = u_0(t,x)$ *on* $(0,T_1) \times D_{R_1}$ *for some* $T_1 > 0$ *and* $R_1 > 0$, *where* $u_0(t,x)$ *is the solution of* (1.1) *obtained in Theorem* 2.1.

If

$$\lim_{t \to +0} \Bigl(\sup_{x \in D_R} |u(t,x)| \Bigr) = 0 \tag{2.4}$$

holds for some $R > 0$, we have (2.3), and so we have

Corollary 2.3. *Suppose the conditions* (2.1), (2.2) *and* $\mathrm{Re}\lambda(0,0) < 0$. *If a solution* $u(t,x) \in \mathscr{X}_1((0,T) \times D_R)$ *of* (1.1) *satisfies* (2.4), *we have* $u(t,x) = u_0(t,x)$ *on* $(0,T_1) \times D_{R_1}$ *for some* $T_1 > 0$ *and* $R_1 > 0$.

If a solution $u(t,x)$ satisfies

$$\sup_{x \in D_R} |u(t,x)| = O(\mu(t)^{\epsilon}) \quad (\text{as } t \longrightarrow +0) \tag{2.5}$$

for some $\epsilon > 0$, we can apply a result in Tahara [**13**]. We note that the condition (2.3) is much weaker than (2.5). In [**13**] higher order equations are dealt with, but it is unclear whether we can generalize Theorem 2.2 to higher order case.

Remark 2.4. (1) In the case $\mathrm{Re}\lambda(0,0) > 0$, we can give many examples in holomorphic category such that the equation has many solutions satisfying (2.4). Therefore, the uniqueness of the solution is not valid in general. See [**8**] and [**15**].

(2) In the case $\mathrm{Re}\lambda(0,0) = 0$, we have the following counter example: the equation

$$t\frac{\partial u}{\partial t} = u\left(\frac{\partial u}{\partial x}\right)^k \quad (k \in \{1,2,\dots\})$$

has a trivial solution $u \equiv 0$ and a family of nontrivial solutions

$$u = \left(\frac{1}{k}\right)^{1/k} \frac{x+\alpha}{(c - \log t)^{1/k}}$$

with arbitrary constants α and c. These solutions satisfy (2.4).

(3) The following example shows that the assumption (2.3) is reasonable: the equation

$$t\frac{\partial u}{\partial t} = -u + \left(\frac{\partial u}{\partial x}\right)^2$$

has a trivial solution $u \equiv 0$ and a nontrivial solution $u = x^2/4$. We note that for $u = x^2/4$ we have

$$\overline{\lim_{R \to +0}} \left[\lim_{\sigma \to +0} \left(\frac{1}{R^2} \sup_{(0,\sigma) \times D_R} |u(t,x)| \right) \right] = \frac{1}{4}.$$

2.2. Proof of Theorem 2.2.

Let $u_0(t,x)$ be the unique solution of (1.1) obtained in Theorem 2.1. Set $v_0(t,x) = (\partial u_0/\partial x)(t,x)$. Then, by setting $w = u - u_0$, our equation (1.1) is reduced to an equation with respect to $w = w(t,x)$:

$$t\frac{\partial w}{\partial t} = H\left(t,x,w,\frac{\partial w}{\partial x}\right) \tag{2.6}$$

where

$$\begin{aligned}H(t,x,w,q) &= F(t,x,w+u_0(t,x),q+v_0(t,x)) \\ &\quad - F(t,x,u_0(t,x),v_0(t,x)).\end{aligned}$$

For $\Omega^* = \{(t,x,u,v) \in [0,\sigma^*] \times D_{R_0^*} \times D_{\rho_0^*} \times D_{\rho_0^*}\}$ we denote by $\mathscr{X}_0(\Omega^*)$ the set of all functions in $C^0(\Omega^*)$ that are holomorphic in the variable (x,w,q) for any fixed t.

Then, we may suppose that $H(t,x,w,q)$ belongs to $\mathscr{X}_0(\Omega^*)$ for sufficiently small $\sigma^* > 0$, $R_0^* > 0$ and $\rho_0^* > 0$. By Taylor expansion in (u,v) we have the expression

$$F(t,x,u,v) = \alpha(t,x) + \lambda(t,x)u + \beta(t,x)v + \sum_{i+j\geq 2} a_{i,j}(t,x)u^i v^j$$

and so we have

$$H(t,x,w,q) = \lambda(t,x)w + \beta(t,x)q + \sum_{i+j\geq 2} a_{i,j}(t,x)\Big[(w+u_0)^i(q+v_0)^j - {u_0}^i {v_0}^j\Big].$$

Hence, it is easy to see that $H(t,x,w,q)$ is expressed in the form

$$H(t,x,w,q) = \lambda(t,x)w + a_1(t,x,w,q)w + b_1(t,x,w,q)q$$

for some functions $a_1(t,x,w,q) \in \mathscr{X}_0(\Omega^*)$ and $b_1(t,x,w,q) \in \mathscr{X}_0(\Omega^*)$. Since (2.2) implies $\beta(t,x) = O(\mu(t))$ (as $t \longrightarrow +0$), and since $u_0 = O(\mu(t))$ (as $t \longrightarrow +0$) and $v_0 = O(\mu(t))$ (as $t \longrightarrow +0$) hold, we may assume:

$$\begin{aligned}&\sup_{x\in D_{R_0^*}} |a_1(t,x,0,0)| = O(\mu(t)) \quad (\text{as } t \longrightarrow +0), \\ &\sup_{x\in D_{R_0^*}} |b_1(t,x,0,0)| = O(\mu(t)) \quad (\text{as } t \longrightarrow +0).\end{aligned}$$

To get Theorem 2.2 it is sufficient to show the following result.

PROPOSITION 2.5. *Suppose* $\mathrm{Re}\lambda(0,0) < 0$. *Let* $w(t,x) \in \mathscr{X}_1((0,\sigma_0) \times D_{R_0})$ *be a solution of* (2.6) *with* $\sigma_0 > 0$ *and* $R_0 > 0$. *If* $w(t,x)$ *satisfies*

$$\varlimsup_{R\to+0}\left[\lim_{\sigma\to+0}\left(\frac{1}{R^2}\sup_{(0,\sigma)\times D_R}|w(t,x)|\right)\right] = 0, \tag{2.7}$$

we have $w(t,x) = 0$ *on* $(0,\sigma) \times D_\delta$ *for some* $\sigma > 0$ *and* $\delta > 0$.

PROOF. Let us prove this step by step.

Step 1: Since $\sigma^* > 0$ and $R_0^* > 0$ are sufficiently small, we may suppose that there is an $a > 0$ satisfying

$$\mathrm{Re}\lambda(t,x) < -2a \quad \text{on } [0,\sigma^*] \times D_{R_0^*}.$$

Since $a_1(t,x,0,0) = O(\mu(t))$ and $b_1(t,x,0,0) = O(\mu(t))$ hold, we have the estimates

$$\begin{aligned}&|a_1(t,x,w,q)| \leq A_0\mu(t) + A_1|w| + A_2|q| \quad \text{on } \Omega^*, \\ &|b_1(t,x,w,q)| \leq B_0\mu(t) + B_1|w| + B_2|q| \quad \text{on } \Omega^*\end{aligned}$$

for some $A_i > 0$ $(i = 0, 1, 2)$ and $B_i > 0$ $(i = 0, 1, 2)$.

Step 2: Let $w(t,x) \in \mathscr{X}_1((0,\sigma_0) \times D_{R_0})$ be a solution of (2.6) for some $0 < \sigma_0 < \sigma^*$ and $0 < R_0 < R_0^*$. We suppose that $w(t,x)$ satisfies (2.7). We set $q(t,x) = (\partial w/\partial x)(t,x)$ and

$$a(t,x) = a_1(t,x,w(t,x),q(t,x)),$$
$$b(t,x) = b_1(t,x,w(t,x),q(t,x)) :$$

these are functions belonging to $\mathscr{X}_0((0,\sigma_0) \times D_{R_0})$. Then, by (2.6) we see that $w(t,x)$ satisfies the following linear partial differential equation:

$$t\frac{\partial w}{\partial t} - b(t,x)\frac{\partial w}{\partial x} = (\lambda(t,x) + a(t,x))w. \tag{2.8}$$

By applying $\partial/\partial x$ to (2.8), we have

$$t\frac{\partial q}{\partial t} - b(t,x)\frac{\partial q}{\partial x} = \gamma(t,x)w + (\lambda(t,x) + a(t,x) + \ell(t,x))q, \tag{2.9}$$

where

$$\gamma(t,x) = (\partial\lambda/\partial x)(t,x) + (\partial a/\partial x)(t,x),$$
$$\ell(t,x) = (\partial b/\partial x)(t,x) :$$

these are also functions belonging to $\mathscr{X}_0((0,\sigma_0) \times D_{R_0})$. For $0 < \sigma < \sigma_0$ and $0 < R < R_0$ we set

$$A = \sup_{(0,\sigma)\times D_R} |a(t,x)|, \quad \Gamma = \sup_{(0,\sigma)\times D_R} |\gamma(t,x)|, \quad L = \sup_{(0,\sigma)\times D_R} |\ell(t,x)|.$$

We set also

$$r_1 = \sup_{(0,\sigma)\times D_R} |w(t,x)|, \quad r_2 = \sup_{(0,\sigma)\times D_R} |q(t,x)|.$$

Lemma 2.6. *By taking $\sigma > 0$ and $R > 0$ sufficiently small, we have the conditions $A + L < a$, and*

$$B_0\varphi(\sigma) + \left(\frac{B_1}{a} + \frac{B_2\Gamma}{a^2}\right) r_1 + \frac{B_2}{a} r_2 < \frac{R}{2}.$$

Proof. By (2.7) we have

$$\lim_{\sigma\to+0} \sup_{(0,\sigma)\times D_R} |w(t,x)| = o(R^2) \quad (\text{as } R \longrightarrow +0). \tag{2.10}$$

By applying Cauchy's integral formula in x to (2.10), we have

$$\lim_{\sigma\to+0} \sup_{(0,\sigma)\times D_R} |q(t,x)| = o(R) \quad (\text{as } R \longrightarrow +0). \tag{2.11}$$

Since $|a_1(t,x,w,q)| \le A_0\mu(t)+A_1|w|+A_2|q|$ and $|b_1(t,x,w,q)| \le B_0\mu(t)+B_1|w|+B_2|q|$ are known, by (2.10) and (2.11) we have

$$\lim_{\sigma\to+0} \sup_{(0,\sigma)\times D_R} |a(t,x)| = o(R) \quad (\text{as } R \longrightarrow +0),$$
$$\lim_{\sigma\to+0} \sup_{(0,\sigma)\times D_R} |b(t,x)| = o(R) \quad (\text{as } R \longrightarrow +0),$$
$$\lim_{\sigma\to+0} \sup_{(0,\sigma)\times D_R} |(\partial b/\partial x)(t,x)| = o(1) \quad (\text{as } R \longrightarrow +0).$$

Therefore, by taking $\sigma > 0$ and $R > 0$ sufficiently small, the numbers A, L, r_1/R and r_2/R will be as small as possible. This proves Lemma 2.6. □

Step 3: Let $\sigma > 0$ and $R > 0$ be as in Lemma 2.6. Take any $t_0 \in (0,\sigma)$ and $\xi \in D_R$; for a while we fix them.

Let us consider the initial value problem

$$t\,\frac{dx}{dt} = -b(t,x), \quad x(t_0) = \xi. \tag{2.12}$$

Here, we regard $b(t,x)$ as a function in $\mathscr{X}_0((0,\sigma)\times D_R)$. Let $x(t)$ be the unique solution in a neighborhood of $t = t_0$. Let $(t_\xi, t_0]$ be the maximal interval of the existence of this solution. Set

$$w^*(t) = w(t,x(t)), \quad q^*(t) = q(t,x(t)).$$

Then, by (2.8) and (2.9) we have

$$t\,\frac{dw^*(t)}{dt} = (\lambda(t,x(t)) + a(t,x(t)))w^*(t), \quad w^*(t_0) = w(t_0,\xi) \tag{2.13}$$

on $(t_\xi, t_0]$, and

$$t\,\frac{dq^*(t)}{dt} = \gamma(t,x(t))w^*(t) + (\lambda(t,x(t)) + a(t,x(t)) + \ell(t,x(t)))q^*(t), \tag{2.14}$$
$$q^*(t_0) = q(t_0,\xi)$$

on $(t_\xi, t_0]$.

Lemma 2.7. *Under the above situation, we have the following estimates for any (t_1,τ) satisfying $t_\xi < t_1 < \tau \le t_0$:*

$$|w^*(\tau)| \le \left(\frac{t_1}{\tau}\right)^a |w^*(t_1)|, \tag{2.15}$$

$$|q^*(\tau)| \le \left(\frac{t_1}{\tau}\right)^a \big(\Gamma|w^*(t_1)|\log(\tau/t_1) + |q^*(t_1)|\big). \tag{2.16}$$

Proof. Let $t_\xi < t_1 < \tau \le t_0$: set

$$\phi(t) = \exp\left[\int_t^\tau \frac{(\lambda(s,x(s)) + a(s,x(s)))}{s}ds\right], \quad t_1 \le t \le \tau.$$

Since $\mathrm{Re}(\lambda(s,x(s))+a(s,x(s)))<-2a+A<-a$, we have

$$|\phi(t)| \le \exp\left[\int_t^\tau \frac{\mathrm{Re}(\lambda(s,x(s))+a(s,x(s)))}{s}ds\right] \le \exp\left[\int_t^\tau \frac{-a}{s}ds\right] = \left(\frac{t}{\tau}\right)^a, \quad t_1 \le t \le \tau.$$

Let us show (2.15). By (2.13) we have

$$\frac{d}{dt}(w^*(t)\phi(t)) = 0$$

and so by integrating this from t_1 to τ we have

$$w^*(\tau)\phi(\tau) = w^*(t_1)\phi(t_1).$$

Since $\phi(\tau)=1$ and $|\phi(t_1)| \le (t_1/\tau)^a$ holds, by applying this to the above equality we have (2.15).

Let us show (2.16). In this case, we set

$$\phi_1(t) = \exp\left[\int_t^\tau \frac{(\lambda(s,x(s))+a(s,x(s))+\ell(t,x(t)))}{s}ds\right], \quad t_1 \le t \le \tau.$$

Since $\mathrm{Re}(\lambda(s,x(s))+a(s,x(s))+\ell(t,x(t)))<-2a+A+L<-a$ we have $|\phi_1(t)| \le (t/\tau)^a$ for $t_1 \le t \le \tau$. Then, we can reduce (2.14) into

$$\frac{d}{dt}(\phi_1(t)q^*(t)) = \frac{\phi_1(t)\gamma(t,x(t))w^*(t)}{t},$$

and so by integrating this from t_1 to τ and by using (2.15) (with τ replaced by t), we have

$$\begin{aligned}|q^*(\tau)| &\le |\phi(t_1)q^*(t_1)| + \int_{t_1}^\tau |\phi_1(t)\gamma(t,x(t))w^*(t)|\frac{dt}{t} \\ &\le \left(\frac{t_1}{\tau}\right)^a |q^*(t_1)| + \int_{t_1}^\tau \left(\frac{t}{\tau}\right)^a \Gamma \left(\frac{t_1}{t}\right)^a |w^*(t_1)|\frac{dt}{t} \\ &= \left(\frac{t_1}{\tau}\right)^a |q^*(t_1)| + \left(\frac{t_1}{\tau}\right)^a \Gamma |w^*(t_1)| \times \log(\tau/t_1).\end{aligned}$$

This proves (2.16). □

Step 4: Recall that $|b_1(t,x,w,q)| \le B_0\mu(t)+B_1|w|+B_2|q|$ holds on Ω^*. We have

LEMMA 2.8. *Under the above situation, we have the following estimate for any* $t_1 \in (t_\xi, t_0)$:

$$|x(t_1)| \le |\xi| + B_0(\varphi(t_0)-\varphi(t_1)) + \left(\frac{B_1}{a}+\frac{B_2\Gamma}{a^2}\right)|w^*(t_1)| + \frac{B_2}{a}|q^*(t_1)|.$$

Proof. Let $t_1 \in (t_\xi, t_0)$. By (2.12) we have

$$x(t_1) = \xi + \int_{t_1}^{t_0} b(\tau, x(\tau)) \frac{d\tau}{\tau}.$$

Since

$$\begin{aligned} |b(\tau, x(\tau))| &\le B_0\mu(\tau) + B_1|w^*(\tau)| + B_2|q^*(\tau)| \\ &\le B_0\mu(\tau) + B_1 \left(\frac{t_1}{\tau}\right)^a |w^*(t_1)| + B_2 \left(\frac{t_1}{\tau}\right)^a \bigl(\Gamma|w^*(t_1)| \log(\tau/t_1) + |q^*(t_1)|\bigr) \end{aligned}$$

holds for any $\tau \in (t_1, t_0]$, we have

$$\begin{aligned} |x(t_1)| \le |\xi| + \int_{t_1}^{t_0} \Biggl(& B_0\mu(\tau) + B_1 \left(\frac{t_1}{\tau}\right)^a |w^*(t_1)| \\ & + B_2 \left(\frac{t_1}{\tau}\right)^a \bigl(\Gamma|w^*(t_1)| \log(\tau/t_1) + |q^*(t_1)|\bigr) \Biggr) \frac{d\tau}{\tau}. \end{aligned} \tag{2.17}$$

Here, we note:

$$\begin{aligned} &\int_{t_1}^{t_0} \left(\frac{t_1}{\tau}\right)^a \frac{d\tau}{\tau} = \frac{1}{a}\left(1 - \frac{{t_1}^a}{{t_0}^a}\right) \le \frac{1}{a}, \\ &\int_{t_1}^{t_0} \left(\frac{t_1}{\tau}\right)^a \log(\tau/t_1)\frac{d\tau}{\tau} = \frac{{t_1}^a}{-a{t_0}^a}\log(t_0/t_1) + \frac{1}{a^2}\left(1 - \frac{{t_1}^a}{{t_0}^a}\right) \le \frac{1}{a^2}. \end{aligned}$$

By applying these estimates to (2.17), we have Lemma 2.8. □

Corollary 2.9. *If $\xi \in D_{R/2}$ we have $t_\xi = 0$.*

Proof. Let $|\xi| < R/2$. Let us show that if $t_\xi > 0$ holds we have a contradiction. Suppose that $t_\xi > 0$ holds. Then, by Lemmas 2.6 and 2.8 we have

$$|x(t_1)| \le \frac{R}{2} + B_0\varphi(\sigma) + \left(\frac{B_1}{2a} + \frac{B_2\Gamma}{a^2}\right) r_1 + \frac{B_2}{a} r_2 = R_1 < R$$

for any $t_1 \in (t_\xi, t_0)$. Since $K = \{x \in \mathbb{C}^n\,;\, |x| \le R_1\}$ is a compact subset of D_R and since $x(t_1) \in K$ for any $t_1 \in (t_\xi, t_0]$, by a theorem in ordinary differential equations (for example, by Theorem 4.1 in Coddington–Levinson [**7**]) we can extend $x(t)$ to $(t_\xi - \varepsilon, t_0]$ for some $\varepsilon > 0$. This contradicts the condition that $(t_\xi, t_0]$ is the maximal interval of the existence of the solution $x(t)$. □

Step 5: Since $t_\xi = 0$, by (2.15) with $\tau = t_0$ we have

$$|w^*(t_0)| \le \left(\frac{t_1}{t_0}\right)^a |w^*(t_1)| \le \left(\frac{t_1}{t_0}\right)^a r_1$$

for any $t_1 \in (0, t_0)$. Since $r_1 > 0$ is independent of t_1, by letting $t_1 \longrightarrow +0$ we have $w^*(t_0) = 0$. Since $w^*(t_0) = w(t_0, \xi)$, we have $w(t_0, x) = 0$ for any $x \in D_{R/2}$. Since $t_0 \in (0, \sigma)$ is taken arbitrarily, we have $w(t, x) = 0$ on $(0, \sigma) \times D_{R/2}$.

This completes the proof of Proposition 2.5. □

2.3. Application.

Let us apply Theorem 2.2 to the problem of analytic continuation of solutions of Briot–Bouquet type partial differential equations.

Let (t, x) be the variables in $\mathbb{C}_t \times \mathbb{C}_x$, and let $F(t, x, u, v)$ be a function in a neighborhood Δ of the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_u \times \mathbb{C}_v$. Set $\Delta_0 = \Delta \cap \{t = 0, u = 0, v = 0\}$. In this subsection, we consider the following equation

$$t\,\frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right) \tag{2.18}$$

(in the germ sense at $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_x$) under the assumptions

B$_1$) $F(t, x, u, v)$ is holomorphic in Δ,
B$_2$) $F(0, x, 0, 0) \equiv 0$ in Δ_0, and
B$_3$) $(\partial F/\partial v)(0, x, 0, 0) \equiv 0$ in Δ_0.

Then, equation (2.18) is called a Briot–Bouquet type partial differential equation with respect to t (by Gérard–Tahara [**8**], [**9**]), and the function

$$\lambda(x) = \frac{\partial F}{\partial u}(0, x, 0, 0)$$

is called the characteristic exponent of (2.18). This equation was studied by [**8**] and Yamazawa [**15**].

By [**8**] we know that if $\lambda(0) \notin \{1, 2, \ldots\}$, equation (2.18) has a unique holomorphic solution $u_0(t, x)$ in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_x$ satisfying $u_0(0, x) = 0$ near $x = 0$. Therefore, by applying Theorem 2.2 (with $\mu(t) = t$) to this case we have

Theorem 2.10. *Suppose the conditions* B$_1$), B$_2$), B$_3$) *and* $\mathrm{Re}\lambda(0) < 0$. *Let* $u(t, x)$ *be a holomorphic solution of* (2.18) *in a neighborhood of* $(0, \sigma_0) \times D_{R_0}$ *for some* $\sigma_0 > 0$ *and* $R_0 > 0$. *If* $u(t, x)$ *satisfies*

$$\varlimsup_{R \to +0} \left[\lim_{\sigma \to +0} \left(\frac{1}{R^2} \sup_{(0,\sigma) \times D_R} |u(t, x)| \right) \right] = 0, \tag{2.19}$$

$u(t, x)$ *can be continued holomorphically up to a neighborhood of* $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_x$.

Remark 2.11. The following example shows that we need some condition like (2.19) in order to get the analytic continuation of solutions: the equation

$$t\,\frac{\partial u}{\partial t} = -2u + xt\left(\frac{\partial u}{\partial x}\right)^2$$

has a solution $u = x/t$.

3. Analysis in Case 2.

Let us consider Case 2 in a little bit general setting. We consider the equation

$$t\,\frac{\partial u}{\partial t} = \alpha(t,x) + \lambda(t,x)u + (\beta(t,x) + xc(t,x))\frac{\partial u}{\partial x} + R_2\left(t, x, u, \frac{\partial u}{\partial x}\right), \tag{3.1}$$

where $\alpha(t,x)$, $\lambda(t,x)$, $\beta(t,x)$ and $c(t,x)$ are continuous functions on $[0,T_0] \times D_{R_0}$ that are holomorphic in x for any fixed t and satisfy

$$\sup_{x \in D_{R_0}} |\alpha(t,x)| = O(\mu(t)) \quad (\text{as } t \longrightarrow +0), \tag{3.2}$$

$$\sup_{x \in D_{R_0}} |\beta(t,x)| = O(\mu(t)) \quad (\text{as } t \longrightarrow +0), \tag{3.3}$$

$$\operatorname{Re} c(t,x) \leq 0 \quad \text{on } [0,T_0] \times D_{R_0}, \tag{3.4}$$

and $R_2(t,x,u,v)$ is a continuous function on Ω (where Ω is the same as in Section 1) which is holomorphic in the variable (x,u,v) for any fixed t and has a Taylor expansion in (u,v) of the form:

$$R_2(t,x,u,v) = \sum_{i+j \geq 2} a_{i,j}(t,x) u^i v^j.$$

As to the existence of a solution, we know a unique solvability result. By Theorem 5.1 in Bacani–Tahara [**1**] we have

Theorem 3.1. *Suppose the conditions* (3.2), (3.3) *and* (3.4). *If* $\operatorname{Re}\lambda(0,0) < 0$ *holds, there are* $T > 0$, $R > 0$, $r > 0$ *and* $M > 0$ *such that equation* (3.1) *has a unique solution* $u_0(t,x) \in \mathscr{X}_1(W_{T,R,r})$ *satisfying*

$$|u_0(t,x)| \leq M\mu(t) \quad \textit{and} \quad \left|\frac{\partial u_0}{\partial x}(t,x)\right| \leq M\mu(t)$$

on $W_{T,R,r}$.

3.1. Uniqueness result in Case 2.

The following theorem is the main result of this section.

Theorem 3.2. *Suppose the conditions* (3.2), (3.3), (3.4) *and* $\operatorname{Re}\lambda(0,0) < 0$. *Let* $u(t,x) \in \mathscr{X}_1((0,T) \times D_R)$ *be a solution of* (3.1) *with* $T > 0$ *and* $R > 0$. *If* $u(t,x)$ *satisfies*

$$\varlimsup_{R \to +0} \left[\lim_{\sigma \to +0} \left(\frac{1}{R^2} \sup_{(0,\sigma) \times D_R} |u(t,x)| \right) \right] = 0, \tag{3.5}$$

we have $u(t,x) = u_0(t,x)$ *on* $(0,T_1) \times D_{R_1}$ *for some* $T_1 > 0$ *and* $R_1 > 0$, *where* $u_0(t,x)$ *is the solution obtained in Theorem* 3.1.

Corollary 3.3. *Suppose the conditions* (3.2), (3.3), (3.4) *and* $\operatorname{Re}\lambda(0,0) < 0$. *If a solution* $u(t,x) \in \mathscr{X}_1((0,T) \times D_R)$ *of* (3.1) *satisfies*

$$\lim_{t \to +0} \Big(\sup_{x \in D_R} |u(t,x)| \Big) = 0,$$

we have $u(t,x) = u_0(t,x)$ *on* $(0,T_1) \times D_{R_1}$ *for some* $T_1 > 0$ *and* $R_1 > 0$.

REMARK 3.4. (1) In the case $\mathrm{Re}\lambda(0,0) > 0$, we have the following counter example: the equation

$$t\frac{\partial u}{\partial t} = 2u - x\frac{\partial u}{\partial x} + u\left(\frac{\partial u}{\partial x}\right)$$

has a trivial solution $u \equiv 0$, a nontrivial solution $u = t^2$ and a family of solutions

$$u = \frac{xt}{c-t}$$

with an arbitrary constant c.

(2) In the case $\mathrm{Re}\lambda(0,0) = 0$, we have the following counter example: the equation

$$t\frac{\partial u}{\partial t} = -x\frac{\partial u}{\partial x} + u^2 + \left(\frac{\partial u}{\partial x}\right)^2$$

has a trivial solution $u \equiv 0$ and a family of nontrivial solutions

$$u = \frac{1}{c - \log t}$$

with an arbitrary constant c.

(3) In the case $\mathrm{Re}\lambda(0,0) < 0$, the following example shows that the condition (3.5) is reasonable: the equation

$$t\frac{\partial u}{\partial t} = -u - x\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2$$

has a trivial solution $u \equiv 0$ and a nontrivial solution $u = 3x^2/4$.

3.2. Proof of Theorem 3.2.

Since the proof of Theorem 3.2 is quite similar to the proof of Theorem 2.2, we give here only a sketch of the proof.

Let $u(t,x) \in \mathscr{X}_1((0,T) \times D_R)$ be a solution of (3.1) satisfying (3.5). Set $w(t,x) = u(t,x) - u_0(t,x)$, where $u_0(t,x)$ is the solution obtained in Theorem 3.1. Then, by the same argument as in (2.8) we see that $w(t,x)$ satisfies a partial differential equation of the form

$$t\frac{\partial w}{\partial t} - (b(t,x) + xc(t,x))\frac{\partial w}{\partial x} = (\lambda(t,x) + a(t,x))w, \tag{3.6}$$

on $(0,\sigma_0) \times D_{R_0}$ for some $\sigma_0 > 0$ and $R_0 > 0$, where $a(t,x)$ and $b(t,x)$ are functions belonging to $\mathscr{X}_0((0,\sigma_0) \times D_{R_0})$ that satisfy

$$\lim_{\sigma\to+0} \sup_{(0,\sigma)\times D_R} |a(t,x)| = o(R) \quad (\text{as } R \longrightarrow +0),$$
$$\lim_{\sigma\to+0} \sup_{(0,\sigma)\times D_R} |b(t,x)| = o(R) \quad (\text{as } R \longrightarrow +0).$$

By applying $\partial/\partial x$ to (3.6) we have

$$t\,\frac{\partial q}{\partial t} - (b(t,x) + xc(t,x))\frac{\partial q}{\partial x}$$
$$= \gamma(t,x)w + (\lambda(t,x) + a(t,x) + c(t,x) + \ell(t,x))q, \tag{3.7}$$

where

$$\gamma(t,x) = (\partial\lambda/\partial x)(t,x) + (\partial a/\partial x)(t,x),$$
$$\ell(t,x) = (\partial b/\partial x)(t,x) + x(\partial c/\partial x)(t,x):$$

these are also functions belonging to $\mathscr{X}_0((0,\sigma_0)\times D_{R_0})$. If we notice the fact that $|x(\partial c/\partial x)(t,x)| \le C_1|x|$ on $(0,\sigma_0)\times D_{R_0}$ for some $C_1 > 0$, by taking $\sigma > 0$ and $R > 0$ sufficiently small we have the same conditions as in Lemma 2.6.

Now, let us consider the initial value problem:

$$t\,\frac{dx}{dt} = -(b(t,x) + xc(t,x)), \quad x(t_0) = \xi. \tag{3.8}$$

Let $x(t)$ be the unique solution in a neighborhood of $t = t_0$. Let $(t_\xi, t_0]$ be the maximal interval of the existence of this solution. Set

$$w^*(t) = w(t, x(t)), \quad q^*(t) = q(t, x(t)).$$

Since $\mathrm{Re}\, c(t,x) \le 0$ is supposed (in (3.4)), we have $\mathrm{Re}\, c(s,x(s)) \le 0$ and so $\mathrm{Re}(\lambda(s,x(s)) + a(s,x(s)) + c(s,x(s)) + \ell(s,x(s))) < -2a + A + 0 + L < -a$. Hence, by the same argument as in the proof of Theorem 2.2 we can show the same conditions as in Lemmas 2.7, 2.8 and Corollary 2.9.

Thus, we have $w(t,x) = 0$ on $(0,\sigma)\times D_{R/2}$ as in Step 5 in the proof of Theorem 2.2. This proves Theorem 3.2. □

3.3. Application.

Let us apply Theorem 3.2 to the problem of analytic continuation of solutions of nonlinear totally characteristic type partial differential equations.

Let us consider the same equation

$$t\,\frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right) \tag{3.9}$$

as in (2.18) in the complex domain Δ under B_1), B_2) and

B_4) $(\partial F/\partial v)(0,x,0,0) = xc(x)$ with $c(0) \ne 0$.

Then, this equation is a typical model of nonlinear totally characteristic partial differential equations discussed by Chen–Tahara [**5**]. As in Subsection 2.3 we set $\lambda(x) = (\partial F/\partial u)(0,x,0,0)$. We write $\mathbb{N}^* = \{1,2,\ldots\}$ and $\mathbb{N} = \{0,1,2,\ldots\}$.

Then, by [**5**] we know the following result: if $c(0) \notin [0,\infty)$ and

$$i - c(0)j - \lambda(0) \ne 0 \quad \text{for any } (i,j) \in \mathbb{N}^* \times \mathbb{N} \tag{3.10}$$

hold, equation (3.9) has a unique holomorphic solution $u_0(t,x)$ in a neighborhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x$ satisfying $u_0(0,x) = 0$ near $x = 0$. Therefore, by applying Theorem 3.2

(with $\mu(t) = t$) to this case we have

THEOREM 3.5. *Suppose the conditions* B_1), B_2), B_4), $\mathrm{Re}\, c(0) < 0$ *and* $\mathrm{Re}\lambda(0) < 0$. *Let* $u(t,x)$ *be a holomorphic solution of* (3.9) *in a neighborhood of* $(0,\sigma_0) \times D_{R_0}$ *for some* $\sigma_0 > 0$ *and* $R_0 > 0$. *If* $u(t,x)$ *satisfies*

$$\overline{\lim_{R \to +0}} \left[\lim_{\sigma \to +0} \left(\frac{1}{R^2} \sup_{(0,\sigma) \times D_R} |u(t,x)| \right) \right] = 0, \tag{3.11}$$

$u(t,x)$ *can be continued holomorphically up to a neighborhood of* $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x$.

REMARK 3.6. The following example shows that we need some condition like (3.11) in order to get the analytic continuation of solutions: the equation

$$t \frac{\partial u}{\partial t} = -2u - x \frac{\partial u}{\partial x} + 2xt \left(\frac{\partial u}{\partial x} \right)^2$$

has a solution $u = x/t$.

4. Analysis in Case 3.

Let us consider Case 3 in a little bit restricted setting. Let $p \in \{1,2,3,\ldots\}$: we consider the equation

$$t \frac{\partial u}{\partial t} = \alpha(t,x) + \lambda(t,x)u + (\beta(t,x) + x^p c(t,x)) \left(x \frac{\partial u}{\partial x} \right) + R_2 \left(t,x,u,x \frac{\partial u}{\partial x} \right) \tag{4.1}$$

where $\alpha(t,x)$, $\lambda(t,x)$, $\beta(t,x)$ and $c(t,x)$ are continuous functions on $[0,T_0] \times D_{R_0}$ that are holomorphic in x for any fixed t and satisfy

$$\sup_{x \in D_{R_0}} |\alpha(t,x)| = O(\mu(t)) \quad (\text{as } t \longrightarrow +0), \tag{4.2}$$

$$\sup_{x \in D_{R_0}} |\beta(t,x)| = O(\mu(t)) \quad (\text{as } t \longrightarrow +0), \tag{4.3}$$

$$c(0,0) \neq 0, \tag{4.4}$$

and $R_2(t,x,u,v)$ is the same as in (3.1). In this case, equations of this type were studied by Chen–Luo–Zhang [**6**], Luo–Chen–Zhang [**12**] and Bacani–Tahara [**2**].

By applying the change of variable $x \longrightarrow e^{i\theta}x$ in equation (4.1) we see that $x^p c(t,x)$ is transformed into $x^p(e^{ip\theta}c(t,e^{i\theta}x))$ and so by taking θ suitably we have the condition: $e^{ip\theta}c(0,0) < 0$. Hence, without loss of generality we may assume

$$c(0,0) < 0 \tag{4.5}$$

from the first. For simplicity, we suppose this condition from now.

As to the existence of a solution, we know a unique solvability result. In order to state the existence result, we prepare some notations: for $T > 0$, $R > 0$, $0 < \theta < \pi/2p$ and $r > 0$ we set

$$
\begin{aligned}
&S = S(\theta, R) = \{x \in \mathbb{C}\,;\, 0 < |x| < R, |\arg x| < \theta\}, \\
&d_S(x) = \min\{\log(R/|x|), \theta - |\arg x|\}, \\
&W_{T,R,\theta,r} = \{(t,x) \in (0,T) \times S\,;\, \varphi(t)/r < d_S(x)\}.
\end{aligned}
$$

Then, by Theorem 8.1 in [**2**] we have

THEOREM 4.1. *Suppose the conditions* (4.2), (4.3) *and* (4.5). *If* $\mathrm{Re}\lambda(0,0) < 0$ *holds, there are* $T > 0$, $R > 0$, $0 < \theta < \pi/2p$, $r > 0$ *and* $M > 0$ *such that equation* (4.1) *has a unique solution* $u_0(t,x) \in \mathscr{X}_1(W_{T,R,\theta,r})$ *satisfying*

$$
|u_0(t,x)| \le M\mu(t) \quad \text{and} \quad \left| x\frac{\partial u_0}{\partial x}(t,x) \right| \le M\mu(t)
$$

on $W_{T,R,\theta,r}$.

REMARK 4.2. (1) In the above theorem we have supposed (4.5), but it is only for simplicity. Even in the case of (4.4) only, we can get the same result as Theorem 4.1 by replacing the definition of $S(\theta, R)$ by

$$
S(\theta, R) = \{x \in \mathbb{C}\,;\, 0 < |x| < R, |\arg x + (\arg c(0,0) - \pi)/p| < \theta\}.
$$

(2) In general, the existence domain of the solution $u_0(t,x)$ depends on the argument of $c(0,0)$. This fact can be illustrated by the following example. Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda < 0$, $c \in \mathbb{C} \setminus \{0\}$, and let us consider

$$
t\frac{\partial u}{\partial t} = tx + \lambda u + cx^2 \frac{\partial u}{\partial x}.
$$

In this case, we have $\lambda(t,x) = \lambda$, $c(t,x) = c$, $p = 1$ and

$$
u_0(t,x) = \frac{t}{c}\int_0^1 \frac{\xi^{-\lambda}}{\log \xi + 1/(cx)} d\xi
$$

which is holomorphic on $\mathbb{C}_t \times (\mathbb{C}_x \setminus \{x\,;\, cx \ge 0\})$. We note that $u_0(t,x)$ is not well-defined on $\{x\,;\, cx \ge 0\}$.

4.1. Uniqueness result in Case 3.

The following theorem is the main result of this section.

THEOREM 4.3. *Suppose the conditions* (4.2), (4.3), (4.5) *and* $\mathrm{Re}\lambda(0,0) < 0$. *Let* $u(t,x) \in \mathscr{X}_1((0,T) \times S(\theta, R))$ *be a solution of* (4.1) *with* $T > 0$, $\theta > 0$ *and* $R > 0$. *If* $u(t,x)$ *satisfies*

$$
\varlimsup_{\eta \to +0} \left[\lim_{\sigma \to +0} \left(\frac{1}{\eta^2} \sup_{(0,\sigma) \times S(\eta\theta, \eta R)} |u(t,x)| \right) \right] = 0, \tag{4.6}
$$

we have $u(t,x) = u_0(t,x)$ *on* $(0,T_1) \times S(\theta_1, R_1)$ *for some* $T_1 > 0$, $\theta_1 > 0$ *and* $R_1 > 0$, *where* $u_0(t,x)$ *is the solution obtained in Theorem* 4.1.

Corollary 4.4. *Suppose the conditions* (4.2), (4.3), (4.5) *and* $\mathrm{Re}\lambda(0,0)<0$. *Let* $u(t,x)\in\mathscr{X}_1((0,T)\times S(\theta,R))$ *be a solution of* (4.1). *If* $u(t,x)$ *satisfies*

$$\lim_{t\to+0}\Bigl(\sup_{x\in S(\theta,R)}|u(t,x)|\Bigr)=0,$$

we have $u(t,x)=u_0(t,x)$ *on* $(0,T_1)\times S(\theta_1,R_1)$ *for some* $T_1>0$, $\theta_1>0$ *and* $R_1>0$.

Remark 4.5. (1) In the case $\mathrm{Re}\lambda(0,0)>0$ we have the following counter example: the equation

$$t\,\frac{\partial u}{\partial t}=2u-x^2\,\frac{\partial u}{\partial x}+\frac{x^2t}{(1-t)}\frac{\partial u}{\partial x}$$

has a trivial solution $u\equiv 0$, a nontrivial solution $u=t^2$ and a family of solutions

$$u=\frac{c\,te^{-1/x}}{1-t}$$

with an arbitrary constant c. In this case we have $p=1$, $\lambda(t,x)=2$, $c(t,x)=-1$, $\beta(t,x)=xt/(1-t)$, $R_2\equiv 0$ and $\mu(t)=t$.

(2) In the case $\mathrm{Re}\lambda(0,0)=0$ we have the following counter example: the equation

$$t\,\frac{\partial u}{\partial t}=-x^2\,\frac{\partial u}{\partial x}+u^2+\left(x\frac{\partial u}{\partial x}\right)^2$$

has a trivial solution $u\equiv 0$ and a family of nontrivial solution

$$u=\frac{1}{c-\log t}$$

with an arbitrary constant c.

(3) We note: the equation

$$t\,\frac{\partial u}{\partial t}=-u-x^2\,\frac{\partial u}{\partial x}+t\left(x\frac{\partial u}{\partial x}\right)^2$$

has a trivial solution $u\equiv 0$ and a nontrivial solution $u=x/t$. This shows that even in the case $\mathrm{Re}\lambda(0,0)<0$, in order to get a uniqueness result we need some condition on the behavior of $u(t,x)$ (as $t\longrightarrow+0$). However, unfortunately the author does not know whether our assumption (4.6) is reasonable or not: he has no good examples.

4.2. Proof of Theorem 4.2.

Let $u(t,x)\in\mathscr{X}_1((0,\sigma_0)\times S(\theta_0,R_0))$ be a solution of (4.1) satisfying (4.6) (with θ and R replaced by θ_0 and R_0, respectively). We may suppose: $0<\theta_0<\pi/2p$. Set

$$w(t,x)=u(t,x)-u_0(t,x),$$

where $u_0(t,x)$ is the solution obtained in Theorem 4.1. We set $v_0(t,x)=x(\partial u_0/\partial x)(t,x)$. By taking σ_0, θ_0 and R_0 sufficiently small we may suppose that $u_0(t,x)$ and $v_0(t,x)$ are defined on $(0,\sigma_0)\times S(\theta_0,R_0)$ and satisfy $|u_0(t,x)|\le M\mu(t)$ and $|v_0(t,x)|\le M\mu(t)$ on $(0,\sigma_0)\times S(\theta_0,R_0)$. Then, $w(t,x)$ satisfies

$$\overline{\lim_{\eta\to+0}}\left[\lim_{\sigma\to+0}\left(\frac{1}{\eta^2}\sup_{(0,\sigma)\times S(\eta\theta_0,\eta R_0)}|w(t,x)|\right)\right]=0 \tag{4.7}$$

and a partial differential equation

$$\begin{aligned} t\frac{\partial w}{\partial t}&=\lambda(t,x)w+(\beta(t,x)+x^p c(t,x))\left(x\frac{\partial w}{\partial x}\right)\\ &\quad+a_1\left(t,x,w,x\frac{\partial w}{\partial x}\right)w+b_1\left(t,x,w,x\frac{\partial w}{\partial x}\right)\left(x\frac{\partial w}{\partial x}\right), \end{aligned} \tag{4.8}$$

where $a_1(t,x,w,q)$ and $b_1(t,x,w,q)$ are suitable functions satisfying

$$\begin{aligned} &a_1(t,x,w,q)w+b_1(t,x,w,q)q\\ &\quad=R_2(t,x,w+u_0(t,x),q+v_0(t,x))-R_2(t,x,u_0(t,x),v_0(t,x)). \end{aligned}$$

We may suppose that $a_1(t,x,w,q)$ and $b_1(t,x,w,q)$ belong to $\mathscr{X}_0(\Omega_0)$ with $\Omega_0=[0,\sigma_0]\times S(\theta_0,R_0)\times D_{\rho_1}\times D_{\rho_1}$ for some $\rho_1>0$. In addition, we have the properties:

$$\begin{aligned} &|\beta(t,x)|\le B\mu(t)\quad\text{on } (0,\sigma_0)\times D_{R_0},\\ &|a_1(t,x,w,q)|\le A_0\mu(t)+A_1|w|+A_2|q|\quad\text{on } \Omega_0,\\ &|b_1(t,x,w,q)|\le B_0\mu(t)+B_1|w|+B_2|q|\quad\text{on } \Omega_0 \end{aligned}$$

for some $B>0$, $A_i>0$ $(i=0,1,2)$ and $B_i>0$ $(i=0,1,2)$. Without loss of generality we may suppose

$$\mathrm{Re}\lambda(t,x)<-2a\quad\text{on } [0,\sigma_0]\times D_{R_0}$$

for some $a>0$. Recall that we have supposed $c(0,0)<0$. Thus, to prove Theorem 4.3 it is sufficient to show the following result.

PROPOSITION 4.6. *In the above situation, we have $w(t,x)=0$ on $(0,T_1)\times S(\theta_1,R_1)$ for some $T_1>0$, $\theta_1>0$ and $R_1>0$.*

Before the proof, we note

LEMMA 4.7. *If a holomorphic function $f(x)$ on $S(\theta,R)$ satisfies*

$$\sup_{S(\eta\theta,\eta R)}|f(x)|=o(\eta^m)\quad(\text{as } \eta\longrightarrow+0)$$

for some $m\ge 1$, we have

$$\sup_{S(\eta\theta,\eta R)}|x(d/dx)f(x)|=o(\eta^{m-1})\quad(\text{as } \eta\longrightarrow+0).$$

Proof. By the assumption, for any $\epsilon > 0$ there is an $\eta_0 \in (0,1)$ such that

$$|f(x)| \le \epsilon\eta^m \quad \text{on } S(\eta\theta, \eta R), \quad 0 < \eta < \eta_0.$$

Take any $0 < \eta < \eta_0$ and fix it. Set $d(x) = \min\{\eta\theta - |\arg x|, \log(\eta R) - \log|x|\}$ for $x \in S(\eta\theta, \eta R)$. Then, by Nagumo's lemma in a sectorial domain (see [**2**, Lemma 4.2]) we have

$$|x(d/dx)f(x)| \le \frac{\epsilon\eta^m}{d(x)} \quad \text{on } S(\eta\theta, \eta R).$$

If $x \in S((\eta/2)\theta, (\eta/2)R)$, we have

$$\begin{aligned} &\eta\theta - |\arg x| > \eta\theta - (\eta/2)\theta = (\eta/2)\theta \ge \min\{(\eta/2)\theta, \log 2\}, \\ &\log(\eta R) - \log|x| \ge \log(\eta R) - \log((\eta/2)R) = \log 2 \ge \min\{(\eta/2)\theta, \log 2\} \end{aligned}$$

and so $d(x) \ge \min\{(\eta/2)\theta, \log 2\}$. If $\eta > 0$ is sufficiently small, we have $d(x) \ge (\eta/2)\theta$, and so

$$|x(d/dx)f(x)| \le \frac{\epsilon\eta^m}{(\eta/2)\theta} = \frac{2^m\epsilon}{\theta}(\eta/2)^{m-1} \quad \text{on } S((\eta/2)\theta, (\eta/2)R).$$

This proves the result in Lemma 4.7. □

Proof of Proposition 4.5. Let us prove Proposition 4.6 step by step.

Step 1: We set $q(t,x) = x(\partial w/\partial x)(t,x)$, and

$$\begin{aligned} a(t,x) &= a_1(t,x,w(t,x),q(t,x)), \\ b(t,x) &= \beta(t,x) + b_1(t,x,w(t,x),q(t,x)) : \end{aligned}$$

we may suppose that these functions belong to $\mathscr{X}_0((0,\sigma_0) \times S(\theta_0, R_0))$. By (4.8) we have the relation

$$t\frac{\partial w}{\partial t} - x(b(t,x) + x^p c(t,x))\frac{\partial w}{\partial x} = (\lambda(t,x) + a(t,x))w. \tag{4.9}$$

By applying $x(\partial/\partial x)$ to (4.9) we have

$$t\frac{\partial q}{\partial t} - x(b(t,x) + x^p c(t,x))\frac{\partial q}{\partial x} = \gamma(t,x)w + (\lambda(t,x) + a(t,x) + \ell(t,x))q, \tag{4.10}$$

where

$$\begin{aligned} \gamma(t,x) &= x(\partial\lambda/\partial x)(t,x) + x(\partial a/\partial x)(t,x), \\ \ell(t,x) &= x(\partial b/\partial x)(t,x) + x(\partial(x^p c)/\partial x)(t,x) : \end{aligned}$$

these are also functions belonging to $\mathscr{X}_0((0,\sigma_0) \times S(\theta_0, R_0))$. For $0 < \sigma_1 < \sigma_0$ and $0 < \eta < 1$ we set

$$A = \sup_{(0,\sigma_1)\times S(\eta\theta_0,\eta R_0)} |a(t,x)|,$$
$$\Gamma = \sup_{(0,\sigma_1)\times S(\eta\theta_0,\eta R_0)} |\gamma(t,x)|,$$
$$L = \sup_{(0,\sigma_1)\times S(\eta\theta_0,\eta R_0)} |\ell(t,x)|.$$

We set also

$$r_1 = \sup_{(0,\sigma_1)\times S(\eta\theta_0,\eta R_0)} |w(t,x)|, \quad r_2 = \sup_{(0,\sigma_1)\times S(\eta\theta_0,\eta R_0)} |q(t,x)|.$$

By (4.7) and by the same argument as in the proof of Lemma 2.6 we have

Lemma 4.8. *By taking* $\sigma_1 > 0$ *and* $\eta > 0$ *sufficiently small we have the following conditions*: $A + L < a$,

$$\delta = (B + B_0)\varphi(\sigma_1) + \left(\frac{B_1}{a} + \frac{B_2\Gamma}{a^2}\right) r_1 + \frac{B_2}{a} r_2 < \log 2,$$

and $0 < \sin^{-1}(2\delta) < \min\{\eta\theta_0/12, \pi/6p\}$.

Step 2: We take $\sigma_1 > 0$ and $\eta > 0$ as in Lemma 4.8, and fix them. After that, we take $0 < \sigma < \sigma_1$ and $0 < R < \eta R_0$ sufficiently small so that

$$\epsilon_1 = \sup_{(0,\sigma)\times S(\eta\theta_0,R)} |\arg(-c(t,x))| < \min\{p(\eta\theta_0)/6, \pi/6\}. \tag{4.11}$$

Since $\arg(-c(0,0)) = 0$ holds, this is possible.

We take such $\sigma > 0$ and $R > 0$ and fix them. Set $\theta = \eta\theta_0$. Then, we have $\epsilon_1/p < \min\{\theta/6, \pi/6p\}$.

Step 3: Take any $t_0 \in (0,\sigma)$ and $\xi \in S(\theta,R)$; for a while we fix them.

Let us consider the initial value problem

$$t\,\frac{dx}{dt} = -x(b(t,x) + x^p c(t,x)), \quad x(t_0) = \xi. \tag{4.12}$$

Here, we regard $b(t,x)$ and $c(t,x)$ as functions in $\mathscr{X}_0((0,\sigma)\times S(\theta,R))$. Let $x(t)$ be the unique solution in a neighborhood of $t = t_0$. Let $(t_\xi, t_0]$ be the maximal interval of the existence of this solution. Set

$$w^*(t) = w(t,x(t)), \quad q^*(t) = q(t,x(t)).$$

Lemma 4.9. (1) *We have* $x(t) \neq 0$ *on* $(t_\xi, t_0]$.

(2) *For any* (t_1,τ) *satisfying* $t_\xi < t_1 < \tau \le t_0$ *we have*

$$|w^*(\tau)| \le \left(\frac{t_1}{\tau}\right)^a |w^*(t_1)|, \tag{4.13}$$

$$|q^*(\tau)| \le \left(\frac{t_1}{\tau}\right)^a \bigl(\Gamma|w^*(t_1)|\log(\tau/t_1) + |q^*(t_1)|\bigr). \tag{4.14}$$

(3) *For any* $t_1 \in (t_\xi, t_0]$ *we have*

$$\left| \int_{t_1}^{t_0} b(\tau, x(\tau)) \frac{d\tau}{\tau} \right| \leq \delta,$$

where δ *is the one in Lemma* 4.8.

Proof. If $x(t_1) = 0$ holds for some $t_1 \in (t_\xi, t_0]$, $x(t)$ is a solution of

$$t\frac{dx}{dt} = -x(b(t,x) + x^p c(t,x)), \quad x(t_1) = 0.$$

Since $x \equiv 0$ is also a solution of this initial value problem, by the uniqueness of the solution we have $x(t) \equiv 0$ and so $\xi = x(t_0) = 0$. This contradicts the condition $\xi \in S(\theta, R)$ (this means $\xi \neq 0$). This proves (1).

By applying the same argument as in the proof of Lemma 2.7 to (4.9) and (4.10) we have the estimates in (2). By using (4.13) and (4.14) we can show

$$\begin{aligned} &\left| \int_{t_1}^{t_0} b(\tau, x(\tau)) \frac{d\tau}{\tau} \right| \\ &\quad \leq (B + B_0)(\varphi(t_0) - \varphi(t_1)) + \left(\frac{B_1}{a} + \frac{B_2 \Gamma}{a^2} \right) r_1 + \frac{B_2}{a} r_2 \end{aligned}$$

in the same way as in the proof of Lemma 2.8. Therefore, by combining this with Lemma 4.8 we have the result (3). □

Lemma 4.10. *We set*

$$\phi(t) = \exp\left[-\int_t^{t_0} b(\tau, x(\tau)) \frac{d\tau}{\tau} \right], \quad t_\xi < t < t_0.$$

Then, we have $1/2 \leq |\phi(t)| \leq 2$ *on* $(t_\xi, t_0]$ *and*

$$\theta_\phi = \sup_{(t_\xi, t_0]} |\arg \phi(t)| < \min\{\theta/12, \pi/6p\}. \tag{4.15}$$

Proof. By (3) of Lemma 4.9 and the condition $\delta < \log 2$ (by Lemma 4.8) we have $|\phi(t)| \leq e^\delta < e^{\log 2} = 2$. Similarly, we have $1/|\phi(t)| \leq e^\delta \leq 2$. This proves the first part. Since

$$|\phi(t) - 1| \leq \sum_{m \geq 1} \frac{1}{m!} \left| \int_t^{t_0} b(\tau, x(\tau)) \frac{d\tau}{\tau} \right|^m \leq \sum_{m \geq 1} \frac{\delta^m}{m!} \leq \delta \sum_{m \geq 0} \frac{\delta^m}{m!} = \delta e^\delta < 2\delta,$$

we have $\phi(t) \in \{z \in \mathbb{C}\,;\, |z - 1| < 2\delta\}$: this yields $\sin|\arg \phi(t)| < 2\delta$. Hence, we have $\sin \theta_\phi \leq 2\delta$, that is, $\theta_\phi \leq \sin^{-1}(2\delta)$. By Lemma 4.8 and $\theta = \eta\theta_0$ (in Step 2) we have $\theta_\phi < \min\{\theta/12, \pi/6p\}$. This proves (4.15). □

Step 4: Let $t_\xi < t_1 < t_0$. By (4.12) we have

$$t\frac{d}{dt}(\phi(t)x(t)) = -(\phi(t)x(t))^{p+1}\frac{c(t,x(t))}{\phi(t)^p}.$$

Since $x(t) \neq 0$ on $(t_\xi, t_0]$, we have

$$\frac{d}{dt}\left(\frac{-1/p}{(\phi(t)x(t))^p}\right) = -\frac{c(t,x(t))}{\phi(t)^p} \times \frac{1}{t}$$

and so by integrating this from t_1 to t_0 we have

$$\frac{-1/p}{(\phi(t_0)x(t_0))^p} - \frac{-1/p}{(\phi(t_1)x(t_1))^p} = -\int_{t_1}^{t_0} \frac{c(\tau,x(\tau))}{\phi(\tau)^p}\frac{d\tau}{\tau},$$

that is,

$$\frac{1}{(\phi(t_1)x(t_1))^p} = \frac{1}{\xi^p} - p\int_{t_1}^{t_0} \frac{c(\tau,x(\tau))}{\phi(\tau)^p}\frac{d\tau}{\tau}.$$

Hence, by solving $x(t_1)$ we have the expression:

$$x(t_1) = \frac{\xi/\phi(t_1)}{\left(1 - p\xi^p \int_{t_1}^{t_0}\big(c(\tau,x(\tau))/\phi(\tau)^p\big)(d\tau/\tau)\right)^{1/p}}, \quad t_\xi < t_1 \leq t_0. \tag{4.16}$$

LEMMA 4.11. *We have the following properties.*
(1) *For any $t_1 \in (t_\xi, t_0]$ we have*

$$|\xi/\phi(t_1)| \leq 2|\xi| \quad \textit{and} \quad |\arg(\xi/\phi(t_1))| \leq |\arg\xi| + \theta_\phi.$$

(2) *If $p|\arg\xi| + \epsilon_1 + p\theta_\phi \leq \pi/2$, we have*

$$\left|\arg\left(-p\xi^p\int_{t_1}^{t_0}\frac{c(\tau,x(\tau))}{\phi(\tau)^p}\frac{d\tau}{\tau}\right)\right| \leq p|\arg\xi| + \epsilon_1 + p\theta_\phi.$$

(3) *If $p|\arg\xi| + \epsilon_1 + p\theta_\phi \leq \pi/2$, for any $t_1 \in (t_\xi, t_0]$ we have*

$$\frac{|\xi|/2}{\big(1 + p|\xi|^p C_0 2^p \log(t_0/t_1)\big)^{1/p}} \leq |x(t_1)| \leq 2|\xi|, \tag{4.17}$$

$$|\arg x(t_1)| \leq 2|\arg\xi| + 2\theta_\phi + \epsilon_1/p, \tag{4.18}$$

where C_0 is a constant satisfying $|c(t,x)| \leq C_0$ on $(0,\sigma) \times S(\theta,R)$.

PROOF. (1) follows from Lemma 4.10. By (4.11) and (4.15) we have $|\arg(-c(t,x))| \leq \epsilon_1$ and $|\arg(1/\phi(t)^p)| \leq p\theta_\phi$. Therefore, we have

$$\left|\arg\left(-p\xi^p\frac{c(\tau,x(\tau))}{\phi(\tau)^p}\right)\right| \leq p|\arg\xi| + \epsilon_1 + p\theta_\phi.$$

If $p|\arg\xi| + \epsilon_1 + p\theta_\phi \leq \pi/2$ holds, the set $\{z \in \mathbb{C}\setminus\{0\}\,;\, |\arg z| \leq p|\arg\xi| + \epsilon_1 + p\theta_\phi\}$ is closed with respect to the addition. This proves (2).

Let us show (3). We know that $|\xi/\phi(t_1)| \geq |\xi|/2$. Since

$$\begin{aligned}\left|1 - p\xi^p \int_{t_1}^{t_0} \frac{c(\tau, x(\tau))}{\phi(\tau)^p} \frac{d\tau}{\tau}\right| &\leq 1 + p|\xi|^p \int_{t_1}^{t_0} \frac{|c(\tau, x(\tau))|}{|\phi(\tau)|^p} \frac{d\tau}{\tau} \\ &\leq 1 + p|\xi|^p \int_{t_1}^{t_0} C_0 2^p \frac{d\tau}{\tau} = 1 + p|\xi|^p C_0 2^p \log(t_0/t_1),\end{aligned}$$

we have the first inequality of (4.17).

If $p|\arg\xi| + \epsilon_1 + p\theta_\phi \leq \pi/2$ holds, by (2) we have

$$\mathrm{Re}\left(-p\xi^p \int_{t_1}^{t_0} \frac{c(\tau, x(\tau))}{\phi(\tau)^p} \frac{d\tau}{\tau}\right) \geq 0 \tag{4.19}$$

and so we have

$$\mathrm{Re}\left(1 - p\xi^p \int_{t_1}^{t_0} \frac{c(\tau, x(\tau))}{\phi(\tau)^p} \frac{d\tau}{\tau}\right) \geq 1,$$

which yields

$$\left|\left(1 - p\xi^p \int_{t_1}^{t_0} \frac{c(\tau, x(\tau))}{\phi(\tau)^p} \frac{d\tau}{\tau}\right)^{1/p}\right| \geq 1.$$

By combining this with (1) we have $|x(t_1)| \leq 2|\xi|$.

Similarly, by (4.19) and the result (2) we have

$$\left|\arg\left(1 - p\xi^p \int_{t_1}^{t_0} \frac{c(\tau, x(\tau))}{\phi(\tau)^p} \frac{d\tau}{\tau}\right)\right| \leq p|\arg\xi| + \epsilon_1 + p\theta_\phi.$$

Hence, we have

$$\begin{aligned}&|\arg x(t_1)| \\ &\quad \leq |\arg\xi| + |\arg\phi(t_1)| + \frac{1}{p}\left|\arg\left(1 - p\xi^p \int_{t_1}^{t_0} \frac{c(\tau, x(\tau))}{\phi(\tau)^p} \frac{d\tau}{\tau}\right)\right| \\ &\quad \leq |\arg\xi| + \theta_\phi + \frac{1}{p}(p|\arg\xi| + \epsilon_1 + p\theta_\phi) = 2|\arg\xi| + 2\theta_\phi + \epsilon_1/p.\end{aligned}$$

This proves (4.18). □

Step 5: We recall that $0 < \theta < \theta_0 < \pi/2p$ holds. By summing up we have

Lemma 4.12. *If $\xi \in S(\theta/3, R/3)$, we have $t_\xi = 0$.*

Proof. Let $\xi \in S(\theta/3, R/3)$. Suppose that $t_\xi > 0$, and let us derive a contradiction. We note:

$$\begin{aligned}p|\arg\xi| + \epsilon_1 + p\theta_\phi &< p(\theta/3) + p\min\{\theta/6, \pi/6p\} + p\min\{\theta/12, \pi/6p\} \\ &< p(\pi/6p) + p(\pi/6p) + p(\pi/6p) = \pi/2.\end{aligned}$$

Therefore, by (3) of Lemma 4.11 we have

$$R_1 = \frac{|\xi|/2}{\left(1 + p|\xi|^p C_0 2^p \log(t_0/t_\xi)\right)^{1/p}} \le |x(t_1)| \le 2|\xi| < 2R/3, \tag{4.20}$$

$$|\arg x(t_1)| \le 2|\arg \xi| + 2\theta_\phi + \epsilon_1/p \le 2(\theta/3) + 2\theta_\phi + \epsilon_1/p. \tag{4.21}$$

If we set $\theta_1 = 2(\theta/3) + 2\theta_\phi + \epsilon_1/p$, we have $\theta_1 < 2(\theta/3) + 2(\theta/12) + \theta/6 = \theta$ and so we see that the set $K = \{x \in S(\theta, R)\,;\, R_1 \le |x| \le 2R/3, |\arg x| \le \theta_1\}$ is a compact subset of $S(\theta, R)$.

By (4.20) and (4.21) we have $x(t_1) \in K$ for any $t_1 \in (t_\xi, t_0]$. Therefore, we can conclude that $x(t)$ can be extended to an interval $(t_\xi - \varepsilon, t_0]$ for some $\varepsilon > 0$. This contradicts the condition that $(t_\xi, t_0]$ is a maximal interval of the existence of the solution $x(t)$. □

Step 6: Since $t_\xi = 0$, by (4.13) with $\tau = t_0$ we have

$$|w^*(t_0)| \le \left(\frac{t_1}{t_0}\right)^a |w^*(t_1)| \le \left(\frac{t_1}{t_0}\right)^a r_1$$

for any $t_1 \in (0, t_0)$. Since $r_1 > 0$ is independent of t_1, by letting $t_1 \longrightarrow +0$ we have $w^*(t_0) = 0$. Since $w^*(t_0) = w(t_0, \xi)$, we have $w(t_0, x) = 0$ for any $x \in S(\theta/3, R/3)$. Since $t_0 \in (0, \sigma)$ is taken arbitrarily, we have $w(t, x) = 0$ on $(0, \sigma) \times S(\theta/3, R/3)$.

This completes the proof of Proposition 4.6 □

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