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Self-avoiding walk on the complete graph

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Abstract. There is an extensive literature concerning self-avoiding walk on infinite graphs, but the subject is relatively undeveloped on finite graphs. The purpose of this paper is to elucidate the phase transition for self-avoiding walk on the simplest finite graph: the complete graph. We make the elementary observation that the susceptibility of the self-avoiding walk on the complete graph is given exactly in terms of the incomplete gamma function. The known asymptotic behaviour of the incomplete gamma function then yields a complete description of the finite-size scaling of the self-avoiding walk on the complete graph. As a basic example, we compute the limiting distribution of the length of a self-avoiding walk on the complete graph, in subcritical, critical, and supercritical regimes. This provides a prototype for more complex unsolved problems such as the self-avoiding walk on the hypercube or on a high-dimensional torus.

1. Introduction and results.

1.1. Self-avoiding walk.

The self-avoiding walk is a mathematical model of interest in combinatorics, in probability, in statistical mechanics, and in polymer science [**10**]. A basic problem in the subject is to count the number of fixed-length non-self-intersecting paths in a transitive graph, starting from some fixed origin. The most common setting is to take the graph to be the d-dimensional integer lattice $\mathbb{Z}^d$ with nearest-neighbour edges. A simple subadditivity argument implies that the number c_N of self-avoiding walks on $\mathbb{Z}^d$ of length N satisfies $c_N^{1/N} \to \mu$ as $N \to \infty$, and also $c_N \geq \mu^N$ (for all N). The *connective constant* μ obeys $\mu \in [d, 2d-1]$, and in particular depends on the dimension d. The generating function $\chi_z = \sum_{N=0}^{\infty} c_N z^N$ of the sequence (c_N) is called the *susceptibility*, and it has a finite radius of convergence equal to the *critical value* $z_c = \mu^{-1}$. For $z \geq z_c$, the susceptibility is infinite and contains no information. Many of the most important mathematical problems for the self-avoiding walk remain unsolved, e.g., the precise asymptotic behaviour on $\mathbb{Z}^2$ or $\mathbb{Z}^3$ of c_N (as $N \to \infty$) or the susceptibility (as $z \uparrow z_c$).

On a finite graph, c_N is zero once N reaches the cardinality of the vertex set, so the susceptibility is a polynomial in z and hence is defined for all $z \in \mathbb{C}$. For large but finite graphs, it can be expected that the divergence of the susceptibility in the infinite setting will be reflected by different behaviour of the susceptibility above and below some *critical scaling window* of positive z values—a phase transition. It is further to be expected that on high-dimensional graphs this transition should share qualitative features with the phase transition for the Erdős–Rényi random graph.

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In this paper, we elucidate the phase transition for self-avoiding walk on the simplest finite graph, the complete graph. We observe that the self-avoiding walk is exactly solvable on the complete graph, in the sense that the susceptibility can be expressed exactly in terms of the upper incomplete gamma function. We use the known asymptotic behaviour of the incomplete gamma function to derive the asymptotic behaviour of the susceptibility. This leads to an identification of the critical scaling window for the phase transition, and of the different behaviours that occur below, above, and within the critical window. In particular, with respect to the probability distribution on the set of all self-avoiding walks (of any length) which assigns probability proportional to z^N to a walk of length N, we identify the different behaviours of the length of a random self-avoiding walk in the different regimes.

For self-avoiding walk on the complete graph, we show that the phase transition corresponds exactly to a transition in the asymptotic behaviour of the incomplete gamma function. Complete information and historical background for the asymptotic behaviour of the incomplete gamma function is provided in [**11**], from which the finite-size scaling of the susceptibility can be read off. The new results of [**11**] are precisely in the region of most interest—what they call the *transition region* and what for us is the critical scaling window—though we do not use anything like the full power of the results of [**11**]. This is an interesting example of a connection between a transition in the asymptotic behaviour of a special function and a phase transition in the finite-size scaling of a statistical mechanical model.

Although the complete graph lacks geometry, it is an example which serves as a prototype for high-dimensional finite graphs which do have interesting geometry, such as the hypercube $\{0,1\}^n$ with nearest-neighbour edges, or high-dimensional discrete tori. Percolation on the hypercube and high-dimensional tori has been well-studied [**1**], [**6**], and it would be of interest to develop the theory of self-avoiding walk on these graphs.

1.2. Exact solution for susceptibility.

Let K_n denote the complete graph on n vertices. An N-step self-avoiding walk on K_n is a sequence of $N+1$ distinct vertices in K_n. Let $\mathcal{S}_N^{(n)}$ denote the set of N-step self-avoiding walks on the complete graph K_n, started from some fixed origin. The cardinality of $\mathcal{S}_N^{(n)}$ is $c_0^{(n)} = 1$ and

$$c_N^{(n)} = (n-1)(n-2)\cdots(n-N) = \frac{(n-1)!}{(n-1-N)!} \qquad (N = 1, \dots, n-1). \tag{1.1}$$

The *susceptibility* is the generating function for $c_N^{(n)}$, namely the polynomial

$$\chi_z^{(n)} = \sum_{N=0}^{n-1} c_N^{(n)} z^N = (n-1)! z^{n-1} \sum_{m=0}^{n-1} \frac{1}{m! z^m}. \tag{1.2}$$

For $x \geq 0$ and $s > 0$, the upper incomplete gamma function is defined by

$$\Gamma(s,x) = \int_x^\infty t^{s-1} e^{-t} dt. \tag{1.3}$$

Analytic continuation in both variables s, x is possible. For $n \in \mathbb{N}$,

$$\Gamma(n,x) = (n-1)!e^{-x}\sum_{k=0}^{n-1}\frac{x^k}{k!}. \tag{1.4}$$

This gives an exact formula for the susceptibility:

$$\chi_z^{(n)} = z^{n-1}e^{1/z}\Gamma\left(n, \frac{1}{z}\right). \tag{1.5}$$

1.3. Random length.

For $z > 0$, we define a probability measure on $\mathcal{S}^{(n)} = \bigcup_{N=0}^{n-1}\mathcal{S}_N^{(n)}$ by assigning probability $z^{|\omega|}/\chi_z^{(n)}$ to $\omega \in \mathcal{S}^{(n)}$, where $|\omega|$ denotes the number of steps in ω. Expectation with respect to this measure is denoted $\mathbb{E}_z^{(n)}$. Given z and n, we define the random variable L to be the length $|\omega|$ of a walk ω drawn randomly from $\mathcal{S}^{(n)}$.

By definition, and by (1.5) and (1.3), the mean of L is

$$\mathbb{E}_z^{(n)}(L) = \frac{1}{\chi_z^{(n)}}\sum_{N=0}^{n-1} N c_N^{(n)} z^N = z\frac{d}{dz}\log\chi_z^{(n)} = (n-1) - \frac{1}{z} + \frac{1}{z\chi_z^{(n)}}. \tag{1.6}$$

Higher moments can also be computed exactly. For example, with $\ell = \mathbb{E}_z^{(n)}(L)$ and $\chi = \chi_z^{(n)}$,

$$\begin{aligned}\mathbb{E}_z^{(n)}(L^2) &= \frac{1}{\chi}\left(z\frac{d}{dz}\right)\left(z\frac{d}{dz}\chi\right) = \frac{1}{\chi}\left(z\frac{d}{dz}\right)(\chi\ell) = \ell^2 + z\frac{d}{dz}\ell \\ &= \ell^2 + \frac{1}{z}\left(1 - \frac{\ell}{\chi} - \frac{1}{\chi}\right), \end{aligned} \tag{1.7}$$

and hence the variance of L is

$$\mathrm{Var}_z^{(n)}(L) = \frac{1}{z}\left(1 - \frac{1}{\chi_z^{(n)}}(\mathbb{E}_z^{(n)}(L)+1)\right). \tag{1.8}$$

Let $z_t = ze^t$. The moment generating function of L is, by definition and by (1.5),

$$M_L(t) = \mathbb{E}_z^{(n)}e^{tL} = \frac{1}{\chi_z^{(n)}}\chi_{z_t}^{(n)} = \frac{z_t^{n-1}e^{1/z_t}\Gamma(n,1/z_t)}{z^{n-1}e^{1/z}\Gamma(n,1/z)} \qquad (t \in \mathbb{R}). \tag{1.9}$$

1.4. Results.

The asymptotic behaviour of the susceptibility is given by the following theorem. The gamma function is $\Gamma(s) = \Gamma(s,0)$. The complementary error function erfc in (1.12) is

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt \qquad (x \in \mathbb{R}), \tag{1.10}$$

and it extends to an entire analytic function on the complex plane. We use the notation $f_n \sim g_n$ to mean that $\lim f_n/g_n = 1$.

THEOREM 1.1. *Let $r > 1/2$ and $y_n = 1/z_n$. If eventually $y_n \geq n + n^r$ (subcritical case) then*

$$\chi_{z_n}^{(n)} = \frac{y_n}{y_n - n}\left(1 - \frac{y_n}{(y_n-n)^2} + O\left(\frac{y_n^2}{(y_n-n)^4}\right)\right). \tag{1.11}$$

Let $\tau_n, \tau \in \mathbb{R}$ with $\tau_n \to \tau$, and let $y_n = n + \tau_n n^{1/2}$ (critical case). Then

$$\chi_{z_n}^{(n)} \sim \sqrt{2\pi n}\, e^{\tau^2/2}\frac{1}{2}\mathrm{erfc}(2^{-1/2}\tau). \tag{1.12}$$

If eventually $y_n \leq n - n^r$ (supercritical case) then

$$\chi_{z_n}^{(n)} \sim y_n^{1-n} e^{y_n}\Gamma(n). \tag{1.13}$$

Special cases of the subcritical case are[1]

$$\chi_z^{(n)} \to \frac{s}{s-1} \qquad \left(z = \frac{1}{sn},\ s > 1\right), \tag{1.14}$$

$$\chi_z^{(n)} \sim \frac{1}{a}n^{1-q} \qquad \left(z = \frac{1}{n + an^q},\ q \in \left(\frac{1}{2}, 1\right)\right). \tag{1.15}$$

By Stirling's formula, special cases of the supercritical case are

$$\log \chi_z^{(n)} = \frac{1}{2}a^2 n^{2q-1} + \log\sqrt{2\pi n} + O(n^{3q-2}) + o(1) \quad \left(z = \frac{1}{n - an^q},\ q \in \left(\frac{1}{2}, 1\right)\right), \tag{1.16}$$

$$\chi_z^{(n)} \sim s\sqrt{2\pi n}\, e^{(|\log s| + s - 1)n} \qquad \left(z = \frac{1}{sn},\ s \in (0,1)\right). \tag{1.17}$$

The exponent in (1.17) is positive for $s < 1$, so there is exponential growth. For $q \in (1/2, 2/3)$, (1.16) yields the asymptotic relation $\chi_z^{(n)} \sim \sqrt{2\pi n}\, e^{a^2 n^{2q-1}/2}$, whereas for $q \in [2/3, 1)$ the error term $O(n^{3q-2})$ for the logarithm is not insignificant. It then follows from (1.8), and since $L < n$ by definition, that $\mathrm{Var}_z^{(n)}(L) \sim z^{-1}$ in the supercritical case.

The crossover between the critical and subcritical regimes, and between the critical and supercritical regimes, is given by an extension of (1.12), namely that as long as $|\tau_n| \leq O(n^{q-1/2})$ with $q \in [1/2, 2/3)$,

$$\chi_{z_n}^{(n)} \sim \sqrt{2\pi n}\, e^{\tau_n^2/2}\frac{1}{2}\mathrm{erfc}(2^{-1/2}\tau_n). \tag{1.18}$$

This reduces to $\chi_z^{(n)} \sim \sqrt{2\pi n}\, e^{a^2 n^{2q-1}/2}$ when $\tau_n = -an^{q-1/2}$ (since then $\mathrm{erfc}(2^{-1/2}\tau_n) \to 2$), and to (1.15) when $\tau_n = an^{q-1/2}$ (since then $\mathrm{erfc}(2^{-1/2}\tau_n) \sim \sqrt{2/\pi}\tau_n^{-1}e^{-\tau_n^2/2}$). The proof of (1.18) is included in the proof of Theorem 1.1.

From Theorem 1.1, it is straightforward to deduce the following asymptotic behaviour for the expected length, by inserting the formulas for the susceptibility into (1.6).

[1] In fact, (1.14) is an immediate consequence of the dominated convergence theorem, because $\lim_{n\to\infty}\sum_{N=0}^{n-1} n^{-N}c_N^{(n)}s^{-N} = (1-s^{-1})^{-1}$ since $n^{-N}c_N^{(n)} \leq 1$ and $\lim_{n\to\infty} n^{-N}c_N^{(n)} = 1$ for each N.

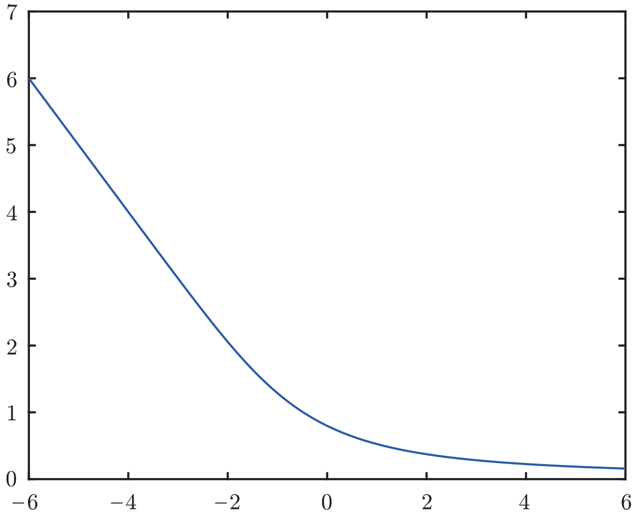

Figure 1. Graph of α_τ vs τ.

The second-order term in (1.11) is needed for the subcritical case of (1.20). The constant in the critical case of (1.20) is defined by

$$\alpha_\tau = -\tau + \sqrt{\frac{2}{\pi}}\, e^{-\tau^2/2}[\mathrm{erfc}(2^{-1/2}\tau)]^{-1}, \tag{1.19}$$

which is the strictly positive decreasing function of $\tau \in \mathbb{R}$ plotted in Figure 1. It obeys $\alpha_\tau \sim -\tau$ as $\tau \to -\infty$ and $\alpha_\tau \sim \tau^{-1}$ as $\tau \to +\infty$.

Corollary 1.2. *Let $r > 1/2$ and $\tau \in \mathbb{R}$. With the subcritical, critical, and supercritical cases as specified in Theorem* 1.1, *the expected length has the asymptotic behaviour, as $n \to \infty$,*

$$\mathbb{E}_{z_n}^{(n)}(L) \sim \begin{cases} \dfrac{n}{1/z_n - n} & (\textit{subcritical}) \\ \alpha_\tau n^{1/2} & (\textit{critical}) \\ n - \dfrac{1}{z_n} & (\textit{supercritical}). \end{cases} \tag{1.20}$$

Proof. Let $y = 1/z$ and $\delta = y - n$. For the subcritical case, by (1.6) and Theorem 1.1,

$$\begin{aligned} \mathbb{E}_z^{(n)}(L) &= n - 1 - y + y\chi^{-1} = -1 - \delta + \delta\left(1 + \frac{y}{\delta^2} + O\left(\frac{y^2}{\delta^4}\right)\right) \\ &= \frac{n}{\delta}\left(1 + O\left(\frac{y^2}{n\delta^2}\right)\right). \end{aligned} \tag{1.21}$$

Since $y^2 = n^2 + 2\delta n + \delta^2$, the error term on the right-hand side is bounded above by a multiple of $n\delta^{-2} + \delta^{-1} + n^{-1}$ and hence goes to zero as $n \to \infty$ since eventually $\delta^2 \geq n^{2r}$. This completes the proof of the subcritical case. The supercritical case follows from (1.6) and the fact that $z\chi \to \infty$ by (1.16) with $q = r$ (and monotonicity of χ in z). The critical case follows from (1.6) and (1.12). $\square$

Special cases of the subcritical and supercritical behaviour are:

$$\mathbb{E}_z^{(n)}(L) \sim \begin{cases} \dfrac{1}{s-1} & \left(z = \dfrac{1}{sn},\ s > 1\right) \\ \dfrac{1}{a} n^{1-q} & \left(z = \dfrac{1}{n + an^q}\right), \end{cases} \tag{1.22}$$

$$\mathbb{E}_z^{(n)}(L) \sim \begin{cases} an^q & \left(z = \dfrac{1}{n - an^q}\right) \\ n(1-s) & \left(z = \dfrac{1}{sn},\ s \in (0,1)\right). \end{cases} \tag{1.23}$$

The crossover from critical to subcritical or supercritical can be seen from (1.6) and (1.18). Indeed, if $1/z = n + \tau_n n^{1/2}$ with $|\tau_n| \leq O(n^{q-1/2})$ for $q \in [1/2, 2/3)$, then we obtain

$$E_z^{(n)}(L) = n - 1 - (n + \tau_n n^{1/2}) + (n + \tau_n n^{1/2})(\chi_z^{(n)})^{-1} \sim \alpha_{\tau_n} \sqrt{n}. \tag{1.24}$$

Since $\alpha_\tau \sim -\tau$ as $\tau \to -\infty$, this is consistent with the first case of (1.23), and since $\alpha_\tau \sim \tau^{-1}$ as $\tau \to +\infty$, it is consistent with the second case of (1.22).

The first case of (1.22) and the last case of (1.23) were proved in [**12**, Theorem 1.5], as was the case $\tau = 0$ of the critical case of (1.20) ($\alpha_0 = \sqrt{2/\pi}$ agrees with the constant α in [**12**]), by using a formula for the expected length in terms of a Poisson random variable rather than employing (1.5) as we do. A proof that $E_z^{(n)}(L) \asymp n^{1-q}$ in the second case of (1.22) was announced in [**13**] (the scaling of z on [**13**, p.185701-3] differs from ours by a factor n).

The next theorem gives the asymptotic distribution of L rescaled as suggested by Corollary 1.2. In its statement, G_p is a geometric random variable with parameter p, W_a is an exponential random variable with mean a^{-1}, and X_τ is the random variable (shown to exist in the proof) with moment generating function

$$M_{X_\tau}(t) = e^{-\tau t + t^2/2} \frac{\operatorname{erfc}(2^{-1/2}(\tau - t))}{\operatorname{erfc}(2^{-1/2}\tau)}. \tag{1.25}$$

Differentiation confirms that the constant α_τ in (1.20) is equal to $\alpha_\tau = EX_\tau = M'_{X_\tau}(0)$.

THEOREM 1.3. *Let $\tau \in \mathbb{R}$, $a > 0$, and $q \in (1/2, 1)$. As $n \to \infty$, the rescaled length converges in distribution as follows:*

$$L \Rightarrow G_{1-1/s} - 1 \qquad \left(z = \frac{1}{sn},\ s > 1\right), \tag{1.26}$$

$$\frac{L}{n^{1-q}} \Rightarrow W_a \qquad \left(z = \frac{1}{n + an^q}\right), \tag{1.27}$$

$$\frac{L}{n^{1/2}} \Rightarrow X_\tau \qquad \left(z = \frac{1}{n + \tau\sqrt{n}}\right), \tag{1.28}$$

$$\frac{L}{n^q} \Rightarrow a \qquad \left(z = \frac{1}{n - an^q}\right), \tag{1.29}$$

$$\frac{L}{n} \Rightarrow 1 - s \qquad \left(z = \frac{1}{sn},\ s \in (0,1)\right). \tag{1.30}$$

The limits (1.29)–(1.30) are special cases of the more general statement that in the supercritical case, with $\ell = \mathbb{E}_z^{(n)}(L)$,

$$\frac{L}{\ell} \Rightarrow 1 \qquad \text{(supercritical)}. \tag{1.31}$$

We will prove the more general statement in the proof of Theorem 1.3.

The proofs of Theorems 1.1 and 1.3 are given in Section 2. We have stated the results above in a simple manner in order to highlight the leading behaviour. Full asymptotic expansions could be obtained by using the expansions of [**11**] for the incomplete gamma function, but our focus is on the leading behaviour and we do not pursue higher precision here.

Note added. After the first version of this paper was posted on arXiv, the author was kindly informed by T. M. Garoni that the forthcoming paper [**3**] contains independently obtained proofs of many of our results, including Theorem 1.3. In [**3**], the distribution of X_τ is identified as that of $-\tau$ plus a standard normal random variable conditioned to exceed τ.

1.5. Critical behaviour.

Summary. The above results can be summarised by saying that there is a *critical scaling window* of z values of the form $n^{-1}(1 + O(n^{-1/2}))$ where the susceptibility is of order $n^{1/2}$, the expected length is of order $n^{1/2}$, and the limiting length has an interesting distribution. Values $z = 1/(sn)$ are *subcritical* when $s > 1$ (L remains bounded) and *supercritical* when $s < 1$ (L is of order n and the self-avoiding walk has positive density). Progression into the critical window from the subcritical side occurs for $z = n^{-1}(1 - n^{-p})$ with $p \in (0, 1/2)$. Similarly, progression out of the critical window to the supercritical side occurs for $z = n^{-1}(1 + n^{-p})$ with $p \in (0, 1/2)$. Both progressions involve power-law behaviour for L.

Percolation on finite graphs. It is instructive to compare our results with the situation for percolation on high-dimensional finite transitive graphs such as the complete graph, the hypercube, or discrete tori. Percolation on the complete graph—the Erdős–Rényi random graph—is of course a fundamental example in probability theory and combinatorics with an extensive literature, e.g., [**8**]. In [**1**], it was shown that the correct definition in the high-dimensional setting is to define the critical probability p_c as the solution to the equation $\chi(p_c) = \lambda V^{1/3}$, where $\chi(p)$ is the expected cluster size of a fixed vertex when the bond occupation probability is p, V is the number of vertices in the graph, and $\lambda > 0$ is an arbitrary fixed constant.[2] For the hypercube and high-dimensional discrete tori, it has been proved[3] [**1**], [**6**] that there is a phase transition with critical scaling window consisting of p values with $|p - p_c|$ of order $\Omega^{-1}V^{-1/3}$, where Ω is the degree ($\Omega = n$ and $V = 2^n$ for the hypercube, $\Omega = 2d$ and $V = r^d$ for a d-dimensional torus of period r). For the hypercube, p_c has an asymptotic expansion to all orders, $p_c \sim \sum_{j=0}^{\infty} a_j n^{-j}$, with rational coefficients a_j that are independent of λ (in particular, $a_1 = a_2 = 1$, $a_3 = 7/2$) [**7**]. We emphasise that the power $V^{1/3}$ is the correct power in the high-dimensional setting, but will not in general be correct for low-dimensional graphs: in [**1**, Section 3.4.2] it is proposed that $V^{1/3}$ corresponds more generally to $V^{(\delta-1)/(\delta+1)}$ where δ is the critical exponent for the magnetisation (with mean-field value $\delta = 2$).

Conjectured picture for self-avoiding walk. For the complete graph, n occurs both as the degree $\Omega = n - 1$ and as the number of vertices $V = n$, but the degree and volume will play distinct roles in other graphs. By analogy with the above picture for percolation, our results suggest that the natural definition[4] of the critical value for self-avoiding walk on a high-dimensional transitive graph with V vertices is the value z_c for which $\chi(z_c) = \lambda V^{1/2}$ with any fixed $\lambda > 0$.[5] Also, our results suggest that the critical scaling window consists of z values with $|z - z_c|$ of order $\Omega^{-1}V^{-1/2}$, where Ω is the degree. For the hypercube, the definition gives z_c according to $\chi_n(z_c) = \lambda 2^{n/2}$ and the scaling window is conjectured to occur for $|z - z_c|$ of order $n^{-1}2^{-n/2}$. For the discrete torus of period r in dimension $d > 4$, instead $\chi(z_c) = \lambda r^{d/2}$ with a scaling window with $|z - z_c|$ of order $(2d)^{-1}r^{-d/2}$. It would be of interest to prove that this conjectured picture does hold and to carry out an analysis similar to what has been done for percolation.

Literature. The idea that for z above a critical value the self-avoiding walk enters a dense phase with walk length of the order of the volume has been explored in previous papers. A sharp transition to a dense phase for self-avoiding walks diagonally crossing a square or higher-dimensional cube was proved in [**9**] and further studied in [**2**]. The existence of a dense phase was established for self-avoiding walk in a two-dimensional domain in [**4**]. The vanishing of the density as the critical point is approached from the supercritical side was proven in [**5**] for weakly self-avoiding walk on a 4-dimensional hierarchical lattice, with a logarithmic correction to the linear $(1-s)$ mean-field behaviour evident in the last case of (1.23). For finite graphs of large girth, a dense phase was proven to exist in [**12**] (where, as mentioned above, the complete graph was also considered). Forthcoming work on the complete graph announced in [**13**] subsequently appeared in [**3**] (see Note added at end of Section 1.4).

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[2] The utility and freedom to vary λ is discussed in [**1**]. There is no notion of "the" critical value on a finite graph, and any value in the critical window will do.

[3] In fact, some open questions remain concerning the supercritical phase on tori; see [**6**].

[4] This definition differs from the proposal in [**12**, Definition 1.2], which calls z_n *subcritical* if $\limsup \chi_{z_n}^{(n)} < \infty$, *supercritical* if $\liminf \chi_{z_n}^{(n)} = \infty$, and *critical* if z_n/s is subcritical when $s > 1$ and supercritical when $s < 1$. By (1.12), (1.14), and (1.17), this prescribes $z = 1/n$ as both critical and supercritical for the complete graph, and thus is insufficiently discerning.

[5] For the complete graph, since the function α_τ is a bijection of $\mathbb{R}$ onto $(0, \infty)$, by the critical case of (1.20) a choice of λ essentially specifies a choice of τ_λ such that the susceptibility at $z = 1/(n + \tau_\lambda n^{1/2})$ has the value $\lambda n^{1/2}$.

2. Proof of results.

Asymptotic formulas for the susceptibility are immediate consequences of the following asymptotic formulas for the incomplete gamma function, taken from [**11**]. Asymptotic expansions to all orders are given in [**11**], but we only need the statements in Lemma 2.1. The variable y_n represents $1/z_n$.

LEMMA 2.1. *Let $r > 1/2$. Suppose that eventually $y_n \geq n + n^r$ (subcritical case). Then*

$$\Gamma(n, y_n) = y_n^n e^{-y_n} \frac{1}{y_n - n} \left(1 - \frac{y_n}{(y_n - n)^2} + O\left(\frac{y_n^2}{(y_n - n)^4}\right)\right). \tag{2.1}$$

Let $\tau_n, \tau \in \mathbb{C}$ with $\tau_n \to \tau$, and let $y_n = n + \tau_n n^{1/2}$. Then

$$\Gamma(n, y_n) \sim \Gamma(n) \frac{1}{2} \mathrm{erfc}(2^{-1/2}\tau). \tag{2.2}$$

Suppose that eventually $y_n \leq n - n^r$ (supercritical case). Then

$$\Gamma(n, y_n) \sim \Gamma(n). \tag{2.3}$$

PROOF. The subcritical case follows from [**11**, (2.2)], together with the observation in [**11**, Appendix A] that [**11**, (2.2)] does hold for $y_n \geq n + n^r$ when $r > 1/2$. The critical case follows from [**11**, Theorem 1.1] (see also [**11**, Proposition 1.1][6]). The supercritical case follows from [**11**, (2.1)], again with the observation in [**11**, Appendix A] that [**11**, (2.1)] does hold under our hypothesis. In more detail for the last case, it follows from [**11**, (2.1)] and Stirling's formula that, with $\delta_n = n - y_n \geq n^r$,

$$\begin{aligned} \left| \frac{\Gamma(n, y_n)}{\Gamma(n)} - 1 \right| &\sim \frac{1}{n - y_n} y_n^n e^{-y_n} \frac{1}{(n-1)^{n-1} e^{-n+1} \sqrt{2\pi n}} \\ &\sim \frac{1}{\sqrt{2\pi}} \frac{n^{1/2}}{\delta_n} \left(1 - \frac{\delta_n}{n}\right)^n e^{\delta_n} \leq \frac{1}{\sqrt{2\pi}\, n^{r-1/2}}, \end{aligned} \tag{2.4}$$

and the right-hand side goes to zero since $r > 1/2$. □

PROOF OF THEOREM 1.1. The asymptotic formulas (1.11)–(1.13) for the susceptibility follow from (1.5) and Lemma 2.1. For (1.12), we also use Stirling's formula and the elementary fact that $(1 + \tau_n n^{-1/2})^{-n} e^{\tau_n n^{1/2}} \to e^{\tau^2/2}$. The extension (1.18) of the critical case is a consequence of the fact that (2.2) continues to hold in the extended setting, by [**11**, Proposition 1.1]. □

It remains to prove Theorem 1.3. The proof uses the fact that convergence in distribution is a consequence of convergence of moment generating functions on an open interval containing 0, except for the critical case (1.28) where we use characteristic functions instead.

[6] This is Proposition 1.2 in the arXiv version (https://arxiv.org/pdf/1803.07841.pdf) of [**11**].

PROOF OF (1.26): $z = 1/(sn)$ with $s > 1$ (subcritical). Fix $s > 1$ and choose $t_0 > 0$ to obey $se^{-t_0} = 1$. Let $t < t_0$; then $se^{-t} > 1$. Let $y = 1/z$ and $y_t = ye^{-t}$. By (1.9) and (1.11), the moment generating function of L obeys

$$M_L(t) \sim \frac{y_t}{y_t - n} \frac{y - n}{y} = e^{-t} \frac{s-1}{se^{-t} - 1} = \frac{1 - 1/s}{1 - e^t/s}. \tag{2.5}$$

The right-hand side is the moment generating function for $G_{1-1/s} - 1$. □

PROOF OF (1.27): $z = 1/(n + an^q)$ (subcritical near critical). Fix $a > 0$, $q \in (1/2, 1)$, and let $t < a$. Let $y = 1/z$ and $y_t = ye^{-t/n^{1-q}}$. By definition, $y - n = an^q$ and

$$y_t - n \sim (n + an^q)(1 - tn^{q-1}) - n \sim (a - t)n^q. \tag{2.6}$$

Since $t < a$, by (1.9) and (1.11) the moment generating function $M_{L/n^{1-q}}(t)$ is

$$M_L\left(\frac{t}{n^{1-q}}\right) \sim \frac{y_t}{y} \frac{y - n}{y_t - n} \sim e^{-t/n^{1-q}} \frac{a}{a-t} \to \frac{a}{a-t}, \tag{2.7}$$

and the right-hand side is the moment generating function for W_a. □

PROOF OF (1.28): $z = 1/(n + \tau\sqrt{n})$ (critical window). Fix $\tau \in \mathbb{R}$ and $t \in \mathbb{C}$. Let $y = 1/z$ and $y_t = ye^{-t/\sqrt{n}}$. Since there is a sequence $x_n \to 0$ such that

$$y_t = n + n^{1/2}(\tau - t + x_n), \tag{2.8}$$

we see from the last member of (1.9) and (1.12) that

$$M_{L/\sqrt{n}}(t) \sim \frac{e^{(\tau-t)^2/2}\mathrm{erfc}(2^{-1/2}(\tau - t))}{e^{\tau^2/2}\mathrm{erfc}(2^{-1/2}\tau)}, \tag{2.9}$$

which agrees with (1.25). In particular, (2.9) holds for $t = i\theta$ (since (2.2) does), in which case $M_{L/\sqrt{n}}(i\theta)$ is the characteristic function of $L/\sqrt{n}$. Since the right-hand side of (2.9) is a continuous function of θ in this case, the right-hand side is the characteristic function of a random variable, and $L/\sqrt{n}$ converges in distribution to this random variable.[7] □

PROOF OF (1.31): (z supercritical). Let $r > 1/2$, $y = 1/z$, and suppose that eventually $y \le n - n^r$. Let $\ell = \mathbb{E}_z^{(n)}(L)$; then $\ell \sim n - y$ by (1.20). Let $t \in \mathbb{R}$ and $y_t = ye^{-t/\ell}$. Then $y_t = y + (y_t - y) \sim y - yt/\ell$. Since eventually

$$\frac{y}{\ell} \sim \frac{y}{n-y} \le \frac{y}{n^r} \le n^{1-r}, \tag{2.10}$$

and since $n^{1-r} = o(n^r)$ because $r > 1/2$, eventually $y_t \le n - n^r + o(n^r) \le n - n^s$ for some $s \in (1/2, r)$. Therefore, by (1.9) and (1.13),

[7] The characteristic function is useful for this last step, as general theory then provides the existence of the limiting random variable.

$$M_{L/\ell}(t) = M_L\left(\frac{t}{\ell}\right) \sim \frac{y_t^{1-n}e^{y_t}}{y^{1-n}e^y} = e^{t(n-1)/\ell}e^{y_t - y}. \tag{2.11}$$

The logarithm of the right-hand side is equal to

$$\frac{t(n-1)}{\ell} + y(e^{-t/\ell} - 1) \sim \frac{t(n-1)}{\ell} - \frac{ty}{\ell} = \frac{t(n-y)}{\ell} - \frac{t}{\ell} \sim t - \frac{t}{\ell} \sim t, \tag{2.12}$$

and hence

$$M_L\left(\frac{t}{\ell}\right) \to e^t. \tag{2.13}$$

The right-hand side is the moment generating function of the constant random variable 1, and the proof is complete. □

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