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Extremal trigonal fibrations on rational surfaces

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Abstract. For a relatively minimal fibration $f : X \to \mathbb{P}^1$ of non-hyperelliptic curves of genus g, we know the Picard number $\rho(X) \leq 3g+8$. We study the case where $\rho(X) = 3g+8$ and the Mordell–Weil group of f is trivial. Such an f occurs only if $g \equiv 0$ or 1 (mod 3), and we describe such $f : X \to \mathbb{P}^1$ explicitly.

1. Introduction.

The theory of the Mordell–Weil lattices are sufficiently developed by Oguiso and Shioda in [**13**] for minimal elliptic rational surfaces. In their work, the even unimodular root lattice E_8 of rank eight played very important role as the predominant frame. For example, it was shown that the Mordell–Weil group is trivial if and only if there exists a singular fibre of type II* in the sense of Kodaira [**7**] (see [**10**, Theorem 4.1]), and its dual graph contains the Dynkin diagram of E_8 as a subgraph. The lattice E_8 also appears in another application by Shioda [**15**] to describe a hierarchy of deformations of rational double points.

Let X be a smooth rational surface defined over $\mathbb{C}$ and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of curves of genus $g \geq 2$ with a section, and let $\mathbb{K}$ be the rational function field of $\mathbb{P}^1$. The Mordell–Weil group of $\mathbb{K}$-rational points on the Jacobian variety of the generic fibre of f is finitely generated, and its rank r is called the Mordell–Weil rank. We know the Picard number $\rho(X) \leq 4g+6$ ([**14**, Theorem 2.8]), further $\rho(X) \leq 3g+8$ if a general fibre F of f is non-hyperelliptic ([**12**, Proposition 2.2]). Saito and Sakakibara showed in [**14**] that $r \leq 4g+4$, and that the fibration with maximal rank $r = 4g+4$ is of hyperelliptic type, and $\rho(X) = 4g+6$. The maximal Mordell–Weil lattice is isomorphic to the unimodular integral lattice D^+_{4g+4}. After that, the second named author gave necessary and sufficient conditions for the Mordell–Weil group of $f : X \to \mathbb{P}^1$ with $\rho(X) = 4g+6$ to be trivial. One of the conditions is the existence of a reducible fibre of f whose dual graph contains, as a subgraph, the extended Dynkin diagram of D^+_{4g+4} as in [**6**, Figure 4].

If F is non-hyperelliptic, then $r \leq 3g+6$ ([**12**, Theorem 1.1]). The fibration with maximal rank $r = 3g+6$ is either of plane quintic or of trigonal type (so Clifford index 1)

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and $\rho(X) = 3g + 8$. Moreover the structure of the corresponding Mordell–Weil lattices are completely determined in [**12**]. There are three types, depending on $0, 1, 2 \pmod 3$.

In this paper we treat the case of trigonal fibrations (i.e., the case where F is a non-hyperelliptic curve, but has a three-to-one map onto $\mathbb{P}^1$, so Clifford index is 1). We consider the case where $\rho(X)$ attains the maximum $3g + 8$. Unlike [**12**], we discuss the other extremal case: Mordell–Weil group is trivial. We prove the following.

Main Theorem (see Theorem 4.1). *Let X be a smooth rational surface, and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of trigonal curves of genus $g \geq 3$, and let $n = [g/3]$, the greatest integer not exceeding $g/3$. Suppose that its Picard number $\rho(X)$ equals maximal possible $3g + 8$.*

(1) *If $g \equiv 0 \pmod 3$, then the Mordell–Weil group of f is trivial if and only if f has a reducible fibre whose dual graph corresponds to the graph as in Figure 1.*

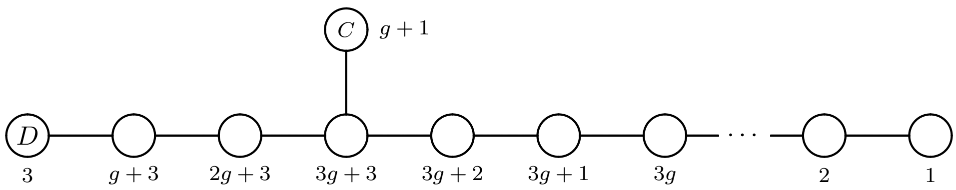

Figure 1.

Here, C is a (-3)-curve, D is a $(-n-1)$-curve, and the other circles denote (-2)-curves. The numbers indicated outside the circles denote the multiplicities of components in the degenerated fibre.

(2) *If $g \equiv 1 \pmod 3$, then the Mordell–Weil group of f is trivial if and only if f has a reducible fibre whose dual graph corresponds to the graph as in Figure 2.*

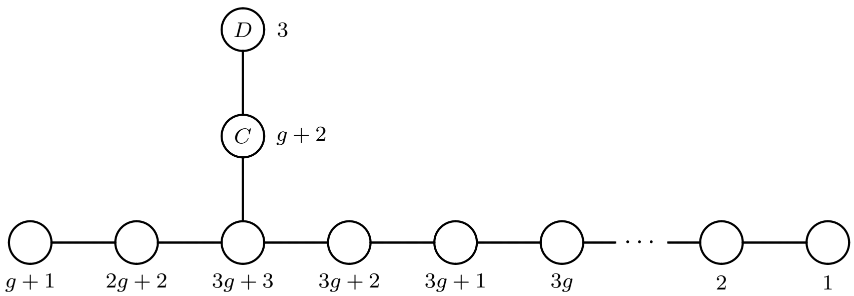

Figure 2.

Here, the same remarks apply as above.

(3) *If $g \equiv 2 \pmod 3$, then the Mordell–Weil group of f cannot be trivial.*

Section 2 is for preliminaries, we review the definition of the *reduction* (see [**4**, Section 1]) of (X, F), and prove some related properties. Consider a relatively minimal fibration of trigonal curves of genus $g \geqq 3$ on a rational surface with $\rho(X) = 3g + 8$ whose Mordell–Weil group is not necessarily trivial. Then we obtain such an f from a pencil Λ on Hirzebruch surface Σ_d of degree d by blowing Σ_d up at $(3g + 6)$ base points (see Theorem 2.4 below).

In Section 3, we restrict ourselves to the case where f has a reducible fibre whose dual graph corresponds to the graph as in Figures 1 (resp. 2). We describe the irreducible components of the reducible fibre in the Néron–Severi group $\mathrm{NS}(X)$ explicitly. As a consequence, we see that the Mordell–Weil group is trivial. In fact, the graphs as in Figures 1 and 2 respectively contain, as subgraphs, the extended Dynkin diagrams of the unimodular lattices Γ^0_{3g+6} and Γ^1_{3g+6} as in [**12**, Figures 1 and 2]. We also give the defining equations of the corresponding pencil Λ.

In Section 4, we prove the main theorem.

In Section 5, we discuss some examples that satisfy with the main theorem. By the main theorem, we know that if $g \equiv 2 \pmod 3$, then the Mordell–Weil group of f cannot be trivial. In this section, we also give a trigonal fibration $f : X \to \mathbb{P}^1$ with $g = 3n + 2$ and $\rho(X) = 3g + 8$ whose Mordell–Weil group is $\mathbb{Z}/3\mathbb{Z}$ for an arbitrary positive integer n.

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2. Preliminaries.

We briefly review basic notation and results on fibred rational surfaces and Mordell–Weil lattices. Here, a fibred rational surface means a smooth projective rational surface $X/\mathbb{C}$ together with a relatively minimal fibration $f : X \to \mathbb{P}^1$ whose general fibre F is a smooth projective curve of genus $g \geq 1$. In particular, any fibre of f is connected and contains no (-1)-curves as components. Since X is rational, the first Betti number of X equals zero. The second Betti number of X is equal to the Picard number $\rho(X)$ since the geometric genus of X is zero. Hence, we see that

$$\rho(X) = 10 - K_X^2 = 4g + 6 - (K_X + F)^2 \tag{2.1}$$

by virtue of Noether's formula. The adjoint divisor $(K_X + F)$ is nef when $g \geq 2$ (see [**4**, Lemma 1.1]). Thus we have that $\rho(X) \leq 4g + 6$. By means of slope inequalities [**8**, Corollary 4.4], we also have that $(K_X + F)^2 \geq g - 2$ and $\rho(X) \leq 3g + 8$ if F is non-hyperelliptic (see [**12**, Proposition 2.2]).

Lemma 2.1 (see [**4**, Lemma 1.2]). *Let C be an irreducible curve on X such that $(K_X + F).C = 0$. If $(K_X + F)^2 > 0$, then C is a smooth rational curve satisfying one of the following*:

(1) *C is a (-2)-curve contained in a fibre.*

(2) *C is a (-1)-section, i.e., a (-1)-curve with $F.C = 1$.*

From now on, we assume that $f : X \to \mathbb{P}^1$ is a relatively minimal fibration of genus $g \geq 2$ such that $(K_X + F)^2 > 0$. Suppose that there exists a (-1)-curve E with $(K_X + F).E = 0$ and let $\mu_1 : X \to X_1$ be its contraction. Since $F.E = 1$, $F_1 := (\mu_1)_*F$ is smooth on X_1. Furthermore, we have $\mu_1^*(K_{X_1} + F_1) = K_X + F$. If there exists a (-1)-curve E_1 with $(K_{X_1} + F_1).E_1 = 0$, then, by contracting it, we get the pair (X_2, F_2) with F_2 smooth and $K_{X_2} + F_2$ pulls back to $K_X + F$. We can continue the procedure until we arrive at a pair (X_n, F_n) such that we cannot find a (-1)-curve E_n with $(K_{X_n} + F_n).E_n = 0$. We put $Y := X_n$ and $G := F_n$. If $\mu : X \to Y$ denotes the natural map, then $\mu^*(K_Y + G) = K_X + F$ and $G = \mu_*F$ is a smooth curve isomorphic to F. The original fibration $f : X \to \mathbb{P}^1$ corresponds to a pencil $\Lambda_f \subset |G|$ with at most simple (but not necessarily transversal) base points. From the assumption $(K_X+F)^2 > 0$, $K_X + F$ is nef and big. This implies that, Y is the minimal resolution of singularities of the surface $\mathrm{Proj}(R(X, K_X+F))$, which has at most rational double points by Lemma 2.1, where $R(X, K_X + F) = \bigoplus_{n\geq 0} H^0(X, n(K_X + F))$. Therefore, such a model is uniquely determined. We call the pair (Y, G) the *reduction* of (X, F).

Assume that $Y = \mathbb{P}^2$. Then G is a smooth plane curve of degree $b \geq 4$. We have $g = (b - 1)(b - 2)/2$ and $(K_X + F)^2 = (K_Y + G)^2 = (b - 3)^2$. In particular, $(K_X + F)^2 = g - 2 + (b - 4)(b - 5)/2 \geq g - 2$. Furthermore, it is well known that the gonality of F is $b - 1$.

Now, for a non-negative integer d, we put

$$\Sigma_d = \{((X_0 : X_1 : X_2), (Y_0 : Y_1)) | X_1 Y_1^d = X_2 Y_0^d\} \subset \mathbb{P}^2 \times \mathbb{P}^1$$

and call it Hirzebruch surface of degree d. The restriction of the second projection to Σ_d gives a structure of $\mathbb{P}^1$-bundle. We also remark that $\Sigma_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Conversely, any $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ is isomorphic to Σ_d for some d. We often consider on the Zariski open subset defined by $X_0 Y_0 \neq 0$ and take $(x, y) = (X_1/X_0, Y_1/Y_0)$ as an affine coordinate. Let $\Delta_{[d]}$ be a minimal section of Σ_d defined by $x = 0$ and $\Gamma_{[d]}$ the fibre defined by $y = 0$. Then we have that $\Delta_{[d]}^2 = -d$, $\Gamma_{[d]}^2 = 0$ and $\Delta_{[d]}.\Gamma_{[d]} = 1$. For any curve C on Σ_d, there exist non-negative integers α and β such that $C \sim \alpha\Delta_{[d]} + \beta\Gamma_{[d]}$, where the symbol $\sim$ means the linear equivalence of divisors. Martens gave a simple proof of the following.

Lemma 2.2 (see [**9**, Corollary 1]). *Let $C \sim \alpha\Delta_{[d]} + \beta\Gamma_{[d]}$ ($\alpha \neq 0$; and $\beta \geq \alpha$ for $d = 0$) be a smooth and irreducible curve on Σ_d, and in the case $d = 1$ let $\alpha \neq \beta$. Then the gonality of C is α.*

We now return to the situation we are interested in, and consider the case where $Y \neq \mathbb{P}^2$. Then we can find at least one base-point-free pencil of rational curves on Y. Among them, we choose a pencil $|R|$ of rational curves with $R^2 = 0$ in such a way that $a := (K_Y + G).R$ is minimal. We call a the *minimal ruling degree of* (Y, G). Note that we have $G.R = a + 2 \geq 3$ from $K_Y.R = -2$ and Lemma 2.1. Let $\psi : Y \to \mathbb{P}^1$ be the morphism defined by $|R|$. We take a relatively minimal model of Y with respect to ψ and consider the image of G. Then we perform a succession of elementary transformations at singular points of the image curve to arrive at a particular relatively minimal model $(Y^\#, G^\#)$, called a #-minimal model in [**3**]. By utilizing the properties of $(Y^\#, G^\#)$, we get lower bounds of $(K_X + F)^2$ according to the Clifford index of F, which is closely related to the minimal ruling degree a of (Y, G), in [**4**, Section 2] and [**5**, Section 2]. The following is an analogy of them.

THEOREM 2.3. *Let X be a smooth rational surface, and $f : X \to \mathbb{P}^1$ a relatively minimal fibration whose general fibre F is a non-hyperelliptic curve of genus $g \geq 3$. Then*

$$\rho(X) \leq 3g + 8.$$

Assume that $\rho(X) = 3g + 8$. Then the reduction (Y, G) of (X, F) satisfies one of the following:

(1) *$Y = \mathbb{P}^2$ and G is a quartic curve.*

(2) *$Y = \mathbb{P}^2$ and G is a quintic curve.*

(3) *$Y = \Sigma_d$ and $G \sim 3\Delta_{[d]} + (1 + (g + 3d)/2)\Gamma_{[d]}$ for some d such that $d \equiv g \pmod 2$ and $0 \leq d \leq (g + 2)/3$.*

In particular, f has at least one (-1)-section. Furthermore, f has at most sixteen (-1)-sections when (Y, G) is as in (1)*, f has at most twenty five (-1)-sections when (Y, G) is as in* (2)*, and f has at most $(3g + 6)$ (-1)-sections otherwise.*

PROOF. Let X be a smooth rational surface and $f : X \to \mathbb{P}^1$ a relatively minimal fibration whose general fibre F is a non-hyperelliptic curve of genus $g \geq 3$. From (2.1) and [**6**, Theorem 2.2], we have $(K_X + F)^2 > 0$. Let (Y, G) denote the reduction of (X, F).

Assume that $Y = \mathbb{P}^2$. Then G is a smooth plane curve of degree $b \geq 4$. By the definition of reduction, f has at least one (-1)-section and has at most b^2 (-1)-sections. Furthermore, we have $g = (b - 1)(b - 2)/2$ and $(K_X + F)^2 = (K_Y + G)^2 = (b - 3)^2$. In particular, $(K_X + F)^2 = g - 2 + (b - 4)(b - 5)/2 \geq g - 2$. This and (2.1) imply $\rho(X) \leq 3g + 8$. If the equality holds, then G is either a quartic curve or a quintic curve.

We next assume that $Y \neq \mathbb{P}^2$. Let a be the minimal ruling degree of (Y, G). When $a \geq 3$, we have $(K_X + F)^2 \geq g + 1$, which is a weaker bound than as in [**5**, Lemmas 2.6 and 2.7], in the same argument as in [**5**, Lemmas 2.6 and 2.7]. Similarly, we get $(K_X + F)^2 \geq g - 1$ when $a = 2$. When $a = 1$, $G^\#$ must be smooth. Hence we have $(Y, G) = (Y^\#, G^\#)$ and $(K_X + F)^2 = g - 2$ (see [**4**, p. 188]). Therefore, we have $\rho(X) \leq 3g + 8$ from (2.1). Furthermore, $\rho(X) = 3g + 8$ leads to $a = 1$ with $Y = \Sigma_d$ for some d. Then we get $G \sim 3\Delta_{[d]} + (1 + (g + 3d)/2)\Gamma_{[d]}$ from $G.(G + K_{\Sigma_d}) = 2g - 2$. Since $\Delta_{[d]}.G$ is a non-negative integer, we show that $d \equiv g \pmod 2$ and $0 \leq d \leq (g + 2)/3$. By the definition of reduction with $X \neq Y$, we see that f has at least one (-1)-section. The number of (-1)-sections of f is less than or equal to $G^2 = 3g + 6$.

This completes the proof of Theorem 2.3. □

As a corollary, we have the following.

THEOREM 2.4. *Let X be a smooth rational surface, and $f : X \to \mathbb{P}^1$ a relatively minimal fibration whose general fibre F is a trigonal curve of genus $g \geq 3$. Assume that $\rho(X) = 3g + 8$. Then there exists a birational morphism $\mu : X \to \Sigma_d$ with $d \equiv g \pmod 2$ and $0 \leq d \leq (g + 2)/3$, and it satisfies conditions* (i), (ii).

(i) *$\mu_* F$ is linearly equivalent to* $(3\Delta_{[d]} + (1 + (g + 3d)/2)\Gamma_{[d]})$.

(ii) *The pull-back to X of a (-1)-curve contracted by μ intersects with F at just one point.*

In particular, f has at least one (-1)-section.

Proof. Let X be a smooth rational surface and $f : X \to \mathbb{P}^1$ a relatively minimal fibration whose general fibre F is a trigonal curve of genus $g \geq 3$. Assume that $\rho(X) = 3g + 8$. Let (Y, G) denote the reduction of (X, F). We first consider the case (3) of Theorem 2.3, the natural map $(X, F) \to (Y, G)$ is a unique birational morphism $\mu : X \to \Sigma_d$ satisfying the desired properties. The case where (Y, G) is as in (2) of Theorem 2.3 does not occur since the smooth quintic curves are tetragonal. Now, we consider the last case where (Y, G) is as in (1) of Theorem 2.3. Let $\nu : \Sigma_1 \to \mathbb{P}^2$ be a blowing-up at a base point of a pencil $\Lambda_f \subset |G|$ corresponding to f. Then the composite map of ν^{-1} and the natural map $(X, F) \to (Y, G)$ satisfies the desired properties of μ. □

Put $n = [g/3]$, the greatest integer not exceeding $g/3$. Let (Σ_d, G) be the image of (X, F) by μ as in Theorem 2.4. We remark that $\Gamma_{[d]}.G = 3$ and $d \equiv g \pmod 2$, $0 \leq d \leq n + 1$,

$$\Delta_{[d]}.G = \frac{g + 2 - 3d}{2} = \begin{cases} 1 + \dfrac{3(n-d)}{2} \geq 1, & \text{when } g = 3n, \\ \dfrac{3(n+1-d)}{2} \geq 0, & \text{when } g = 3n+1, \\ 2 + \dfrac{3(n-d)}{2} \geq 2, & \text{when } g = 3n+2. \end{cases}$$

Therefore, we have the following.

Corollary 2.5. *Put $n = [g/3]$. Let (Σ_d, G) be the image of (X, F) by μ as in Theorem 2.4 and D an irreducible reduced curve on Σ_d. Then the following holds.*

(1) *$D.G = 0$ if and only if $D = \Delta_{[n+1]}$ with $d = n + 1$ and $g = 3n + 1$.*
(2) *$D.G = 1$ if and only if $D = \Delta_{[n]}$ with $d = n$ and $g = 3n$.*
(3) *$D.G = 2$ if and only if $D = \Delta_{[n]}$ with $d = n$ and $g = 3n + 2$.*

Remark 2.6. Put $n = [g/3]$. Let (Σ_d, G) be the image of (X, F) by μ as in Theorem 2.4. Then G is very ample except when $(d, g) = (n+1, 3n+1)$. In the exceptional case, f have at least one reducible fibre (cf. [**12**, (3.1)]). Furthermore, $K_{\Sigma_d} + G$ is also very ample except when $(d, g) = (2, 4)$ (cf. [**12**, Lemma 3.1]).

Via f, we can regard X as a smooth projective curve of genus g defined over the rational function field $\mathbb{K} = f^*\mathbb{C}(\mathbb{P}^1)$. We assume that it has a $\mathbb{K}$-rational point O. Let $\mathcal{J}_{\mathcal{F}}/\mathbb{K}$ be the Jacobian variety of the generic fibre $\mathcal{F}/\mathbb{K}$ of f. The Mordell–Weil group of f is the group of $\mathbb{K}$-rational points $\mathcal{J}_{\mathcal{F}}(\mathbb{K})$. It is a finitely generated Abelian group, since $X/\mathbb{C}$ is a rational surface. The rank $\mathrm{rk}\mathcal{J}_{\mathcal{F}}(\mathbb{K})$ of the group is called the *Mordell–Weil rank*. There is a formula, often referred as the Shioda–Tate formula, relating the Mordell–Weil rank and the Picard number:

$$\mathrm{rk}\mathcal{J}_{\mathcal{F}}(\mathbb{K}) = \rho(X) - 2 - \sum_{t\in\mathbb{P}^1}(v_t - 1), \tag{2.2}$$

where v_t denotes the number of irreducible components of the fibre $f^{-1}(t)$. There is a natural one-to-one correspondence between the set of $\mathbb{K}$-rational points $\mathcal{F}(\mathbb{K})$ and the set of sections of f. For $P \in \mathcal{F}(\mathbb{K})$, we denote by (P) the section corresponding to P which is regarded as a horizontal curve on X. In particular, (O) corresponding to the origin O of $\mathcal{J}_{\mathcal{F}}(\mathbb{K})$ is called the *zero section*. Let T be the subgroup of $\mathrm{NS}(X)$ generated by (O) and the irreducible components of the fibres of f. In [**16**, Theorem 3], we have the natural isomorphism of groups $\mathcal{J}_{\mathcal{F}}(\mathbb{K}) \simeq \mathrm{NS}(X)/T$. As a corollary, we have the following.

LEMMA 2.7. *Keep the notation and assumptions as above. $\mathcal{J}_{\mathcal{F}}(\mathbb{K})$ is trivial if and only if $T = \mathrm{NS}(X)$.*

3. Explicit constructions.

Let X be a smooth projective rational surface and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of genus $g \geq 1$. Assume that $g = 1$ and f has a section. Then Miranda and Persson [**10**] studied extremal rational elliptic surfaces, where "extremal" means that the Mordell–Weil rank of f is zero (see also [**11**]). We next assume that $g \geq 2$ and $\rho(X) = 4g + 6$. Then the second named author gave necessary and sufficient conditions for the Mordell–Weil group of f to be trivial. One of the conditions is the existence of a special fibre of f, which is an extension of a singular fibre of type II^* in the sense of Kodaira [**7**]. In particular, its dual graph contains, as a subgraph, the extended Dynkin diagram of the unimodular integral lattice D^+_{4g+4} as in [**6**, Figure 4]. From now on we assume that f is trigonal and $\rho(X) = 3g + 8$. In order to construct $f : X \to \mathbb{P}^1$ whose Mordell–Weil group is trivial, we consider a reducible fibre whose dual graph contains, as a subgraph, the extended Dynkin diagram of the unimodular lattice Γ^0_{3g+6} as in [**12**, Figure 1].

PROPOSITION 3.1. *Let n be a positive integer, X a smooth rational surface and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of trigonal curves of genus $g = 3n$. Assume that $\rho(X) = 3g + 8$. Then the following conditions are equivalent.*

(1) *f has a reducible fibre whose dual graph corresponds to the graph as in Figure* 1.

(2a) *$f : X \to \mathbb{P}^1$ can be obtained from Σ_n by eliminating the base points of the following pencil Λ: Let $\Delta_{[n]}$ be the minimal section of Σ_n and $\Gamma_{[n]}$ a fibre of Σ_n. Take a curve $D_{[n],0}$ satisfying the following three conditions.*

   (i) $D_{[n],0} \sim 3\Delta_{[n]} + (3n+1)\Gamma_{[n]}$.

   (ii) *$D_{[n],0}$ is smooth at the intersection point of $\Gamma_{[n]}$ and $\Delta_{[n]}$.*

   (iii) *$D_{[n],0}$ has a contact of order* 3 *with $\Gamma_{[n]}$ at the above intersection point.*

   Then the pencil Λ is defined by $D_{[n],0}$ and $(3\Delta_{[n]} + (3n+1)\Gamma_{[n]})$.

(2b) *$f : X \to \mathbb{P}^1$ can be obtained from Σ_n by eliminating the base points of the following pencil Λ: For $t \in \mathbb{C}$, each member of Λ is defined by*

$$tx^3y^{3n+1} = y + x^3 + \sum_{i=1}^{3}\sum_{j=1}^{in+1} c_{i,j}x^iy^j, \tag{3.3}$$

where $c_{i,j}$ are complex numbers. The member of Λ corresponding to ∞ is $(3\Delta_{[n]} + (3n+1)\Gamma_{[n]})$, which is defined by $x^3y^{3n+1} = 0$.

PROOF. (1) $\Rightarrow$ (2a): We denote by F a general fibre of f. Let Θ_k, $k = 0, 1, \ldots, 9n+6$ be components of the reducible fibre F_∞ that satisfy the following condition:

$$(\Theta_{i-1}.\Theta_{j-1})_{1\le i,j\le 9n+7} = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ddots & \ddots & 1 & 0 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 1 & 0 & -n-1 \end{pmatrix}.$$

We know that f has a (-1)-section E_{9n+6} by the last assertion of Theorem 2.4. Since Θ_0 is a unique component whose multiplicity in F_∞ is one, E_{9n+6} intersects with Θ_0. Let μ be the birational morphism contracting $E_{9n+6}, \Theta_0, \Theta_1, \ldots, \Theta_{9n+4}$ in turn. Then $(\mu_*\Theta_{9n+5})^2 = 0$ and $(\mu_*\Theta_{9n+6})^2 = -n$. Since $\rho(X) = 3g + 8 = 9n + 8$, the image of X by μ is Σ_n with the minimal section $\Delta_{[n]} = \mu_*\Theta_{9n+6}$ and a fibre $\Gamma_{[n]} = \mu_*\Theta_{9n+5}$ of Σ_n. Furthermore, multiplicities of Θ_{9n+6} and Θ_{9n+5} in F_∞ imply that $\mu_*F_\infty = 3\Delta_{[n]} + (3n+1)\Gamma_{[n]}$, which provides the assertion (i). Let $D_{[n],0}$ be the image by μ of $f^{-1}(0)$. By the Shioda–Tate formula (2.2) and its non-negativity, $D_{[n],0}$ is an irreducible curve. The original fibration $f: X \to \mathbb{P}^1$ corresponds to a pencil Λ generated by $D_{[n],0}$ and $(3\Delta_{[n]} + (3n+1)\Gamma_{[n]})$. Since E_{9n+6} intersects with $f^{-1}(0)$ at one point transversely, the assertion (ii) follows. The assertion (iii) also follows from the configuration of $E_{9n+6}, \Theta_0, \Theta_1, \ldots, \Theta_{9n+5}$ and Θ_{9n+6}.

(2a) $\Rightarrow$ (2b): Let $\Gamma_{[n]}$ be the fibre of Σ_n defined by $y = 0$. The assertion (i) implies that the defining equation of $D_{[n],0}$ can be

$$\sum_{i=0}^{3}\sum_{j=0}^{in+1} c_{i,j}x^iy^j = 0$$

for some complex numbers $c_{i,j}$. Since $D_{[n],0}$ is an irreducible curve, $\Gamma_{[n]}$ is not a component of $D_{[n],0}$. This and (iii) yield that $c_{2,0} = c_{1,0} = c_{0,0} = 0$ and $c_{3,0} \neq 0$. Furthermore, (ii) implies $c_{0,1} \neq 0$. We may put $c_{0,1} = c_{3,0} = 1$ without loss of generality.

(2b) $\Rightarrow$ (1): We consider a pencil Λ on Σ_n defined by (3.3), namely, each member $D_{[n],t}$ in Λ is defined by (3.3) for $t \in \mathbb{C}$ and the member $D_{[n],\infty}$ in Λ corresponding to ∞ is $(3\Delta_{[n]} + (3n+1)\Gamma_{[n]})$, which is defined by $x^3y^{3n+1} = 0$. Then $D_{[n],t}$ is smooth at the intersection point of $\Gamma_{[n]}$ with $\Delta_{[n]}$ for all $t \in \mathbb{C}$. Furthermore, $D_{[n],t}$ has a contact of order 3 with $\Gamma_{[n]}$ at the smooth point $(x, y) = (0, 0)$. Thus any two members in Λ are disjoint on $\Sigma_n \setminus \{(0, 0)\}$. In particular, the $(9n + 6)$ base points of Λ consist of the point $(0, 0)$ and its infinitely near points. Hence a general member in Λ is a smooth and irreducible curve of genus $g = 3n$, which is trigonal by Lemma 2.2. Therefore, we obtain a relatively minimal fibration $f : X \to \mathbb{P}^1$ of trigonal curves of genus $g = 3n$ from $\Phi_\Lambda : \Sigma_n \dashrightarrow \mathbb{P}^1$ by eliminating the base points of Λ as follows:

Let $\mu_{9n+6} : W_1 \to \Sigma_n$ be the blowing-up at the point $(x, y) = (0, 0)$ with the exceptional curve E_1, i.e., $\mu_{9n+6}(E_1) = (0, 0)$. Let P_2 be the intersection point of E_1 and the strict transform to W_1 of $\Gamma_{[n]}$. The strict transform to W_1 of $D_{[n],t}$ has a contact of order 2 with that of $\Gamma_{[n]}$ at P_2 for all $t \in \mathbb{C}$. Next let $\mu_{9n+5} : W_2 \to W_1$ be the blowing-up at the base point P_2 with $E_2 = \mu_{9n+5}^{-1}(P_2)$. Let P_3 denote the intersection point of E_2 and the strict transform to W_2 of $\Gamma_{[n]}$. For all $t \in \mathbb{C}$ the strict transform to W_2 of $D_{[n],t}$ meets that of $\Gamma_{[n]}$ transversally at P_3. Denote the pull-back of curves by the same symbols for simplicity. Then we get the irreducible decomposition

$$D_{[n],\infty} - E_1 - E_2 = 3(\Delta_{[n]} - E_1) + (3n+1)(\Gamma_{[n]} - E_1 - E_2) + (3n+3)(E_1 - E_2) + (6n+3)E_2.$$

Furthermore, $D_{[n],t} - E_1 - E_2$ has a contact of order $(9n + 4)$ with the other members at P_3 for all $t \in \mathbb{C}$. Denote by $\mu_{9n+4} : W_3 \to W_2$ the blowing-up at the base point P_3 with $E_3 = \mu_{9n+4}^{-1}(P_3)$. Let P_4 be the intersection point of E_3 and the strict transform to W_3 of $D_{[n],t}$. In fact, P_4 corresponds to a tangent direction of $D_{[n],t} - E_1 - E_2$ at P_3 on W_2 by μ_{9n+4}, and $D_{[n],t} - E_1 - E_2 - E_3$ has a contact of order $(9n + 3)$ with the other members at P_4 for all $t \in \mathbb{C}$. In the same way, for $i = 4, 5, \dots, 9n + 5$, after the blowing-up $\mu_{9n+7-i} : W_i \to W_{i-1}$ at the base point P_i with $E_i = \mu_{9n+7-i}^{-1}(P_i)$, $D_{[n],t} - E_1 - E_2 - \cdots - E_i$ has a contact of order $(9n + 6 - i)$ with the other members at P_{i+1}. Let $\mu_1 : X \to W_{9n+5}$ be the blowing-up at the base point P_{9n+6} with $E_{9n+6} = \mu_1^{-1}(P_{9n+6})$. Put $f = \Phi_\Lambda \circ \mu_{9n+6} \circ \mu_{9n+5} \circ \cdots \circ \mu_1$. Then $f : X \to \mathbb{P}^1$ is a relatively minimal fibration whose general fibre F is $D_{[n],t} - E_1 - E_2 - \cdots - E_{9n+6}$ for general $t \in \mathbb{C}$ and $f^{-1}(\infty) = D_{[n],\infty} - E_1 - E_2 - \cdots - E_{9n+6}$ is a reducible fibre. We remark that E_{9n+6} is a (-1)-section of f. The configuration of E_{9n+6}, F and the irreducible components of $f^{-1}(\infty)$ is as in Figure 3. Furthermore, we see that the dual graph of the reducible fibre $f^{-1}(\infty)$ corresponds to the graph as in Figure 1. □

COROLLARY 3.2. *Let $f : X \to \mathbb{P}^1$ be as in Proposition* 3.1. *Then the Mordell–Weil group of f is trivial.*

PROOF. We use the same notation as in Proof of Proposition 3.1. The irreducible components of $f^{-1}(\infty)$ are $\Delta_{[n]} - E_1$, $\Gamma_{[n]} - E_1 - E_2 - E_3$ and $E_i - E_{i+1}, i = 1, 2, \dots, 9n + 5$. These and E_{9n+6}, which is a (-1)-section of f, generate $\Delta_{[n]}, \Gamma_{[n]}$ and $E_j, j = 1, 2, \dots, 9n+6$, and form a $\mathbb{Z}$-basis of $\mathrm{NS}(X)$. Therefore the Mordell–Weil group of f is trivial by Lemma 2.7. □

Now, we consider a reducible fibre whose dual graph contains, as a subgraph, the extended Dynkin diagram of the unimodular lattice Γ^1_{3g+6} as in [**12**, Figure 2]. The proof of Proposition 3.3 below is similar to that of Proposition 3.1. For the convenience of the reader, we still give the details of the proof.

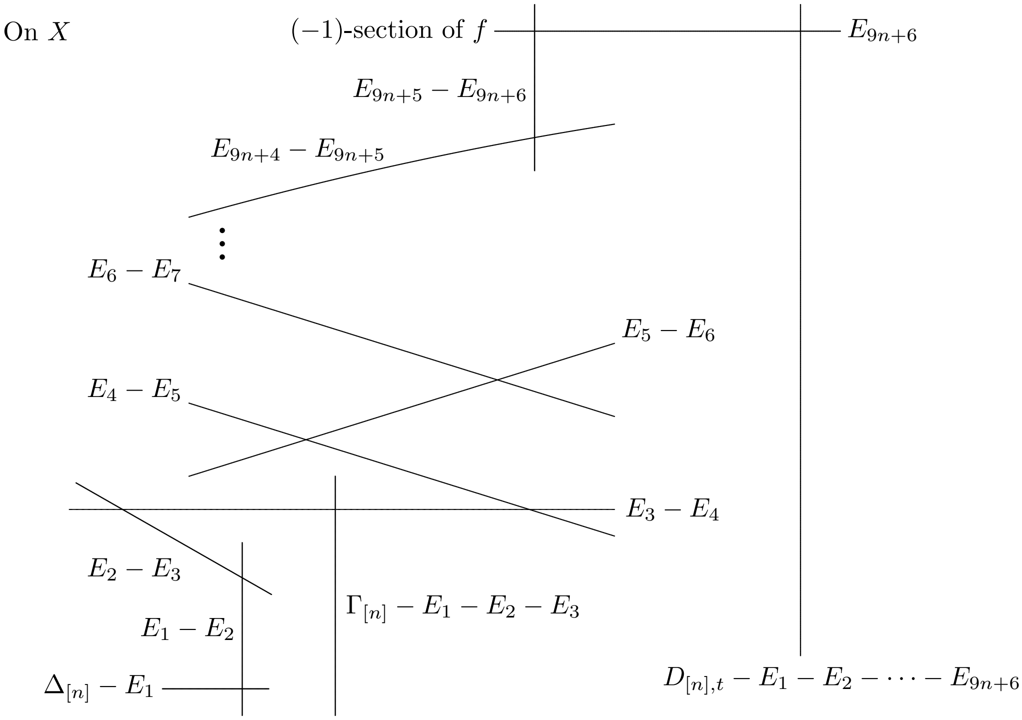

Figure 3.

Proposition 3.3. *Let n be a positive integer, X a smooth rational surface and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of trigonal curves of genus $g = 3n+1$. Assume that $\rho(X) = 3g + 8$. Then the following conditions are equivalent.*

(1) *f has a reducible fibre whose dual graph corresponds to the graph as in Figure 2.*

(2a) *$f : X \to \mathbb{P}^1$ can be obtained from Σ_{n+1} by eliminating the base points of the following pencil Λ: Let $\Delta_{[n+1]}$ be the minimal section of Σ_{n+1} and $\Gamma_{[n+1]}$ a fibre of Σ_{n+1}. Let p denote a point on $\Gamma_{[n+1]}$ except the intersection point of $\Gamma_{[n+1]}$ and $\Delta_{[n+1]}$. Take a curve $D_{[n+1],0}$ satisfying the following three conditions.*

- (i) $D_{[n+1],0} \sim 3\Delta_{[n+1]} + (3n+3)\Gamma_{[n+1]}$.
- (ii) *$D_{[n+1],0}$ is smooth at p.*
- (iii) *$D_{[n+1],0}$ has a contact of order 3 with $\Gamma_{[n+1]}$ at p.*

Then the pencil Λ is defined by $D_{[n+1],0}$ and $(3\Delta_{[n+1]} + (3n+3)\Gamma_{[n+1]})$.

(2b) *$f : X \to \mathbb{P}^1$ can be obtained from Σ_{n+1} by eliminating the base points of the following pencil Λ: For $t \in \mathbb{C}$, each member of Λ is defined by*

$$tx^3y^{3n+3} = 1 + x^3y + \sum_{i=1}^{3} \sum_{j=1}^{i(n+1)} c_{i,j}x^iy^j, \tag{3.4}$$

where $c_{i,j}$ are complex numbers with $c_{3,1} \neq -1$. The member of Λ corresponding to ∞ is $(3\Delta_{[n+1]} + (3n+3)\Gamma_{[n+1]})$, which is defined by $x^3 y^{3n+3} = 0$.

PROOF. (1) $\Rightarrow$ (2a): We denote by F a general fibre of f. Let Θ_k, $k = 0, 1, \ldots, 9n+9$ be components of the reducible fibre F_∞ that satisfy the following condition:

$$(\Theta_{i-1}.\Theta_{j-1})_{1 \le i,j \le 9n+10} = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ddots & \ddots & 1 & 0 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 1 & -n-1 \end{pmatrix}.$$

We know that f has a (-1)-section E_{9n+9} by the last assertion of Theorem 2.4. Since Θ_0 is a unique component whose multiplicity in F_∞ is one, E_{9n+9} intersects with Θ_0. Let μ be the birational morphism contracting $E_{9n+9}, \Theta_0, \Theta_1, \ldots, \Theta_{9n+7}$ in turn. Then $(\mu_*\Theta_{9n+8})^2 = 0$ and $(\mu_*\Theta_{9n+9})^2 = -n-1$. Since $\rho(X) = 3g+8 = 9n+11$, the image of X by μ is Σ_{n+1} with the minimal section $\Delta_{[n+1]} = \mu_*\Theta_{9n+9}$ and a fibre $\Gamma_{[n+1]} = \mu_*\Theta_{9n+8}$ of Σ_{n+1}. Furthermore, multiplicities of Θ_{9n+9} and Θ_{9n+8} in F_∞ imply that $\mu_*F_\infty = 3\Delta_{[n+1]} + (3n+3)\Gamma_{[n+1]}$, which provides the assertion (i). Let $D_{[n+1],0}$ be the image by μ of $f^{-1}(0)$. By the Shioda–Tate formula (2.2) and its non-negativity, $D_{[n+1],0}$ is an irreducible curve. The original fibration $f : X \to \mathbb{P}^1$ corresponds to a pencil Λ generated by $D_{[n+1],0}$ and $(3\Delta_{[n+1]} + (3n+3)\Gamma_{[n+1]})$. Remark that base points of Λ are not on $\Delta_{[n+1]}$. Since E_{9n+9} intersects with $f^{-1}(0)$ at one point transversely, the assertion (ii) follows. The assertion (iii) also follows from the configuration of $E_{9n+9}, \Theta_0, \Theta_1, \ldots, \Theta_{9n+8}$ and Θ_{9n+9}.

(2a) $\Rightarrow$ (2b): Let $\Gamma_{[n+1]}$ be the fibre of Σ_{n+1} defined by $y = 0$. The assertion (i) implies that the defining equation of $D_{[n+1],0}$ can be

$$\sum_{i=0}^{3} \sum_{j=0}^{i(n+1)} c_{i,j} x^i y^j = 0$$

for some complex numbers $c_{i,j}$. We define the intersection point p of $\Gamma_{[n+1]}$ with $D_{[n+1],0}$ by $(x, y) = (-1/o, 0)$. Since $D_{[n+1],0}$ is an irreducible curve, $\Gamma_{[n+1]}$ is not a component of $D_{[n+1],0}$. This and (iii) yield that $c_{3,0} = c_{0,0}o^3$, $c_{2,0} = 3c_{0,0}o^2$, $c_{1,0} = 3c_{0,0}o$ and $c_{0,0} \neq 0$. Furthermore, (ii) implies $c_{3,1} \neq 0$. We may put $c_{0,0} = c_{3,1} = 1$ and $o = 0$ without loss of generality.

(2b) $\Rightarrow$ (1): We consider a pencil Λ on Σ_{n+1} defined by (3.4), namely, each member $D_{[n+1],t}$ in Λ is defined by (3.4) for $t \in \mathbb{C}$ and the member $D_{[n+1],\infty}$ in Λ corresponding to

∞ is $(3\Delta_{[n+1]}+(3n+3)\Gamma_{[n+1]})$, which is defined by $x^3y^{3n+3}=0$. Then $D_{[n+1],t}$ is smooth at the point $((X_0:X_1:X_2),(Y_0:Y_1))=((0:1:0),(1:0))$ for all $t\in\mathbb{C}$. Furthermore, $D_{[n+1],t}$ has a contact of order 3 with $\Gamma_{[n+1]}$ at the smooth point $((0:1:0),(1:0))$. Thus any two members in Λ are disjoint on $\Sigma_{n+1}\setminus\{((0:1:0),(1:0))\}$. In particular, the $(9n+9)$ base points of Λ consist of the point $((0:1:0),(1:0))$ and its infinitely near points. Hence a general member in Λ is a smooth and irreducible curve of genus $g=3n+1$, which is trigonal by Lemma 2.2. Therefore, we obtain a relatively minimal fibration $f:X\to\mathbb{P}^1$ of trigonal curves of genus $g=3n+1$ from $\Phi_\Lambda:\Sigma_{n+1}\dashrightarrow\mathbb{P}^1$ by eliminating the base points of Λ as follows:

Let $\mu_{9n+9}:W_1\to\Sigma_{n+1}$ be the blowing-up at the point $((X_0:X_1:X_2),(Y_0:Y_1))=((0:1:0),(1:0))$ with the exceptional curve E_1, i.e., $\mu_{9n+9}(E_1)=((0:1:0),(1:0))$. Let P_2 be the intersection point of E_1 and the strict transform to W_1 of $\Gamma_{[n+1]}$. The strict transform to W_1 of $D_{[n+1],t}$ has a contact of order 2 with that of $\Gamma_{[n+1]}$ at P_2 for all $t\in\mathbb{C}$. Next let $\mu_{9n+8}:W_2\to W_1$ be the blowing-up at the base point P_2 with $E_2=\mu_{9n+8}^{-1}(P_2)$. Let P_3 denote the intersection point of E_2 and the strict transform to W_2 of $\Gamma_{[n+1]}$. For all $t\in\mathbb{C}$ the strict transform to W_2 of $D_{[n+1],t}$ meets that of $\Gamma_{[n+1]}$ transversally at P_3. Denote the pull-back of curves by the same symbols for simplicity. Then we get the irreducible decomposition

$$D_{[n+1],\infty}-E_1-E_2=3\Delta_{[n+1]}+(3n+3)(\Gamma_{[n+1]}-E_1-E_2)+(3n+2)(E_1-E_2)+(6n+4)E_2.$$

Furthermore, $D_{[n+1],t}-E_1-E_2$ has a contact of order $(9n+7)$ with the other members at P_3 for all $t\in\mathbb{C}$. Denote by $\mu_{9n+7}:W_3\to W_2$ the blowing-up at the base point P_3 with $E_3=\mu_{9n+7}^{-1}(P_3)$. Let P_4 be the intersection point of E_3 and the strict transform to W_3 of $D_{[n+1],t}$. In fact, P_4 corresponds to a tangent direction of $D_{[n+1],t}-E_1-E_2$ at P_3 on W_2 by μ_{9n+7}, and $D_{[n+1],t}-E_1-E_2-E_3$ has a contact of order $(9n+6)$ with the other members at P_4 for all $t\in\mathbb{C}$. In the same way, for $i=4,5,\dots,9n+8$, after the blowing-up $\mu_{9n+10-i}:W_i\to W_{i-1}$ at the base point P_i with $E_i=\mu_{9n+10-i}^{-1}(P_i)$, $D_{[n+1],t}-E_1-E_2-\cdots-E_i$ has a contact of order $(9n+9-i)$ with the other members at P_{i+1}. Let $\mu_1:X\to W_{9n+8}$ be the blowing-up at the base point P_{9n+9} with $E_{9n+9}=\mu_1^{-1}(P_{9n+9})$. Put $f=\Phi_\Lambda\circ\mu_{9n+9}\circ\mu_{9n+8}\circ\cdots\circ\mu_1$. Then $f:X\to\mathbb{P}^1$ is a relatively minimal fibration whose general fibre F is $D_{[n+1],t}-E_1-E_2-\cdots-E_{9n+9}$ for general $t\in\mathbb{C}$ and $f^{-1}(\infty)=D_{[n+1],\infty}-E_1-E_2-\cdots-E_{9n+9}$ is a reducible fibre. We remark that E_{9n+9} is a (-1)-section of f. The configuration of E_{9n+9}, F and the irreducible components of $f^{-1}(\infty)$ is as in Figure 4. Furthermore, we see that the dual graph of the reducible fibre $f^{-1}(\infty)$ corresponds to the graph as in Figure 2. □

Corollary 3.4. *Let $f:X\to\mathbb{P}^1$ be as in Proposition 3.3. Then the Mordell–Weil group of f is trivial.*

Proof. We use the same notation as in Proof of Proposition 3.3. The irreducible components of $f^{-1}(\infty)$ are $\Delta_{[n+1]}$, $\Gamma_{[n+1]}-E_1-E_2-E_3$ and $E_i-E_{i+1}, i=1,2,\dots,9n+8$. These and E_{9n+9}, which is a (-1)-section of f, generate $\Delta_{[n+1]},\Gamma_{[n+1]}$ and $E_j, j=1,2,\dots,9n+9$, and form a $\mathbb{Z}$-basis of $\mathrm{NS}(X)$. Therefore the Mordell–Weil group of f is trivial by Lemma 2.7. □

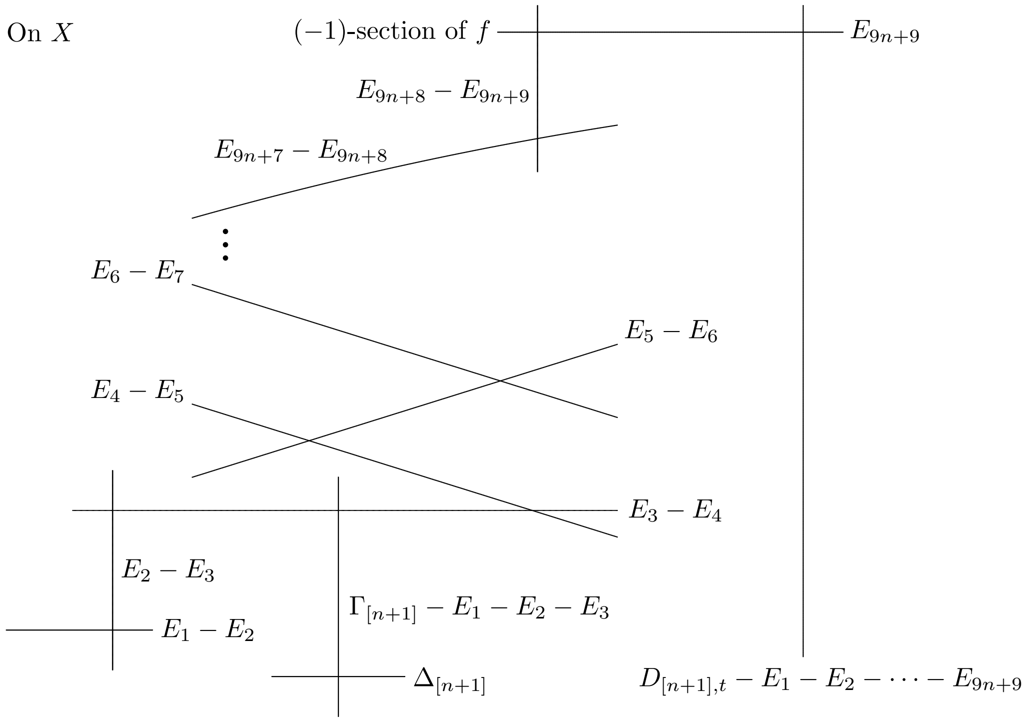

Figure 4.

4. Main theorem.

Let X be a smooth projective rational surface and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of genus $g \geq 1$. Assume that $\rho(X) = 4g + 6$. Then, for all g, there exists $f : X \to \mathbb{P}^1$ whose Mordell–Weil group is trivial. From now on we assume that f is trigonal and $\rho(X) = 3g + 8$. Then we see a difference in existence of $f : X \to \mathbb{P}^1$ whose Mordell–Weil group is trivial according to g as follows.

THEOREM 4.1. *Let X be a smooth rational surface, and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of trigonal curves of genus $g \geq 3$. Suppose that its Picard number $\rho(X)$ equals maximal possible $3g + 8$.*

(1) *If $g \equiv 0 \pmod 3$, then the Mordell–Weil group of f is trivial if and only if f is as in Proposition* 3.1.

(2) *If $g \equiv 1 \pmod 3$, then the Mordell–Weil group of f is trivial if and only if f is as in Proposition* 3.3.

(3) *If $g \equiv 2 \pmod 3$, then the Mordell–Weil group of f cannot be trivial.*

PROOF. Combining Corollaries 3.2 and 3.4, it is sufficient to show the following to prove Theorem 4.1.

Lemma 4.2. *Let X be a smooth rational surface, and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of trigonal curves of genus $g \geq 3$. Let $n = [g/3]$, the greatest integer not exceeding $g/3$. Assume that $\rho(X) = 3g+8$ and the Mordell–Weil group of f is trivial. Then $g = 3n$ or $3n+1$. Furthermore, if $g = 3n$ (resp. $3n+1$), f has a reducible fibre whose dual graph corresponds to the graph as in Figures 1 (resp. 2).*

Proof. Let X be a smooth rational surface and $f : X \to \mathbb{P}^1$ a relatively minimal fibration whose general fibre F is a trigonal curve of genus $g \geq 3$. Assume that $\rho(X) = 3g+8$. Let (Σ_d, G) be the image of (X, F) by μ as in Theorem 2.4. We know $G \sim 3\Delta_{[d]} + (1 + (g+3d)/2)\Gamma_{[d]}$. Assume that the Mordell–Weil group of f is trivial. Then a section of f is unique. We shall denote by E_{3g+6} the (-1)-section of f. Furthermore, in the process of contracting by μ, we may assume that E_{i+1} corresponds to an infinitely near point of the point corresponding to E_i for $i = 1, 2, \ldots, 3g+5$. Since $(3g+5)$ (-2)-curves $E_i - E_{i+1}$, $i = 1, 2, \ldots, 3g+5$ are connected, a reducible singular fibre F_∞ of f contains all of them. However, they do not generate F_∞. By the Shioda–Tate formula (2.2) and $\rho(X) = 3g+8$, the number of reducible fibres of f is at most two.

We suppose that f has a reducible fibre F_1 other than F_∞. Let $\Theta_{1,0}$ be the identity component of F_1. By the Shioda–Tate formula (2.2), we show that F_1 has exactly one component $\Theta_{1,1}$ other than $\Theta_{1,0}$. Since $\Theta_{1,1}$ is disjoint from the zero section $(O) = E_{3g+6}$ and the $(3g+5)$ irreducible components $(E_i - E_{i+1})$ of the other reducible fibre F_∞, we can regard $\Theta_{1,1}$ as an irreducible curve on Σ_d by μ. Thus, $\Theta_{1,1}.G = \Theta_{1,1}.F = 0$ by μ. Therefore, by virtue of Corollary 2.5, we conclude that $\Theta_{1,1}$ must be identified with the minimal section $\Delta_{[n+1]}$ of Σ_d with $d = n+1$ and $g = 3n+1$. Similarly, F_∞ also has exactly one component $\Theta_{\infty,1}$ other than the $(3g+5)$ irreducible components $(E_i - E_{i+1})$. Furthermore, the image of $\Theta_{\infty,1}$ on Σ_{n+1} is linearly equivalent to $3\Delta_{[n+1]} + (3n+3)\Gamma_{[n+1]}$ by Theorem 2.4. Hence, we have

$$\begin{aligned} T &\simeq \mathbb{Z}\Theta_{1,1} \oplus \mathbb{Z}\Theta_{\infty,1} \oplus \bigoplus_{i=1}^{3g+5} \mathbb{Z}(E_i - E_{i+1}) \oplus \mathbb{Z}(O) \\ &\simeq \mathbb{Z}\Delta_{[n+1]} \oplus (3n+3)\mathbb{Z}\Gamma_{[n+1]} \oplus \bigoplus_{i=1}^{3g+6} \mathbb{Z}E_i. \end{aligned}$$

This contradicts the assumption that the Mordell–Weil group of f is trivial by Lemma 2.7.

In this way, F_∞ is the unique reducible fibre of f. By the Shioda–Tate formula (2.2) and $\rho(X) = 3g+8$, the number of irreducible components of F_∞ is $3g+7$. Let $\Theta_{3g+5-i} = E_i - E_{i+1}$, $i = 1, 2, \ldots, 3g+5$. We denote by $\Theta_{3g+6}, \Theta_{3g+5}$ the two components other than them. We remark that μ does not contract Θ_{3g+6} and Θ_{3g+5}. Let D_1 and D_2, respectively, be the images of Θ_{3g+6} and Θ_{3g+5} on Σ_d. Since $\bigoplus_{i=0}^{3g+4} \mathbb{Z}\Theta_i \oplus \mathbb{Z}(O) \simeq \bigoplus_{i=1}^{3g+6} \mathbb{Z}E_i$, by Lemma 2.7 we have

$$\mathbb{Z}D_1 \oplus \mathbb{Z}D_2 \simeq \mathbb{Z}\Delta_{[d]} \oplus \mathbb{Z}\Gamma_{[d]}. \tag{4.5}$$

If one of the two curves D_1 and D_2 is a trisection of Σ_d, which means a horizontal curve meeting a fibre of Σ_d at three points, then the other one must be a fibre of Σ_d. Therefore, $\mathbb{Z}D_1 \oplus \mathbb{Z}D_2 \simeq 3\mathbb{Z}\Delta_{[d]} \oplus \mathbb{Z}\Gamma_{[d]}$, which contradicts (4.5). Thus, at least one of them is a section of Σ_d.

In what follows, we assume that D_1 is a section of Σ_d. In particular, $D_1 \sim \Delta_{[d]} + \gamma\Gamma_{[d]}$ for some non-negative integer γ and is smooth. For $h = 0, 1, \ldots, 3g+4$ and $j = 3g+5$, $3g+6$, we know $\Theta_h.\Theta_j \geq 0$ and $F.\Theta_j = (O).\Theta_j = 0$. Hence we have

$$\Theta_{3g+6} = \Delta_0 + \gamma\Gamma - E_1 - E_2 - \cdots - E_{D_1.G}, \quad D_1.G \leq 3g+5,$$

$$\Theta_{3g+5} = \alpha\Delta_0 + \beta\Gamma - \sum_{i=1}^{3g+5} \delta_i E_i, \quad 2 \geq \alpha \geq \delta_1 \geq \delta_2 \geq \cdots \geq \delta_{3g+5} \geq 0, \quad \sum_{i=1}^{3g+5} \delta_i = D_2.G,$$

$\delta_{D_2.G+1} = 0$ for some non-negative integers α, β. Therefore we have

$$0 \leq \Theta_{3g+6}.\Theta_{3g+5} = \begin{cases} D_1.D_2 - \sum_{i=1}^{D_2.G} \delta_i = D_1.D_2 - D_2.G & \text{if } D_2.G \leq D_1.G, \\ D_1.D_2 - \sum_{i=1}^{D_1.G} \delta_i \leq D_1.D_2 - D_1.G & \text{otherwise.} \end{cases}$$

Thus,

$$\min\{D_1.G,\ D_2.G\} \leq D_1.D_2 \tag{4.6}$$

is a necessary condition for the Mordell–Weil group of f to be trivial.

At first, we concentrate on the case of $\Gamma_{[d]}.D_2 = 1$. Without loss of generality, we may assume that $G = D_1 + 2D_2$. When $D_2.G \leq D_1.G$, we have $D_1.D_2 + 2D_2^2 = D_2.G \leq D_1.D_2$ from (4.6). This implies $D_2^2 \leq 0$. Thus $D_2 = \Delta_{[d]}$. However, $\mathbb{Z}D_1 \oplus \mathbb{Z}D_2 \simeq \mathbb{Z}\Delta_{[d]} \oplus (1 + (g+3d)/2)\mathbb{Z}\Gamma_{[d]}$, which contradicts (4.5). When $D_1.G \leq D_2.G$, we have $D_1^2 + 2D_1.D_2 = D_1.G \leq D_1.D_2$ from (4.6). This and $D_1.D_2 \geq 0$ imply $D_1^2 \leq 0$. Hence $D_1 = \Delta_{[d]}$, which also leads a contradiction to (4.5). Therefore, the case of $\Gamma_{[d]}.D_2 = 1$ does not occur.

By an argument similar to the previous case, we can show that the case of $\Gamma_{[d]}.D_2 = 2$ does not occur again.

At last, we consider the case of $\Gamma_{[d]}.D_2 = 0$. Then D_2 must be a fibre of Σ_d. This implies that $D_1.D_2 = 1$ and $D_2.G = 3$. With the aid of (4.6), we have $D_1.G \leq 1$. Therefore, by Corollary 2.5 we conclude that $\gamma = 0$ and

$$(D_1.G, g, d) = (1, 3n, n), (0, 3n+1, n+1).$$

Conversely, the both situations provide (4.5). Furthermore, since $\Theta_0, \Theta_1, \ldots, \Theta_{3g+6}$ and (O) form a $\mathbb{Z}$-basis of $\mathrm{NS}(X)$, their multiplicities in F_∞ are uniquely determined. Thus we see that F_∞ is a reducible fibre whose dual graph corresponds to the graph as in Figures 1 (resp. 2), when $g = 3n$ (resp. $3n+1$). □

This completes the proof of Theorem 4.1. □

5. Examples.

In [**1**], Beauville pointed out that the minimum number of singular fibres is two over $\mathbb{P}^1$, if $f : X \to \mathbb{P}^1$ is not a trivial fibration. There are many interesting arithmetic and geometric properties in this extreme case (see [**2**]). In this section, first we will discuss some examples of fibrations with only two singular fibres on rational surfaces X. In these examples, the general fibre F is a trigonal curve of genus $g \geq 3$ and $\rho(X) = 3g + 8$.

EXAMPLE 5.1. Let $f : X \to \mathbb{P}^1$ be as in Proposition 3.1. Consider the case where $c_{i,j} = 0$ for the defining equation (3.3). Let $D_{[n],t}$ be a curve on Σ_n defined by $tx^3y^{3n+1} = y + x^3$. We consider $D_{[n],t}$ on the Zariski open subset of Σ_n defined by $X_2Y_1 \neq 0$ and take $(u, z) = (X_0/X_2, Y_0/Y_1)$ as an affine coordinate. Then a local equation of $D_{[n],t}$ on $X_2Y_1 \neq 0$ is $t = z^{3n+1} + u^3$, which is the transition function u^3z times that on $X_0Y_0 \neq 0$ with putting $x = z^n/u$ and $y = 1/z$. Hence $D_{[n],t}$ is smooth unless $t = 0, \infty$, namely, the number of singular fibres of f is two.

EXAMPLE 5.2. Let $f : X \to \mathbb{P}^1$ be as in Proposition 3.3. Consider the case where $c_{i,j} = 0$ for the defining equation (3.4). Let $D_{[n+1],t}$ be a curve on Σ_{n+1} defined by $tx^3y^{3n+3} = 1 + x^3y$. We consider $D_{[n+1],t}$ on the Zariski open subset of Σ_{n+1} defined by $X_2Y_1 \neq 0$ and use the same affine coordinate (u, z) as in Example 5.1. Then a local equation of $D_{[n+1],t}$ on $X_2Y_1 \neq 0$ is $t = u^3 + z^{3n+2}$, which is the transition function u^3 times that on $X_0Y_0 \neq 0$ with putting $x = z^{n+1}/u$ and $y = 1/z$. Therefore, $D_{[n+1],t}$ is smooth unless $t = 0, \infty$, namely, the number of singular fibres of f is also two.

For the converse of Examples 5.1 and 5.2, we have the following.

THEOREM 5.3. *Let X be a smooth rational surface, and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of trigonal curves of genus $g \geq 3$. Assume that $\rho(X) = 3g + 8$, that the Mordell–Weil group of f is trivial and that the number of singular fibres of f is two. Then $g \equiv 0, 1 \pmod 3$ and f is defined explicitly by the equation*: $u^3 = z^{g+1} + t$.

PROOF. We can check that such an equation of f satisfies the following conditions: X is rational, $\rho(X) = 3g + 8$, and $f^{-1}(0)$ and $f^{-1}(\infty)$ are the only two singular fibres. Especially, $f^{-1}(\infty)$ is the fibre appearing in Proposition 3.1 or 3.3 according as $g \equiv 0$ or $1 \pmod 3$, respectively. In particular, the Mordell–Weil group of f is trivial.

We claim that such f is unique. Because the number of singular fibres of f is two, f is isotrivial by [**1**]. It is enough to prove that a general fibre of f is uniquely determined. Let $F_0 = f^{-1}(0)$ be the singular fibre with a topological monodromy $\sigma_1/\lambda_1 + \sigma_2/\lambda_2 + \sigma_3/\lambda_3$ and a principal component $\Gamma \cong \mathbb{P}^1$. Consider the base change $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ defined by $t = w^n$. Then we get a trivial fibration $\tilde{f} : F \times \mathbb{P}^1 \to \mathbb{P}^1$ where F is isomorphic to a general fibre of f. Restricting $\tilde{f}$ to the fibre $F = \tilde{f}^{-1}(0)$, we has a cyclic cover $\tilde{f}|_F : F \to \Gamma$ ramified over three points $0, 1, \infty$ on Γ by taking a suitable Möbius transform. More precisely, the cyclic cover is defined by $z^n = x^{n\sigma_1/\lambda_1}(x - 1)^{n\sigma_2/\lambda_2}$. So F is uniquely determined by the topological monodromy of F_0. □

By Theorem 4.1, we know that if $g \equiv 2 \pmod 3$, then the Mordell–Weil group of f cannot be trivial. Here, for an arbitrary positive integer n, we give a trigonal fibration $f : X \to \mathbb{P}^1$ with $g = 3n + 2$ and $\rho(X) = 3g + 8$ whose Mordell–Weil group is $\mathbb{Z}/3\mathbb{Z}$.

EXAMPLE 5.4. On Σ_n we consider the pencil Λ whose members $D_{[n],t}$ are defined by $y + x^3 + tx^3y^{3n+2} = 0$ on $X_0Y_0 \neq 0$, and $u^3z + z^{3n+2} + t = 0$ on $X_2Y_1 \neq 0$ with $t \in \mathbb{P}^1$, where (u, z) denotes the same affine coordinate as in Example 5.1. In particular, $D_{[n],\infty} = 3\Delta_{[n]} + (3n+2)\Gamma_{[n]}$. Let $G_{0,0}$ be a unicuspidal curve defined by $u^3 + z^{3n+1} = 0$. Remark that $D_{[n],0} = \Gamma_{[n],\infty} + G_{0,0}$ and $G_{0,0}$ is linearly equivalent to $3\Delta_{[n]} + (3n+1)\Gamma_{[n]}$, where $\Gamma_{[n],\infty}$ is the fibre of Σ_n defined by $z = 0$. Furthermore, it has a contact of order 3 with $\Gamma_{[n]}$ at the point P_4 defined by $(x, y) = (0, 0)$. So the other members $D_{[n],t}$ (with $t \neq 0, \infty$) has a contact of order 3 with $\Gamma_{[n]}$ at P_4 and also has a contact of order 3 with $\Gamma_{[n],\infty}$ at the point P_1 defined by $((X_0 : X_1 : X_2), (Y_0 : Y_1)) = ((1 : 0 : 0), (0 : 1))$. Furthermore, we can check the other members $D_{[n],t}$ are smooth curves.

Any two members in Λ are disjoint on $\Sigma_n \setminus \{P_1, P_4\}$. In particular, the $(9n + 12)$ base points of Λ consist of the point P_1, P_4 and their infinitely near points. In addition, $D_{[n],t}$ with $t \neq 0, \infty$ are trigonal curves of genus $g = 3n + 2$. Therefore, we obtain a relatively minimal fibration $f : X \to \mathbb{P}^1$ of trigonal curves of genus $g = 3n + 2$ from $\Phi_\Lambda : \Sigma_n \dashrightarrow \mathbb{P}^1$ by eliminating the base points of Λ as follows:

Let $\mu_{9n+12} : W_1 \to \Sigma_n$ be the blowing-up at the point P_1 with the exceptional curve E_1, i.e., $\mu_{9n+12}(E_1) = P_1$. Let P_2 be the intersection point of E_1 and the strict transform to W_1 of $\Gamma_{[n],\infty}$. The strict transform to W_1 of $D_{[n],t}$ has a contact of order 2 with that of $\Gamma_{[n],\infty}$ at P_2 for all $t \in \mathbb{C} \setminus \{0\}$. Next let $\mu_{9n+11} : W_2 \to W_1$ be the blowing-up at the base point P_2 with $E_2 = \mu^{-1}_{9n+11}(P_2)$. Let P_3 denote the intersection point of E_2 and the strict transform to W_2 of $\Gamma_{[n],\infty}$. For all $t \in \mathbb{C} \setminus \{0\}$ the strict transform to W_2 of $D_{[n],t}$ meets that of $\Gamma_{[n],\infty}$ transversally at P_3. Denote the pull-back of curves by the same symbols for simplicity. Then we get the irreducible decomposition

$$D_{[n],\infty} - E_1 - E_2 = 3(\Delta_{[n]} - E_1) + (3n + 2)\Gamma_{[n]} + 2(E_1 - E_2) + E_2.$$

Furthermore, $D_{[n],t} - E_1 - E_2$ meets the other members at P_3 transversally for all $t \in \mathbb{P}^1$. Denote by $\mu_{9n+10} : W_3 \to W_2$ the blowing-up at the base point P_3 with $E_3 = \mu^{-1}_{9n+10}(P_3)$.

We can regard P_4 as a point on W_3 by $\mu_{9n+12} \circ \mu_{9n+11} \circ \mu_{9n+10}$. Any two of the strict transform to W_3 of the members in Λ are disjoint on $W_3 \setminus \{P_4\}$. In fact, $D_{[n],t} - E_1 - E_2 - E_3$ has a contact of order $(9n + 9)$ with the other members at P_4 for all $t \in \mathbb{C}$. Let $\mu_{9n+9} : W_4 \to W_3$ be the blowing-up at the point P_4 with the exceptional curve E_4, i.e., $\mu_{9n+9}(E_4) = P_4$. Let P_5 be the intersection point of E_4 and the strict transform to W_4 of $\Gamma_{[n]}$. The strict transform to W_4 of $D_{[n],t}$ has a contact of order 2 with that of $\Gamma_{[n]}$ at P_5 for all $t \in \mathbb{C}$. Next let $\mu_{9n+8} : W_5 \to W_4$ be the blowing-up at the base point P_5 with $E_5 = \mu^{-1}_{9n+8}(P_5)$. Let P_6 denote the intersection point of E_5 and the strict transform to W_5 of $\Gamma_{[n]}$. For all $t \in \mathbb{C}$ the strict transform to W_5 of $D_{[n],t}$ meets that of $\Gamma_{[n]}$ transversally at P_6. Denote the pull-back of curves by the same symbols for simplicity. Then we get the irreducible decomposition

$$\begin{aligned} &D_{[n],\infty} - E_1 - E_2 - E_3 - E_4 - E_5 \\ &\quad = (3n + 4)(E_4 - E_5) + (6n + 5)E_5 + (3n + 2)(\Gamma_{[n]} - E_4 - E_5) \\ &\qquad + 2(E_1 - E_2) + (E_2 - E_3) + 3(\Delta_{[n]} - E_1 - E_4). \end{aligned}$$

Furthermore, $D_{[n],t} - E_1 - E_2 - E_3 - E_4 - E_5$ has a contact of order $(9n + 7)$ with the other members at P_6 for all $t \in \mathbb{C}$. Denote by $\mu_{9n+7} : W_6 \to W_5$ the blowing-up at the base point P_6 with $E_6 = \mu_{9n+7}^{-1}(P_6)$. Let P_7 be the intersection point of E_6 and the strict transform to W_6 of $D_{[n],t}$. In fact, P_7 corresponds to a tangent direction of $D_{[n],t}-E_1-E_2-E_3-E_4-E_5$ at P_6 on W_5 by μ_{9n+1}, and $D_{[n],t}-E_1-E_2-E_3-E_4-E_5-E_6$ has a contact of order $(9n+6)$ with the other members at P_7 for all $t \in \mathbb{C}$.

In the same way, for $i = 7, 8, \ldots, 9n+11$, after the blowing-up $\mu_{9n+13-i} : W_i \to W_{i-1}$ at the base point P_i with $E_i = \mu_{9n+13-i}^{-1}(P_i)$, $D_{[n],t} - E_1 - E_2 - \cdots - E_i$ has a contact of order $(9n+12-i)$ with the other members at P_{i+1}. Let $\mu_1 : X \to W_{9n+11}$ be the blowing-up at the base point P_{9n+12} with $E_{9n+12} = \mu_1^{-1}(P_{9n+12})$. Put $f = \Phi_\Lambda \circ \mu_{9n+12} \circ \mu_{9n+11} \circ \cdots \circ \mu_1$. Then $f : X \to \mathbb{P}^1$ is a relatively minimal fibration whose general fibre F is $D_{[n],t} - E_1 - E_2 - \cdots - E_{9n+12}$ for $t \in \mathbb{C} \setminus \{0\}$, and the reducible fibres are $f^{-1}(0)$ and $f^{-1}(\infty)$. We remark that E_3 and E_{9n+12} are (-1)-sections of f. The configuration of the two (-1)-sections, F, the irreducible components of $f^{-1}(0)$ and that of $f^{-1}(\infty)$ is as in Figure 5.

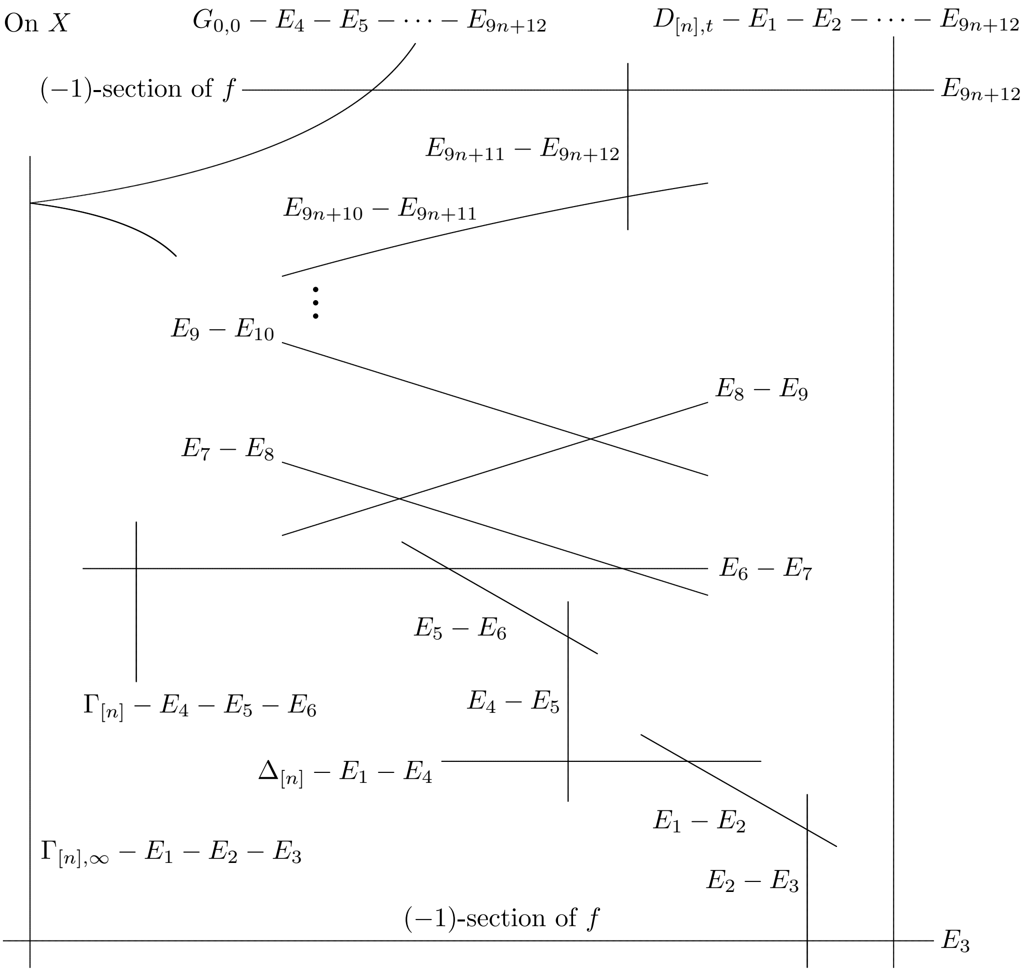

Figure 5.

Let E_{3g+6} be the zero section and $\Theta_{0,0} = G_{0,0} - E_4 - E_5 - \cdots - E_{3g+6}$, $\Theta_{0,1} = \Gamma_{[n],\infty} - E_1 - E_2 - E_3$, $\Theta_{\infty,3g+5-i} = E_i - E_{i+1}$, $i = 4, 5, \ldots, 3g+5$, $\Theta_{\infty,3g+2} = \Gamma_{[n]} - E_4 - E_5 - E_6$, $\Theta_{\infty,3g+3} = E_2 - E_3$, $\Theta_{\infty,3g+4} = E_1 - E_2$, $\Theta_{\infty,3g+5} = \Delta_{[n]} - E_1 - E_4$. Then we have the irreducible decompositions $f^{-1}(0) = \Theta_{0,0} + \Theta_{0,1}$ and

$$\begin{aligned} f^{-1}(\infty) &= 3\Theta_{\infty,3g+5} + 2\Theta_{\infty,3g+4} + \Theta_{\infty,3g+3} + (3n+2)\Theta_{\infty,3g+2} \\ &\quad + (3n+4)\Theta_{\infty,3g+1} + (6n+5)\Theta_{\infty,3g} + \sum_{j=1}^{3g}(3g+1-j)\Theta_{\infty,3g-j}. \end{aligned}$$

Furthermore, we see that the dual graph of the reducible fibre $f^{-1}(\infty)$ corresponds to the graph as in Figure 6. Here, C is a (-3)-curve, A is a $(-n-2)$-curve, and the other circles

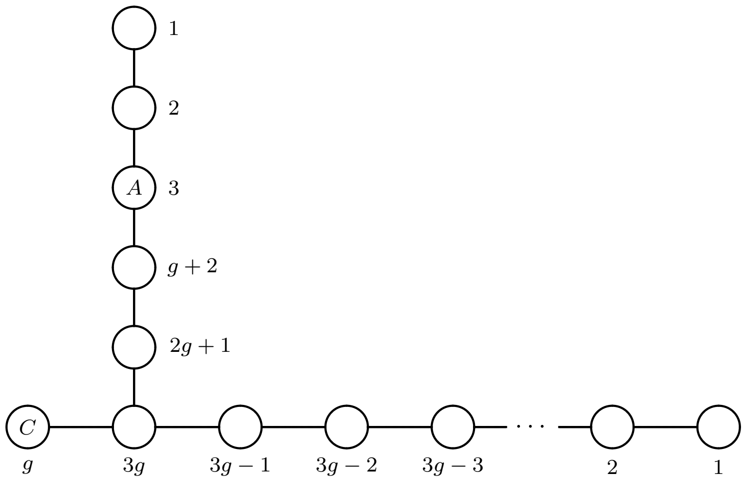

Figure 6.

denote (-2)-curves. The numbers indicated outside the circles denote the multiplicities of components in the degenerated fibre. We also have

$$\begin{aligned} T &= \mathbb{Z}E_{3g+6} \oplus \mathbb{Z}F \oplus \mathbb{Z}\Theta_{0,1} \oplus \bigoplus_{i=1}^{3g+5} \mathbb{Z}\Theta_{\infty,i} \simeq \mathbb{Z}E_{3g+6} \oplus \mathbb{Z}\Theta_{0,1} \oplus \bigoplus_{i=0}^{3g+5} \mathbb{Z}\Theta_{\infty,i} \\ &\simeq 3\mathbb{Z}E_3 \oplus \mathbb{Z}(\Delta_{[n]} - E_1) \oplus \mathbb{Z}(E_1 - E_2) \oplus \mathbb{Z}(E_2 - E_3) \oplus \mathbb{Z}\Gamma_{[n]} \oplus \bigoplus_{i=4}^{3g+6} \mathbb{Z}E_i. \end{aligned}$$

If $2E_3 \in T$, then $3E_3 - 2E_3 = E_3 \in T$. This implies the Mordell–Weil group of f is trivial by Lemma 2.7. However, f has two (-1)-sections, which is a contradiction. Therefore the Mordell–Weil group of f is $\mathbb{Z}/3\mathbb{Z}$ from [**16**, Theorem 3].

Question. Let X be a smooth rational surface, and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of trigonal curves of genus $g \geq 3$. Suppose that $\rho(X) = 3g + 8$, and $g \equiv 2 \pmod 3$, is there a fibration whose Mordell–Weil group is $\mathbb{Z}/2\mathbb{Z}$?

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