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A mass transportation proof of the sharp one-dimensional Gagliardo–Nirenberg inequalities

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Abstract. The aim of this paper is to give a mass transportation proof for a full family of sharp Gagliardo–Nirenberg inequalities in dimension one. In fact, we shall establish a duality principle which derives this family of inequalities as a consequence. We also characterize all optimizers for these inequalities via the mass transportation method.

1. Introduction.

In this paper, we consider the following family of one-dimensional Gagliardo–Nirenberg inequalities that can be written as

$$\|f\|_{L^m(\mathbb{R})} \le C_{GN}(p,q,m)\|f'\|_{L^p(\mathbb{R})}^{\theta}\|f\|_{L^q(\mathbb{R})}^{1-\theta} \tag{1.1}$$

for $p > 1$ and $1 \le q < m < \infty$, where $\theta = (m-q)p/m(p+pq-q)$ and

$$C_{GN}(p,q,m) = \left(\frac{2^p(p-1)^{1-p}((p-1)m+p)^{\frac{(p-1)m+p}{m-q}}}{(m-q)^{2p-1}((p-1)q+p)^{\frac{(p-1)q+p}{m-q}}}\, B\left(\frac{(p-1)q+p}{p(m-q)}, \frac{2p-1}{p}\right)^p\right)^{\frac{q-m}{m((p-1)q+p)}}. \tag{1.2}$$

Here we use the notation $\|g\|_{L^r(\mathbb{R})} = (\int_{\mathbb{R}} |g|^r dx)^{1/r}$ for any $r \ge 1$ and for any measurable function g on $\mathbb{R}$. This family was obtained by Szőkefalvi-Nagy [**26**]. The equality holds in (1.1) for function f of the form

$$f(x) = \left(1 - B^{-1}\left(|x|; 1-\frac{1}{p}, \frac{p-q}{p(m-q)}\right)\right)^{1/(m-q)},$$

if $q \ge p$, and

$$f(x) = \left(B^{-1}\left(B\left(\frac{p-q}{p(m-q)}, \frac{p-1}{p}\right)(1-|x|)_+; \frac{p-q}{p(m-q)}, 1-\frac{1}{p}\right)\right)^{1/(m-q)},$$

if $q < p$. Here we use the notation $a_+ = \max\{a, 0\}$, and $B^{-1}(x;a,b)$ with $a > 0$ denotes the inverse function of the incomplete beta function $B(x;a,b)$ which is defined as

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$$B(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt, \quad 0 \le x < 1. \tag{1.3}$$

A new proof of (1.1) was recently given by Liu and Wang [**19**]. Indeed, a simple variational argument shows that a minimizer for the functional

$$J(f) = \frac{\|f'\|_{L^p(\mathbb{R})}^p \|f\|_{L^q(\mathbb{R})}^{p(1-\theta)/\theta}}{\|f\|_{L^m(\mathbb{R})}^{p/\theta}}$$

exists and satisfies an Euler–Lagrange equation. Using the positivity and uniqueness of solutions to this Euler–Lagrange equation (see [**24**], [**25**]), Liu and Wang found out the explicit form of the solution to this Euler–Lagrange equation in terms of incomplete beta function (1.3) above and hence compute exactly the sharp constant in (1.1).

The Gagliardo–Nirenberg inequalities were established in the higher dimensions independently by Gagliardo and Nirenberg. We refer the readers to [**15**], [**16**], [**23**] for the original papers. Among the Gagliardo–Nirenberg inequalities, there are only a few cases for which best constants are explicit and optimal functions can be characterized. For example, the sharp constant and optimal functions in Nash's inequality (see [**20**]) were found by Carlen and Loss [**8**] and some sharp interpolation inequalities on the sphere were established by Beckner [**5**] and by Bidaut-Véron and Véron [**6**]. Another subfamily of Gagliardo–Nirenberg inequalities for which the best constants and the optimal functions are explicit was obtained by Del Pino and Dolbeault [**11**], [**12**],

$$\|f\|_{L^{\alpha p}(\mathbb{R}^n)} \le D_{GN} \|\nabla f\|_{L^p(\mathbb{R}^n)}^{\theta} \|f\|_{L^{\alpha(p-1)+1}(\mathbb{R}^n)}^{1-\theta},$$

if $\alpha \in (1, n/(n-p))$ when $n > p$ and $\alpha < \infty$ when $n \le p$, and

$$\|f\|_{L^{\alpha(p-1)+1}(\mathbb{R}^n)} \le D_{GN} \|\nabla f\|_{L^p(\mathbb{R}^n)}^{\theta} \|f\|_{L^{\alpha p}(\mathbb{R}^n)}^{1-\theta},$$

if $\alpha \in (0,1)$ with appropriate values of θ. Another proof of these inequalities was given by Cordero-Erausquin, Nazaret and Villani by using the mass transportation method [**10**]. A systematical study on the best constants and the optimal functions of the Gagliardo–Nirenberg inequalities can be found in the paper of Liu and Wang [**19**]. In that paper, Liu and Wang use the variational method to obtain some explicit results for the best constants and optimal functions of the Gagliardo–Nirenberg inequality which includes the one of Del Pino and Dolbeault. The weighted Gagliardo–Nirenberg inequalities in the half space were proved by the author in [**21**] via the mass transportation method.

The aim of this paper is to show that the mass transportation method can be applied to give an alternative proof of (1.1). A proof by using the mass transportation method of (1.1) in the case $p = 2$ was given by Dolbeault, Esteban, Laptev and Loss [**13**]. In that paper, they also gave another proof of (1.1) in this case by using the nonlinear flows. We also refer the readers to the papers [**1**], [**2**] for an earlier proof of (1.1) with $p = 2$ by exploiting the relation between the Gagliardo–Nirenberg inequalities and mass transport theory. In particular, in those papers, Agueh investigated how Barenblatt functions are transformed into optimal functions for the inequalities, and gave and expression of the explicit transport map in the case of optimal functions.

Let us denote by $\mathcal{D}_q^{1,p}(\mathbb{R})$ the space of all functions $f \in L^q(\mathbb{R})$ such that f' (in the distributional sense) belongs to $L^p(\mathbb{R})$. By this notation, we have $\mathcal{D}_p^{1,p}(\mathbb{R}) = W^{1,p}(\mathbb{R})$ the usual Sobolev space on $\mathbb{R}$. We also denote by $L_q^1(\mathbb{R})$ the space of $G \in L^1(\mathbb{R})$ such that $\int_{\mathbb{R}} |G||x|^q dx < \infty$. We define the constant

$$c_{p,q,m} = \left(\frac{m(p-1)+p}{p}\right)^{p(m-q)/(m(2p-1)-p(q-1))}. \tag{1.4}$$

Using the mass transportation method, we shall establish the following duality principle related to the Gagliardo–Nirenberg inequality (1.1).

THEOREM 1.1. *Let $p > 1$ and $1 \le q < m < \infty$. Then the following relation holds,*

$$\sup_{G \in L^1_{p'}(\mathbb{R}), G \ge 0} \frac{\int_{\mathbb{R}} G^{\frac{m(p-1)+p}{m(2p-1)-p(q-1)}} dy}{\left(\int_{\mathbb{R}} G|y|^{\frac{p}{p-1}} dy\right)^{\frac{(m-q)(p-1)}{m(2p-1)-p(q-1)}} \left(\int_{\mathbb{R}} G dy\right)^{\frac{p+q(p-1)}{m(2p-1)-p(q-1)}}}$$
$$= c_{p,q,m} \inf_{f \in \mathcal{D}_q^{1,p}(\mathbb{R}), f \ge 0} \frac{\|f'\|_{L^p(\mathbb{R})}^{\frac{p(m-q)}{m(2p-1)-p(q-1)}} \|f\|_{L^q(\mathbb{R})}^{\frac{q(m(p-1)+p)}{m(2p-1)-p(q-1)}}}{\|f\|_{L^m(\mathbb{R})}^{\frac{m(p+q(p-1))}{m(2p-1)-p(q-1)}}} \tag{1.5}$$

with $p' = p/(p-1)$.

Moreover, we have the following conclusions:

(i) *If $q \ge p$, then the right-hand side of* (1.5) *is minimized by*

$$f(x) = \left(1 - B^{-1}\left(|x|; 1 - \frac{1}{p}, \frac{p-q}{p(m-q)}\right)\right)^{1/(m-q)},$$

while the left-hand side of (1.5) *is maximized by*

$$G(y) = (1 + |y|^{p'})^{-(m(2p-1)-p(q-1))/p(m-q)}.$$

(ii) *If $1 \le q < p$, then the right-hand side of* (1.5) *is minimized by*

$$f(x) = \left(B^{-1}\left(B\left(\frac{p-q}{p(m-q)}, \frac{p-1}{p}\right)(1-|x|)_+; \frac{p-q}{p(m-q)}, 1 - \frac{1}{p}\right)\right)^{1/(m-q)},$$

while the left-hand side of (1.5) *is maximized by*

$$G(y) = (1 + |y|^{p'})^{-(m(2p-1)-p(q-1))/p(m-q)}.$$

In the case $p = 2$, Theorem 1.1 was established by Dolbeault, Esteban, Laptev and Loss (see Theorem 1.1 in [**13**]). From Theorem 1.1, we see that all variational problems in (1.5) have explicit extremal functions. This fact gives us an efficient method for computing the sharp constant $C_{GN}(p, q, m)$ in (1.1).

The rest of this paper is organized as follows. In Section 2, we give the proof of Theorem 1.1 via the mass transportation method. We also consider the threshold case

$m = q = p$ to establish a dual principle for the general one-dimensional logarithmic Sobolev inequality in this section. Section 3 is devoted to compute the sharp constant in (1.1) and to discuss the optimal functions for (1.1). In particular, we shall characterize all the optimal functions for (1.1) up to a translation, a dilation and a multiplicative constant by using the mass transportation method in this section.

2. Proof of Theorem 1.1 via mass transportation method.

This section is devoted to prove Theorem 1.1. We also investigate the thresholds case corresponding to $m = p = q$ to establish a dual principle for the general one-dimensional logarithmic Sobolev inequality. Our proof is based on the mass transportation method in dimension one. The mass transportation method is now an useful tool to prove several sharp inequalities analysis and geometry (e.g., see [**1–4**], [**7**], [**9**], [**10**], [**14**], [**21**], [**22**] and references therein). We refer the readers to the book [**27**] for more background on this method and its developments.

PROOF OF THEOREM 1.1. We first prove (1.5). By density, it is enough to consider the infimum on the right-hand side of (1.5) for functions $f \in C_0^\infty(\mathbb{R})$, $f \geq 0$. We start the proof by recalling some basic facts from optimal transportation theory in dimension one. Let F, G be two probability densities on $\mathbb{R}$, i.e., F and G are non-negative functions, $F \in C_0^1(\mathbb{R})$ and $\int_{\mathbb{R}} F(x)dx = \int_{\mathbb{R}} G(x)dx = 1$. For any $t \in \mathbb{R}$, define

$$\Phi(t) = \int_{-\infty}^t F(x)dx, \quad \text{and} \quad \Psi(t) = \int_{-\infty}^t G(x)dx.$$

Then $\Phi, \Psi : \mathbb{R} \to [0,1]$ are non-decreasing functions. Define $\varphi(t) = \Psi^{-1}(\Phi(t))$. Then φ is increasing function and we have

$$\int_{-\infty}^t F(x)dx = \int_{-\infty}^{\varphi(t)} G(x)dx.$$

Differentiating in t, we get

$$F(t) = G(\varphi(t))\varphi'(t), \quad \text{for almost everywhere } t \in \mathbb{R}. \tag{2.1}$$

Let $\theta \in (0,1)$ be fixed later, by making the change of variable $y = \varphi(x)$ and using (2.1), we have

$$\int_{\mathbb{R}} G(y)^\theta dy = \int_{\mathbb{R}} G(\varphi(x))^\theta \varphi'(x)dx = \int_{\mathbb{R}} F(x)^\theta \varphi'(x)^{1-\theta} dx. \tag{2.2}$$

For any $\alpha \in (0, \theta)$, by using Hölder's inequality, we get

$$\begin{aligned}\int_{\mathbb{R}} F(x)^\theta \varphi'(x)^{1-\theta} dx &= \int_{\mathbb{R}} F(x)^{\theta-\alpha} F(x)^\alpha \varphi'(x)^{1-\theta} dx \\ &\leq \left(\int_{\mathbb{R}} F(x)^{1-\alpha/\theta} dx\right)^\theta \left(\int_{\mathbb{R}} F(x)^{\alpha/(1-\theta)} \varphi'(x) dx\right)^{1-\theta}. \end{aligned} \tag{2.3}$$

Considering the last integral and using integration by parts, we have

$$\int_{\mathbb{R}} F(x)^{\alpha/(1-\theta)}\varphi'(x)dx = -\frac{\alpha}{1-\theta}\int_{\mathbb{R}} F(x)^{\alpha/(1-\theta)-1}F'(x)\varphi(x)dx. \tag{2.4}$$

We now choose θ and α such that

$$1-\frac{\alpha}{\theta} = \frac{q}{m}, \quad \text{and} \quad \frac{\alpha}{1-\theta} - \frac{1}{m} = \frac{p-1}{p}.$$

A simple computation shows that

$$\alpha = \frac{m-q}{m}\frac{m(p-1)+p}{m(2p-1)-p(q-1)}, \quad \text{and} \quad \theta = \frac{m(p-1)+p}{m(2p-1)-p(q-1)}. \tag{2.5}$$

Applying Hölder's inequality to (2.4) and using (2.1), we get

$$\begin{aligned}\int_{\mathbb{R}} F(x)^{\frac{\alpha}{1-\theta}}\varphi'(x)dx &= -\frac{p(m+1)-m}{p}\int_{\mathbb{R}} F(x)^{\frac{p-1}{p}}\varphi(x)(F^{\frac{1}{m}})'(x)dx \\ &\le \frac{p(m+1)-m}{p}\left(\int_{\mathbb{R}} F(x)|\varphi(x)|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}} |(F^{\frac{1}{m}})'|^p dx\right)^{\frac{1}{p}} \\ &= \frac{p(m+1)-m}{p}\left(\int_{\mathbb{R}} G(y)|y|^{\frac{p}{p-1}}dy\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}} |(F^{\frac{1}{m}})'|^p dx\right)^{\frac{1}{p}}. \end{aligned} \tag{2.6}$$

By taking $F = f^m$, we have from (2.2), (2.3) and (2.6) and the choice (2.5) of α and θ

$$\frac{\int_{\mathbb{R}} G^\theta dy}{\left(\int_{\mathbb{R}} G(y)|y|^{\frac{p}{p-1}}dy\right)^{\frac{(1-\theta)(p-1)}{p}}} \le \left(\frac{p(m+1)-m}{p}\right)^{1-\theta}\left(\int_{\mathbb{R}} f^q dx\right)^{\theta}\left(\int_{\mathbb{R}} |f'|^p dx\right)^{\frac{1-\theta}{p}} \tag{2.7}$$

for any non-negative function f, G such that $\int_{\mathbb{R}} G dy = \int_{\mathbb{R}} f^m dx = 1$ with θ given by (2.5). Taking into account the homogeneity, the inequality (2.7) yields

$$\frac{\int_{\mathbb{R}} G^{\frac{m(p-1)+p}{m(2p-1)-p(q-1)}}dy}{\left(\int_{\mathbb{R}} G|y|^{\frac{p}{p-1}}dy\right)^{\frac{(m-q)(p-1)}{m(2p-1)-p(q-1)}}\left(\int_{\mathbb{R}} G dy\right)^{\frac{p+q(p-1)}{m(2p-1)-p(q-1)}}} \le c_{p,q,m}\frac{\|f'\|_{L^p(\mathbb{R})}^{\frac{p(m-q)}{m(2p-1)-p(q-1)}}\|f\|_{L^q(\mathbb{R})}^{\frac{q(m(p-1)+p)}{m(2p-1)-p(q-1)}}}{\|f\|_{L^m(\mathbb{R})}^{\frac{m(p+q(p-1))}{m(2p-1)-p(q-1)}}},$$

for any non-negative functions $G \in L^1_{p'}(\mathbb{R})$ and $f \in \mathcal{D}^{1,p}_q(\mathbb{R})$. In other words, we have the following inequality

$$\sup_{G\in L^1_{p'}(\mathbb{R}), G\ge 0}\frac{\int_{\mathbb{R}} G^{\frac{m(p-1)+p}{m(2p-1)-p(q-1)}}dy}{\left(\int_{\mathbb{R}} G|y|^{\frac{p}{p-1}}dy\right)^{\frac{(m-q)(p-1)}{m(2p-1)-p(q-1)}}\left(\int_{\mathbb{R}} G dy\right)^{\frac{p+q(p-1)}{m(2p-1)-p(q-1)}}}$$

$$\leq c_{p,q,m} \inf_{f\in\mathcal{D}^{1,p}_q(\mathbb{R}), f\geq 0} \frac{\|f'\|_{L^p(\mathbb{R})}^{\frac{p(m-q)}{m(2p-1)-p(q-1)}} \|f\|_{L^q(\mathbb{R})}^{\frac{q(m(p-1)+p)}{m(2p-1)-p(q-1)}}}{\|f\|_{L^m(\mathbb{R})}^{\frac{m(p+q(p-1))}{m(2p-1)-p(q-1)}}}. \tag{2.8}$$

We next prove that (2.8) is indeed an equality. To do this, we trace back to the equality case in the previous applications of Hölder's inequality. We have the following observations. Firstly, equality holds in (2.3) if

$$\lambda F(x)^{1-\alpha/\theta} = \varphi'(x)F(x)^{\alpha/(1-\theta)},$$

for some $\lambda > 0$, or equivalently

$$\varphi'(x) = \lambda F(x)^{1-\alpha/\theta(1-\theta)}. \tag{2.9}$$

Equality holds in (2.6) if

$$\varphi(x)|\varphi(x)|^{p'-2} = \mu(-F'(x))F(x)^{1/m-1/p-1}, \tag{2.10}$$

for some $\mu > 0$. Hence, it holds

$$\varphi'(x)\varphi(x)|\varphi(x)|^{p'-2} = c(-F'(x))F(x)^{1/m-1/p-\alpha/\theta(1-\theta)},$$

for some $c > 0$. Let x_0 be a point where $\varphi(x_0) = 0$, the previous equality implies

$$|\varphi(x)|^{p'} = c_1\left(F(x)^{-(m-q)/m} - F(x_0)^{-(m-q)/m}\right),$$

for some $c_1 > 0$, here we used $1/m - 1/p - \alpha/\theta(1-\theta) + 1 = -(m-q)/m$. Hence we have

$$F(x) = \left(F(x_0)^{-(m-q)/m} + \frac{|\varphi(x)|^{p'}}{c_1}\right)^{-m/(m-q)}.$$

The preceding expression of F together with (2.9) and (2.1) suggests us to consider the function G having the form $G(y) = (1+|y|^{p'})^{-(m(2p-1)-p(q-1))/p(m-q)}$. On the other hand, from (2.10), we have

$$(-F'(x))F(x)^{\frac{p-m}{mp}-1}\left(F(x)^{-\frac{m-q}{m}} - F(x_0)^{-\frac{m-q}{m}}\right)^{-\frac{1}{p}} = c_2 \mathrm{sign}(x-x_0), \tag{2.11}$$

for some $c_2 > 0$. If $q \geq p$, integrating the equality (2.11) from x_0 to x, we get

$$F(x) = F(x_0)\left(1 - B^{-1}\left(c_3|x-x_0|; \frac{1}{p'}, \frac{p-q}{p(m-q)}\right)\right)^{m/(m-q)},$$

for some $c_3 > 0$. If $q < p$, integrating the equality (2.11) from x_0 to x, we get

$$c_2|x-x_0| = \int_{x_0}^x (-F'(y))F(y)^{\frac{p-m}{mp}-1+\frac{m-q}{mp}}\left(1 - \left(\frac{F(y)}{F(x_0)}\right)^{\frac{m-q}{m}}\right)^{-\frac{1}{p}} dy$$

$$= \frac{m}{m-q} F(x_0)^{\frac{p-q}{mp}} \int_{\left(\frac{F(x)}{F(x_0)}\right)^{\frac{m-q}{m}}}^{1} (1-s)^{-\frac{1}{p}} s^{\frac{p-q}{p(m-q)}-1} ds$$

$$= \frac{m}{m-q} F(x_0)^{\frac{p-q}{mp}} \left(B\left(\frac{p-q}{p(m-q)}, \frac{1}{p'}\right) - B\left(\left(\frac{F(x)}{F(x_0)}\right)^{\frac{m-q}{m}}; \frac{p-q}{p(m-q)}, \frac{1}{p'}\right)\right).$$

Since $p > q$, then the function F must have compact support and have the form

$$F(x) = F(x_0) \left(B^{-1} \left(B\left(\frac{p-q}{p(m-q)}, \frac{1}{p'}\right) \left(1 - \frac{|x-x_0|}{R}\right)_+ ; \frac{p-q}{p(m-q)}, \frac{1}{p'}\right)\right)^{m/(m-q)},$$

for some $R > 0$. The observations above suggest us the functions which are optimizers for the variational problems on the left-hand side and right-hand side of (1.5).

Proof of (1.5) for $p \le q$ and part (i). Following the previous observations, let us consider the function

$$f_*(x) = C \left(1 - B^{-1}\left(|x|; 1 - \frac{1}{p}, \frac{p-q}{p(m-q)}\right)\right)^{1/(m-q)},$$

where C is chosen such that $\int_{\mathbb{R}} f_*^m dx = 1$. A simple computation shows that

$$C^m = \frac{1}{2} \left[B\left(\frac{p-1}{p}, \frac{p(m+1)-q}{p(m-q)}\right)\right]^{-1}.$$

We next define the function $G_* \in L^1_{p'}(\mathbb{R})$ by

$$G_*(x) = p' C^m (1 + |x|^{p'})^{-(m(2p-1)-p(q-1))/p(m-q)}.$$

Obviously, $\int_{\mathbb{R}} G_* dy = 1$. Denote $F_*(x) = f_*(x)^m$. It is easy to check that

$$F_*(x) = G_*(\varphi(x))\varphi'(x),$$

with

$$\varphi(x) = \operatorname{sign}(x) \left(\frac{B^{-1}\big(|x|; 1 - 1/p, (p-q)/p(m-q)\big)}{1 - B^{-1}\big(|x|; 1-1/p, (p-q)/p(m-q)\big)} \right)^{1/p'}.$$

With this choice of φ, we have equalities in (2.3) and (2.6) for F, G replaced by F_*, and G_* respectively. Thus, (2.7) becomes an equality for G_* and f_* which implies the equality in (2.8). This proves (1.5) for $q \ge p$ and part (i).

Proof of (1.5) for $q < p$ and part (ii). Following the previous observations, let us consider the function

$$f_\star(x) = D \left(B^{-1} \left(B\left(\frac{p-q}{p(m-q)}, \frac{1}{p'}\right) (1 - |x|)_+ ; \frac{p-q}{p(m-q)}, 1 - \frac{1}{p}\right)\right)^{1/(m-q)},$$

where D is chosen such that $\int_{\mathbb{R}} f_\star^m dx = 1$. A simple computation shows that

$$D^m = B\left(\frac{p-q}{p(m-q)}, \frac{1}{p'}\right) C^m.$$

We next define the function $G_* \in L^1_{p'}(\mathbb{R})$ by

$$G_*(x) = p' C^m (1 + |x|^{p'})^{-(m(2p-1)-p(q-1))/p(m-q)}.$$

Obviously, $\int_{\mathbb{R}} G_* dy = 1$. Denote $F_\star(x) = f_\star(x)^m$. It is easy to check that

$$F_\star(x) = G_*(\varphi(x))\varphi'(x), \quad -1 < x < 1,$$

with

$$\varphi(x) = \mathrm{sign}(x) \left(\frac{1 - B^{-1}\left(B\left(\frac{p-q}{p(m-q)}, \frac{1}{p'}\right)(1-|x|)_+; \frac{p-q}{p(m-q)}, \frac{1}{p'}\right)}{B^{-1}\left(B\left(\frac{p-q}{p(m-q)}, \frac{1}{p'}\right)(1-|x|)_+; \frac{p-q}{p(m-q)}, \frac{1}{p'}\right)}\right)^{\frac{1}{p'}}.$$

With this choice of φ, we have equalities in (2.3) and (2.6) for F, G replaced by $F_\star$, and G_* respectively (notice that the integral of $F_\star$ is taken in $(-1,1)$). Thus, (2.7) becomes an equality for G_* and $f_\star$ which implies the equality in (2.8). This proves (1.5) for $q < p$ and part (ii). ☐

We next investigate the threshold case $p = q$ and $m \downarrow p$. If we apply (1.5) for $p = q$, take the logarithm of both sides of the obtained inequality, multiply by $(m(2p-1) - p(p-1))/(m-p)(p-1)$ and pass to the limit as $m \downarrow p$, then we have

$$\begin{aligned} &- p' \frac{\int_{\mathbb{R}} G \ln G dy}{\int_{\mathbb{R}} G dy} - \ln \int_{\mathbb{R}} G|y|^{p'} dy + \frac{2p-1}{p-1} \ln \int_{\mathbb{R}} G dy \\ &\quad \le \frac{1}{p-1} \ln \int_{\mathbb{R}} |f'|^p dx + \ln \int_{\mathbb{R}} f^p dx - p' \frac{\int_{\mathbb{R}} f^p \ln f^p dx}{\int_{\mathbb{R}} f^p dx} + p' \ln p. \end{aligned}$$

Thus, we obtain the following result.

PROPOSITION 2.1. *If $p > 1$, then the following relation holds,*

$$\begin{aligned} &\sup_{G \in L^1_{p'}(\mathbb{R}) \setminus \{0\}, G \ge 0} \left(\ln \frac{\int_{\mathbb{R}} G dy}{\int_{\mathbb{R}} G|y|^{p'} dy} - p' \frac{\int_{\mathbb{R}} G \ln G dy - \int_{\mathbb{R}} G dy \ln \int_{\mathbb{R}} G dy}{\int_{\mathbb{R}} G dy}\right) \\ &= p' \inf_{f \in W^{1,p}(\mathbb{R}) \setminus \{0\}, f \ge 0} \left(\ln\left(p \frac{\|f'\|_{L^p(\mathbb{R})}}{\|f\|_{L^p(\mathbb{R})}}\right) - \frac{\int_{\mathbb{R}} f^p \ln f^p dx - \int_{\mathbb{R}} f^p dx \ln \int_{\mathbb{R}} f^p dx}{\int_{\mathbb{R}} f^p dx}\right). \end{aligned} \tag{2.12}$$

Moreover, the functions $G(y) = e^{-|y|^{p'}}$ and $f(x) = e^{-|x|^{p'}/p}$ solve both the variational problems on the left-hand side and right-hand side of (2.12) respectively.

The equality (2.12) provides a dual principle for the general one-dimensional L^p logarithmic Sobolev inequality. The general L^p logarithmic Sobolev inequality was first proved by Del Pino and Dolbeault (see [**11**] for $p = 2$ and [**12**] for $1 < p < n$) by considering it as the limiting case of their sharp Gagliardo–Nirenberg inequalities. This inequality then was extended to any $1 < p < \infty$ by Gentil [**17**] by another method based on the hypercontractivity of the solution of a special Jacobi–Hamilton type equation. The equality (2.12) with $p = 2$ was recently proved in [**13**]. A direct proof based on optimal transportation method, in any dimension, can be found in [**9**].

3. Sharp constants and optimal functions.

This section is devoted to compute the best constants in the Gagliardo–Nirenberg inequality (1.1) and to discuss the optimal functions for (1.1).

3.1. Sharp constants in the Gagliardo–Nirenberg inequality.

Let us compute the sharp constant $C_{GN}(p,q,m)$ in (1.1). Notice that

$$\frac{1}{C_{GN}(p,q,m)} = \inf_{f\in\mathcal{D}^{1,p}_q(\mathbb{R}), f\geq 0} \frac{\|f'\|^{\theta}_{L^p(\mathbb{R})}\|f\|^{1-\theta}_{L^q(\mathbb{R})}}{\|f\|_{L^m(\mathbb{R})}}$$
$$= \left(\inf_{f\in\mathcal{D}^{1,p}_q(\mathbb{R}), f\geq 0} \frac{\|f'\|^{\frac{p(m-q)}{m(2p-1)+p(q-1)}}_{L^p(\mathbb{R})}\|f\|^{\frac{q(m(p-1)+p)}{m(2p-1)-p(q-1)}}_{L^q(\mathbb{R})}}{\|f\|^{\frac{m(p+q(p-1))}{m(2p-1)-p(q-1)}}_{L^m(\mathbb{R})}}\right)^{\frac{m(2p-1)-p(q-1)}{m(q(p-1)+p)}}.$$

Combining the previous equality together with (1.5), we get

$$\frac{1}{C_{GN}(p,q,m)}$$
$$= \left(\frac{1}{c_{p,q,m}} \sup_{\substack{G\in L^1_{p'}(\mathbb{R}),\\ G\geq 0}} \frac{\int_{\mathbb{R}} G^{\frac{m(p-1)+p}{m(2p-1)-p(q-1)}}\,dy}{\left(\int_{\mathbb{R}} G|y|^{\frac{p}{p-1}}dy\right)^{\frac{(m-q)(p-1)}{m(2p-1)-p(q-1)}}\left(\int_{\mathbb{R}} Gdy\right)^{\frac{p+q(p-1)}{m(2p-1)-p(q-1)}}}\right)^{\frac{m(2p-1)-p(q-1)}{m(q(p-1)+p)}}. \tag{3.1}$$

It follows from Theorem 1.1 that the right-hand side of (3.1) is maximized by the function

$$G(y) = (1+|y|^{p'})^{-(m(2p-1)-p(q-1))/p(m-q)}.$$

The direct computations show that

$$\int_{\mathbb{R}} G(y)dy = \frac{2}{p'}B\left(\frac{m(2p-1)-p(q-1)}{p(m-q)}, \frac{1}{p'}\right) = 2\frac{q(p-1)+p}{p(m-q)}B\left(\frac{q(p-1)+p}{p(m-q)}, \frac{2p-1}{p}\right),$$

$$\int_{\mathbb{R}} G(y)|y|^{p'}dy = 2\frac{p-1}{p}B\left(\frac{q(p-1)+p}{p(m-q)}, \frac{2p-1}{p}\right),$$

and

$$\int_{\mathbb{R}} G(y)^{(m(2p-1)-p(q-1))/p(m-q)}dy = 2\frac{m(p-1)+p}{p(m-q)}B\left(\frac{q(p-1)+p}{p(m-q)}, \frac{2p-1}{p}\right).$$

Inserting the preceding integrals into (3.1), we obtain the value of $C_{GN}(p,q,m)$ as given in (1.2).

In the threshold case $m=p=q$, Proposition 2.1 gives the following one-dimension L^p logarithmic Sobolev inequality (see [**17**] for more general inequalities in any dimensional)

$$\int_{\mathbb{R}} f^p \ln f^p dx - \int_{\mathbb{R}} f^p dx \ln \int_{\mathbb{R}} f^p dx \le \frac{1}{p}\|f\|_{L^p(\mathbb{R})}^p \ln\left(\frac{p^p}{(ep')^{p-1}\left(2\Gamma(1+\frac{1}{p'})\right)^p}\frac{\|f'\|_{L^p(\mathbb{R})}^p}{\|f\|_{L^p(\mathbb{R})}^p}\right),$$

for any $f \in W^{1,p}(\mathbb{R})$. In the special case $p=2$, we get

$$\int_{\mathbb{R}} f^2 \ln f^2 dx - \int_{\mathbb{R}} f^2 dx \ln \int_{\mathbb{R}} f^2 dx \le \frac{1}{2}\|f\|_{L^2(\mathbb{R})}^2 \ln\left(\frac{2}{\pi e}\frac{\|f'\|_{L^2(\mathbb{R})}^2}{\|f\|_{L^2(\mathbb{R})}^2}\right),$$

for any function $f \in H^1(\mathbb{R})$, which is equivalent to Gross's famous logarithmic Sobolev inequality for Gaussian measure (see [**18**]).

3.2. The optimal functions.

Let us discuss about the optimal functions for the sharp Gagliardo–Nirenberg inequality (1.1). Suppose that f is an optimizer for the sharp Gagliardo–Nirenberg inequality (1.1). We write f as $f = f_+ - f_-$ where $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$. Notice that $\|f\|_{L^r(\mathbb{R})}^r = \|f_+\|_{L^r(\mathbb{R})}^r + \|f_-\|_{L^r(\mathbb{R})}^r$ for any $r \in [1,\infty)$. An simple argument based on convexity shows that either $f_+ = 0$ or $f_- = 0$ almost everywhere in $\mathbb{R}$. Hence up to a multiplicative constant ± 1, we can assume that f is a nonnegative function. By the homogeneity of (1.1), we can assume $\int_{\mathbb{R}} f^m dx = 1$. Denote $F(x) = f^m(x)$ and $G(y) = p'C^m(1+|y|^{p'})^{-(m(2p-1)-p(q-1))/p(m-q)}$ with the constant C given

$$C^m = \frac{1}{2}\left[B\left(\frac{p-1}{p}, \frac{p(m+1)-q}{p(m-q)}\right)\right]^{-1}$$

and chosen such that $\int_{\mathbb{R}} G(y)dy = 1$. Notice that G is positive and continuous on $\mathbb{R}$. Define

$$\Phi(t) = \int_{-\infty}^t F(x)dx, \quad \text{and} \quad \Psi(t) = \int_{-\infty}^t G(y)dy.$$

Then Φ is a non-decreasing and continuous function from $\mathbb{R}$ to $[0,1]$ and Ψ is a diffeomorphism form $\mathbb{R}$ to $(0,1)$. Let $a = \inf\{t : \Phi(t) > 0\}$ and $b = \sup\{t : \Phi(t) < 1\}$. Notice that if $a > -\infty$ then $F(x) = 0$ almost everywhere in $\{x \le a\}$, similarly if $b < \infty$ then $F(x) = 0$ for almost everywhere $x \ge b$. Let $\varphi(t) = \Psi^{-1}(\Phi(t))$ with $t \in (a,b)$. Notice that $\varphi : (a,b) \to \mathbb{R}$ is an increasing function. Moreover, for almost everywhere $t \in (a,b)$, we have

$$F(t) = G(\varphi(t))\varphi'(t). \tag{3.2}$$

Let θ, α be defined by (2.5). Using (3.2) and making the change of variable $y = \varphi(t)$, we have

$$\int_{\mathbb{R}} G^\theta dy = \int_a^b G(\varphi(t))^\theta \varphi'(t) dt = \int_a^b F(t)^\theta \varphi'(t)^{1-\theta} dt.$$

Using Hölder inequality, we get the following form of (2.3)

$$\int_a^b F(t)^\theta \varphi'(t)^{1-\theta} dt \leq \left(\int_a^b F(t)^{1-\alpha/\theta} dt \right)^\theta \left(\int_a^b F(t)^{\alpha/(1-\theta)} \varphi'(t) dt \right)^{1-\theta}. \tag{3.3}$$

Our next aim is to apply integration by parts in the last integral in the right-hand side of (3.3). We shall follow the argument in [**10**, Lemma 7]. Let $x_0 = \min\{x \in (a, b) : \varphi(x) = 0\}$. We have $x_0 \in (a, b)$ and $\varphi(x_0) = 0$. For $\epsilon > 0$, define $f_\epsilon(x) = \min\{f(x_0 + (x - x_0)/(1-\epsilon)), f(x)\chi(\epsilon(x - x_0))\}$, where χ is cut-off function, i.e., $\chi \in C_0^\infty((-2, 2))$ is radial function, $0 \leq \chi \leq 1$, and $\chi(x) = 1$ if $|x| \leq 1$. For $\epsilon > 0$ small enough, we have the support of f_ϵ is contained in $I_\epsilon = (x_0 + (1-\epsilon)(a - x_0), x_0 + (1-\epsilon)(b - x_0)) \cap (x_0 - 2\epsilon^{-1}, x_0 + 2\epsilon^{-1})$. Notice that $\overline{I}_\epsilon \subset (a, b)$. For $\delta > 0$ small enough (smaller the distance from ∂I_ϵ to $\{a, b\}$), we define $f_{\epsilon,\delta} = f_\epsilon \star \phi_\delta$, where $\phi_\delta = \delta^{-1}\phi(\cdot/\delta)$, and $\phi \in C_0^\infty(\mathbb{R})$ is radial nonnegative function such that $\int_{\mathbb{R}} \phi dx = 1$. We then have $f_{\epsilon,\delta} \in C_0^\infty((a, b))$. Denote $F_{\epsilon,\delta} = f_{\epsilon,\delta}^m$ and $F_\epsilon = f_\epsilon^m$ then $F_\epsilon \leq F$. By integration by parts, we have

$$\int_a^b F_{\epsilon,\delta}(t)^{\alpha/(1-\theta)} \varphi'(t) dt = -\frac{p(m+1) - m}{p} \int_a^b f_{\epsilon,\delta}(t)^{m(p-1)/p} \varphi(t) (f_{\epsilon,\delta})'(t) dt. \tag{3.4}$$

Notice that $f_{\epsilon,\delta} \to f_\epsilon$ in $L^q(\mathbb{R}) \cap L^m(\mathbb{R})$ and $f'_{\epsilon,\delta} \to f'_\epsilon$ in $L^p(\mathbb{R})$ as $\delta \to 0$. Let $I \subset (a, b)$ be an interval such that $\overline{I}_\epsilon \subset I \subset \overline{I} \subset (a, b)$, then the support of $f_{\epsilon,\delta}$ is contained in I for $\delta > 0$ small enough. Moreover, φ is bounded in I. Letting $\delta \to 0$ in (3.4), and using Fatou's lemma (notice that $\varphi' \geq 0$), we get

$$\int_a^b F_\epsilon(t)^{\alpha/(1-\theta)} \varphi'(t) dt \leq -\frac{p(m+1) - m}{p} \int_a^b f_\epsilon(t)^{m(p-1)/p} \varphi(t) (f_\epsilon)'(t) dt. \tag{3.5}$$

It is easy to check that $f_\epsilon \to f$ in $L^m(\mathbb{R}) \cap L^q(\mathbb{R})$ as $\epsilon \to 0$, and $\|f'_\epsilon\|_{L^p(\mathbb{R})} = (1 - \epsilon)^{-1/p'} \|f'\|_{L^p(\mathbb{R})}$ is bounded as $\epsilon \to 0$. Hence by extracting a subsequence $\epsilon_k \to 0$, we have $f_{\epsilon_k} \to f$ in $L^m(\mathbb{R}) \cap L^q(\mathbb{R})$ and almost everywhere in $\mathbb{R}$ as $k \to \infty$, and $f'_{\epsilon_k} \rightharpoonup f'$ weakly in $L^p(\mathbb{R})$. Notice that $\int_{\mathbb{R}} G|y|^{p'} dy < \infty$, which implies $\int_a^b f^m(t)|\varphi(t)|^{p'} dt < \infty$. Since $f_\epsilon \leq f$, hence by dominated convergence theorem we have $f_\epsilon^{m/p'} \varphi \to f^{m/p'} \varphi$ in $L^{p'}(\mathbb{R})$ as $\epsilon \to 0$. Consequently, applying (3.5) for sequence ϵ_k and letting $k \to \infty$ and using again Fatou's lemma, we have

$$\int_a^b F(t)^{\alpha/(1-\theta)} \varphi'(t) dt \leq -\frac{p(m+1) - m}{p} \int_a^b f(t)^{m(p-1)/p} \varphi(t) f'(t) dt. \tag{3.6}$$

Now, applying Hölder's inequality to (3.6), we get

$$\int_a^b F(t)^{\frac{\alpha}{1-\theta}}\varphi'(t)dt \le \frac{p(m+1)-m}{p}\left(\int_a^b f(t)^m|\varphi(t)|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}}|f'|^p dt\right)^{\frac{1}{p}}$$
$$= \frac{p(m+1)-m}{p}\left(\int_{\mathbb{R}} G(y)|y|^{\frac{p}{p-1}}dy\right)^{\frac{p-1}{p}}\left(\int_a^b |f'|^p dx\right)^{\frac{1}{p}}. \tag{3.7}$$

Combining (3.3) and (3.7), we get

$$\frac{\|f'\|_{L^p(\mathbb{R})}^{\theta}\|f\|_{L^q(\mathbb{R})}^{1-\theta}}{\|f\|_{L^m(\mathbb{R})}} \ge \frac{1}{C_{GN}(p,q,m)}.$$

Since f is an optimizer then it holds

$$\frac{\|f'\|_{L^p(\mathbb{R})}^{\theta}\|f\|_{L^q(\mathbb{R})}^{1-\theta}}{\|f\|_{L^m(\mathbb{R})}} = \frac{1}{C_{GN}(p,q,m)}.$$

Consequently, we must have equality in (3.3) and (3.7). Equality holds true in (3.3) if and only if

$$\varphi'(t) = \lambda F(t)^{1-\alpha/\theta(1-\theta)} \tag{3.8}$$

for almost everywhere $t \in (a,b)$ and for some $\lambda > 0$. Equality holds true in (3.7) if and only if

$$\varphi(t)|\varphi(t)|^{p'-2} = \mu(-F'(t))F(t)^{1/m-1/p-1},$$

for almost everywhere $t \in (a,b)$ and for some $\mu > 0$. Notice that $\varphi(x_0) = 0$. Combining the previous two equalities, we get

$$c|\varphi(t)|^{p'} = F(t)^{-(m-q)/m} - F(x_0)^{-(m-q)/m}, \quad t \in (a,b),$$

for some $c > 0$, or equivalently,

$$F(t) = \left(F(x_0)^{-(m-q)/m} + c|\varphi(t)|^{p'}\right)^{-m/(m-q)}, \quad t \in (a,b). \tag{3.9}$$

From (3.2), we get

$$\varphi'(t) = C^m(1+|\varphi(t)|^{p'})^{-\frac{m(2p-1)-p(q-1)}{p(m-q)}}\left(F(x_0)^{-\frac{m-q}{m}} + c|\varphi(t)|^{p'}\right)^{-\frac{m}{m-q}}, \quad t \in (a,b).$$

Since $\varphi(t) \to -\infty$ as $t \to a^+$ (and $\varphi(t) \to \infty$ as $t \to b^-$), then $\varphi'(t) \sim |\varphi(t)|^{1+\frac{p-q}{(p-1)(m-q)}}$ as $t \to a^+$ (and $t \to b^-$). Fix a number $t_0 \in (a, x_0)$, then

$$\int_t^{t_0} \varphi'(s)(-\varphi(s))^{-1+(q-p)/(p-1)(m-q)}ds \sim (t_0 - t),$$

as $t \to a^+$. Consequently, we must have $a = -\infty$ if $q \ge p$, and $a > -\infty$ if $q < p$. Indeed, if $q \ge p$, we have

$$\int_t^{t_0} \varphi'(s)(-\varphi(s))^{-1+(q-p)/(p-1)(m-q)} ds \sim \begin{cases} (-\varphi(t))^{(q-p)/(p-1)(m-q)} & \text{if } q > p, \\ \ln(-\varphi(t)) & \text{if } q = p, \end{cases}$$

as $t \to a^+$, which forces $a = -\infty$ since $\varphi(t) \to -\infty$ as $t \to a^+$. A similar argument shows $a > -\infty$ if $q < p$. By the same arguments, we have $b = +\infty$ if $q \geq p$ and $b < +\infty$ if $q < p$. To continue, we divide our arguments into two cases according to $q \geq p$ or $q < p$.

CASE $q \geq p$. In this case we have $(a, b) = \mathbb{R}$ and $F(t) > 0$ for any t. From (3.8) and (3.9), we have

$$\varphi'(t) \left(F(x_0)^{-(m-q)/m} + c|\varphi(t)|^{p'} \right)^{-(m(p-1)-p(q-1))/p(m-q)} = \lambda.$$

Integrating this equality implies

$$cF(x_0)^{(m-q)/m} |\varphi(x)|^{p'} = \frac{B^{-1}\left(C|x-x_0|; 1/p', (p-q)/p(m-q)\right)}{1 - B^{-1}\big(C|x-x_0|; 1/p', (p-q)/p(m-q)\big)},$$

for some $C > 0$. Inserting this expression of φ into (3.9), we get

$$F(x) = F(x_0) \left(1 - B^{-1} \left(C|x - x_0|; \frac{1}{p'}, \frac{p-q}{p(m-q)} \right) \right)^{m/(m-q)},$$

as desired.

CASE $1 \leq q < p$. In this case we have $-\infty < a < b < \infty$ and $F(t) > 0$ for $t \in (a, b)$. From (3.8) and (3.9), we have

$$\varphi'(t) \left(F(x_0)^{-(m-q)/m} + c|\varphi(t)|^{p'} \right)^{-(m(p-1)-p(q-1))/p(m-q)} = \lambda, \quad t \in (a, b).$$

Integrating this equality implies

$$C(b-x) = B\left(1 - \frac{cF(x_0)^{(m-q)/m}\varphi(x)^{p'}}{1 + cF(x_0)^{(m-q)/m}\varphi(x)^{p'}}; \frac{p-q}{p(m-q)}, \frac{1}{p'} \right), \quad x \in [x_0, b)$$

and

$$C(x-a) = B\left(1 - \frac{cF(x_0)^{(m-q)/m}(-\varphi(x))^{p'}}{1 + cF(x_0)^{(m-q)/m}(-\varphi(x))^{p'}}; \frac{p-q}{p(m-q)}, \frac{1}{p'} \right), \quad x \in (a, x_0]$$

for some $C > 0$. Taking $x = x_0$, we get $x_0 = (a + b)/2$ and

$$C\frac{b-a}{2} = B\left(\frac{p-q}{p(m-q)}, \frac{1}{p'} \right).$$

Hence, it holds

$$cF(x_0)^{\frac{m-q}{m}}|\varphi(x)|^{p'} = \frac{1}{B^{-1}\left(C(\frac{b-a}{2} - |x-x_0|); \frac{p-q}{p(m-q)}, \frac{1}{p'}\right)} - 1, \quad x \in (a,b).$$

Inserting this expression of φ into (3.9), we get

$$F(x) = F(x_0)\left(B^{-1}\left(B\left(\frac{p-q}{p(m-q)}, \frac{1}{p'}\right)\frac{(\frac{b-a}{2} - |x-x_0|)_+}{a}; \frac{p-q}{p(m-q)}, \frac{1}{p'}\right)\right)^{\frac{m}{m-q}},$$

as desired.

We have thus shown the following result.

THEOREM 3.1. *Let $p > 1$ and $m > q \geq 1$. Suppose f is an optimizer for the sharp Gagliardo–Nirenberg inequality* (1.1). *Then we have the following*:

(i) *If $q \geq p$, then*

$$f(x) = c\left(1 - B^{-1}\left(C|x-x_0|; \frac{1}{p'}, \frac{p-q}{p(m-q)}\right)\right)^{1/(m-q)},$$

for some $c \in \mathbb{R}$, $C > 0$ and $x_0 \in \mathbb{R}$.

(ii) *If $q < p$, then*

$$f(x) = c\left(B^{-1}\left(B\left(\frac{p-q}{p(m-q)}, \frac{1}{p'}\right)\frac{(a - |x-x_0|)_+}{a}; \frac{p-q}{p(m-q)}, \frac{1}{p'}\right)\right)^{1/(m-q)}$$

for some $c \in \mathbb{R}$, $a > 0$ and $x_0 \in \mathbb{R}$.

Theorem 3.1 was proved in [**19**] by solving explicitly the solution of the Euler–Lagrange equations related to the Gagliardo–Nirenberg inequality (1.1) (see also [**13**, Appendix B] for the case $p = 2$). Here, we give another proof of this result via the mass transportation method. Notice that our proof above of Theorem 3.1 does not use any decreasing rearrangement argument as done in [**19**] and in [**13**, Appendix B]. We only use the equality case in Hölder's inequality. This is one of the advantages of the mass transportation method (compared with the variational method) in proving the sharp functional inequalities and characterizing their optimal functions.

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