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Linking forms, finite orthogonal groups and periodicity of links

By Maciej Borodzik, Przemysław Grabowski, Adam Król and Maria Marchwicka

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Abstract. For a prime number $q \neq 2$ and $r > 0$, we study whether there exists an isometry of order q^r acting on a free $\mathbb{Z}_{p^k}$-module equipped with a scalar product. We investigate whether there exists such an isometry with no non-zero fixed points. Both questions are completely answered in this paper if $p \neq 2, q$. As an application, we refine Naik's criterion for periodicity of links in S^3. The periodicity criterion we obtain is effectively computable and gives concrete restrictions for periodicity of low-crossing knots.

1. Introduction.

1.1. General overview.

Let $L \subset S^3$ be a link. We say that L is m-periodic if there exists an orientation-preserving diffeomorphism $\phi\colon S^3 \to S^3$ such that $\phi^m = id$, $\phi^j \neq id$ for $j < m$, $\phi(L) = L$ and L is disjoint from the rotation axis of ϕ. One of the oldest questions in knot theory is to determine which links are periodic. The existing obstructions for periodicity can be divided in a few classes.

(1) The first such class consists of classical criteria, that is, criteria based on the Seifert matrix. The classical criteria are usually easy to apply and to implement on a computer. They are also quite effective. However, these classical criteria do not obstruct periodicity of any knot which has the same Seifert matrix as a periodic knot. Examples of such criteria include:

- Murasugi's criterion [**11**] based on Alexander polynomial;
- a refinement of Murasugi's criterion by Davis and Livingston [**6**];
- Naik's homological criterion [**12**], [**13**] based on first homology of branched cover.

(2) The second class consists of criteria based on Jones and HOMFLYPT polynomials. Like the classical criteria, these are also effective and algorithmically computable. These criteria include:

- Traczyk's criterion on Jones polynomial [**18**], [**19**];
- Przytycki's criterion on HOMFLYPT polynomial [**15**].

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(3) The third class consists of criteria based on twisted invariants. The class includes:
   - Naik's criterion on Casson–Gordon invariant;
   - a criterion of Hillman, Livingston and Naik [**7**] based on twisted Alexander polynomial.

(4) Finally, recently many criteria have been developed from homological invariants of knots and 3-manifolds. Examples include:
   - the Jabuka and Naik criterion [**8**] based on Heegaard Floer homology of branched cover;
   - criteria based on Khovanov homology given by Politarczyk [**14**], and the first author and Politarczyk [**1**].

There are other tools to obstruct link periodicity which do not fit into any of the four classes above. We can mention here:

- Sakuma's work [**17**] on periodicity of non-prime knots;
- SnapPy, a computer program [**4**] that enable to verify periodicity of hyperbolic knots based on results of [**22**];
- Chen's criterion for fundamental group of link complement [**3**].

The criterion we develop in this paper fits into the first class of the criteria in the above list.

1.2. Overview of the paper.

Our criterion is built on top of Naik's homological criterion. That is, only if a knot passes Naik's criterion, we can apply our criterion and (sometimes) obstruct periodicity.

To be more specific, let K be an m-periodic knot where $m = q^r$ is an odd prime power. Let $K/\mathbb{Z}_m$ be the quotient knot. Consider the double branched cover $\Sigma(K)$ and let T_p be the p-torsion part of $H_1(\Sigma(K);\mathbb{Z})$. For any odd prime number $p \neq q$, the $\mathbb{Z}_m$ action on K induces a group action on T_p. Furthermore, Naik [**13**] shows that the fixed point set of this action is the p-torsion part of $H_1(\Sigma(K/\mathbb{Z}_m);\mathbb{Z})$.

The rank of $H_1(\Sigma(K/\mathbb{Z}_m);\mathbb{Z})$ is the absolute value of the Alexander polynomial $\Delta_{K/\mathbb{Z}_m}(-1)$. By Murasugi's criterion [**11**], the Alexander polynomial of $K/\mathbb{Z}_m$ is a factor of the Alexander polynomial of K. In some cases, for all the factors Δ' of Δ_K that can be Alexander polynomials of some knot, the prime number p does not divide $\Delta'(-1)$. This translates into the statement that $\mathbb{Z}_m$ acts on T_p without non-zero fixed points.

The action of $\mathbb{Z}_m$ on T_p preserves the linking form on $H_1(\Sigma(K);\mathbb{Z})$. We show that T_p can be decomposed into the orthogonal sum of spaces $T_{p,1},\dots,T_{p,s}$, where each of the $T_{p,i}$ is of form $(\mathbb{Z}_{p^{a_i}})^{n_i}$ for some a_i, n_i. Moreover, the decomposition is preserved by the group action; see Proposition 2.2. From the linking form on $T_{p,i}$ we construct a bilinear form $T_{p,i}\times T_{p,i}\to\mathbb{Z}_{p^{a_i}}$; see Section 3. The $\mathbb{Z}_m$-action preserves this form, too.

Let $O(T_{p,i})$ be the group of invertible matrices over $\mathbb{Z}_{p^{a_i}}$ which preserve the bilinear form on $T_{p,i}$. From what was said, there is a subgroup of $O(T_{p,i})$ isomorphic to $\mathbb{Z}_m$ acting on $T_{p,i}$ without non-zero fixed points. From this observation Naik [**13**] deduces that the rank n_i of $T_{p,i}$ must be a multiple of a number depending on q and p only.

The refinement that we propose goes into a deeper study of the group $O(T_{p,i})$. First, Wall in [**21**] proves that on any $T_{p,i}$ there are precisely two non-isomorphic bilinear forms with values in $\mathbb{Z}_{p^{a_i}}$. We distinguish them by an index $\epsilon = \pm 1$; see Theorem 3.3 below. The structure of the group $O(T_{p,i})$ depends on this index. We invoke a result of Weir [**23**] which determines the Sylow q-group of $O(T_{p,i})$ depending on the index ϵ. Weir's result uses so-called wreath product; see Definition 5.3 below. We unfold the definition of the wreath product to show in which cases this Sylow q-group contains an element of order q^r.

To complete the picture we need to study whether an element of order q^r of $O(T_{p,i})$ can act on $T_{p,i}$ without non-zero fixed points. For this purpose we consider an orthogonal decomposition $T_{p,i} = T'_{p,i} \oplus \mathbb{Z}_{p^{a_i}}$ for some submodule $T'_{p,i}$. This decomposition also induces an embedding $O(T'_{p,i}) \hookrightarrow O(T_{p,i})$. We show that if this embedding induces an isomorphism on Sylow q-groups, then every element of order q^r in $O(T_{p,i})$ has a fixed subspace of dimension at least 1. Therefore, the only chance that an element of $O(T_{p,i})$ of order q^r acts without non-zero fixed points is that the q-Sylow groups of $T_{p,i}$ are strictly larger than those of $T'_{p,i}$. With a little extra effort, we obtain a full characterization of groups $O(T_{p,i})$ that contain an element of order q^r acting on $T_{p,i}$ without non-zero fixed points; see Theorem 1.3.

Retracing the path from the periodic knot K to subgroups of $O(T_{p,i})$ we obtain a refinement of Naik's periodicity criterion. It is stated as Theorem 1.4 below.

1.3. Main results.

Before we state Theorem 1.3, we need to introduce some terminology.

Definition 1.1. Let p be a prime number and m an odd number coprime with p. We denote by $[m|p]$ the minimal positive exponent s such that either $m|(p^s - 1)$ or $m|(p^s + 1)$. We define $\eta(m) = 1$ if $m|(p^{[m|p]} - 1)$ and $\eta(m) = -1$ if $m|(p^{[m|p]} + 1)$.

Remark 1.2. For any $n > 0$, m divides $p^{n[m|p]} - \eta(m)^n$. We also note that $\eta(m) = -1$ if and only if p is a root of -1 in $\mathbb{Z}_m$, that is, if some power of p is equal to -1 in $\mathbb{Z}_m$.

We recall also that if $p \neq 2$ and $k > 0$, a free $\mathbb{Z}_{p^k}$-module of rank n can be equipped with two non-equivalent symmetric bilinear forms with values in $\mathbb{Z}_{p^k}$. These two forms are distinguished by an index ϵ, which we introduce rigorously in Definition 3.5.

The following result is the main technical result in the present paper.

Theorem 1.3. *Let $p \neq q$ be odd prime numbers. Let B be a free $\mathbb{Z}_{p^k}$-module of rank $n > 0$, for some $k > 0$. Let $\beta\colon B \times B \to \mathbb{Z}_{p^k}$ be a non-degenerate symmetric bilinear form. Let s be the maximal integer such that $q^s|(p^{2[q|p]} - 1)$. Fix an integer $r > 0$ and set $\widetilde{r} = \max(r - s, 0)$.*

(a) *There exists an isometry of (B, β) of order q^r if and only if $n \geq 2[q|p]q^{\widetilde{r}} + 1$, or $n = 2[q|p]q^{\widetilde{r}}$ and $\epsilon(B, \beta) = \eta(q)$.*

(b) *There exists an isometry of* (B, β) *of order* q^r *with no non-zero fixed points if and only if* $n = 2[q|p]d$ *for* $d \geq q^{\widetilde{r}}$ *and* $\epsilon(B, \beta) = \eta(q)^d$.

The main application of Theorem 1.3 in our paper is the following refinement of Naik's theorem [**13**, Theorem 5].

Theorem 1.4. *Let* L *be a* q^r*-periodic link with* $r \geq 1$ *and* q *an odd prime. Let* T_p *be the* p*-torsion subgroup of* $H_1(\Sigma_k(L); \mathbb{Z})$, *where* $\Sigma_k(L)$ *is a* k*-fold branched cyclic cover for* $k > 1$ *and* $p \neq q$ *an odd prime.*

(a) *If* T_p *is non-trivial, then* T_p *splits as a sum* $T_{p,1} \oplus T_{p,2} \oplus \cdots$ *of pairwise orthogonal summands. The summand* $T_{p,i}$ *is a free* $\mathbb{Z}_{p^i}$*-module with linking form* $\lambda_{p,i}$. *The* $\mathbb{Z}_{q^r}$*-symmetry of* L *induces an action of* $\mathbb{Z}_{q^r}$ *on* T_p *preserving the orthogonal splitting.*

(b) *Suppose the first homology of the* k*-fold cover of the quotient link* $\Sigma_k(L/\mathbb{Z}_{q^r})$ *has no* p*-torsion. If* $T_{p,i}$ *is non-trivial, then there exists an integer* $d_i \geq q^{r-s}$ *such that* $\operatorname{rank} T_{p,i} = 2[q|p]d_i$ *and* $\epsilon(T_{p,i}) = \eta(q)^{d_i}$.

Remark 1.5. In Section 3 we explain how to extend the definition of the ϵ index from symmetric bilinear forms to linking forms, so that the index $\epsilon(T_{p,i})$ in item (b) above makes sense.

Our result extends [**13**, Theorem 5] in the following two ways.

- While Naik's result deals with links admitting a $\mathbb{Z}_q$-action with q prime, we extend it to links that admit an action of $\mathbb{Z}_{q^r}$ for $r > 1$.
- Our new condition $\epsilon(T_{p,i}) = \eta(q)^{d_i}$ rules out approximately half of linking forms that can appear as linking forms of a periodic link such that the quotient has no p-torsion. This is a substantial strengthening of Naik's criterion, as shown in Section 11.

Example 1.6. As explained in detail in Section 11, the knots $13n3659$, $14n908$, $14n913$, $14n2451$, $14n2458$, $14n6565$, $14n9035$, $14n11989$, $14n14577$, $14n23051$ and $14n24618$ pass Naik's periodicity criterion for period 3, but can be shown not to be 3-periodic by Theorem 1.4.

1.4. Structure of the paper.

The structure of the paper is as follows. In Section 2 we recall Naik's result in details. Section 3 recalls basics on linking forms and pairings. This section gives a translation between Theorem 1.3 and Theorem 1.4. We explain this translation in Subsection 1.6 below. Sections 3–6 contain the proof of Theorem 1.3. We review the content of these sections in Subsection 1.5 below.

Section 9 shows an explicit way of applying Theorem 1.4 to obstruct periodicity. An elaborated example of a knot that actually passes the criterion is given in Section 10. A comparison of our criterion with other periodicity criteria is given in Section 11.

1.5. Plan of the proof of Theorem 1.3.

The proof of Theorem 1.3 stretches for Sections 3–8. Sections 3–6 deal with part (a), while Sections 7 and 8 address part (b).

In more detail, in Section 3 we recall Wall's result on the classification of symmetric bilinear forms on free $\mathbb{Z}_{p^k}$-modules. We recall the definition of the index $\epsilon(B,\beta)$, which is crucial in our applications. In Section 4 we recall a classical result on the rank of the group of isometries of a symmetric bilinear form on a $\mathbb{Z}_p$-module. We introduce the concept of a (mod p)-reduction, which allows us to translate various results on forms on $\mathbb{Z}_p$-modules to forms on $\mathbb{Z}_{p^k}$-modules for $k > 1$. One of such results is the calculation of the Sylow q-group of isometries due to Weir [**23**], stated in Section 5, which we show holds for isometries of forms on $\mathbb{Z}_{p^k}$-modules by Theorem 5.2.

The description of the Sylow q-groups given in Section 5 is phrased in terms of the wreath product, see Definition 5.3. In Section 6 we use elementary properties of the wreath product to find the maximal r such that the Sylow q-group of the group of isometries contains an element of order q^r. Section 6 concludes with Theorem 6.2, which states precise conditions under which the group of isometries of a p^k form B contains an element of order q^r. The proof of Theorem 1.3(a) follows.

Part (b) of Theorem 1.3 is proved in Sections 7 and 8. It is essentially done on a case-by-case analysis. We begin by showing that if the conditions $n = 2[q|p]d$, $d \geq q^{\widetilde{r}}$ and $\epsilon(B,\beta) = \eta(q)^d$ are not satisfied, than any isometry of order $q^{\widetilde{r}}$ must have a non-zero fixed point. This is the statement of Theorem 7.1 in Section 7. Section 8 is devoted to a construction of an isometry with no non-zero fixed points when the conditions do hold. We perform the construction separately in the case $\widetilde{r} = r - s$ and in the case $\widetilde{r} = 0$ (i.e. $r \leq s$). The key results are Lemmas 8.3 and 8.7, which show that if $n = 2[q|p]q^{r-s}$ and $\epsilon(B,\beta) = \eta(q)^{q^{r-s}}$, then every isometry of order q^{r-s} has only zero as its fixed point. Lemma 8.3 deals with the case of $\mathbb{Z}_p$-forms and Lemma 8.7 takes care of general $\mathbb{Z}_{p^k}$-forms via the (mod p)-reduction. Lemma 8.8 deals with the case $\widetilde{r} = 0$, but the key argument is essentially the same as in Lemma 8.3. The three lemmas are used to give a proof of Theorem 8.1. This theorem concludes the proof of Theorem 1.3.

1.6. Plan of proof of Theorem 1.4.

Part (a) is proved as Proposition 2.3. To prove part (b) we use Proposition 2.2 to conclude that $\mathbb{Z}_{q^r}$ acts on T_p with no non-zero fixed points. Then by part (a), we know that $\mathbb{Z}_{q^r}$ acts on each of the $T_{p,i}$ preserving the linking form and with no non-zero fixed points.

Now $T_{p,i}$ is a free $\mathbb{Z}_{p^i}$-module. By the discussion of Section 3, a linking form $T_{p,i} \times T_{p,i} \to \mathbb{Q}/\mathbb{Z}$ induces a symmetric bilinear form on $T_{p,i}$ with values in $\mathbb{Z}_{p^i}$. Isometries of the linking form are isometries of the symmetric bilinear forms, as discussed in Lemma 3.1. Thus, we obtain an action of $\mathbb{Z}_{q^r}$ on $T_{p,i}$ equipped with a symmetric bilinear form such that the action is by isometries and with no non-zero fixed points. We conclude by Theorem 1.3.

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2. Review of Naik's criterion.

Let L be an m-periodic link and let $L/\mathbb{Z}_m$ be the quotient link. For $k > 1$ consider the branched covers $\Sigma_k(L)$ and $\Sigma_k(L/\mathbb{Z}_m)$. We write $\Sigma(L), \Sigma(L/\mathbb{Z}_m)$ instead of $\Sigma_2(L)$ and $\Sigma_2(L/\mathbb{Z}_m)$.

For simplicity, we will always assume that k is such that $\Sigma_k(L)$ is a rational homology sphere. In particular, there is a non-degenerate linking form $H_1(\Sigma_k(L);\mathbb{Z}) \times H_1(\Sigma_k(L);\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$.

We have the following observation, see [**12**, Section 2].

LEMMA 2.1. *An action of $\mathbb{Z}_m$ on S^3 that preserves L and whose fixed points are disjoint from L lifts to an action of $\mathbb{Z}_m$ on $\Sigma_k(L)$. The quotient $\Sigma_k(L)/\mathbb{Z}_m$ is diffeomorphic to the branched cover $\Sigma_k(L/\mathbb{Z}_m)$ of the quotient link.*

The following result relates Lemma 2.1 to the group action on homology.

PROPOSITION 2.2 (see [**12**, Proposition 2.5]). *If p is a prime number such that $p \nmid m$ and $\phi\colon \Sigma_k(L) \to \Sigma_k(L)$ is a generator of the action of $\mathbb{Z}_m$ on $\Sigma_k(L)$, then ϕ_* is an isometry of $H_1(\Sigma_k(L))$ (with respect to the linking form) and the fixed point set*

$$\operatorname{Fix} \phi_*\colon H_1(\Sigma_k(L))_p \to H_1(\Sigma_k(L))_p$$

is equal to $H_1(\Sigma_k(L/\mathbb{Z}_m))_p$. Here the subscript p denotes the p-primary part.

Proposition 2.2 is non-trivial and the condition $p \nmid m$ cannot in general be relaxed. For example, if $K = T(2,5)$, then K is clearly 5-periodic. We have $H_1(\Sigma_2(K)) = \mathbb{Z}_5$. The only action of $\mathbb{Z}_5$ on $\mathbb{Z}_5$ by isometries is trivial (because any isometry of $\mathbb{Z}_5$ has order 1 or 2), hence $\mathbb{Z}_5$ is a fixed subspace of this action. However, the quotient knot $K/\mathbb{Z}_5$ is trivial, $H_1(\Sigma_2(K/\mathbb{Z}_5);\mathbb{Z}) = 0$.

In order to apply Proposition 2.2 we use the following result.

PROPOSITION 2.3. *For an odd prime number p coprime with m the group $T_p = H_1(\Sigma_k(L))_p$ decomposes as a sum $T_{p,1} \oplus T_{p,2} \oplus \cdots$, where each of the $T_{p,i}$ is a free $\mathbb{Z}_{p^i}$-module. This decomposition is orthogonal with respect to the linking form. Moreover, if $\mathbb{Z}_m$ acts on T_p by isometries, then the decomposition can be made invariant with respect to this action.*

PROOF. This result is rather standard; we present a quick proof for the reader's convenience. Write $T_p = H_1(\Sigma_k(L))_p$ and consider the linking form $\lambda\colon T_p \times T_p \to \mathbb{Q}/\mathbb{Z}$.

For an element $x \in T_p$ let $r(x)$ be the minimal positive integer such that $p^{r(x)}x = 0 \in T_p$. Take $x \in T$ for which $r(x)$ is maximal among all elements in T. As λ is non-degenerate, there exists $y \in T_p$ such that $p^{r(x)-1}\lambda(x,y) \neq 0 \in \mathbb{Q}/\mathbb{Z}$. Indeed, if for all $y \in T_p$ we have $p^{r(x)-1}\lambda(x,y) = 0$, then $p^{r(x)-1}x = 0$ pairs trivially with all $y \in T_p$, contradicting non-degeneracy of λ. Note also that we have $r(y) = r(x)$.

Consider now $p^{r(x)-1}\lambda(x,x)$, $p^{r(x)-1}\lambda(y,y)$ and $p^{r(x)-1}\lambda(x+y,x+y)$. If all three expressions are zero in $\mathbb{Q}/\mathbb{Z}$, we conclude that $p^{r(x)-1}\lambda(x,y)=0$, contradicting the assumptions (here we use the assumption that 2 is invertible modulo p). Thus, there is an element $z\in T_p$ such that $r(z)$ is maximal and $p^{r(z)-1}\lambda(z,z)\neq 0$.

Let T_z be the $\mathbb{Z}$-submodule of T_p generated by $z,\phi_*(z),\dots,\phi_*^{m-1}(z)$. The number of generators of T_z is equal to n_z, where n_z is the minimal n such that $\phi_*^n(z)$ belongs to the subgroup generated by $z,\phi_*(z),\dots,\phi_*^{n-1}(z)$. We have an isomorphism of $\mathbb{Z}$-modules $T_z\cong\mathbb{Z}_{p^{r(z)}}^{n_z}$. In particular, T_z is a free $\mathbb{Z}_{p^{r(z)}}$-module.

Let T' be the orthogonal complement of T_z in T_p. As the linking form is invariant under ϕ_*, and T_z is invariant, T' is also invariant.

Now T' has smaller number of generators than T_p. As a result of a recursive application of the procedure, we present T_p as an orthogonal sum of modules $T_{z_1}\oplus\cdots\oplus T_{z_m}$. We set

$$T_{p,i}=\bigoplus_{j\colon r(z_j)=i}T_{z_j}. \qquad\square$$

The next result gives a number theoretical criterion for applying Proposition 2.2. It is due to Davis [**5**] (see also [**13**]).

THEOREM 2.4. *If $q\neq p$ is an odd prime number and $\mathbb{Z}_q$ acts on $T_{p,i}$ by isometries with no non-zero fixed points, then the rank of $T_{p,i}$ as a $\mathbb{Z}_{p^i}$-module is a multiple of $2[q|p]$.*

Theorem 1.3 of the present paper is a generalization of Theorem 2.4.

The condition described in Theorem 2.4 requires some knowledge of fixed point set of the action of $\mathbb{Z}_q$. By Proposition 2.2 this fixed point set is the homology of the cover of the quotient link. As explained in [**12**], [**13**], it is often possible to check whether $H_1(\Sigma_k(L/Z_q);\mathbb{Z})_p=0$ using Murasugi's criterion, which we now recall.

THEOREM 2.5 (Murasugi's criterion for knots, see [**11**]). *Suppose $K\subset S^3$ is a q^r-periodic knot with q prime and $r>0$. Let Δ be the Alexander polynomial of K and Δ' be the Alexander polynomial of the quotient knot $K'=K/\mathbb{Z}_q$. Let ℓ be the absolute value of the linking number of K with the symmetry axis. Then $\Delta'|\Delta$ and up to multiplication by a power of t we have*

$$\Delta\equiv\Delta'^{q^r}(1+t+\cdots+t^{\ell-1})^{q^r-1}\bmod q. \tag{2.6}$$

There are various versions of Murasugi's criterion for links, see [**16**], [**20**, Theorem 1.10.1]. Precise statements depend on the action of the symmetry group on the set of components of the symmetric link.

Naik's homological criterion relies on combining Proposition 2.2, Theorem 2.4 and Murasugi's criterion (Theorem 2.5). We will show two variations of such criterion, both due to Naik [**13**]. As Propositions 2.7 and 2.8 are instructive and indicate how Theorem 1.3 can be applied, we include short proofs, not claiming any originality.

PROPOSITION 2.7. *Let L be a q-periodic link with q an odd prime. Let Δ be the Alexander polynomial of L. Suppose Δ' is the Alexander polynomial of the quotient. For* *any odd prime number $p \neq q$, if $s = s(p)$ is the maximal positive integer such that p^s divides $\Delta(-1)/\Delta'(-1)$, then s is a multiple of $2[q|p]$.*

Proof. Let T be the p-torsion part of $H_1(\Sigma(L);\mathbb{Z})$. The rank of T is equal to p^{s_0}, where s_0 is the maximal positive integer such that p^{s_0} divides

$$|H_1(\Sigma(L);\mathbb{Z})| = |\det(L)| = |\Delta(-1)|.$$

By Proposition 2.2 the fixed point set of the $\mathbb{Z}_q$ action has rank equal to $|H_1(\Sigma(L/\mathbb{Z}_q);\mathbb{Z})| = |\Delta'(-1)|$. Therefore, the action of $\mathbb{Z}_q$ on $H_1(\Sigma(L);\mathbb{Z})_p$ induces an action on the quotient $H_1(\Sigma(L);\mathbb{Z})_p/H_1(\Sigma(L/\mathbb{Z}_q);\mathbb{Z})_p$ with only trivial fixed points, which has rank p^s. Applying Theorem 2.4 shows that s is a multiple of $2[q|p]$. □

Proposition 2.8. *Let L be a q-periodic link and suppose Δ' is the Alexander polynomial of the quotient. If $p \neq 2, q$ is a prime number such that p does not divide $\Delta'(-1)$, then the p-torsion part $H_1(\Sigma(L);\mathbb{Z})_p$ decomposes as an orthogonal sum of modules $T_{p,1},\dots,$ where each of the $T_{p,i}$ is a free $\mathbb{Z}_{p^i}$-module whose rank is divisible by $2[q|p]$.*

Proof. The assumptions of the proposition imply via Proposition 2.2 that $\mathbb{Z}_q$ acts on $H_1(\Sigma(L);\mathbb{Z})_p$ with no non-zero fixed points. Again, Theorem 2.4 implies that s is a multiple of $2[q|p]$. □

While Propositions 2.7 and 2.8 do not exhaust potential applications of Naik's criterion, they give a very good balance between applicability and generality. We demonstrate the result of implementing Proposition 2.7 and Proposition 2.8 in Section 11.

3. Linking forms and symmetric forms.

Let L be an m-periodic link and suppose $k > 1$ is such that $\Sigma_k(L)$ is a rational homology sphere. We let T denote the group $H_1(\Sigma_k(L);\mathbb{Z})$. Finally, let $\lambda\colon T\times T \to \mathbb{Q}/\mathbb{Z}$ be the linking form.

For a prime number p we write T_p for the p-torsion part of T, so that $T = \bigoplus T_p$, where the sum is over all prime numbers p. The summands are pairwise orthogonal with respect to the linking form. In order to study the linking form on T_p in more detail, we split T_p into summands $T_{p,k}$ as in Proposition 2.3. Each of the $T_{p,k}$ is a free $\mathbb{Z}_{p^k}$-module and the linking form restricts over $T_{p,k}$ to a form

$$\lambda\colon \mathbb{Z}_{p^k}^n \times \mathbb{Z}_{p^k}^n \to \mathbb{Q}/\mathbb{Z},$$

where n is the rank of $T_{p,k}$.

Let $\lambda\colon \mathbb{Z}_{p^k}^n \times \mathbb{Z}_{p^k}^n \to \mathbb{Q}/\mathbb{Z}$ be a non-degenerate linking form. As for any $x, y \in \mathbb{Z}_{p^k}^n$ we have $p^k x = 0$, we infer that $p^k\lambda(x,y) = 0 \in \mathbb{Q}/\mathbb{Z}$. It follows that

$$\lambda(x,y) = \frac{\beta(x,y)}{p^k}$$

for some $\beta(x,y) \in \mathbb{Z}$ well-defined modulo p^k. The following observation is straightforward.

LEMMA 3.1. *The form $\beta(x,y)\colon \mathbb{Z}_{p^k}^n \times \mathbb{Z}_{p^k}^n \to \mathbb{Z}_{p^k}$ is a symmetric, non-degenerate bilinear form. Moreover, any automorphism ψ preserving the linking form is an isometry of the bilinear form λ. Conversely, any isometry of the bilinear form β gives an automorphism of the linking form.*

Lemma 3.1 gives a translation from linking forms to symmetric bilinear forms. The reverse passage is also possible: a form $\beta(x,y)$ induces a linking form $\lambda(x,y) = \beta(x,y)/p^k$. In particular, the classification and the symmetries of linking forms correspond to the classification and the symmetries of bilinear forms.

DEFINITION 3.2. A *p^k-form* or, if the context is clear, simply a *form*, is a pair (B,β), where B is a free $\mathbb{Z}_{p^k}$-module of finite rank and $\beta\colon B \times B \to \mathbb{Z}_{p^k}$ is a symmetric, non-degenerate bilinear form. The *rank* of the form is the rank of B as a $\mathbb{Z}_{p^k}$-module.

The classification of p^k-forms is well-known. We recall a classical result of Wall [**21**]; see also [**9**].

THEOREM 3.3. *Let $p > 2$ be a prime, $k, n > 0$. Any p^k-form of rank n is isometric to one of the two following diagonal forms*:

- *the standard form, having $+1$ on the diagonal, i.e. $\beta(x,y) = \sum x_i y_i$, where $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$;*
- *the non-standard form, $\beta(x,y) = \tau x_1 y_1 + \sum_{i>1} x_i y_i$, where τ is not a square modulo p^k.*

REMARK 3.4. It is an immediate consequence of Theorem 3.3 that for every p^k-form (B,β) of rank n there exists a basis $x_1,\dots,x_n$ of B such that $\beta(x_i,x_j) = c_i\delta_{ij}$. Here, $c_i = 1$ for $i = 2,\dots,n$. For the standard form $c_1 = 1$; if the form is not standard, c_1 is not a square modulo p^k.

We will now introduce a notion of an index of a p^k-form. It distinguishes the standard form from the non-standard one. For reasons that will become clear later, the index of a standard form is not always equal to $+1$. Instead, a correction term (denoted by ϵ_2 below) depending on the rank and the prime number p is needed.

DEFINITION 3.5. If (B,β) is a p^k-form of rank n, the *index* $\epsilon(B,\beta)$ (written also $\epsilon(B)$ if no risk of confusion arises) is defined as $\epsilon_1\epsilon_2$, where $\epsilon_1 = 1$ if the form is standard (using the terminology of Theorem 3.3) and $\epsilon_1 = -1$ otherwise. We set $\epsilon_2 = -1$ if $p \equiv 3 \bmod 4$ and $n \equiv 2 \bmod 4$, otherwise $\epsilon_2 = 1$.

If M is a free $\mathbb{Z}_{p^k}$-module and $\lambda\colon M \times M \to \mathbb{Q}/\mathbb{Z}$ is a linking form, then the index $\epsilon(M)$ is the index of the p^k-form associated with (M,λ) via Lemma 3.1.

REMARK 3.6. In the present paper, we use the index of a form to distinguish isomorphism classes of its group of isometries. By Theorem 4.2 below, the groups of isometries of forms of odd rank do not depend on the index. Our interest will be mainly the index of a form of even rank. The index of a form of an odd rank is defined for completeness.

Lemma 3.7. *Suppose that $(B,\beta) = (B',\beta') \oplus (B'',\beta'')$ and B', B'' have even rank. Then $\epsilon(B,\beta) = \epsilon(B',\beta')\epsilon(B'',\beta'')$.*

Proof. Clearly $\epsilon_1(B,\beta) = \epsilon_1(B',\beta')\epsilon_1(B'',\beta'')$. Thus, we have to check that the same holds for ϵ_2. This is obvious if $p \equiv 1 \bmod 4$, because then $\epsilon_2 = 1$. If $p \equiv 3 \bmod 4$, then we check $\epsilon_2(B,\beta) = \epsilon_2(B',\beta')\epsilon_2(B'',\beta'')$ on a (trivial) case by case basis. □

4. Ranks of groups of isometries.

Definition 4.1. For a form (B,β) we let $O(B)$ be the group of isometries of B. More specifically, the group $O(p^k, n, \epsilon)$ is the group of isometries of the p^k-form of rank n with index ϵ.

We have the following result, for which we refer to the book of Wilson [**24**, Section 3.7.2].

Theorem 4.2. *The rank of $O(p,n,\epsilon)$ is equal to*

- $2p^m(p^2-1)(p^4-1)\cdots(p^{2m}-1)$ *if* $n = 2m+1$;
- $2p^{m(m-1)}(p^2-1)(p^4-1)\cdots(p^{2m-2}-1)(p^m-\epsilon)$ *if* $n = 2m$.

Moreover, if n is odd, the groups $O(p,n,+1)$ and $O(p,n,-1)$ are isomorphic.

We will extend Theorem 4.2 to compute the order of $O(p^k,n,\epsilon)$ for $k > 1$. First, we need the following construction.

Definition 4.3. Let (B,β) be a p^k-form. A (mod p)-*reduction* (or just: a *reduction*) of (B,β) is a symmetric bilinear form (B_{red}, β_{red}), where $B_{red} = B/p^{k-1}B$ is a free $\mathbb{Z}_p$-module and for $x, y \in B_{red}$ we set

$$\beta_{red}(x,y) = [\beta(\widetilde{x},\widetilde{y})] \in B/p^{k-1}B,$$

where $\widetilde{x}, \widetilde{y} \in B$ are lifts of x, y to B and $[\cdot]$ denotes the class in the quotient.

Informally, one may think of B_{red} as 'B mod p' and of β_{red} as 'β mod p'. We have an obvious observation.

Lemma 4.4. *The form (B_{red}, β_{red}) is non-degenerate. In particular, a reduction of a p^k-form is a p-form.*

Let (B,β) be a p^k-form and (B_{red},β_{red}) be a (mod p)-reduction. Suppose $\phi \in O(B)$. We define the isometry $\pi(\phi) \in O(B_{red})$ by the formula

$$\pi(\phi)(x) = [\phi(\widetilde{x})]\,, \tag{4.5}$$

where $x \in B_{red}$ and $\widetilde{x}$ is any lift of x to B. Formula (4.5) defines a group homomorphism

$$\pi\colon O(B) \to O(B_{red}),$$

which is also referred to as the (mod p)-reduction of an isometry.

Lemma 4.6. *The kernel of π has cardinality $p^{(k-1)\binom{n}{2}}$. The map π is a surjection.*

Proof. We focus on the case where B is standard. The other case is analogous. Choose a basis $\{e_i\}_{i=1}^n$ such that β is the identity matrix in this base. By $\overline{e}_i$ we will denote the basis of B_{red} obtained by reducing the basis $e_1, \dots, e_n$. Take $\phi_1 \in O(B_{red})$. Using the basis $\overline{e}_1, \dots, \overline{e}_n$ we represent ϕ_1 as a matrix Φ_p with coefficients in $\mathbb{Z}_p$. Choose a lift of the matrix Φ_p to a matrix Φ over $\mathbb{Z}$, that is, lift all the coefficients to $\mathbb{Z}$. The matrix Φ defines also a linear map $\phi\colon B \to B$. Obviously, ϕ depends on the choice of a lift of Φ_p to Φ.

We want to find vectors $v_1, \dots, v_n$ in $\mathbb{Z}_{p^k}^n$ such that $\widetilde{\phi}$ defined by $\widetilde{\phi}(e_i) = \phi(e_i) + pv_i$ is an isometry. Note that regardless of the choice of v_i we have $\pi(\widetilde{\phi}) = \pi(\phi)$. Moreover, if the vectors $v_1, \dots, v_n$ are replaced by $v_1', \dots, v_n'$ such that $v_i - v_i'$ is a multiple of p^{k-1}, then $\widetilde{\phi}$ does not change.

The map $\widetilde{\phi}$ is an isometry if and only if the vectors v_i satisfy the following condition for all $1 \leq i \leq j \leq n$

$$(\phi(e_i) + pv_i)^T(\phi(e_j) + pv_j) = \delta_{ij}, \tag{4.7}$$

where δ_{ij} is the Kronecker's delta. Here and afterward in the proof, we write $x^T y$ for the scalar product of x and y, that is, for $\beta(x, y)$.

As ϕ_1 is an isometry, we infer that $\phi(e_i)^T\phi(e_j) = \delta_{ij} + pc_{ij}$ for some c_{ij}, hence (4.7) becomes

$$p(c_{ij} + v_i^T\phi(e_j) + \phi(e_i)^T v_j) = 0. \tag{4.8}$$

The equation (4.8) taken for all $i \leq j$ gives a system of $n(n+1)/2$ linear independent equations with n^2 variables: the variables are coefficients of the vectors v_i, which should be considered as elements of the space over $\mathbb{Z}_{p^{k-1}}$. Indeed, changing coordinates of v_i by a multiple of p^{k-1} does not change $\widetilde{\phi}$.

The independence of the system (4.8) follows easily from the independence of vectors $\phi(e_j)$. The space of solutions has dimension $n^2 - n(n+1)/2 = \binom{n}{2}$. Therefore, the number of solutions is equal to $p^{(k-1)\binom{n}{2}}$ as desired. □

As a corollary, we obtain the following fact.

Corollary 4.9. *Let B be a p^k-form of rank n and B_{red} its* (mod p)*-reduction. Then*

$$|O(B)| = p^{(k-1)\binom{n}{2}}|O(B_{red})|.$$

5. Sylow groups of the orthogonal group $O(B)$.

In this section we let q be a fixed odd prime number different from p.

Definition 5.1. For a group G, we denote by $\mathrm{Syl}_q\, G$ a Sylow q-subgroup of G.

We have the following relation between Sylow groups of the orthogonal group of a form and Sylow groups of the orthogonal group of the (mod p)-reduction.

Theorem 5.2. *The* (mod p)*-reduction* $\pi\colon O(B)\to O(B_{red})$ *takes a Sylow group* $\mathrm{Syl}_q\,O(B)$ *isomorphically to a Sylow group of* $O(B_{red})$.

Proof. By definition, $\mathrm{Syl}_q\,O(B)$ has rank equal to q^s, where s is the maximum integer such that q^s divides $|O(B)|$. Likewise, the rank of $\mathrm{Syl}_q\,O(B_{red})$ is equal to q^{s_1} for s_1 the maximal positive integer such that q^{s_1} divides $|O(B_{red})|$. By Corollary 4.9, we obtain $|\mathrm{Syl}_q\,O(B)|=|\mathrm{Syl}_q\,O(B_{red})|$.

Now let $G=\pi(\mathrm{Syl}_q\,O(B))$, where $\pi\colon O(B)\to O(B_{red})$. Clearly, G is a q-subgroup of $O(B_{red})$. The map $\pi|_{\mathrm{Syl}_q\,O(B)}\colon \mathrm{Syl}_q\,O(B)\to G$ is a homomorphism of groups. The kernel of $\pi|_{\mathrm{Syl}_q\,O(B)}$ is a subgroup of $\mathrm{Syl}_q\,O(B)$ and also a subgroup of $\ker\pi$. However, $\ker\pi$ is a p-group by Lemma 4.6 and so its intersection with $\mathrm{Syl}_q\,O(B)$ is trivial. In particular, $\pi|_{\mathrm{Syl}_q\,O(B)}$ is an isomorphism onto its image.

By Sylow's theorem, there exists a Sylow q-group H of $O(B_{red})$, such that $G\subset H$. However, $|G|=|\mathrm{Syl}_q\,O(B)|$ and $|H|=|\mathrm{Syl}_q\,O(B_{red})|=|\mathrm{Syl}_q\,O(B)|$. Hence, $|G|=|H|$ and thus $G=H$. □

To understand the structure of $\mathrm{Syl}_q\,O(B)$ we will use Weir's theorem [**23**]. We need to recall the definition of a regular wreath product.

Definition 5.3. Let H and G be finite groups. Let $K=\Pi_{g\in G}H$. Let ψ be the action of G on K by left multiplication of indices. We call a semidirect product of K and G a *regular wreath product* of H and G. We denote it by $H\boldsymbol{\iota_r}G=K\rtimes_\psi G$.

Remark 5.4. The subscript r in the symbol $\boldsymbol{\iota_r}$ is not a parameter. It is a shorthand for 'regular'.

The following result is due to Weir [**23**]. Note that the original statement is for p-forms. By Theorem 5.2, the result carries through to p^k-forms for general $k\geq 1$.

Theorem 5.5 (see [**23**]). *Let d be a natural number with q-adic expansion given by $d=a_0+a_1q+a_2q^2+\cdots$. Let (B,β) be a p-form of rank $2[q|p]d$ such that $\epsilon(B)=\eta(q)^d$. Any Sylow q-subgroup of $O(B)$ is isomorphic to*:

$$\mathbb{Z}_{q^s}^{a_0}\times(\mathbb{Z}_{q^s}\boldsymbol{\iota_r}\mathbb{Z}_q)^{a_1}\times((\mathbb{Z}_{q^s}\boldsymbol{\iota_r}\mathbb{Z}_q)\boldsymbol{\iota_r}\mathbb{Z}_q)^{a_2}\times(((\mathbb{Z}_{q^s}\boldsymbol{\iota_r}\mathbb{Z}_q)\boldsymbol{\iota_r}\mathbb{Z}_q)\boldsymbol{\iota_r}\mathbb{Z}_q)^{a_3}\times\cdots, \tag{5.6}$$

where s is such that $q^s|p^{2[q|p]}-1$ and $q^{s+1}\nmid p^{2[q|p]}-1$.

The general case of (B,β) not satisfying the assumptions $\mathrm{rk}\,B=2[q|p]d$, $\epsilon(B)=\eta(q)^d$ reduces to Theorem 5.5. The approach is implicit in [**23**], but it will be used in several places in the present paper, so we sketch it briefly below.

Let B be a form of rank $2[q|p]d+R$ with $0\leq R<2[q|p]$. We want to find a decomposition of B into an orthogonal sum of forms B' and B'' in such a way that the map induced by inclusion $O(B')\to O(B)$ (extending an isometry of B' by the identity on B'') induces an isomorphism of Sylow q-groups and B' satisfies the hypotheses of

Theorem 5.5. To this end, we let $\mathbf{1}_+$ denote the one-dimensional p-form with $\epsilon = 1$ and $\mathbf{1}_-$ denote the one-dimensional form with $\epsilon = -1$. We have the following possibilities:

- Case 1: $R = 0$ and $\epsilon(B) = \eta(q)^d$. Then B already satisfies the assumptions of Theorem 5.5. We set $B' = B$ and let B'' be trivial (zero-dimensional).
- Case 2: $R > 0$. We define B' to be the unique form of rank $2[q|p]d$ for which $\epsilon(B') = \eta(q)^d$. We let B'' be a form of rank R that is a direct sum of $\mathbf{1}_+$ and $\mathbf{1}_-$ arranged in such a way that $\epsilon(B' \oplus B'') = \epsilon(B)$. Then B and $B' \oplus B''$ have the same rank and the same index ϵ, so they are isomorphic.
- Case 3: $R = 0$ but $\epsilon(B) \neq \eta(q)^d$. We define B' to be the unique form of rank $2[q|p](d-1)$ such that $\epsilon(B') = \eta(q)^{d-1}$. As in the previous case, we choose B' to be a direct sum of $2[q|p]$ forms $\mathbf{1}_+$ and $\mathbf{1}_-$ in such a way that $\epsilon(B' \oplus B'') = \epsilon(B)$.

Remark 5.7. In the above construction, we do not use the fact that $\epsilon(B' \oplus B'') = \epsilon(B')\epsilon(B'')$. Multiplicativity of ϵ was proved only if both summands have even rank. This is not necessarily true in Case 2 above.

Lemma 5.8. *The inclusion $O(B') \to O(B)$ induces an isomorphism $\operatorname{Syl}_q O(B') \cong \operatorname{Syl}_q O(B)$.*

Proof. Let G be a Sylow q-group of $O(B')$. Write $i\colon O(B') \to O(B)$ for the map induced by inclusion. Then $i(G)$ is a q subgroup of $O(B)$ and as such, it is contained in some Sylow group H of $O(B)$. Now $|H|/|G|$ is equal to the maximal exponent s such that q^s divides $|O(B)|/|O(B')|$. However, by Theorem 4.2 and Corollary 4.9, q does not divide $|O(B)|/|O(B')|$. Therefore, $|H|/|G| = 1$ and $i(G) \subset H$, so $i(G) = H$. □

Definition 5.9. The form B' is called the *maximal q-regular subform of B*. The form B'' is called the *complementary* form.

6. Elements of maximal order in Sylow groups.

Let G be a finite group. For a prime q we define $\mu_q(G)$ to be the maximal order of those elements of G whose order is a power of q. We have the following result, which is well-known to the experts in group theory. For the reader's convenience, we present a proof.

Lemma 6.1. *Let H be a finite q-group and $\mu_q(H) = q^t$ for some $t > 0$. Then $\mu_q(H \boldsymbol{\imath_r} \mathbb{Z}_q) = q^{t+1}$.*

Proof. Let h be an element of H such that $\operatorname{ord}(h) = \mu_q(H)$ and let e be the identity of H. Then $h^* = ((h, e, e, \dots, e), 1) \in H \boldsymbol{\imath_r} \mathbb{Z}_q$ has order equal to q^{t+1}. Therefore, $\mu_q(H \boldsymbol{\imath_r} \mathbb{Z}_q) \geq q^{t+1}$.

To obtain the opposite inequality, we first observe that if $G = K \rtimes_\psi H$, then $\mu_q(G) \leq \mu_q(K)\mu_q(H)$. Indeed, consider $x = (k, h) \in K \rtimes_\psi H$, then

$$x^u = (k\psi_h(k) \cdots \psi_{h^{u-1}}(k), h^u).$$

Suppose u is the minimal power such that $h^u = e \in H$. Set $y = k\psi_h(k)\cdots\psi_{h^{u-1}}(k)$. Then we have $x^u = (y, e)$ for some y. Thus, $x^{ru} = (y^r, e)$. From this it easily follows that $\mu_q(G) \le \mu_q(K)\mu_q(H)$.

Given the last inequality, by Definition 5.3, we write $H \boldsymbol{\iota_r} \mathbb{Z}_q$ as $K \rtimes_\psi \mathbb{Z}_q$, where $K = \Pi_{g\in\mathbb{Z}_q} H$. We have $\mu_q(K) = \mu_q(H)$, hence $\mu(H\boldsymbol{\iota_r}\mathbb{Z}_q) \le \mu_q(H)\mu_q(\mathbb{Z}_q)$ as desired. □

Lemma 6.1 combined with Weir's theorem (Theorem 5.5) allows us to describe all forms whose orthogonal groups have elements of order q^r.

Theorem 6.2. *Let s be a natural number such that $q^s | p^{2[q|p]}-1$ and $q^{s+1} \nmid p^{2[q|p]}-1$. Let B be a p^k-form.*

- *For $r \ge s$, the group $O(B)$ contains an element of order q^r if and only if B has rank $2[q|p]q^{r-s}$ and $\epsilon(B) = \eta(q)^{q^{r-s}}$, or B has rank strictly greater than $2[q|p]q^{r-s}$.*
- *For $r < s$, the group $O(B)$ contains an element of order q^r if and only if either B has rank $2[q|p]$ with $\epsilon(B) = \eta(q)$, or rank B is greater than $2[q|p]$.*

Proof. Let B' be a maximal q-regular subform of B as in Definition 5.9. By Lemma 5.8, we have an isomorphism of Sylow q-groups of B and B'. Therefore, it suffices to restrict to the case where B has rank $2[q|p]d$ and $\epsilon(B) = \eta(q)^d$. We clearly have $\mu_q(G \times H) = \max(\mu_q(G), \mu_q(H))$. Hence, by (5.6):

$$\mu_q(O(B)) = \max_{i\colon a_i\neq 0} \mu_q(\underbrace{(((\mathbb{Z}_{q^s}\boldsymbol{\iota_r}\mathbb{Z}_q)\boldsymbol{\iota_r}\mathbb{Z}_q)\cdots)\boldsymbol{\iota_r}\mathbb{Z}_q}_{i \text{ times}}),$$

where a_i form the q-adic presentation of d, that is, $d = a_0 + a_1 q + a_2 q^2 + \cdots$.

By Lemma 6.1 we obtain that $\mu_q((\mathbb{Z}_{q^s}\boldsymbol{\iota_r}\mathbb{Z}_q)\cdots\boldsymbol{\iota_r}\mathbb{Z}_q) = q^{s+i}$ if the wreath product is taken i times. In particular

$$\mu_q(O(B)) = \max_{i\colon a_i\neq 0} q^{s+i}. \tag{6.3}$$

Suppose $r \ge s$. We conclude that $\mu_q(O(B)) \ge q^r$ if and only if some of the $a_t > 0$ for $t \ge r - s$. This amounts to saying that $d \ge q^{r-s}$, if $r \ge s$, or $d \ge 1$ if $r < s$. □

Proof of Theorem 1.3(a). Theorem 1.3(a) follows immediately from Theorem 6.2. □

7. Fixed points of isometries of order q^r.

Let B be a p^k-form and let $\phi\colon B \to B$ be an isometry. We study the fixed point set

$$\operatorname{Fix}\phi = \{x \in B\colon \phi(x) = x\}.$$

The following result implies the 'only if' direction of Theorem 1.3(b).

Theorem 7.1. *Let B be a p^k-form of rank $n = 2[q|p]d + R$ with $0 \le R < 2[q|p]$. Let $\phi \in \operatorname{Syl}_q O(B)$ have order q^r. Let s be the maximal integer such that $q^s | p^{2[q|p]} - 1$.*

(a) *If $R > 0$, then ϕ has a fixed subspace of rank at least R. More precisely there is an orthogonal decomposition of B into submodules B_1 and B_2 such that B_1 has rank divisible by $2[q|p]$ and $\phi|_{B_2}$ is the identity.*

(b) *If $R = 0$ and $\eta(q)^n \neq \epsilon(B)$, then ϕ has a fixed subspace of rank at least $2[q|p]$.*

Proof. Write $B = B' \oplus B''$, where B' and B'' are as in Definition 5.9. Note that in cases (a) and (b) the subspace B'' is non-trivial. In fact, case (a) of Theorem 7.1 corresponds to Case 2 of the list preceding Remark 5.7. Case (b) of Theorem 7.1 corresponds to Case 3 of the list.

Let $i\colon O(B') \to O(B)$ be the map induced by inclusion. Recall that i takes an element $\psi \in O(B')$ and extends it by the identity on B''. This implies, in particular, that B'' is contained in the fixed point set of any element of $O(B)$ that is in the image of i. Let G be a Sylow group of $O(B')$. By Lemma 5.8 we have that $i(G)$ is a Sylow group of $O(B)$. Take $\phi \in O(B)$ of rank q^r and let $H \subset O(B)$ be a Sylow group containing ϕ. As all Sylow groups are conjugate, we have that $i(G) = gHg^{-1}$ for some $g \in O(B)$. In particular, $\phi = g\phi' g^{-1}$ for some $\phi' \in i(G)$. Define $B_1 = gB'$ and $B_2 = gB''$. Then B_1 has rank divisible by $2[q|p]$ and B_2 contained in the fixed point set of ϕ. □

8. Isometries with no non-zero fixed points.

The goal of this section is to provide the 'if' implication of Theorem 1.3.

Theorem 8.1. *Suppose we have a p^k-form (B, β) of rank $n = 2[q|p]d$ and $\eta(q)^d = \epsilon(B)$. Suppose also that $\mathrm{Syl}_q\, O(B)$ contains an element of order q^r. There exists $\psi \in O(B)$ of order q^r such that ψ has no non-zero fixed points.*

The remaining part of Section 8 is devoted to the proof of Theorem 8.1. We first prove some auxiliary results, then give the proof of Theorem 8.1.

The first step in the proof of Theorem 8.1 is the following result, which may be of independent interest.

Proposition 8.2. *Let (B, β) be a p-form and let $\phi \in O(B)$ be of order q^r for some $r > 0$. Then there exists an orthogonal decomposition of B into free $\mathbb{Z}_p$-modules B_{fix} and B_{oth}, such that ϕ is the identity of B_{fix} and ϕ acts with no non-zero fixed points on B_{oth}.*

Proof. Let ζ be a primitive root of unity of order q^r and consider the ring $\Lambda = \mathbb{Z}_p[\zeta]$. By Maschke's theorem, Λ is semisimple. On the other hand, B has the structure of a Λ-module, where the multiplication by ζ corresponds to the action of ϕ. Hence, B can be written as a direct sum of cyclic Λ-modules. Let B_{fix} be the sum of those cyclic modules on which ζ acts trivially and B_{oth} be the sum of all the others. The decomposition is orthogonal. Indeed, consider the map $B_{fix} \to B_{oth}$ given by the inclusion of B_{fix} to B followed by the orthogonal projection to B_{oth}. This map is trivial by Schur's lemma, and therefore B_{fix} is orthogonal to B_{oth}. By the construction, B_{fix} is the fixed point set of ϕ and ϕ acts on B_{oth} with no non-zero fixed points. □

Proposition 8.2 allows us to prove the following lemma, which is the key argument in the proof of Theorem 8.1.

Lemma 8.3. *Suppose* (B,β) *is a p-form. Let s be the maximal integer such that* $q^s|(p^{2[q|p]}-1)$. *Assume* $r \geq s$ *and the rank of B equals* $2[q|p]q^{r-s}$. *If* $\epsilon(B)=\eta(q)^{q^{r-s}}$ *then an isometry* $\psi \in O(B)$ *of order* q^r *has no non-zero fixed points on B.*

Proof. Suppose towards contradiction that there exists $\psi \in O(B)$ of order q^r with a non-trivial fixed subspace. By Proposition 8.2, we infer that B splits as a direct sum of B_{fix} and B_{oth}. The map ψ preserves the splitting. Write ψ' for the restriction of ψ to B_{oth}. Then $\psi' \in O(B_{oth})$ has order q^r. However, the rank of B_{oth} is smaller than $2[q|p]q^{r-s}$, so by Theorem 6.2, $O(B_{oth})$ cannot contain an element of order q^r. This contradiction shows that ψ itself cannot have a non-zero fixed point. □

Proposition 8.2 is true for p-forms, not p^k-forms in general. As an example, one can take the module $\mathbb{Z}_{p^3}^2$ with the standard linking form and a morphism ϕ given by the matrix

$$A=\begin{pmatrix}1-2p^2 & -2p\\ 2p & 1-2p^2\end{pmatrix}. \tag{8.4}$$

Over $\mathbb{Z}_{p^3}$ we have $A^TA=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, hence ϕ is an isometry. The point (p^2,p^2) is a non-zero fixed point of A. However, there is no free $\mathbb{Z}_{p^3}$-submodule on which ϕ acts trivially.

To deal with p^k-forms we apply again the (mod p)-reduction.

Lemma 8.5. *Suppose* (B,β) *is a* p^k*-form,* ψ *is an isometry and* $z \in B$ *is a non-zero fixed point of* ψ. *Let* (B_{red},β_{red}) *be the* (mod p)*-reduction and let* ψ_{red} *be the reduction of* ψ. *Then* ψ_{red} *has a non-trivial fixed point.*

Proof. If the reduction $\pi(z)$ is non-trivial, then clearly $\pi(z)$ is a fixed point of ψ_{red}. Otherwise, if $\pi(z)$ is trivial, then z can be written as $z=p^\ell y$ for some $\ell>0$ and y such that $\pi(y)\neq 0$. As $\psi(z)-z=0$ and ψ is linear, we infer that $\psi(y)-y=p^{k-\ell}u$ for some $u\in B$. However, this means that $\psi_{red}(\pi(y))-\pi(y)=0$. Hence, $\pi(y)$ is a non-trivial fixed point of ψ_{red}. □

Example 8.6. If A is the matrix as in (8.4) defining a morphism ϕ of $\mathbb{Z}_{p^3}^2$, then the (mod p)-reduction $\pi(\phi)$ is the identity matrix.

Given Lemma 8.5 we quickly generalize Lemma 8.3 to p^k-forms.

Lemma 8.7. *Let s be as in Lemma* 8.3 *and* $r\geq s$. *Let* (B,β) *be a* p^k*-form of rank* $2[q|p]q^{r-s}$ *and* $\epsilon(B)=\eta(q)^{q^{r-s}}$. *Any element* $\psi\in O(B)$ *of order* q^r *has no non-zero fixed points.*

Proof. If such an isometry ψ has a non-trivial fixed point, then $\pi(\psi)$ has a non-trivial fixed point (and hence a fixed subspace) by Lemma 8.5. The isometry $\pi(\psi)$ has the same order q^r because by Theorem 5.2 the reduction operation yields an isomorphism on Sylow q-groups. However, this contradicts Lemma 8.3. □

The method of the proof of Lemma 8.7 gives the following simple result, which is needed to complete the proof of Theorem 8.1.

Lemma 8.8. *Suppose B has rank $2[q|p]$ and $\epsilon(B) = \eta(q)$. For any ℓ such that $1 \leq \ell \leq s$, there exists an isometry of B of order q^ℓ with no non-zero fixed points.*

Proof. We act as in the proof of Lemma 8.3. Suppose B is a p-form of rank $2[q|p]$. From Theorem 6.2 we deduce that there exists an isometry ψ of order q^ℓ. If it has a fixed subspace, the form splits as an orthogonal sum of B_{fix} and B_{oth}, but B_{oth} has rank less than $2[q|p]$ and hence it does not admit any isometry of order q^ℓ for $\ell > 0$.

For p^k-forms we use the same argument as in Lemma 8.7. □

Now we can give a proof of Theorem 8.1.

Proof of Theorem 8.1. First, assume that $r \geq s$. We must have $d \geq q^{r-s}$, for otherwise there is no element in $O(B)$ of order q^r by Theorem 6.2. Let B_0 be the unique form of rank $2[q|p]q^{r-s}$ such that $\epsilon(B_0) = \eta(q)^{q^{r-s}}$. Consider also the forms $B_1, \dots, B_{d-q^{r-s}}$ that have rank $2[q|p]$ and index $\eta(q)$. The form $\widetilde{B} = B_0 \oplus B_1 \oplus \cdots \oplus B_{d-q^{r-s}}$ has the same rank as B and, by Lemma 3.7 we have that $\epsilon(B) = \epsilon(\widetilde{B}) = \eta(q)^d$. Hence, B and $\widetilde{B}$ are isometric.

We construct now an isometry on $\widetilde{B}$ of order q^r which does not have non-zero fixed points. It is a block sum of isometries on B_0 and $B_1, \dots, B_{d-q^{r-s}}$. On B_0 we use an isometry $\psi_0 \colon B_0 \to B_0$ with no non-zero fixed points, which is provided by Lemma 8.7. For $j = 1, \dots, d - q^{r-s}$ we take an isometry $\psi_j \colon B_j \to B_j$ which is of order q and which is provided by Lemma 8.8. The block sum of such isometries gives an isometry of B with no non-zero fixed points and of order q^r.

If $r < s$, the proof is analogous. Namely, as (B, β) has rank $2[q|p]d$, we can present it as a direct sum of d copies of a form (B_0, β_0) such that $\epsilon(B_0, \beta_0) = \eta(q)$. The form (B_0, β_0) admits an isometry ϕ_0 of order q^r with no non-zero fixed points by Lemma 8.8. The isometry on B is a direct sum of d copies of ϕ_0. □

9. Computing $\epsilon(B)$ for forms on the double branched cover.

In order to effectively apply Theorem 1.4, we need a way to compute the indices $\epsilon(B)$ of forms associated with linking forms on branched covers of links. We present such an algorithm now. A detailed example is given in Section 10.

Suppose L is a link and S a Seifert matrix. Write $A = S + S^T$. If $\det(A) \neq 0$, then the double branched cover $\Sigma(L)$ is a rational homology sphere. For the rest of this section, we shall assume that this is the case. Let n denote the size of matrix A. Then

$$H_1(\Sigma(L); \mathbb{Z}) \cong \mathbb{Z}^n / A\mathbb{Z}^n.$$

Under this identification, the linking form is given by

$$\begin{aligned} \lambda_A \colon \mathbb{Z}^n / A\mathbb{Z}^n \times \mathbb{Z}^n / A\mathbb{Z}^n &\to \mathbb{Q}/\mathbb{Z} \\ (x, y) &\mapsto x^T A^{-1} y. \end{aligned} \tag{9.1}$$

Put A into Smith normal form, that is, write

$$A = CDE,$$

where C and E are invertible over $\mathbb{Z}$ and D is diagonal with integer entries $(d_1, \dots, d_n)$ on the diagonal such that $d_i | d_{i+1}$ for $i = 1, \dots, n-1$.

The following simple result gives an effective way of computing the linking form on $\Sigma(L)$.

Lemma 9.2. *The pairing* (9.1) *is isometric to the pairing*

$$\begin{aligned} \lambda_D \colon \mathbb{Z}^n/D\mathbb{Z}^n \times \mathbb{Z}^n/D\mathbb{Z}^n &\to \mathbb{Q}/\mathbb{Z} \\ (x,y) &\mapsto x^T C^T E^{-1} D^{-1} y. \end{aligned} \tag{9.3}$$

Proof. The map $\mathbb{Z}^n \xrightarrow{x \mapsto Cx} \mathbb{Z}^n$ descends to an isomorphism of abelian groups from $\mathbb{Z}^n/D\mathbb{Z}^n$ to $\mathbb{Z}^n/A\mathbb{Z}^n$. Now if $x, y \in \mathbb{Z}^n/D\mathbb{Z}^n$, then $Cx, Cy \in \mathbb{Z}^n/A\mathbb{Z}^n$. Then,

$$\lambda_A(Cx, Cy) = (Cx)^T (CDE)^{-1} Cy = \lambda_D(x,y). \qquad \square$$

Let p be an odd prime. Let $0 = \alpha_0 \leq \alpha_1 \leq \cdots$ be the integers such that if $\alpha_k \leq i < \alpha_{k+1}$, then $p^k | d_i$ but $p^{k+1} \nmid d_i$. Let h_i be the vectors having 0 at all places except for the i-th coordinate, which is equal to d_i/p^k. Let $T_{p,k}$ be the submodule of $\mathbb{Z}^n/D\mathbb{Z}^n$ generated by $h_{\alpha_k}, \dots, h_{\alpha_{k+1}-1}$. Then $T_{p,k}$ is a free $\mathbb{Z}_{p^k}$-module. The sum $T_{p,1} \oplus T_{p,2} \oplus \cdots$ is the p-torsion part of the module $\mathbb{Z}^n/D\mathbb{Z}^n$. Despite the similarity of this notation to the decomposition in Proposition 2.3, in this case, the summands need not be pairwise orthogonal.

The linking form λ_D restricts to a linking form on each of the $T_{p,k}$. Write $\beta_{p,k}$ for the associated bilinear form. In the basis $h_{\alpha_k}, \dots, h_{\alpha_{k+1}-1}$, the form can be written by a matrix, which we denote B_k. Note that the ij-entry of B_k is given by $p^k \lambda_D(h_{i+\alpha_k-1}, h_{j+\alpha_k-1})$.

Write $\widetilde{T}_{p,k}$ for the (mod p)-reduction of $T_{p,k}$ and let $\widetilde{B}_k$ be the matrix of the reduced form. Clearly we have

$$\det \widetilde{B}_k \equiv \det B_k \pmod{p}.$$

Proposition 9.4. *The linking form on the p^k-torsion part of $H_1(\Sigma(K);\mathbb{Z})$ has $\epsilon_1 = +1$ (see Definition 3.5) if and only if $\det \widetilde{B}_k$ is a square modulo p.*

Proof. We first give the proof under an extra assumption that $T_{p,k}$ is orthogonal to $T_{p,k'}$ with respect to the form λ_D for all k, k' such that $k \neq k'$. By Theorem 3.3 there exists a matrix U with coefficients in $\mathbb{Z}_{p^k}$ such that UB_kU^T is a diagonal matrix whose all the diagonal entries but the top-left one are equal to 1, and the top-left entry is either 1 (if $\epsilon_1 = 1$) or a non-square modulo p^k (if $\epsilon_1 = -1$). Thus, the index ϵ_1 depends on whether $\det UB_kU^T$ is a square modulo p^k or not. However, $\det B_k$ and $\det UB_kU^T$ differ by a square. Therefore, $\epsilon_1 = 1$ if and only if $\det B_k$ is a square modulo p^k. By Hensel's lemma, this is the same as saying that $\det \widetilde{B}_k$ is a square modulo p.

Now consider the general case, where the summands $T_{p,k}$ are not necessarily orthogonal. Write $T_p = T_{p,1} \oplus T_{p,2} \oplus \cdots$. By Proposition 2.3, T_p splits as a sum $T'_{p,1} \oplus T'_{p,2} \oplus \cdots$ of pairwise orthogonal summands (with respect to the linking form λ_D) such that $T'_{p,j}$ is a free $\mathbb{Z}_{p^j}$-module. We have thus an isometry

$$\psi \colon T_{p,1} \oplus T_{p,2} \oplus \cdots \to T'_{p,1} \oplus T'_{p,2} \oplus \cdots .$$

Write ψ_{ij} for the part of ψ mapping from $T_{p,i}$ to $T'_{p,j}$. Choose a basis $e_{i1}, \dots, e_{ik_i}$ for $T_{p,i}$ and also a basis $f_{jk_j}, \dots, f_{jk_j}$ for $T'_{p,j}$. For e_{ij} we can take vectors h_d defined before, but here we need a more concise notation. With respect to these bases, we denote by κ_i the determinant of the intersection form on $T_{p,i}$ and by κ'_j the determinant of the intersection form on $T'_{p,j}$.

Lemma 9.5. *There is a congruence $\kappa'_i \equiv q_i^2 \kappa_i \pmod p$ for some q_i invertible modulo p.*

Given Lemma 9.5 we quickly finish the proof of Proposition 9.4. Namely,

- The first part of the proof of Proposition 9.4 applied to $T'_{p,i}$ tells that $\epsilon_1 = 1$ if and only if κ'_i is a square modulo p;
- Lemma 9.5 implies that $\epsilon_1 = 1$ if and only if κ_i is a square modulo p;
- The bases $e_{i1}, \dots, e_{ik_i}$ and $h_{\alpha_i}, \dots, h_{\alpha_{i+1}-1}$ differ by an invertible matrix U_i. Hence, $\kappa_i = \det U_i \det B_i \det U_i^T$, so $\kappa_i = \det B_i (\det U_i)^2$. That is, κ_i is a square modulo p if and only if $\det B_i$ is a square modulo p. □

It remains to prove Lemma 9.5.

Proof of Lemma 9.5. Take $x, y \in T_{p,i}$. We have

$$p^i \lambda_D(x,y) = p^i \lambda_D(\psi x, \psi y) = p^i \lambda_D\Bigl(\sum_j \psi_{ji} x, \sum_k \psi_{ki} y\Bigr).$$

The spaces $T'_{p,j}$ are pairwise orthogonal, so we rewrite the last sum as

$$p^i \lambda_D(x,y) = p^i \sum_j \lambda_D(\psi_{ji} x, \psi_{ji} y). \tag{9.6}$$

Claim. *If $j \neq i$, we have*

$$p^i \lambda_D(\psi_{ji} x, \psi_{ji} y) \equiv 0 \pmod p. \tag{9.7}$$

To prove the claim, we consider two cases. Either $j < i$ or $j > i$. Suppose $j < i$. We have $\psi_{ji} x, \psi_{ji} y \in T'_{p,j}$, which is a p^j-torsion part. In particular $\lambda_D(\psi_{ji} x, \psi_{ji} y) = c/p^j$ for some integer c. As $i > j$ we obtain (9.7) immediately.

Suppose $j > i$. Both x and y are annihilated by p^i. Consequently, $\psi_{ji} x, \psi_{ji} y$ are annihilated by p^i. As $T'_{p,j}$ is a free $\mathbb{Z}_{p^j}$-module we infer that $\psi_{ji} x = p^{j-i} x_j$, $\psi_{ji} y = p^{j-i} y_j$ for some elements $x_j, y_j \in T'_{p,j}$. We also have $\lambda_D(x_j, y_j) = c'/p^j$ for $c' \in \mathbb{Z}$. Hence

$$\lambda_D(\psi_{ji}x, \psi_{ji}y) = p^{2j-2i}\lambda_D(x_j, y_j) = c'p^{j-2i}.$$

Therefore, $p^i\lambda_D(\psi_{ji}x, \psi_{ji}y) = c'p^{j-i} \equiv 0 \pmod p$, so (9.7) holds also for $j > i$.

Having established (9.7), we obtain

$$p^i\lambda_D(x, y) \equiv p^i\lambda_D(\psi_{ii}x, \psi_{ii}y) \pmod p. \tag{9.8}$$

Write $\widetilde{T}_{p,i}$, $\widetilde{T}'_{p,i}$ for the reductions of the forms on $T_{p,i}$ and $T'_{p,i}$. Equation (9.8) implies that ψ_{ii} induces an isometry between $\widetilde{T}_{p,i}$ and $\widetilde{T}'_{p,i}$. As the forms $\widetilde{T}_{p,i}$ and $\widetilde{T}'_{p,i}$ are isometric, their determinants differ by a square modulo p. □

We conclude this section with the following remark. In the proof of Proposition 9.4, we did not assume that the splitting of $T_p = T_{p,1} \oplus \cdots$ is invariant with respect to the group action. Indeed, our argument implies that the sign of the linking form does not depend on the particular choice of splitting.

10. Example: the knot 10_{123}.

Our aim is now to illustrate the algorithm described in Section 9 on a concrete knot. Many knots pass Naik's obstructions but fail to the obstruction provided by Theorem 1.4; see Section 11. However all such knots that are known to us have Seifert matrices of size at least 14×14. In order to make the discussion in this section more transparent, we provide an example of a knot that actually is 5-periodic. Its Seifert matrix has size only 8×8. We will show, how we did verify that it passes our criterion.

Consider the knot $K = 10_{123}$. It is well known to be 5-periodic. A 5-periodic diagram can be found in [**8**, Figure 2] or on the KnotInfo webpage [**2**]. The Alexander polynomial of K is equal to

$$\Delta = t^8 - 6t^7 + 15t^6 - 24t^5 + 29t^4 - 24t^3 + 15t^2 - 6t + 1 = (t^4 - 3t^3 + 3t^2 - 3t + 1)^2.$$

There are two factors of the Alexander polynomial over $\mathbb{Z}[t, t^{-1}]$. One is 1, the other one is $t^4 - 3t^3 + 3t^2 - 3t + 1$. We check that the latter does not satisfy (2.6) for any ℓ. On the other hand, we have a congruence

$$\Delta \equiv (1 + t + t^2)^4 \bmod 5,$$

so Δ (rather unsurprisingly) passes Murasugi's criterion. The polynomial 1 is the only candidate for the Alexander polynomial of the quotient. In particular, by Proposition 2.2, the $\mathbb{Z}_5$ symmetry should act on $H_1(\Sigma(K); \mathbb{Z})_p$ with only 0 as a fixed point for all $p \neq 5$.

We study the homology of the double branched cover. The Seifert matrix of K is

$$S = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have

$$A = S + S^T = CDE,$$

where D is the diagonal matrix with the diagonal vector $(1, 1, 1, 1, 1, 1, 11, 11)$ and C and E are invertible over $\mathbb{Z}$. We see that the homology of the double branched cover is equal to $\mathbb{Z}_{11}^2$. Now for $p = 11$ and $q = 5$, we have $[q|p] = 1$, because $5|(11 - 1)$. In particular, 10_{123} passes Naik's criterion. Note also, for future use, that $\eta(q) = 1$.

We calculate the index of the linking form on $\mathbb{Z}_{11}^2$. The generators for the module $\mathbb{Z}^8/D\mathbb{Z}^8$ are $h_1 = (0, 0, 0, 0, 0, 0, 1, 0)$ and $h_2 = (0, 0, 0, 0, 0, 0, 0, 1)$. The matrix of the linking form in this basis is given by the 2×2 square submatrix of $C^T E^{-1} D^{-1}$ in the bottom right corner:

$$\begin{pmatrix} -46 & -20 & -60 & -52 & -62 & -51 & -4 & -5 \\ -20 & -12 & -35 & -34 & -39 & -31 & -3 & -2 \\ -60 & -35 & -98 & -93 & -107 & -85 & -8 & -6 \\ -52 & -34 & -93 & -90 & -103 & -81 & -8 & -5 \\ -62 & -39 & -107 & -103 & -118 & -93 & -9 & -6 \\ -51 & -31 & -85 & -81 & -93 & -74 & -7 & -5 \\ -4 & -3 & -8 & -8 & -9 & -7 & -8/11 & -4/11 \\ -5 & -2 & -6 & -5 & -6 & -5 & -4/11 & -6/11 \end{pmatrix}.$$

The associated bilinear form (B, β) on $\mathbb{Z}_{11}^2$ has the matrix

$$\begin{pmatrix} -8 & -4 \\ -4 & -6 \end{pmatrix}.$$

The determinant of this matrix is equal to $32 \equiv 10 \bmod 11$, so it is not a square modulo 11. Thus, the form has $\epsilon_1 = -1$. However, the rank $n = 2$ is congruent to $2 \bmod 4$ and $11 \equiv 3 \bmod 4$, so according to the definition of the index (Definition 3.5) we have $\epsilon_2 = -1$. Hence, $\epsilon(B) = 1$. Then $\epsilon(B) = \eta(q)$. The knot passes our criterion.

11. Theorem 1.4 for low crossing knots.

We applied the following criteria for knots up to 15 crossings and for periods $q = 3, 5, 7, 11, 13$. The Sage script that we used is available in [**10**].

- Przytycki's criterion for HOMFLYPT polynomials [**15**];
- Murasugi's criterion as stated in Theorem 2.5;
- Naik's homological criterion as stated in Propositions 2.7 and 2.8;
- Theorem 1.4.

We applied these criteria as follows. For a given knot K and period q we checked Przytycki's criterion. Independently, we considered all candidates Δ' for the Alexander polynomial of the quotient, that is, those that satisfy the statement of Theorem 2.5. For each such Δ' we checked if the criterion of Proposition 2.7 was satisfied for all odd prime numbers $p \neq q$ that divided $\Delta(-1)$. If Δ' passed the criterion of Proposition 2.7 we checked whether K passed Proposition 2.8 with this Δ' for all prime numbers $p \neq 2, q$ that divided $\Delta(-1)$ but did not divide $\Delta'(-1)$.

If Δ' passed the criterion of Proposition 2.8 we looked at Theorem 1.4 for all prime numbers $p \neq 2, q$ such that $p|\Delta(-1)$ but p does not divide $\Delta'(-1)$.

If at some point in this algorithm Δ' did not pass the criterion, it was discarded from a list of potential Alexander polynomials of the quotient. If the list was empty, we concluded that the knot K was not q-periodic. We recorded whether this conclusion is achieved using Proposition 2.7 only, or one needs Proposition 2.8 or even Theorem 1.4 to obstruct q-periodicity.

It turns out that for periods greater than 5, Theorem 1.4 did not obstruct any case that was not obstructed by the combination of Murasugi's and Naik's criteria.

For period 5, our criterion obstructed the knots $14n26993$, $15a80526$, $15n83514$ and $15n95792$, but all of these knots are not 5-periodic by Przytycki's criterion.

The most interesting situation was for period 3. Here we were able to obstruct the knots $12a100$ and $12a348$, which can also be obstructed using Jabuka and Naik's d-invariants criterion. See [**8**, Section 2.4] for a detailed discussion of the $12a100$ knot.

There are also 19 alternating knots with crossing numbers from 13 to 15 whose 3-periodicity is obstructed by Theorem 1.4, but not by Naik's criterion nor by Przytycki's criterion. These are:

$13a4648$	$14a7583$	$14a7948$	$14a8670$	$14a9356$
$14a14971$	$14a16311$	$14a17173$	$14a17260$	$14a18647$
$15a6030$	$15a6066$	$15a10622$	$15a15077$	$15a33910$
$15a36983$	$15a46768$	$15a72333$	$15a82771$.	

Among non-alternating knots with 12–15 crossings, there are 57 knots whose 3-periodicity can be obstructed by Theorem 1.4, but for which Naik's criterion and Przytycki's criterion do not obstruct 3-periodicity. These are $13n3659$, $14n908$, $14n913$, $14n2451$, $14n2458$, $14n6565$, $14n9035$, $14n11989$, $14n14577$, $14n23051$ and $14n24618$, as well as further 46 knots with 15 crossings.

We note that for these non-alternating examples the Jabuka–Naik criterion [**8**] cannot be easily applied, because it requires calculating d-invariants of double branched covers of non-alternating knots, for which no algorithm presently exists. Another criterion involving knot homology, namely Khovanov homology, see [**1**], does not work for period 3, so it cannot obstruct periodicity of these 57 knots.

We have also applied Theorem 1.4 to obstruct 3^2-periodicity of knots, but we could not find any example where Theorem 1.4 could obstruct periodicity of any knot that passed Murasugi's criterion for period 9. To find interesting examples, one probably needs to consider knots with many more crossings.

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Maciej Borodzik
Institute of Mathematics
University of Warsaw
ul. Banacha 2
02-097 Warsaw, Poland
E-mail: mcboro@mimuw.edu.pl

Przemysław Grabowski
Institute of Mathematics
University of Warsaw
ul. Banacha 2
02-097 Warsaw, Poland
E-mail: pg370922@students.mimuw.edu.pl

Adam Król
Institute of Mathematics
University of Warsaw
ul. Banacha 2
02-097 Warsaw, Poland
E-mail: adam.krol@student.uw.edu.pl

Maria Marchwicka
Faculty of Mathematics and Computer Science
Adam Mickiewicz University in Poznań
ul. Uniwersytetu Poznańskiego 4
61-614 Poznań, Poland
E-mail: maria.marchwicka@amu.edu.pl