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An extension of the characterization of CMO and its application to compact commutators on Morrey spaces

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Abstract. In 1978 Uchiyama gave a proof of the characterization of $\mathrm{CMO}(\mathbb{R}^n)$ which is the closure of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n)$. We extend the characterization to the closure of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in the Campanato space with variable growth condition. As an application we characterize compact commutators $[b,T]$ and $[b,I_\alpha]$ on Morrey spaces with variable growth condition, where T is the Calderón–Zygmund singular integral operator, I_α is the fractional integral operator and b is a function in the Campanato space with variable growth condition.

1. Introduction.

Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and T be a Calderón–Zygmund singular integral operator. In 1976 Coifman, Rochberg and Weiss [**12**] proved that the commutator $[b,T] = bT - Tb$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, that is,

$$\|[b,T]f\|_{L^p} = \|bTf - T(bf)\|_{L^p} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

where C is a positive constant independent of b and f. For the fractional integral operator I_α, Chanillo [**5**] proved the boundedness of $[b,I_\alpha]$ in 1982. Coifman, Rochberg and Weiss [**12**] and Chanillo [**5**] also gave the necessary conditions for the boundedness, that is, if the commutator $[b,T]$ or $[b,I_\alpha]$ is bounded, then b is in $\mathrm{BMO}(\mathbb{R}^n)$. These results were extended to Morrey and generalized Morrey spaces by Di Fazio and Ragusa [**13**] in 1991 and Mizuhara [**23**] in 1999, respectively. In 1978 Janson [**19**] investigated the commutator $[b,T]$ with a function b in BMO_ϕ which is a kind of generalized Campanato spaces. For other extensions and generalizations of [**5**], [**12**], see [**15**], [**17**], [**21**], [**36**], [**43**], [**44**], etc.

On the other hand, Uchiyama [**45**] considered the compactness of the commutator $[b,T]$ on $L^p(\mathbb{R}^n)$ in 1978, where T is a Calderón–Zygmund singular integral operator with convolution type of smooth kernel $K \not\equiv 0$. He proved that $[b,T]$ is compact on $L^p(\mathbb{R}^n)$ if and only if $b \in \mathrm{CMO}(\mathbb{R}^n)$, where $\mathrm{CMO}(\mathbb{R}^n)$ is the closure of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n)$. In its proof he used the following characterization of $\mathrm{CMO}(\mathbb{R}^n)$, which was mentioned by Neri [**37**, Remark 2.6] without proof.

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Theorem 1.1 ([**45**]). *Let $f \in \mathrm{BMO}(\mathbb{R}^n)$, and let $\mathrm{MO}(f, B(x,r))$ be the mean oscillation of f on the ball $B(x,r)$ centered at $x \in \mathbb{R}^n$ and of radius $r > 0$. Then $f \in \mathrm{CMO}(\mathbb{R}^n)$ if and only if f satisfies the following three conditions*:

(i) $\displaystyle\lim_{r\to+0} \sup_{x\in\mathbb{R}^n} \mathrm{MO}(f, B(x,r)) = 0.$

(ii) $\displaystyle\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \mathrm{MO}(f, B(x,r)) = 0.$

(iii) $\displaystyle\lim_{|y|\to\infty} \mathrm{MO}(f, B(x+y,r)) = 0$ *for each ball $B(x,r)$.*

After that, using this characterization, many authors gave the characterization of various compact commutators on several function spaces. For example, Chen, Ding and Wang [**7**], [**9**] gave the characterization of the compact commutators $[b,T]$ and $[b,I_\alpha]$ on Morrey spaces. For the others, see [**4**], [**6**], [**8**], [**10**], [**11**], [**22**], etc.

In this paper we extend Theorem 1.1 to $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$ which is the closure of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in the generalized Campanato space $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$ with variable growth condition. To prove the extension of Theorem 1.1 we improve the proof of Uchiyama [**45**] by using the mollifier and a smooth cut-off method. As a corollary we give a characterization of the space $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathrm{Lip}_\alpha(\mathbb{R}^n)}$ which is the closure of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in the Lipschitz space $\mathrm{Lip}_\alpha(\mathbb{R}^n)$, $0 < \alpha < 1$. Moreover, as an application of the extension of Theorem 1.1 we give a characterization of compact commutators $[b,T]$ and $[b,I_\alpha]$ on generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth condition. We shall give the definitions of the function spaces $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $L^{(p,\varphi)}(\mathbb{R}^n)$ in Sections 2 and 4, respectively.

Recently, the authors [**2**] proved that, under suitable conditions, the commutator $[b,T]$ or $[b,I_\rho]$ is bounded on $L^{(p,\varphi)}(\mathbb{R}^n)$ if and only if b is in $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$, where T is the Calderón–Zygmund operator and I_ρ is the generalized fractional integral operator, see Section 4 for their definitions. Moreover, using Sawano and Shirai's method in [**41**], the authors [**3**] proved that, if b is in $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$, then $[b,T]$ and $[b,I_\rho]$ are compact on $L^{(p,\varphi)}(\mathbb{R}^n)$. In this paper, as an application of the extension of Theorem 1.1, we prove that, if the commutator $[b,T]$ or $[b,I_\alpha]$ is compact, then b is in $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$.

The organization of this paper is as follows. We state the definition of $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and the main result (Theorem 2.1) in Section 2 and prove it in Section 3. Next, in Section 4, we state the results (Theorems 4.5 and 4.6) on the commutators $[b,T]$ and $[b,I_\rho]$ on $L^{(p,\varphi)}(\mathbb{R}^n)$ together with known results. Then we give proofs of the results on commutators in Sections 5 and 6.

At the end of this section, we make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p, is dependent on the subscripts. If $f \le Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

2. Generalized Campanato spaces with variable growth condition and main results.

In this paper we denote by $B(x,r)$ the open ball centered at $x \in \mathbb{R}^n$ and of radius r, that is,

$$B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}.$$

For a ball $B = B(x,r)$ and a positive constant k we denote $B(x,kr)$ by kB. For a measurable set $G \subset \mathbb{R}^n$, we denote by $|G|$ and χ_G the Lebesgue measure of G and the characteristic function of G, respectively. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a ball B, let

$$f_B = \fint_B f = \fint_B f(y)\,dy = \frac{1}{|B|}\int_B f(y)\,dy.$$

First we recall the definition of generalized Campanato spaces $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ for $p \in [1,\infty)$ and variable growth function $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. For a ball $B = B(x,r)$ we write $\phi(B) = \phi(x,r)$.

Definition 2.1. For $p \in [1,\infty)$ and $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ be the set of all functions f such that the following functional is finite:

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_B \frac{1}{\phi(B)}\left(\fint_B |f(y) - f_B|^p\,dy\right)^{1/p},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Then $\|f\|_{\mathcal{L}_{p,\phi}}$ is a norm modulo constant functions and thereby $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ is a Banach space. If $p = 1$ and $\phi \equiv 1$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. If $p = 1$ and $\phi(x,r) \equiv r^\alpha$ $(0 < \alpha \le 1)$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ coincides with $\mathrm{Lip}_\alpha(\mathbb{R}^n)$.

Generalized Campanato spaces $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ with variable growth condition were introduced in [**35**] to characterize pointwise multipliers on $\mathrm{BMO}(\mathbb{R}^n)$ and studied in [**24**], [**30**], [**32**], etc. Moreover, it has been proved that $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ is the dual space of the Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent in [**34**].

We say that a function $\theta : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ satisfies the doubling condition if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,

$$\frac{1}{C} \le \frac{\theta(x,r)}{\theta(x,s)} \le C, \quad \text{if } \frac{1}{2} \le \frac{r}{s} \le 2. \tag{2.1}$$

We say that θ is almost increasing (resp. almost decreasing) if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,

$$\theta(x,r) \le C\theta(x,s) \quad (\text{resp. } \theta(x,s) \le C\theta(x,r)), \quad \text{if } r < s. \tag{2.2}$$

We also consider the following nearness condition; there exists a positive constant C such that, for all $x, y \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$\frac{1}{C} \le \frac{\theta(x,r)}{\theta(y,r)} \le C, \quad \text{if } \ |x-y| \le r. \tag{2.3}$$

For two functions $\theta, \kappa : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, we write $\theta \sim \kappa$ if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$\frac{1}{C} \le \frac{\theta(x,r)}{\kappa(x,r)} \le C. \tag{2.4}$$

Let $1 \le p < \infty$ and $\phi, \tilde{\phi} : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. If $\phi \sim \tilde{\phi}$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}_{p,\tilde{\phi}}(\mathbb{R}^n)$ with equivalent norms.

In this paper we consider the following class of ϕ:

Definition 2.2. Let $\mathcal{G}^{\mathrm{inc}}$ be the set of all functions $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that ϕ is almost increasing and that $r \mapsto \phi(x,r)/r$ is almost decreasing. That is, there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,

$$\phi(x,r) \le C\phi(x,s), \quad C\phi(x,r)/r \ge \phi(x,s)/s, \quad \text{if } r < s.$$

If $\phi \in \mathcal{G}^{\mathrm{inc}}$, then ϕ satisfies the doubling condition (2.1).

Remark 2.1. It is known that, if $\phi \in \mathcal{G}^{\mathrm{inc}}$ and ϕ satisfies (2.3), then $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ with equivalent norms for each $p \in [1,\infty)$, see [**31**, Theorem 3.1]. In particular, for each $p \in [1,\infty)$, $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$ if $\phi \equiv 1$ and $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$ if $\phi(x,r) \equiv r^\alpha$, $0 < \alpha \le 1$. For the relation between $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and Hölder (Lipschitz) spaces $\Lambda_\phi(\mathbb{R}^n)$ with variable growth condition, see [**30**, Theorem 2.4].

For a measurable function f and a ball B, we denote by $\mathrm{MO}(f,B)$ the mean oscillation of f on B, that is,

$$\mathrm{MO}(f,B) = \fint_B |f(y) - f_B| \, dy. \tag{2.5}$$

Then our main results are the following:

Theorem 2.1. *Let ϕ be in $\mathcal{G}^{\mathrm{inc}}$ and satisfy (2.3). Assume that*

$$\lim_{r \to +0} \inf_{x \in \mathbb{R}^n} \frac{\phi(x,r)}{r} = \infty, \quad \lim_{r \to \infty} \inf_{x \in \mathbb{R}^n} r^n \phi(x,r) = \infty. \tag{2.6}$$

Let $f \in \mathcal{L}_{1,\phi}(\mathbb{R}^n)$. Then $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$ if and only if f satisfies the following three conditions:

(i) $\displaystyle\lim_{r \to +0} \sup_{x \in \mathbb{R}^n} \frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)} = 0.$

(ii) $\displaystyle\lim_{r \to \infty} \sup_{x \in \mathbb{R}^n} \frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)} = 0.$

(iii) $\displaystyle\lim_{|x| \to \infty} \frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)} = 0$ *for each $r > 0$.*

Remark 2.2. We do not need (2.6) to prove that, if f satisfies (i)–(iii), then $f \in \overline{C^\infty_{\rm comp}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$. We do not need (2.3) to prove that, if $f \in \overline{C^\infty_{\rm comp}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$, then f satisfies (i)–(iii).

If $\phi \equiv 1$, then the theorem above is the same as Theorem 1.1. If $\phi(x,r) \equiv r^\alpha$, then we have the following corollary.

Corollary 2.2 ([**38**]). *Let $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, $0<\alpha<1$. Then $f \in \overline{C^\infty_{\rm comp}(\mathbb{R}^n)}^{\mathrm{Lip}_\alpha(\mathbb{R}^n)}$ if and only if f satisfies the following three conditions*:

(i) $\displaystyle \lim_{r\to+0} \sup_{x\in\mathbb{R}^n} \frac{\mathrm{MO}(f,B(x,r))}{r^\alpha} = 0.$

(ii) $\displaystyle \lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \frac{\mathrm{MO}(f,B(x,r))}{r^\alpha} = 0.$

(iii) $\displaystyle \lim_{|x|\to\infty} \mathrm{MO}(f,B(x,r)) = 0$ *for each $r>0$.*

As another corollary, we consider the Lipschitz (Hölder) space with variable exponent. For $\alpha(\cdot) : \mathbb{R}^n \to [0,\infty)$ and $\alpha_* \in [0,\infty)$, let $\mathrm{Lip}^{\alpha_*}_{\alpha(\cdot)}(\mathbb{R}^n)$ be the set of all functions f such that the following functional is finite:

$$\|f\|_{\mathrm{Lip}^{\alpha_*}_{\alpha(\cdot)}} = \max\left\{ \sup_{0<|x-y|<1} \frac{2|f(x)-f(y)|}{|x-y|^{\alpha(x)}+|x-y|^{\alpha(y)}}, \sup_{|x-y|\ge 1} \frac{|f(x)-f(y)|}{|x-y|^{\alpha_*}} \right\},$$

see [**32**, Definition 2.1 and Remark 2.2]. For these $\alpha(\cdot)$ and α_*, let

$$\phi(x,r) = \begin{cases} r^{\alpha(x)}, & 0<r<1, \\ r^{\alpha_*}, & 1 \le r < \infty. \end{cases} \tag{2.7}$$

If

$$0 \le \inf_{x\in\mathbb{R}^n} \alpha(x) \le \sup_{x\in\mathbb{R}^n} \alpha(x) < 1, \quad 0 \le \alpha_* < 1, \tag{2.8}$$

then ϕ is in $\mathcal{G}^{\rm inc}$ and satisfies (2.6). If $\alpha(\cdot)$ is log-Hölder continuous also, that is, there exists a positive constant C such that, for all $x,y \in \mathbb{R}^n$,

$$|\alpha(x)-\alpha(y)| \le \frac{C}{\log(e/|x-y|)} \quad \text{if} \quad 0<|x-y|<1,$$

then ϕ satisfies (2.3), see [**32**, Proposition 3.3]. Moreover, if $\inf_{x\in\mathbb{R}^n} \alpha(x) > 0$ and $\alpha_* > 0$, then $\mathcal{L}_{1,\phi}(\mathbb{R}^n) = \mathrm{Lip}^{\alpha_*}_{\alpha(\cdot)}(\mathbb{R}^n)$ with equivalent norms, see [**32**, Corollary 3.5]. Hence we have the following corollary.

Corollary 2.3. *Let $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be defined by* (2.7). *Assume that $\alpha(\cdot)$ and α_* satisfy* (2.8) *and that $\alpha(\cdot)$ is log-Hölder continuous. Let $f \in \mathcal{L}_{1,\phi}(\mathbb{R}^n)$. Then $f \in \overline{C^\infty_{\rm comp}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$ if and only if f satisfies the following three conditions*:

(i) $\displaystyle\lim_{r\to+0}\sup_{x\in\mathbb{R}^n}\frac{\mathrm{MO}(f,B(x,r))}{r^{\alpha(x)}}=0.$

(ii) $\displaystyle\lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\frac{\mathrm{MO}(f,B(x,r))}{r^{\alpha_*}}=0.$

(iii) $\displaystyle\lim_{|x|\to\infty}\mathrm{MO}(f,B(x,r))=0$ *for each* $r>0$.

Moreover, if $\inf_{x\in\mathbb{R}^n}\alpha(x)>0$ *and* $\alpha_*>0$, *then* $f\in\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathrm{Lip}^{\alpha_*}_{\alpha(\cdot)}(\mathbb{R}^n)}$ *if and only if* f *satisfies the above three conditions.*

3. Proof of Theorem 2.1.

In this section we first show three lemmas and one proposition to prove Theorem 2.1. Let η be a function on $\mathbb{R}^n$ such that

$$\operatorname{supp}\eta\subset\overline{B(0,1)},\quad 0\le\eta\le 2\quad\text{and}\quad\int_{B(0,1)}\eta(y)\,dy=|B(0,1)|,\tag{3.1}$$

and let $\bar\eta_r(x)=|B(0,r)|^{-1}\eta(x/r)$. Then, for $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\bar\eta_r*f(x)=⨍_{B(x,r)}\eta((x-y)/r)f(y)\,dy.\tag{3.2}$$

If $\eta=\chi_{B(0,1)}$, then $\bar\eta_r*f(x)=f_{B(x,r)}$. If $\eta\in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, then (3.2) is a mollifier. We can choose $\eta\in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ which satisfies (3.1) and

$$\|\nabla\eta\|_{L^\infty}\le c_n\tag{3.3}$$

for some positive constant c_n dependent only on n.

For two balls B_1 and B_2, if $B_1\subset B_2$, then

$$|f_{B_1}-f_{B_2}|\le\frac{|B_2|}{|B_1|}\mathrm{MO}(f,B_2),\tag{3.4}$$

and

$$\mathrm{MO}(f,B_1)\le 2\frac{|B_2|}{|B_1|}\mathrm{MO}(f,B_2).\tag{3.5}$$

The first lemma is an extension of (3.4).

Lemma 3.1. *If* $B_1=B(x,r)\subset B_2$, *then*

$$|\bar\eta_r*f(x)-f_{B_2}|\le 2\frac{|B_2|}{|B_1|}\mathrm{MO}(f,B_2).\tag{3.6}$$

Proof. From (3.1) and (3.2) it follows that

$$|\bar\eta_r*f(x)-f_{B_2}|=\left|⨍_{B_1}\eta((x-y)/r)f(y)\,dy-f_{B_2}\right|$$

$$= \left| \fint_{B_1} \eta((x-y)/r)\big(f(y) - f_{B_2}\big)\,dy \right|$$
$$\le 2 \fint_{B_1} \big|f(y) - f_{B_2}\big|\,dy \le 2\frac{|B_2|}{|B_1|} \fint_{B_2} |f(y) - f_{B_2}|\,dy,$$

which shows the conclusion. □

LEMMA 3.2. *For any ball $B(x,r)$,*

$$\fint_{B(x,r)} |f(y) - \bar{\eta}_r * f(y)|\,dy \le 2^{n+2}\mathrm{MO}(f, B(x,2r)). \tag{3.7}$$

PROOF. Let $B = B(x,r)$. From Lemma 3.1 it follows that

$$\begin{aligned} \fint_B |f(y) - \bar{\eta}_r * f(y)|\,dy &\le \fint_B \big(|f(y) - f_{2B}| + |f_{2B} - \bar{\eta}_r * f(y)|\big)\,dy \\ &\le \fint_B |f(y) - f_{2B}|\,dy + 2^{n+1}\mathrm{MO}(f, 2B) \\ &\le 2^{n+2}\mathrm{MO}(f, 2B), \end{aligned}$$

which shows the conclusion. □

LEMMA 3.3. *Let η be in $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ and satisfy* (3.1)*. If $y, z \in B(x,r)$, then*

$$|\bar{\eta}_r * f(y) - \bar{\eta}_r * f(z)| \le 2^n \|\nabla\eta\|_{L^\infty} \frac{|y-z|}{r}\mathrm{MO}(f, B(x,2r)). \tag{3.8}$$

PROOF. Letting $\tilde{f}(x) = f(x) - f_{B(x,2r)}$, we have

$$\begin{aligned} &|\bar{\eta}_r * f(y) - \bar{\eta}_r * f(z)| = |\bar{\eta}_r * \tilde{f}(y) - \bar{\eta}_r * \tilde{f}(z)| \\ &\quad = \left| \frac{1}{|B(x,r)|} \int_{B(x,2r)} \big(\eta((y-w)/r) - \eta((z-w)/r)\big)\tilde{f}(w)\,dw \right| \\ &\quad \le 2^n \fint_{B(x,2r)} \big|\big(\eta((y-w)/r) - \eta((z-w)/r)\big)\tilde{f}(w)\big|\,dw \\ &\quad \le 2^n \frac{|y-z|}{r} \|\nabla\eta\|_{L^\infty} \fint_{B(x,2r)} |\tilde{f}(w)|\,dw, \end{aligned}$$

which shows the conclusion. □

PROPOSITION 3.4. *Let η be in $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ and satisfy* (3.1) *and* (3.3)*. Let ϕ be in $\mathcal{G}^{\mathrm{inc}}$ and satisfy* (2.3)*. Then there exists a positive constant C, dependent only on n and ϕ, such that, for all $r > 0$,*

$$\|f - \bar{\eta}_r * f\|_{\mathcal{L}_{1,\phi}} \le C \sup_{x\in\mathbb{R}^n,\ 0<t\le 2r} \frac{\mathrm{MO}(f, B(x,t))}{\phi(x,t)}. \tag{3.9}$$

Before we prove Proposition 3.4 we state its corollary, which is a variant of Theorem 2.1.

Corollary 3.5. *Let η be in $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ and satisfy* (3.1) *and* (3.3)*. Let ϕ be in $\mathcal{G}^{\mathrm{inc}}$ and satisfy* (2.3)*. Then there exists a positive constant C, dependent only on n and ϕ, such that, for all $f \in \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ and $r > 0$,*

$$\|\bar{\eta}_r * f\|_{\mathcal{L}_{1,\phi}} \le C\|f\|_{\mathcal{L}_{1,\phi}}. \tag{3.10}$$

Moreover, if f satisfies (i) *in Theorem* 2.1*, then $\bar{\eta}_r * f \to f$ in $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$ as $r \to +0$.*

Proof of Proposition 3.4. We show that

$$\frac{\mathrm{MO}(f - \bar{\eta}_r * f, B(x,t))}{\phi(x,t)}$$

is dominated by the right hand side of (3.9) for each ball $B(x,t)$.

Case 1: $0 < t \le r$: From Lemma 3.3 it follows that

$$\begin{aligned}
&\frac{1}{\phi(x,t)} ⨍_{B(x,t)} |\bar{\eta}_r * f(y) - (\bar{\eta}_r * f)_{B(x,t)}|\,dy \\
&\quad\le \frac{1}{\phi(x,t)} ⨍_{B(x,t)} ⨍_{B(x,t)} |\bar{\eta}_r * f(y) - \bar{\eta}_r * f(z)|\,dz\,dy \\
&\quad\le \frac{2^n\|\nabla\eta\|_{L^\infty}}{\phi(x,t)} \left(⨍_{B(x,t)} ⨍_{B(x,t)} \frac{|y-z|}{r}\,dz\,dy \right) \mathrm{MO}(f, B(x,2r)) \\
&\quad\le 2^n c_n \frac{2t}{r\phi(x,t)} \mathrm{MO}(f, B(x,2r)) \le C_{n,\phi} \frac{\mathrm{MO}(f, B(x,2r))}{\phi(x,2r)}.
\end{aligned}$$

In the above we used the almost decreasingness of $r \mapsto \phi(x,r)/r$ for the last inequality. Hence

$$\begin{aligned}
&\frac{\mathrm{MO}(f - \bar{\eta}_r * f, B(x,t))}{\phi(x,t)} \\
&\quad= \frac{1}{\phi(x,t)} ⨍_{B(x,t)} |f(y) - \bar{\eta}_r * f(y) - (f - \bar{\eta}_r * f)_{B(x,t)}| \\
&\quad\le \frac{1}{\phi(x,t)} ⨍_{B(x,t)} |f(y) - f_{B(x,t)}|\,dy + \frac{1}{\phi(x,t)} ⨍_{B(x,t)} |\bar{\eta}_r * f(y) - (\bar{\eta}_r * f)_{B(x,t)}|\,dy \\
&\quad\le \frac{\mathrm{MO}(f, B(x,t))}{\phi(x,t)} + C_{n,\phi} \frac{\mathrm{MO}(f, B(x,2r))}{\phi(x,2r)}.
\end{aligned}$$

Case 2: $t > r$: Take balls $\{B(x_j, r)\}_j$ such that

$$B(x,t) \subset \bigcup_j B(x_j, r) \subset B(x,2t), \qquad \sum_j |B(x_j,r)| \le C_n |B(x,t)|,$$

where C_n is a positive constant depending only on n. Then, using Lemma 3.2, we have

$$\begin{aligned}
\mathrm{MO}(f-\bar{\eta}_r * f, B(x,t)) &\leq \frac{2}{|B(x,t)|}\int_{B(x,t)} |f(y)-\bar{\eta}_r * f(y)|\,dy \\
&\leq \frac{2}{|B(x,t)|}\sum_j \int_{B(x_j,r)} |f(y)-\bar{\eta}_r * f(y)|\,dy \\
&\leq \frac{2}{|B(x,t)|}\sum_j |B(x_j,r)|2^{n+2}\mathrm{MO}(f,B(x_j,2r)) \\
&\leq 2^{n+3}C_n \sup_j \mathrm{MO}(f,B(x_j,2r)).
\end{aligned}$$

By the almost increasingness of ϕ, (2.3) and the doubling condition of ϕ we have

$$\phi(x_j,2r) \lesssim \phi(x_j,2t) \sim \phi(x,2t) \lesssim \phi(x,t).$$

Therefore,

$$\frac{\mathrm{MO}(f-\bar{\eta}_r * f, B(x,t))}{\phi(x,t)} \leq C'_{n,\phi}\sup_j \frac{\mathrm{MO}(f,B(x_j,2r))}{\phi(x_j,2r)}.$$

The proof is complete. □

Proof of Theorem 2.1. Part 1: Let $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$. Then, from the inequality

$$⨍_{B(x,r)} |f(y)-f_B|\,dy \leq 2r\|\nabla f\|_{L^\infty}$$

and (2.6) it follows that

$$\lim_{r\to+0}\sup_{x\in\mathbb{R}^n}\frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)} \leq \lim_{r\to+0}\sup_{x\in\mathbb{R}^n}\frac{2r}{\phi(x,r)}\|\nabla f\|_{L^\infty} = 0.$$

On the other hand, from the inequality

$$⨍_{B(x,r)} |f(y)-f_B|\,dy \leq \frac{2|\operatorname{supp} f|\|f\|_{L^\infty}}{|B(x,r)|}$$

and (2.6) it follows that

$$\lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)} \leq \lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\frac{2|\operatorname{supp} f|\|f\|_{L^\infty}}{\phi(x,r)|B(x,r)|} = 0.$$

For each $r>0$, take $x\in\mathbb{R}^n$ such that $\operatorname{supp} f \cap B(x,r)=\emptyset$. Then

$$\frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)} = 0.$$

That is, f satisfies (i), (ii) and (iii).

Let $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$. Then, for any $\epsilon>0$, there exists $g\in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ such that, $\sup_{x\in\mathbb{R}^n,\ r>0}\mathrm{MO}(f-g,B(x,r))/\phi(x,r)<\epsilon$. Therefore, f satisfies (i), (ii) and (iii).

Part 2: Let f satisfy (i), (ii) and (iii). For any $\epsilon > 0$, from (i) and (ii) there exist integers i_ϵ and k_ϵ ($i_\epsilon < k_\epsilon$) such that

$$\sup\left\{\frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)} : x \in \mathbb{R}^n,\ 0 < r \le 2^{i_\epsilon}\right\} < \epsilon$$

and

$$\sup\left\{\frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)} : x \in \mathbb{R}^n,\ r \ge 2^{k_\epsilon}\right\} < \epsilon.$$

From (iii) it follows that

$$\lim_{|x|\to\infty} \max\left\{\frac{\mathrm{MO}(f,B(x,2^\ell))}{\phi(x,2^\ell)} : \ell = i_\epsilon, i_\epsilon+1, \dots, k_\epsilon\right\} = 0.$$

By (3.5) and the doubling condition of ϕ we have

$$\sup_{2^{\ell-1}\le r\le 2^\ell} \frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)} \le C\frac{\mathrm{MO}(f,B(x,2^\ell))}{\phi(x,2^\ell)}, \quad \ell = i_\epsilon, i_\epsilon+1,\dots,k_\epsilon,$$

where the positive constant C is dependent only on n and ϕ. Consequently,

$$\lim_{|x|\to\infty} \sup_{2^{i_\epsilon}\le r\le 2^{k_\epsilon}} \frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)} = 0.$$

Then there exists an integer j_ϵ such that $j_\epsilon > k_\epsilon (> i_\epsilon)$ and

$$\sup\left\{\frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)} : B(x,r)\cap B(0,2^{j_\epsilon}) = \emptyset\right\} < \epsilon.$$

Using i_ϵ, k_ϵ and j_ϵ, we set

$$\begin{aligned}
\mathcal{B}_1 &= \left\{B(x,r) : x \in \mathbb{R}^n,\ 0 < r \le 2^{i_\epsilon}\right\},\\
\mathcal{B}_2 &= \left\{B(x,r) : x \in \mathbb{R}^n,\ r \ge 2^{k_\epsilon}\right\},\\
\mathcal{B}_3 &= \left\{B(x,r) : B(x,r)\cap B(0,2^{j_\epsilon}) = \emptyset\right\}.
\end{aligned}$$

Then $\mathrm{MO}(f,B)/\phi(B) < \epsilon$ if $B \in \mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3$.

We define a C^∞-function f_1 as follows: Let η be in $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ and satisfy (3.1) and (3.3), and let

$$f_1 = \bar{\eta}_{r_1} * f, \quad r_1 = 2^{i_\epsilon - 1}.$$

Then, from Proposition 3.4 it follows that

$$\|f - f_1\|_{\mathcal{L}_{1,\phi}} \le C_{n,\phi} \sup_{B\in\mathcal{B}_1} \frac{\mathrm{MO}(f,B)}{\phi(B)} \le C_{n,\phi}\,\epsilon, \tag{3.11}$$

where the positive constant $C_{n,\phi}$ is dependent only on n and ϕ, and independent of r_1.

This also shows that

$$\frac{\mathrm{MO}(f_1,B)}{\phi(B)} \le \|f-f_1\|_{\mathcal{L}_{1,\phi}} + \frac{\mathrm{MO}(f,B)}{\phi(B)} \le (C_{n,\phi}+1)\epsilon \quad \text{for} \quad B \in \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3. \tag{3.12}$$

Next we define a C^∞-function f_2 as follows: Let $h \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ satisfy

$$\chi_{B(0,1)} \le h \le \chi_{B(0,2)}, \quad \|\nabla h\|_{L^\infty} \le 2,$$

and let

$$f_2 = (f_1 - (f_1)_{B(0,4r_2)})h_{r_2} + (f_1)_{B(0,4r_2)}, \quad h_{r_2}(x) = h(x/r_2), \quad r_2 = 2^{j_\epsilon+1}.$$

Then $f_2 - (f_1)_{B(0,4r_2)} \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, that is,

$$\min_{g\in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)} \|f_2 - g\|_{\mathcal{L}_{1,\phi}} = 0. \tag{3.13}$$

In the following, using (3.12), we will show that there exists a positive constant $\widetilde{C}_{n,\phi}$, dependent only on n and ϕ, such that

$$\|f_1 - f_2\|_{\mathcal{L}_{1,\phi}} \le \widetilde{C}_{n,\phi}\,\epsilon. \tag{3.14}$$

Once we show (3.14), combining this with (3.11) and (3.13), we obtain that $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$.

Now, take a ball $B = B(z,r)$ arbitrarily.

Case 1: $r \ge r_2/2$: In this case $B \in \mathcal{B}_2$.

Case 1-1: If $B \cap B(0,2r_2) = \emptyset$, then $f_2 = (f_1)_{B(0,4r_2)}$ on B, that is, $\mathrm{MO}(f_2,B) = 0$. Hence, by (3.12) we have

$$\frac{\mathrm{MO}(f_1-f_2,B)}{\phi(B)} = \frac{\mathrm{MO}(f_1,B)}{\phi(B)} \le (C_{n,\phi}+1)\epsilon.$$

Case 1-2: If $B \cap B(0,2r_2) \neq \emptyset$, then, using the almost increasingness, the nearness condition (2.3) and the doubling condition (2.1) of ϕ, we have

$$\phi(0,4r_2) \lesssim \phi(0,8r) \sim \phi(z,8r) \sim \phi(B), \quad |B(0,4r_2)| \le 8^n|B|,$$

and then

$$\begin{aligned}\frac{\mathrm{MO}(f_2,B)}{\phi(B)} &= \frac{\mathrm{MO}((f_1-(f_1)_{B(0,4r_2)})h_{r_2},B)}{\phi(B)}\\ &\le \frac{2}{\phi(B)} \fint_B |(f_1(y)-(f_1)_{B(0,4r_2)})h_{r_2}|\,dy\\ &\le \frac{2}{\phi(B)|B|}\int_{B(0,4r_2)} |f_1(y)-(f_1)_{B(0,4r_2)}|\,dy\end{aligned}$$

$$\lesssim \frac{\mathrm{MO}(f_1, B(0,4r_2))}{\phi(B(0,4r_2))}.$$

Since both B and $B(0,4r_2)$ are in $\mathcal{B}_2$, from (3.12) it follows that

$$\frac{\mathrm{MO}(f_1 - f_2, B)}{\phi(B)} \le \frac{\mathrm{MO}(f_1, B)}{\phi(B)} + \frac{\mathrm{MO}(f_2, B)}{\phi(B)} \le C'_{n,\phi}\,\epsilon,$$

where $C'_{n,\phi}$ is dependent only on n and ϕ.

Case 2: $r < r_2/2$:

Case 2-1: If $B \subset B(0,r_2)$, then $\mathrm{MO}(f_1 - f_2, B) = 0$, since

$$f_1 - f_2 = \big(f_1 - (f_1)_{B(0,4r_2)}\big)\big(1 - h_{r_2}\big) = 0 \quad \text{on} \quad B(0,r_2).$$

Case 2-2: If $B \cap B(0,2r_2) = \emptyset$, then $B \in \mathcal{B}_3$ and $f_2 = (f_1)_{B(0,4r_2)}$ on B. Hence

$$\frac{\mathrm{MO}(f_1 - f_2, B)}{\phi(B)} = \frac{\mathrm{MO}(f_1, B)}{\phi(B)} \le (C_{n,\phi} + 1)\epsilon.$$

Case 2-3: If $B \cap (B(0,2r_2) \setminus B(0,r_2)) \ne \emptyset$, then $B \subset B(0,4r_2) \setminus B(0,r_2/2)$, since $r < r_2/2$, and hence $B \in \mathcal{B}_3$. Choose a sequence of balls $\{B_\ell\}_{\ell=0}^{m+1}$ such that

$$\begin{cases} B(0,4r_2) = B_0 \supset B_1 \supset \cdots \supset B_m \supset B_{m+1} = B, \\ |B_\ell| = 2^n |B_{\ell+1}|, & \ell = 0, \dots, m-1, \\ |B_m| \le 2^n |B_{m+1}|, \\ B_\ell \in \mathcal{B}_2, & \ell = 0,1,2,3, \\ B_\ell \in \mathcal{B}_3, & \ell = 4, \dots, m+1. \end{cases}$$

Note that the balls above are not concentric. Then, using (3.4) and (3.12), we have

$$\begin{aligned} |(f_1)_{B(0,4r_2)} - (f_1)_B| &\le \sum_{\ell=0}^{m} |(f_1)_{B_\ell} - (f_1)_{B_{\ell+1}}| \\ &\le 2^n \sum_{\ell=0}^{m} \phi(B_\ell) \max\left\{ \frac{\mathrm{MO}(f_1, B_\ell)}{\phi(B_\ell)} : \ell = 0,1,\dots,m \right\} \\ &\le 2^n (C_{n,\phi} + 1) \sum_{\ell=0}^{m} \phi(B_\ell)\,\epsilon. \end{aligned}$$

Since ϕ is in $\mathcal{G}^{\mathrm{inc}}$ and satisfies the nearness condition (2.3), the inequalities

$$\phi(B_\ell)/(2^{2-\ell} r_2) \le C_\phi\, \phi(B)/r, \quad \ell = 0,1,\dots,m,$$

hold for some positive constant C_ϕ dependent only on ϕ. Then

$$\sum_{\ell=0}^{m} \phi(B_\ell) \le \sum_{\ell=0}^{m} C_\phi \frac{(2^{2-\ell} r_2)\phi(B)}{r} \le 2^3 C_\phi \frac{r_2 \phi(B)}{r}.$$

Hence,

$$|(f_1)_{B(0,4r_2)} - (f_1)_B| \le C''_{n,\phi} \frac{r_2\phi(B)}{r}\,\epsilon, \tag{3.15}$$

where $C''_{n,\phi} = 2^{n+3}(C_{n,\phi}+1)C_\phi$. Next, let

$$C_{f_1} = \big((f_1)_B - (f_1)_{B(0,4r_2)}\big)\big(1-(h_{r_2})_B\big).$$

Then

$$\begin{aligned}
&\big(f_1(y) - f_2(y)\big) - C_{f_1} \\
&\quad = \big(f_1(y) - (f_1)_{B(0,4r_2)}\big)\big(1 - h_{r_2}(y)\big) - \big((f_1)_B - (f_1)_{B(0,4r_2)}\big)\big(1-(h_{r_2})_B\big) \\
&\quad = \Big(\big(f_1(y) - (f_1)_B\big)\big(1 - h_{r_2}(y)\big)\Big) + \Big(\big(h_{r_2}(y) - (h_{r_2})_B\big)\big((f_1)_{B(0,4r_2)} - (f_1)_B\big)\Big),
\end{aligned}$$

and then, for $y \in B = B(z,r)$,

$$\begin{aligned}
\Big|\big(f_1(y) - f_2(y)\big) - C_{f_1}\Big| &\le \big|f_1(y) - (f_1)_B\big| + 2r\|\nabla h_{r_2}\|_{L^\infty}\big|(f_1)_{B(0,4r_2)} - (f_1)_B\big| \\
&\le \big|f_1(y) - (f_1)_B\big| + 2r\frac{2}{r_2} \times C''_{n,\phi}\frac{r_2\phi(B)}{r}\,\epsilon,
\end{aligned}$$

where we used (3.15) in the last inequality. Hence,

$$\frac{1}{\phi(B)}\fint_B \big|\big(f_1(y) - f_2(y)\big) - C_f\big|\,dy \le \frac{\mathrm{MO}(f_1,B)}{\phi(B)} + 2^2 C''_{n,\phi}\,\epsilon \le C'''_{n,\phi}\,\epsilon,$$

where $C'''_{n,\phi}$ is dependent only on n and ϕ, which shows

$$\frac{\mathrm{MO}(f_1 - f_2, B)}{\phi(B)} \le 2C'''_{n,\phi}\,\epsilon.$$

The proof is complete. □

4. Commutators on Morrey spaces.

In this section, as an application of Theorem 2.1, we give a characterization of compact commutators $[b,T]$ and $[b,I_\alpha]$ with $b \in \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ on generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth condition. First we state the definition of the Morrey space $L^{(p,\varphi)}(\mathbb{R}^n)$ in Subsection 4.1. Next we state known results on the boundedness and compactness of the commutators $[b,T]$ and $[b,I_\rho]$ in Subsections 4.2 and 4.3, respectively, where I_ρ is the generalized fractional integral operator. Then we state the characterization in Subsection 4.4.

4.1. Generalized Morrey spaces with variable growth condition.

First we recall the definition of generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ for $p \in [1,\infty)$ and variable growth function $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Recall also that, for a ball $B = B(x,r)$, we write $\varphi(B) = \varphi(x,r)$.

Definition 4.1. For $p \in [1,\infty)$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let $L^{(p,\varphi)}(\mathbb{R}^n)$ be the set of all functions f such that the following functional is finite:

$$\|f\|_{L^{(p,\varphi)}} = \sup_B \left(\frac{1}{\varphi(B)} \fint_B |f(y)|^p \, dy \right)^{1/p},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Then $\|f\|_{L^{(p,\varphi)}}$ is a norm and $L^{(p,\varphi)}(\mathbb{R}^n)$ is a Banach space. Let $\varphi_\lambda(x,r) = r^\lambda$ for $\lambda \in [-n, 0]$. Then $L^{(p,\varphi_\lambda)}(\mathbb{R}^n)$ is the classical Morrey space. That is,

$$\|f\|_{L^{(p,\varphi_\lambda)}} = \sup_B \left(\frac{1}{\varphi_\lambda(B)} \fint_B |f(y)|^p \, dy \right)^{1/p} = \sup_{B=B(x,r)} \left(\frac{1}{r^\lambda} \fint_B |f(y)|^p \, dy \right)^{1/p}.$$

Note that $L^{(p,\varphi_{-n})}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ if $\lambda = -n$ and that $L^{(p,\varphi_0)}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ if $\lambda = 0$. Generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth function φ were introduced in [**25**] and studied in [**26**], [**30**], [**33**], etc.

We consider the following class of φ:

Definition 4.2. Let $\mathcal{G}^{\mathrm{dec}}$ be the set of all functions $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that φ is almost decreasing and that $r \mapsto \varphi(x,r)r^n$ is almost increasing. That is, there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,

$$C\varphi(x,r) \ge \varphi(x,s), \quad \varphi(x,r)r^n \le C\varphi(x,s)s^n, \quad \text{if } r < s.$$

4.2. Calderón–Zygmund operators.

We recall the definition of Calderón–Zygmund operators following [**46**]. Let Ω be the set of all nonnegative nondecreasing functions ω on $(0,\infty)$ such that $\int_0^1 (\omega(t)/t)dt < \infty$.

Definition 4.3 (standard kernel). Let $\omega \in \Omega$. A continuous function $K(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) \in \mathbb{R}^{2n}\}$ is said to be a standard kernel of type ω if the following conditions are satisfied;

$$|K(x,y)| \le \frac{C}{|x-y|^n} \quad \text{for} \quad x \ne y,$$

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \le \frac{C}{|x-y|^n}\, \omega\!\left(\frac{|y-z|}{|x-y|}\right) \quad \text{for} \quad 2|y-z| \le |x-y|.$$

Definition 4.4 (Calderón–Zygmund operator). Let $\omega \in \Omega$. A linear operator T from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ is said to be a Calderón–Zygmund operator of type ω, if T is bounded on $L^2(\mathbb{R}^n)$ and there exists a standard kernel K of type ω such that, for $f \in L^2_{\mathrm{comp}}(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) \, dy, \quad x \notin \operatorname{supp} f.$$

It is known by [**46**, Theorem 2.4] that any Calderón–Zygmund operator of type

$\omega \in \Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. This result was extended to generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth function φ by [**25**] as follows: Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and that there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$\int_r^\infty \frac{\varphi(x,t)}{t}\,dt \le C\varphi(x,r). \tag{4.1}$$

For $f \in L^{(p,\varphi)}(\mathbb{R}^n)$, $1 < p < \infty$, we define Tf on each ball B by

$$Tf(x) = T(f\chi_{2B})(x) + \int_{\mathbb{R}^n\setminus 2B} K(x,y)f(y)\,dy, \quad x \in B.$$

Then the first term in the right-hand side is well defined, since $f\chi_{2B} \in L^p(\mathbb{R}^n)$, and the integral of the second term converges absolutely. Moreover, $Tf(x)$ is independent of the choice of the ball B containing x. By this definition we can show that T is a bounded operator on $L^{(p,\varphi)}(\mathbb{R}^n)$, see [**25**].

For the boundedness of the commutator $[b,T]$ on $L^{(p,\varphi)}(\mathbb{R}^n)$, we have the following theorem.

THEOREM 4.1 ([**2**]). *Let $1 < p \le q < \infty$ and $\varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and $\psi \in \mathcal{G}^{\mathrm{inc}}$. Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and T be a Calderón–Zygmund operator of type $\omega \in \Omega$.*

(i) *Assume that ψ satisfies (2.3), that φ satisfies (4.1), that $\int_0^1 (\omega(t)\log(1/t)/t)dt < \infty$ and that there exists a positive constant C_0 such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$\psi(x,r)\varphi(x,r)^{1/p} \le C_0\varphi(x,r)^{1/q}. \tag{4.2}$$

If $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, then, for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$,

$$[b,T]f(x) = [b,T](f\chi_{2B})(x) + \int_{\mathbb{R}^n\setminus 2B} (b(x)-b(y))K(x,y)f(y)\,dy, \quad x \in B,$$

is well defined for each ball B and there exists a positive constant C, independent of b and f, such that

$$\|[b,T]f\|_{L^{(q,\varphi)}} \le C\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(p,\varphi)}}.$$

(ii) *Conversely, assume that φ satisfies (2.3) and that there exists a positive constant C_0 such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$C_0\psi(x,r)\varphi(x,r)^{1/p} \ge \varphi(x,r)^{1/q}. \tag{4.3}$$

If T is a convolution type such that

$$Tf(x) = p.v.\int_{\mathbb{R}^n} K(x-y)f(y)\,dy \tag{4.4}$$

with homogeneous kernel $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ satisfying $K(x) = |x|^{-n}K(x/|x|)$, $\int_{S^{n-1}} K = 0$, $K \in C^\infty(S^{n-1})$ and $K \not\equiv 0$, and if $[b,T]$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, then $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b, such that

$$\|b\|_{\mathcal{L}_{1,\psi}} \le C\|[b,T]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}},$$

where $\|[b,T]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}$ is the operator norm of $[b,T]$ from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.

Remark 4.1. For the well-definedness of $[b,T]f$ under the assumption in Theorem 4.1, see [**2**, Remark 4.2].

Next we state sufficient conditions for the compactness. To do this we consider the following condition on ψ: There exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$\int_r^\infty \frac{\psi(x,t)}{t^2}\,dt \le C\frac{\psi(x,r)}{r}. \tag{4.5}$$

Theorem 4.2 ([**3**]). *Let $1 < p \le q < \infty$ and $\varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume the same condition as Theorem* 4.1 (i). *Assume also that, for all $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$,*

$$Tf(x) = \lim_{\epsilon\to+0}\int_{|x-y|\ge\epsilon} K(x,y)f(y)\,dy \quad a.e.\, x \in \mathbb{R}^n, \tag{4.6}$$

and that φ and ψ satisfy (2.3) *and* (4.5), *respectively. If $b \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\psi}(\mathbb{R}^n)}$, then the commutator $[b,T]$ is compact from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.*

4.3. Generalized fractional integral operators.

Let I_α be the fractional integral operator of order $\alpha \in (0,n)$, that is,

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

Then it is known as the Hardy–Littlewood–Sobolev theorem that I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, if $\alpha \in (0,n)$, $p,q \in (1,\infty)$ and $-n/p + \alpha = -n/q$. This boundedness was extended to Morrey spaces by Adams [**1**] as follows: If $\alpha \in (0,n)$, $p,q \in (1,\infty)$, $\lambda \in [-n,0)$ and $\lambda/p + \alpha = \lambda/q$, then I_α is bounded from $L^{(p,\varphi_\lambda)}(\mathbb{R}^n)$ to $L^{(q,\varphi_\lambda)}(\mathbb{R}^n)$. See also [**39**] for the boundedness of I_α on Morrey and Campanato spaces.

For a function $\rho : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, we consider the generalized fractional integral operator I_ρ defined by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(x,|x-y|)}{|x-y|^n} f(y)\,dy, \tag{4.7}$$

where we always assume that

$$\int_0^1 \frac{\rho(x,t)}{t}\,dt < \infty \quad \text{for each } x \in \mathbb{R}^n, \tag{4.8}$$

and that there exist positive constants C, K_1 and K_2 with $K_1 < K_2$ such that

$$\sup_{r \le t \le 2r} \rho(x,t) \le C \int_{K_1 r}^{K_2 r} \frac{\rho(x,t)}{t}\,dt \quad \text{for all } x \in \mathbb{R}^n \text{ and } r > 0. \tag{4.9}$$

Condition (4.8) is necessary for the integral in (4.7) to converge for bounded functions f with compact support. Condition (4.9) was considered in [**40**].

If $\rho(x,r) = r^\alpha$, $0 < \alpha < n$, then I_ρ is the usual fractional integral operator I_α. If $\alpha(\cdot) : \mathbb{R}^n \to (0,n)$ and $\rho(x,r) = r^{\alpha(x)}$, then I_ρ is a generalized fractional integral operator $I_{\alpha(x)}$ with variable order. For the boundedness of I_ρ, see [**14**], [**27**], [**28**], [**29**], [**42**], etc.

Assume that ρ satisfies (4.8) and (4.9). Let $1 < p < \infty$ and $\varphi \in \mathcal{G}^{\mathrm{dec}}$. Then, for $f \in L^{(p,\varphi)}(\mathbb{R}^n)$, under some suitable condition, the integral in (4.7) converges absolutely and we can show that I_ρ is a bounded operator from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, see [**33**, Corollary 2.13].

For the boundedness of the commutator $[b, I_\rho]$ on $L^{(p,\varphi)}(\mathbb{R}^n)$, we have the following theorem.

THEOREM 4.3 ([**2**]). *Let $1 < p < q < \infty$ and $\rho, \varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and $\psi \in \mathcal{G}^{\mathrm{inc}}$. Assume also that ρ satisfies* (4.8) *and* (4.9). *Let $\rho^*(x,r) = \int_0^r (\rho(x,t)/t)dt$.*

(i) *Assume that ρ, ρ^* and ψ satisfy* (2.3), *that φ satisfies* (4.1) *and that there exist positive constants ϵ, C_ρ, C_0, C_1 and an exponent $\tilde{p} \in (p,q]$ such that, for all $x, y \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,*

$$C_\rho \frac{\rho(x,r)}{r^{n-\epsilon}} \ge \frac{\rho(x,s)}{s^{n-\epsilon}}, \quad \text{if} \quad r < s, \tag{4.10}$$

$$\left| \frac{\rho(x,r)}{r^n} - \frac{\rho(y,s)}{s^n} \right| \le C_\rho \left(|r-s| + |x-y|\right) \frac{\rho^*(x,r)}{r^{n+1}}, \quad \text{if} \quad \frac{1}{2} \le \frac{r}{s} \le 2 \quad \text{and} \quad |x-y| < \frac{r}{2}, \tag{4.11}$$

$$\int_0^r \frac{\rho(x,t)}{t}\,dt\, \varphi(x,r)^{1/p} + \int_r^\infty \frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\,dt \le C_0 \varphi(x,r)^{1/\tilde{p}}, \tag{4.12}$$

$$\psi(x,r)\varphi(x,r)^{1/\tilde{p}} \le C_1 \varphi(x,r)^{1/q}. \tag{4.13}$$

If $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, then $[b, I_\rho]f$ is well defined for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b and f, such that

$$\|[b, I_\rho]f\|_{L^{(q,\varphi)}} \le C \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(p,\varphi)}}.$$

(ii) *Conversely, assume that φ satisfies* (2.3), *that $\rho(x,r) = r^\alpha$, $0 < \alpha < n$, and that*

$$C_0 \psi(x,r) r^\alpha\, \varphi(x,r)^{1/p} \ge \varphi(x,r)^{1/q}. \tag{4.14}$$

If $[b,I_\alpha]$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, then $b\in\mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b, such that

$$\|b\|_{\mathcal{L}_{1,\psi}}\le C\|[b,I_\alpha]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}},$$

where $\|[b,I_\alpha]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}$ is the operator norm of $[b,I_\alpha]$ from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.

Remark 4.2. For the well-definedness of $[b,I_\rho]f$ under the assumption in Theorem 4.3, see [**2**, Remark 4.3].

Next we state a sufficient condition for the compactness of the commutator $[b,I_\rho]$ on $L^{(p,\varphi)}(\mathbb{R}^n)$.

Theorem 4.4 ([**3**]). *Let $1<p<q<\infty$ and $\rho,\varphi,\psi:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$. Assume the same condition as Theorem* 4.3 (i). *Assume also that φ and ψ satisfy* (2.3) *and* (4.5), *respectively. If $b\in\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\psi}(\mathbb{R}^n)}$, then the commutator $[b,I_\rho]$ is compact from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.*

4.4. Characterization of compact commutators.

In the previous subsections we state sufficient conditions for the compactness of the commutators $[b,T]$ and $[b,I_\rho]$ on $L^{(p,\varphi)}(\mathbb{R}^n)$. In this subsection, to characterize the compactness, we give necessary conditions. To prove the results we apply Theorem 2.1 in the final section.

Theorem 4.5. *Let $1<p\le q<\infty$ and $\varphi,\psi:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$. Let T be a Calderón–Zygmund operator of convolution type with kernel $K:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$. Assume the same condition on φ, ψ and T as Theorem* 4.1 (ii). *Assume also that there exists a positive constant μ_0 such that*

$$\limsup_{r\to+0}\sup_{x\in\mathbb{R}^n}\varphi(x,r)^{1/p}\psi(x,r)r^{n/q}\le\mu_0\inf_{x\in\mathbb{R}^n,\ r\in(0,1]}\varphi(x,r)^{1/p}\psi(x,r)r^{n/q},\tag{4.15}$$

$$\sup_{x\in\mathbb{R}^n,\ r\in[1,\infty)}\varphi(x,r)^{1/p}\psi(x,r)r^{n/q}\le\mu_0\liminf_{r\to\infty}\inf_{x\in\mathbb{R}^n}\varphi(x,r)^{1/p}\psi(x,r)r^{n/q},\tag{4.16}$$

$$\limsup_{|x|\to\infty}\varphi(x,r)^{1/p}\psi(x,r)\le\mu_0\liminf_{|x|\to\infty}\varphi(x,r)^{1/p}\psi(x,r)\ \textit{for every}\ r>0.\tag{4.17}$$

Let b be a real valued function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If $[b,T]$ is well defined on $L^{(p,\varphi)}(\mathbb{R}^n)$ and compact from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, then b is in $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\psi}(\mathbb{R}^n)}$.

We note that the Riesz transforms fall under the scope of Theorem 4.5.

Remark 4.3. If φ and ψ satisfy

$$\begin{cases} \lim_{r \to +0} \sup_{x \in \mathbb{R}^n} \varphi(x,r)^{1/p} \psi(x,r) r^{n/q} = 0, \\ \lim_{r \to \infty} \inf_{x \in \mathbb{R}^n} \varphi(x,r)^{1/p} \psi(x,r) r^{n/q} = \infty, \\ \lim_{|x| \to \infty} \varphi(x,r)^{1/p} \psi(x,r) \text{ exists for every } r > 0, \end{cases} \tag{4.18}$$

or

$$\mu_0^{-1} \le \varphi(x,r)^{1/p} \psi(x,r) r^{n/q} \le \mu_0 \text{ for all } x \in \mathbb{R}^n, r \in (0,\infty), \tag{4.19}$$

then the conditions (4.15), (4.16) and (4.17) hold.

Example 4.1. Let $1 < p \le q < \infty$ and $\beta(\cdot), \lambda(\cdot) : \mathbb{R}^n \to (-\infty, \infty)$. Assume that

$$\begin{gathered} 0 \le \inf_{x \in \mathbb{R}^n} \beta(x) \le \sup_{x \in \mathbb{R}^n} \beta(x) \le 1, \quad 0 \le \beta_* \le 1, \\ -n \le \inf_{x \in \mathbb{R}^n} \lambda(x) \le \sup_{x \in \mathbb{R}^n} \lambda(x) < 0, \quad -n \le \lambda_* < 0. \end{gathered}$$

Let

$$\psi(x,r) = \begin{cases} r^{\beta(x)}, \\ r^{\beta_*}, \end{cases} \quad \varphi(x,r) = \begin{cases} r^{\lambda(x)}, & 0 < r < 1, \\ r^{\lambda_*}, & 1 \le r < \infty. \end{cases}$$

Assume that $\lambda(\cdot)$ is log-Hölder continuous. Assume also that $\beta(x)$ and $\lambda(x)$ have finite limits as $|x| \to \infty$ respectively and that

$$\begin{gathered} \inf_{x \in \mathbb{R}^n} (\beta(x) + \lambda(x)/p) > -n/q, \quad \beta_* + \lambda_*/p > -n/q, \\ \beta(x) + \lambda(x)/p \le \lambda(x)/q, \quad \beta_* + \lambda_*/p \ge \lambda_*/q. \end{gathered}$$

Then φ satisfies (2.3) and φ and ψ satisfy (4.3) and (4.18). Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If a Calderón–Zygmund operator T satisfies the assumption in Theorem 4.5, and if $[b,T]$ is compact from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, then b is in $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\psi}(\mathbb{R}^n)}$.

We also take the cases

$$\psi(x,r) = \begin{cases} r^{\beta(x)} (1/\log(e/r))^{\beta_1(x)}, & 0 < r < 1, \\ r^{\beta_*} (\log(er))^{\beta_{**}}, & 1 \le r < \infty, \end{cases}$$

etc.

Theorem 4.6. *Let $1 < p < q < \infty$, $0 < \alpha < n$ and $\varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume the same condition on φ, ψ and α as Theorem* 4.3 (ii)*. Assume also that there exists a positive constant μ_0 such that*

$$\limsup_{r \to +0} \sup_{x \in \mathbb{R}^n} \varphi(x,r)^{1/p} \psi(x,r) r^{\alpha + n/q} \le \mu_0 \inf_{x \in \mathbb{R}^n,\ r \in (0,1]} \varphi(x,r)^{1/p} \psi(x,r) r^{\alpha + n/q}, \tag{4.20}$$

$$\sup_{x \in \mathbb{R}^n,\ r \in [1,\infty)} \varphi(x,r)^{1/p} \psi(x,r) r^{\alpha + n/q} \le \mu_0 \liminf_{r \to \infty} \inf_{x \in \mathbb{R}^n} \varphi(x,r)^{1/p} \psi(x,r) r^{\alpha + n/q}, \tag{4.21}$$

$$\limsup_{|x|\to\infty} \varphi(x,r)^{1/p}\psi(x,r) \le \mu_0 \liminf_{|x|\to\infty} \varphi(x,r)^{1/p}\psi(x,r) \text{ for every } r>0. \tag{4.22}$$

Let b be a real valued function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If $[b,I_\alpha]$ is well defined on $L^{(p,\varphi)}(\mathbb{R}^n)$ and compact from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, then b is in $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\psi}(\mathbb{R}^n)}$.

We can take similar examples to Example 4.1 for the compactness of $[b,I_\alpha]$.

We will prove Theorems 4.5 and 4.6 in the following sections by using Theorem 2.1.

5. Lemmas.

In this section we show several lemmas to prove Theorems 4.5 and 4.6 in Section 6.

Lemma 5.1 ([**24**, Corollary 2.4]). *There exists a positive constant c_n dependent only on n such that, for all $x\in\mathbb{R}^n$ and $r,s\in(0,\infty)$,*

$$|f_{B(x,r)} - f_{B(x,s)}| \le c_n \int_r^{2s} \frac{\mathrm{MO}(f,B(x,t))}{t}\,dt, \quad \text{if } r<s.$$

The next lemma is well known as the John–Nirenberg inequality.

Lemma 5.2 ([**20**]). *For all cubes Q_0 and all $t>0$,*

$$|\{x\in Q_0 : |f(x)-f_{Q_0}|>t\}| \le e|Q_0|\exp\left(-At/\sup\{\mathrm{MO}(f,Q) : Q\subset Q_0\}\right),$$

with $A=(2^n e)^{-1}$.

For the constants e and A in the above lemma, see [**16**, Theorem 3.1.6].

Corollary 5.3. *Assume that $\psi\in\mathcal{G}^{\mathrm{inc}}$. Let $\nu>1$ and $f\in\mathcal{L}_{1,\psi}(\mathbb{R}^n)$ with $\|f\|_{\mathcal{L}_{1,\psi}}=1$. Then, for all balls B_0 and all $t>0$,*

$$|\{x\in \nu B_0 : |f(x)-f_{B_0}|>t+A_0\nu\psi(B_0)\}| \le A_1\nu^n|B_0|\exp\left(-A_2 t/(\nu\psi(B_0))\right),$$

where the constants A_0, A_1 and A_2 are dependent only on n and ψ.

Proof. We denote by v_n the volume of the unit ball. Let Q_0 be the smallest cube containing νB_0. Then

$$\nu B_0 \subset Q_0 \subset \sqrt{n}\nu B_0, \quad \frac{|Q_0|}{|B_0|} = \frac{(2\nu)^n}{v_n}.$$

By this relation, Lemma 5.1 and $\|f\|_{\mathcal{L}_{1,\psi}}=1$ we have

$$\begin{aligned}|f_{B_0}-f_{Q_0}| &\le |f_{B_0}-f_{\sqrt{n}\nu B_0}| + |f_{\sqrt{n}\nu B_0}-f_{Q_0}| \\ &\le c_n\int_1^{2\sqrt{n}\nu} \frac{\mathrm{MO}(f,tB_0)}{t}\,dt + \frac{|\sqrt{n}\nu B_0|}{|Q_0|}\mathrm{MO}(f,\sqrt{n}\nu B_0)\end{aligned}$$

$$\begin{aligned}
&\le c_n \int_1^{2\sqrt{n}\nu} \frac{\psi(tB_0)}{t}\,dt + (\sqrt{n}/2)^n v_n \psi(\sqrt{n}\nu B_0)\\
&\le A_0 \nu \psi(B_0),
\end{aligned}$$

where the constant A_0 is dependent only on n and ψ. Since

$$\begin{aligned}
&|f(x) - f_{B_0}| > t + A_0\nu\psi(B_0)\\
&\qquad\qquad \Rightarrow \quad |f(x) - f_{B_0}| > t + |f_{B_0} - f_{Q_0}| \quad \Rightarrow \quad |f(x) - f_{Q_0}| > t,
\end{aligned}$$

we have

$$\begin{aligned}
&|\{x \in \nu B_0 : |f(x) - f_{B_0}| > t + A_0\nu\psi(B_0)\}|\\
&\quad \le |\{x \in \nu B_0 : |f(x) - f_{Q_0}| > t\}|\\
&\quad \le |\{x \in Q_0 : |f(x) - f_{Q_0}| > t\}|\\
&\quad \le e|Q_0| \exp\left(-At/\sup\{\mathrm{MO}(f,Q) : Q \subset Q_0\}\right)\\
&\quad = \frac{e(2\nu)^n}{v_n}|B_0| \exp\left(-At/\sup\{\mathrm{MO}(f,Q) : Q \subset Q_0\}\right) \quad \text{with } A = (2^n e)^{-1}.
\end{aligned}$$

In the above the third inequality follows from the John–Nirenberg inequality. For any cube $Q \subset Q_0$, take the smallest ball B containing Q. Then

$$Q \subset B \subset \sqrt{n}\nu B_0, \qquad \frac{|B|}{|Q|} = (\sqrt{n}/2)^n v_n.$$

Hence

$$\mathrm{MO}(f,Q) \le \frac{2|B|}{|Q|}\mathrm{MO}(f,B) = 2(\sqrt{n}/2)^n v_n \mathrm{MO}(f,B).$$

That is,

$$\begin{aligned}
\sup\{\mathrm{MO}(f,Q) : Q \subset Q_0\} &\le 2(\sqrt{n}/2)^n v_n \sup\left\{\mathrm{MO}(f,B) : B \subset \sqrt{n}\nu B_0\right\}\\
&\le 2(\sqrt{n}/2)^n v_n \sup\left\{\psi(B) : B \subset \sqrt{n}\nu B_0\right\}\\
&\le A_2' \nu \psi(B_0),
\end{aligned}$$

where the constant A_2' is dependent only on n and ψ. Letting $A_1 = e2^n/v_n$ and $A_2 = A/A_2'$, we have the conclusion. □

In the following lemma we used the idea in [**7**].

Lemma 5.4. *Let b be a real valued function. For any ball B, let*

$$f^B(z) = \varphi(B)^{1/p}\Big(\mathrm{sgn}(b(z) - b_B) - c_0\Big)\chi_B(z), \ \text{where } c_0 = \fint_B \mathrm{sgn}(b(z) - b_B)\,dz. \tag{5.1}$$

Then

$$\operatorname{supp} f^B \subset B, \quad \int_{\mathbb{R}^n} f^B(z)\,dz = 0, \tag{5.2}$$

$$f^B(z)(b(z) - b_B) \ge 0, \tag{5.3}$$

$$\int_{\mathbb{R}^n} f^B(z)(b(z) - b_B)\,dz = \varphi(B)^{1/p}|B|\mathrm{MO}(b,B), \tag{5.4}$$

$$\|f^B\|_{L^{(p,\varphi)}} \le C, \tag{5.5}$$

where C is a constant dependent only on n and φ.

Proof. The first assertion (5.2) is clear. Since $\int_B (b(z) - b_B)\,dz = 0$, it is easy to check $|c_0| < 1$. Then we have

$$f^B(z)(b(z) - b_B) = \varphi(B)^{1/p}\Big(|b(z) - b_B| - c_0(b(z) - b_B)\Big)\chi_B(z) \ge 0$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} f^B(z)(b(z) - b_B)\,dz &= \varphi(B)^{1/p}\int_B \Big(|b(z) - b_B| - c_0(b(z) - b_B)\Big)\,dz \\ &= \varphi(B)^{1/p}\int_B |b(z) - b_B|\,dz \\ &= \varphi(B)^{1/p}|B|\mathrm{MO}(b,B).\end{aligned}$$

Finally, let $B = B(x,r)$. We show that, for any $B' = B(x',r')$,

$$\frac{1}{\varphi(B')}\fint_{B'} |f^B(z)|^p\,dz \le C.$$

If $B \cap B' \neq \emptyset$ and $r' \le r$, then $\varphi(x,r) \sim \varphi(x,2r) \sim \varphi(x',2r) \lesssim \varphi(x',r')$ by (2.1), (2.3) and the almost decreasingness of φ. Hence

$$\frac{1}{\varphi(B')}\fint_{B'} |f^B(z)|^p\,dz \le \frac{\varphi(B)}{\varphi(B')} \le C.$$

If $B \cap B' \neq \emptyset$ and $r' > r$, then $\varphi(x,r)r^n \lesssim \varphi(x,2r')(2r')^n \sim \varphi(x',2r')(2r')^n \sim 2^n\varphi(x',r')(r')^n$ by the almost increasingness of $t \mapsto \varphi(x,t)t^n$, (2.3) and (2.1). Hence

$$\frac{1}{\varphi(B')}\fint_{B'} |f^B(z)|^p\,dz \le \frac{\varphi(B)|B|}{\varphi(B')|B'|} \le C. \qquad \square$$

Lemma 5.5. *Let $p, q \in (1,\infty)$. Let T be a convolution type singular integral operator such that*

$$Tf(x) = p.v.\int_{\mathbb{R}^n} K(x-y)f(y)\,dy \tag{5.6}$$

with homogeneous kernel $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ satisfying $K(x) = |x|^{-n}K(x/|x|)$, $\int_{S^{n-1}} K = 0$, $K \in C^\infty(S^{n-1})$ and $K \not\equiv 0$, Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and $\psi \in \mathcal{G}^{\mathrm{inc}}$. Assume also that ψ satisfies (4.5). Let b be a real valued function and $\|b\|_{\mathcal{L}_{1,\psi}} = 1$. For any ball

B, define f^B by (5.1). Then, for any constants $\epsilon_0, \mu_0 \in (0,\infty)$, there exist constants $\nu_1, \nu_2 \in [2,\infty)$ ($\nu_1 < \nu_2$), $\nu_3 \in (0,\infty)$ and $\nu_4 \in (0,1)$ such that, for all balls B satisfying $\mathrm{MO}(b,B)/\psi(B) \ge \epsilon_0$, the following three inequalities hold:

$$\left(\frac{1}{|B|}\int_{\nu_2 B\setminus \nu_1 B} \left|[b,T]f^B(y)\right|^q dy\right)^{1/q} \ge \nu_3 \varphi(B)^{1/p}\psi(B), \tag{5.7}$$

$$\left(\frac{1}{|B|}\int_{\mathbb{R}^n\setminus \nu_2 B} \left|[b,T]f^B(y)\right|^q dy\right)^{1/q} \le \frac{\nu_3}{4\mu_0} \varphi(B)^{1/p}\psi(B), \tag{5.8}$$

and, for any measurable set $E \subset \nu_2 B \setminus \nu_1 B$ satisfying $|E|/|B| \le \nu_4$,

$$\left(\frac{1}{|B|}\int_{E} \left|[b,T]f^B(y)\right|^q dy\right)^{1/q} \le \frac{\nu_3}{4} \varphi(B)^{1/p}\psi(B). \tag{5.9}$$

The Riesz transforms fall under the scope of Lemma 5.5.

PROOF. Step 1: Since $K \in C^\infty(S^{n-1})$ and $K \not\equiv 0$, we may assume that $|K(y') - K(z')| \le |y'-z'|$ for all $y', z' \in S^{n-1}$ and that

$$\sigma(\{x' \in S^{n-1} : K(x') \ge 2\epsilon_1\}) > 0.$$

for some constant $\epsilon_1 \in (0,1)$, where σ is the area measure on S^{n-1}. Let

$$\Lambda = \{x' \in S^{n-1} : K(x') \ge 2\epsilon_1\}.$$

Then

$$y' \in \Lambda,\ z' \in S^{n-1} \text{ and } |y'-z'| \le \epsilon_1 \quad \Rightarrow \quad K(z') \ge \epsilon_1, \tag{5.10}$$

since $K(y') \ge 2\epsilon_1$ and $|K(y') - K(z')| \le |y'-z'| \le \epsilon_1$. Set $\ell = 2/\epsilon_1 > 2$.

Step 2: Let $B = B(x,r)$ satisfy $\mathrm{MO}(b,B)/\psi(B) \ge \epsilon_0$. We show that

$$|T((b-b_B)f^B)(y)| \ge \frac{\varphi(B)^{1/p}\psi(B)|B|}{(2|y-x|)^n}\epsilon_1\epsilon_0 \quad \text{for } y \notin \ell B \text{ and } \frac{y-x}{|y-x|} \in \Lambda, \tag{5.11}$$

$$|T((b-b_B)f^B)(y)| \le 2^n C_K \frac{\varphi(B)^{1/p}\psi(B)|B|}{|y-x|^n} \quad \text{for } y \notin \ell B, \tag{5.12}$$

$$|(b(y)-b_B)T(f^B)(y)| \le C_K \frac{r|b(y)-b_B|\varphi(B)^{1/p}|B|}{|y-x|^{n+1}} \quad \text{for } y \notin \ell B, \tag{5.13}$$

where the constant C_K is dependent only on the kernel K.

Now, for $y \notin \ell B$ and $z \in B$, we have

$$\left|\frac{y-x}{|y-x|} - \frac{y-z}{|y-z|}\right| \le \left|\frac{y-x}{|y-x|} - \frac{y-z}{|y-x|}\right| + \left|\frac{y-z}{|y-x|} - \frac{y-z}{|y-z|}\right| \le \frac{2|z-x|}{|y-x|} \le \frac{2}{\ell} = \epsilon_1.$$

In this case, if $(y-x)/|y-x| \in \Lambda$ also, then $K((y-z)/|y-z|) \ge \epsilon_1$ by (5.10), and then

$$K(y-z) \geq \frac{\epsilon_1}{|y-z|^n} \geq \frac{\epsilon_1}{(2|y-x|)^n}.$$

Hence, from (5.3) and (5.4) it follows that, for $y \notin \ell B$ and $(y-x)/|y-x| \in \Lambda$,

$$|T((b-b_B)f^B)(y)| = \int_B K(y-z)(b(z)-b_B)f^B(z)\,dz \geq \frac{\varphi(B)^{1/p}|B|\mathrm{MO}(b,B)}{(2|y-x|)^n}\epsilon_1,$$

which shows (5.11), since $\mathrm{MO}(b,B) \geq \psi(B)\epsilon_0$. On the other hand, for $y \notin \ell B$ and $z \in B$, we have

$$|K(y-z)| \leq \frac{C_K}{|y-z|^n} \leq \frac{2^n C_K}{|y-x|^n}.$$

Then, from (5.3) and (5.4) it follows that, for $y \notin \ell B$,

$$|T((b-b_B)f^B)(y)| \leq 2^n C_K \frac{\varphi(B)^{1/p}|B|\mathrm{MO}(b,B)}{|y-x|^n},$$

which shows (5.12), since $\|b\|_{\mathcal{L}_{1,\psi}} = 1$. Finally, from (5.2) and (5.5) it follows that, for $y \notin \ell B$,

$$\begin{aligned}
|(b(y)-b_B)T(f^B)(y)| &= \left|(b(y)-b_B)\int_B \big(K(y-z)f^B(z) - K(y-x)f^B(z)\big)\,dz\right| \\
&\leq |b(y)-b_B| \int_B \frac{C_K|z-x|}{|y-x|^{n+1}}|f^B(z)|\,dz \\
&\leq C_K \frac{r|b(y)-b_B|\varphi(B)^{1/p}|B|}{|y-x|^{n+1}},
\end{aligned}$$

which is (5.13).

Step 3: Let $\kappa = n - n/q > 0$. From the condition (4.5) it follows that $t \mapsto \psi(x,t)/t^{1-\theta}$ is almost decreasing for some constant $\theta \in (0,1)$, see [**25**, Lemma 2] or [**31**, Lemma 7.1]. In this step, using (5.13), we show

$$\left(\int_{\mathbb{R}^n \setminus 2^{j_0}B} |(b(y)-b_B)T(f^B)(y)|^q\,dy\right)^{1/q} \leq C_1 (2^{j_0})^{-\kappa-\theta}\varphi(B)^{1/p}|B|^{1/q}\psi(B), \tag{5.14}$$

where the constant C_1 is independent of B and $j_0 \in \mathbb{Z}$ satisfying $j_0 \geq \log_2 \ell$.

By Lemma 5.1 and $\|b\|_{\mathcal{L}_{1,\psi}} = 1$ we have

$$\begin{aligned}
\left(⨍_{2^{j+1}B} |b(y)-b_B|^q\,dy\right)^{1/q} &\leq \left(⨍_{2^{j+1}B} |b(y)-b_{2^{j+1}B}|^q\,dy\right)^{1/q} + |b_{2^{j+1}B} - b_B| \\
&\leq c_n \int_r^{2^{j+2}r} \frac{\psi(x,t)}{t}\,dt, \quad j = 1,2,\dots.
\end{aligned}$$

Then, for $j_0 \geq \log_2 \ell$, by (5.13),

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^n\setminus 2^{j_0}B} |(b(y)-b_B)T(f^B)(y)|^q\,dy\right)^{1/q}\\
&\le C_K r\varphi(B)^{1/p}|B|\sum_{j=j_0}^{\infty}\left(\int_{2^{j+1}B\setminus 2^jB}\frac{|b(y)-b_B|^q}{|y-x|^{q(n+1)}}\,dy\right)^{1/q}\\
&\lesssim r\varphi(B)^{1/p}|B|\sum_{j=j_0}^{\infty}\frac{|2^{j+1}B|^{1/q}}{(2^jr)^{n+1}}\int_r^{2^{j+2}r}\frac{\psi(x,t)}{t}\,dt\\
&\lesssim r\varphi(B)^{1/p}|B|\int_{2^{j_0}r}^{\infty}s^{-n+n/q-2}\left(\int_r^s\frac{\psi(x,t)}{t}\,dt\right)ds.
\end{aligned}
$$

Recall that $\kappa=n-n/q>0$, and let

$$
I_1=\int_{2^{j_0}r}^{\infty}s^{-\kappa-2}\left(\int_r^{2^{j_0}r}\frac{\psi(x,t)}{t}\,dt\right)ds,\quad I_2=\int_{2^{j_0}r}^{\infty}s^{-\kappa-2}\left(\int_{2^{j_0}r}^{s}\frac{\psi(x,t)}{t}\,dt\right)ds.
$$

Then

$$
\left(\int_{\mathbb{R}^n\setminus 2^{j_0}B}|(b(y)-b_B)T(f^B)(y)|^q\,dy\right)^{1/q}\lesssim r\varphi(B)^{1/p}|B|(I_1+I_2). \tag{5.15}
$$

Using the almost decreasingness of $t\mapsto\psi(x,t)/t^{1-\theta}$, we have

$$
\begin{aligned}
I_1&=\frac{(2^{j_0}r)^{-\kappa-1}}{\kappa+1}\int_r^{2^{j_0}r}\frac{\psi(x,t)}{t}\,dt\lesssim(2^{j_0}r)^{-\kappa-1}\frac{\psi(x,r)}{r^{1-\theta}}\int_r^{2^{j_0}r}t^{-\theta}\,dt\\
&\lesssim(2^{j_0}r)^{-\kappa-1}\frac{\psi(x,r)}{r^{1-\theta}}(2^{j_0}r)^{1-\theta}\sim(2^{j_0})^{-\kappa-\theta}\frac{\psi(B)}{r}|B|^{-1+1/q}.
\end{aligned}
$$

and

$$
\begin{aligned}
I_2&=\int_{2^{j_0}r}^{\infty}\frac{\psi(x,t)}{t}\left(\int_t^{\infty}s^{-\kappa-2}\,ds\right)dt=\int_{2^{j_0}r}^{\infty}\frac{\psi(x,t)}{t}\frac{t^{-\kappa-1}}{\kappa+1}\,dt\\
&\lesssim\frac{\psi(x,2^{j_0}r)}{(2^{j_0}r)^{1-\theta}}\int_{2^{j_0}r}^{\infty}t^{-\kappa-\theta-1}\,dt\lesssim\frac{\psi(x,r)}{r^{1-\theta}}(2^{j_0}r)^{-\kappa-\theta}\sim(2^{j_0})^{-\kappa-\theta}\frac{\psi(B)}{r}|B|^{-1+1/q}.
\end{aligned}
$$

Hence, combining (5.15) with the estimates of I_1 and I_2, we have (5.14).

Step 4: Recall that $\kappa=n-n/q>0$. We show (5.7) and (5.8). From (5.11) and (5.14) it follows that, for $j_1>j_0$,

$$
\begin{aligned}
&\left(\int_{2^{j_1}B\setminus 2^{j_0}B}|[b,T]f^B(y)|^q\,dy\right)^{1/q}\\
&\ge\left(\int_{2^{j_1}B\setminus 2^{j_0}B}|T((b(y)-b_B)f^B)(y)|^q\,dy\right)^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\int_{\mathbb{R}^n \setminus 2^{j_0} B} |(b(y) - b_B) T(f^B)(y)|^q \, dy \right)^{1/q} \\
&\geq \varphi(B)^{1/p} \psi(B) |B| \left(\int_{(2^{j_1} B \setminus 2^{j_0} B) \cap \{y : (y-x)/|y-x| \in \Lambda\}} \frac{1}{(2|y-x|)^{nq}} \, dy \right)^{1/q} \epsilon_1 \epsilon_0 \\
&\quad - C_1 (2^{j_0})^{-\kappa-\theta} \varphi(B)^{1/p} |B|^{1/q} \psi(B) \\
&\geq \varphi(B)^{1/p} |B|^{1/q} \psi(B) \left(C_2 \left((2^{j_0})^{-\kappa q} - (2^{j_1})^{-\kappa q} \right)^{1/q} \epsilon_1 \epsilon_0 - C_1 (2^{j_0})^{-\kappa-\theta} \right),
\end{aligned}
$$

where the constant C_2 is independent of B, j_0 and j_1. From (5.12) and (5.14) it follows that

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^n \setminus 2^{j_1} B} |[b, T] f^B(y)|^q \, dy \right)^{1/q} \\
&\leq 2^n C \varphi(B)^{1/p} \psi(B) |B| \left(\int_{\mathbb{R}^n \setminus 2^{j_1} B} \frac{1}{|y-x|^{nq}} \, dy \right)^{1/q} + C_1 (2^{j_1})^{-\kappa-\theta} \varphi(B)^{1/p} |B|^{1/q} \psi(B) \\
&\leq \varphi(B)^{1/p} |B|^{1/q} \psi(B) \left(C_3 (2^{j_1})^{-\kappa} + C_1 (2^{j_1})^{-\kappa-\theta} \right),
\end{aligned}
$$

where the constant C_3 is independent of B, j_0 and j_1. Therefore, we can choose $\nu_1 = 2^{j_0}$, $\nu_2 = 2^{j_1}$ and $\nu_3 > 0$ such that (5.7) and (5.8) hold.

Step 5: We show (5.9). Let $E \subset \nu_2 B \setminus \nu_1 B$. From (5.12) and (5.13) it follows that

$$
\begin{aligned}
&\left(\int_E |[b, T] f^B(y)|^q \, dy \right)^{1/q} \\
&\leq 2^n C_K \varphi(B)^{1/p} \psi(B) |B| \left(\int_E \frac{1}{|y-x|^{nq}} \, dy \right)^{1/q} \\
&\quad + C_K r \varphi(B)^{1/p} |B| \left(\int_E \frac{|b(y) - b_B|^q}{|y-x|^{(n+1)q}} \, dy \right)^{1/q} \\
&\leq C_{K,n} (\nu_1)^{-n} \varphi(B)^{1/p} \psi(B) |E|^{1/q} \\
&\quad + C_{K,n} (\nu_1)^{-n-1} \varphi(B)^{1/p} \left(\int_E |b(y) - b_B|^q \, dy \right)^{1/q}.
\end{aligned}
\tag{5.16}
$$

Let $\tilde{b} = b - b_B$, and let

$$
\lambda(\omega) = |\{x \in E : |\tilde{b}(x)| > \omega\}| \quad \text{and} \quad \tilde{b}^*(t) = \inf\{\omega > 0 : \lambda(\omega) \leq t\}.
$$

Since $E \subset \nu_2 B$, by Corollary 5.3 we have

$$
\lambda(\omega + A_0 \nu_2 \psi(B)) \leq A_1 {\nu_2}^n |B| \exp\left(-A_2 \omega / (\nu_2 \psi(B)) \right).
$$

Hence

$$
\lambda(\omega) \leq A_1 {\nu_2}^n |B| \exp\left(-A_2 (\omega - A_0 \nu_2 \psi(B)) / (\nu_2 \psi(B)) \right).
$$

Since

$$t = A_1 {\nu_2}^n |B| \exp\left(-A_2(\omega - A_0\nu_2\psi(B))/(\nu_2\varphi(B))\right)$$
$$\Leftrightarrow \quad \omega = \nu_2\psi(B)\left(A_0 + \frac{1}{A_2}\log\frac{A_1{\nu_2}^n|B|}{t}\right),$$

we see that

$$\tilde{b}^*(t) \le \nu_2\psi(B)\left(A_0 + \frac{1}{A_2}\log\frac{A_1{\nu_2}^n|B|}{t}\right) \le A_3\nu_2\psi(B)\left(1 + \log\frac{A_1{\nu_2}^n|B|}{t}\right),$$

with $A_3 = \max(1, A_0)/\min(1, A_2)$. Then

$$\begin{aligned}\int_E |b(x) - b_B|^q\,dx &\le \int_0^{|E|} (\tilde{b}^*(t))^q\,dt \\ &\le (A_3\nu_2\psi(B))^q \int_0^{|E|}\left(1 + \log\frac{A_1{\nu_2}^n|B|}{t}\right)^q dt \\ &\le (A_3\nu_2\psi(B))^q A_1{\nu_2}^n|B| \int_0^{|E|/(A_1{\nu_2}^n|B|)}\left(1 + \log\frac{1}{t}\right)^q dt.\end{aligned} \tag{5.17}$$

Since

$$\left(1 + \log\frac{1}{t}\right)^q \le 2\frac{d}{dt}\left(t\left(1 + \log\frac{1}{t}\right)^q\right), \quad 0 < t \le e^{-2q},$$

if $|E|/(A_1{\nu_2}^n|B|) \le e^{-2q}$, then

$$\int_0^{|E|/(A_1{\nu_2}^n|B|)}\left(1 + \log\frac{1}{t}\right)^q dt \le \frac{2|E|}{A_1{\nu_2}^n|B|}\left(1 + \log\frac{A_1{\nu_2}^n|B|}{|E|}\right)^q. \tag{5.18}$$

Combining (5.16), (5.17) and (5.18), we have

$$\left(\int_E |[b,T]f^B(y)|^q\,dy\right)^{1/q} \le C\varphi(B)^{1/p}|B|^{1/q}\psi(B)\left(\frac{|E|}{|B|}\right)^{1/q}\left(1 + \log\frac{A_1{\nu_2}^n|B|}{|E|}\right),$$

where C is dependent only on n, A_0, A_2, ν_1 and ν_2. Therefore, we can choose $\nu_4 \in (0,1)$ such that (5.9) holds whenever $|E|/|B| \le \nu_4$. □

LEMMA 5.6. *Let $p, q \in (1,\infty)$ and $\alpha \in (0,n)$. Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and $\psi \in \mathcal{G}^{\mathrm{inc}}$. Assume also that ψ satisfies* (4.5) *and that $n - \alpha - n/q > 0$. Let b be a real valued function and $\|b\|_{\mathcal{L}_{1,\psi}} = 1$. For any ball B, define f^B by* (5.1)*. Then, for any constants $\epsilon_0, \mu_0 \in (0,\infty)$, there exist constants $\nu_1, \nu_2 \in [2,\infty)$ $(\nu_1 < \nu_2)$, $\nu_3 \in (0,\infty)$ and $\nu_4 \in (0,1)$ such that, for all balls B satisfying* $\mathrm{MO}(b,B)/\psi(B) \ge \epsilon_0$*, the following three inequalities hold*:

$$\left(\frac{1}{|B|}\int_{\nu_2B\setminus\nu_1B}\left|[b,I_\alpha]f^B(y)\right|^q dy\right)^{1/q} \ge \nu_3\varphi(B)^{1/p}|B|^{\alpha/n}\psi(B), \tag{5.19}$$

$$\left(\frac{1}{|B|}\int_{\mathbb{R}^n\setminus\nu_2 B}\left|[b,I_\alpha]f^B(y)\right|^q\,dy\right)^{1/q}\le\frac{\nu_3}{4\mu_0}\varphi(B)^{1/p}|B|^{\alpha/n}\psi(B),\tag{5.20}$$

and, for any measurable set $E\subset\nu_2 B\setminus\nu_1 B$ satisfying $|E|/|B|\le\nu_4$,

$$\left(\frac{1}{|B|}\int_{E}\left|[b,I_\alpha]f^B(y)\right|^q\,dy\right)^{1/q}\le\frac{\nu_3}{4}\varphi(B)^{1/p}|B|^{\alpha/n}\psi(B).\tag{5.21}$$

Proof. Let $B=B(x,r)$ satisfy $\mathrm{MO}(b,B)/\psi(B)\ge\epsilon_0$. For $y\notin 2B$ and $z\in B$, we have

$$\frac{1}{(2|y-x|)^{n-\alpha}}\le\frac{1}{|y-z|^{n-\alpha}}\le\frac{1}{(|y-x|/2)^{n-\alpha}}.$$

From (5.3), (5.4), $\|b\|_{\mathcal{L}_{1,\psi}}=1$ and $\mathrm{MO}(b,B)\ge\psi(B)\epsilon_0$ it follows that, for $y\notin 2B$,

$$|I_\alpha((b-b_B)f^B)(y)|=\int_B\frac{(b(z)-b_B)f^B(z)}{|y-z|^{n-\alpha}}\,dz\le\frac{\varphi(B)^{1/p}\psi(B)|B|}{(|y-x|/2)^{n-\alpha}},\tag{5.22}$$

$$|I_\alpha((b-b_B)f^B)(y)|=\int_B\frac{(b(z)-b_B)f^B(z)}{|y-z|^{n-\alpha}}\,dz\ge\frac{\varphi(B)^{1/p}\psi(B)|B|}{(2|y-x|)^{n-\alpha}}\epsilon_0.\tag{5.23}$$

From (5.2) and (5.5) it follows that, for $y\notin 2B$,

$$\begin{aligned}|(b(y)-b_B)I_\alpha(f^B)(y)|&=\left|(b(y)-b_B)\int_B\frac{f^B(z)}{|y-z|^{n-\alpha}}\,dz\right|\\&=\left|(b(y)-b_B)\int_B\left(\frac{f^B(z)}{|y-z|^{n-\alpha}}-\frac{f^B(z)}{|y-x|^{n-\alpha}}\right)dz\right|\\&\le\frac{r|b(y)-b_B|}{(n-\alpha)(|y-x|/2)^{n-\alpha+1}}\int_B|f^B(z)|\,dz\\&\le\frac{r|b(y)-b_B|\varphi(B)^{1/p}|B|}{(n-\alpha)(|y-x|/2)^{n-\alpha+1}}.\end{aligned}\tag{5.24}$$

Next, let $\kappa=n-\alpha-n/q>0$. Then in a similar way to Step 3 in the proof of Lemma 5.5, instead of (5.14), we have that

$$\left(\int_{\mathbb{R}^n\setminus 2^{j_0}B}|(b(y)-b_B)I_\alpha(f^B)(y)|^q\,dy\right)^{1/q}\le C_1(2^{j_0})^{-\kappa-\theta}\varphi(B)^{1/p}|B|^{\alpha/n+1/q}\psi(B),\tag{5.25}$$

for some $\theta\in(0,1)$, where the constant C_1 is independent of B and j_0. Moreover, in a similar way to Steps 4 and 5 in the proof of Lemma 5.5, using (5.22)–(5.25), we have (5.19), (5.20) and (5.21). □

6. Proofs of Theorems 4.5 and 4.6.

In this section, we prove Theorem 4.5 by using Theorem 2.1 and Lemma 5.5. We omit the proof of Theorem 4.6, since we can prove it in the same way as Theorem 4.5 by using Lemma 5.6 instead of Lemma 5.5.

Proof of Theorem 4.5. Since $[b,T]$ is compact from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, then $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$ by Theorem 4.1 (ii). We may assume that $\|b\|_{\mathcal{L}_{1,\psi}} = 1$. Below we show that b must satisfy the conditions (i), (ii) and (iii) in Theorem 2.1.

Part 1: Firstly, we show that, if b does not satisfy the condition (i), then $[b,T]$ is not compact. Since b does not satisfy the condition (i), there exist $\epsilon_0 > 0$ and a sequence of balls $\{B_j\}_{j=1}^{\infty} = \{B(x_j,r_j)\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} r_j = 0$ such that, for every j,

$$\frac{\mathrm{MO}(b,B_j)}{\psi(B_j)} > \epsilon_0. \tag{6.1}$$

For every B_j, we define $f_j = f^{B_j}$ by (5.1). Then $\sup_j \|f_j\|_{L^{(p,\varphi)}} \le C$ by Lemma 5.4. If we can choose a subsequence $\{f_{j(k)}\}_{k=1}^{\infty}$ such that $\{[b,T]f_{j(k)}\}_{k=1}^{\infty}$ has no any convergence subsequence in $L^{(q,\varphi)}(\mathbb{R}^n)$, then we have the conclusion.

Now, for the constant ϵ_0 in (6.1), let ν_i $(i = 1,2,3,4)$ be the constants defined by Lemma 5.5. By $\lim_{j\to\infty} r_j = 0$ and the assumption (4.15) we may choose a subsequence $\{B_{j(k)}\}$ such that

$$\frac{|B_{j(k+1)}|}{|B_{j(k)}|} < \frac{\nu_4}{{\nu_2}^n} \tag{6.2}$$

and

$$\varphi(B_{j(k+1)})^{1/p}\psi(B_{j(k+1)})|B_{j(k+1)}|^{1/q} \le \mu_0\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})|B_{j(k)}|^{1/q}. \tag{6.3}$$

Then the subsequence $\{f_{j(k)}\}$ associated with $\{B_{j(k)}\}$ is just what we request. Namely, there exists a positive constant δ such that, for any $k,\ell \in \mathbb{N}$ with $k < \ell$,

$$\|[b,T]f_{j(k)} - [b,T]f_{j(\ell)}\|_{L^{(q,\varphi)}} \ge \delta. \tag{6.4}$$

In fact, for fixed $k,\ell \in \mathbb{N}$ with $k < \ell$, denote

$$G = \nu_2 B_{j(k)} \setminus \nu_1 B_{j(k)}, \quad E = G \cap \nu_2 B_{j(\ell)}.$$

Then by (6.2) we have

$$\frac{|E|}{|B_{j(k)}|} \le \frac{|\nu_2 B_{j(\ell)}|}{|B_{j(k)}|} < \nu_4.$$

From the relation $G \setminus E = G \setminus \nu_2 B_{j(\ell)} \subset \nu_2 B_{j(k)} \cap (\nu_2 B_{j(\ell)})^{\complement}$ it follows that

$$\begin{aligned}\left(\int_G |[b,T]f_{j(k)}|^q\,dx - \int_E |[b,T]f_{j(k)}|^q\,dx\right)^{1/q} &= \left(\int_{G\setminus \nu_2 B_{j(\ell)}} |[b,T]f_{j(k)}|^q\,dx\right)^{1/q} \\ \le \left(\int_{\nu_2 B_{j(k)}} |[b,T]f_{j(k)} - [b,T]f_{j(\ell)}|^q\,dx\right)^{1/q} &+ \left(\int_{(\nu_2 B_{j(\ell)})^{\complement}} |[b,T]f_{j(\ell)}|^q\,dx\right)^{1/q}. \end{aligned} \tag{6.5}$$

By (5.7), (5.8), (5.9) and (6.3) we have

$$\int_G |[b,T]f_{j(k)}|^q\,dx \ge \Big(\nu_3\,\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})\Big)^q |B_{j(k)}|, \tag{6.6}$$

$$\begin{aligned}\left(\int_{(\nu_2 B_{j(\ell)})^{\complement}} |[b,T]f_{j(\ell)}|^q\,dx\right)^{1/q} &\le \frac{\nu_3}{4\mu_0}\varphi(B_{j(\ell)})^{1/p}\psi(B_{j(\ell)})|B_{j(\ell)}|^{1/q} \\ &\le \frac{\nu_3}{4}\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})|B_{j(k)}|^{1/q},\end{aligned} \tag{6.7}$$

$$\int_E |[b,T]f_{j(k)}|^q\,dx \le \Big(\frac{\nu_3}{4}\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})\Big)^q |B_{j(k)}|. \tag{6.8}$$

Combining (6.5)–(6.8), we have

$$\begin{aligned}&\Big({\nu_3}^q - \big(\nu_3/4\big)^q\Big)^{1/q}\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})|B_{j(k)}|^{1/q} \\ &\quad\le \left(\int_{\nu_2 B_{j(k)}} |[b,T]f_{j(k)} - [b,T]f_{j(\ell)}|^q\,dx\right)^{1/q} + \frac{\nu_3}{4}\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})|B_{j(k)}|^{1/q},\end{aligned}$$

which shows

$$\delta_0\,\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})|B_{j(k)}|^{1/q} \le \left(\int_{\nu_2 B_{j(k)}} |[b,T]f_{j(k)} - [b,T]f_{j(\ell)}|^q\,dx\right)^{1/q},$$

where $\delta_0 = \Big({\nu_3}^q - \big(\nu_3/4\big)^q\Big)^{1/q} - \nu_3/4 > 0$. Thus, using (4.3) and the almost decreasingness of φ, we have

$$\left(\frac{1}{\varphi(\nu_2 B_{j(k)})} \fint_{\nu_2 B_{j(k)}} |[b,T]f_{j(k)} - [b,T]f_{j(\ell)}|^q\,dx\right)^{1/q} \ge \delta,$$

where δ is independent on m and ℓ, which shows (6.4).

Part 2: Secondly, we show that, if b does not satisfy the condition (ii), then $[b,T]$ is not compact. Since b does not satisfy the condition (ii), there exist $\epsilon_0 > 0$ and a sequence of balls $\{B_j\}_{j=1}^\infty = \{B(x_j,r_j)\}_{j=1}^\infty$ with $\lim\limits_{j\to\infty} r_j = \infty$ such that, for every j,

$$\frac{\mathrm{MO}(b,B_j)}{\psi(B_j)} > \epsilon_0.$$

For every B_j, we define $f_j = f^{B_j}$ by (5.1). Then $\sup_j \|f_j\|_{L^{(p,\varphi)}} \le C$ by Lemma 5.4. By $\lim\limits_{j\to 0} r_j = \infty$ and the assumption (4.16) we may choose a subsequence $\{B_{j(k)}\}_{k=1}^\infty$ such that

$$\frac{|B_{j(k)}|}{|B_{j(k+1)}|} < \frac{\nu_4}{{\nu_2}^n}$$

and

$$\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})|B_{j(k)}|^{1/q} \le \mu_0\varphi(B_{j(k+1)})^{1/p}\psi(B_{j(k+1)})|B_{j(k+1)}|^{1/q}.$$

Then, in a similar way to Step 1 we conclude that there exists a positive constant δ such that, for all $k, \ell \in \mathbb{N}$ with $k < \ell$,

$$\left(\frac{1}{\varphi(\nu_2 B_{j(\ell)})} \fint_{\nu_2 B_{j(\ell)}} |[b,T]f_{j(\ell)} - [b,T]f_{j(k)}|^q \, dx\right)^{1/q} \ge \delta.$$

That is, $[b,T]$ is not compact.

Part 3: Finally, we show that, if b does not satisfy the condition (iii), then $[b,T]$ is not compact. Since b does not satisfy the condition (iii), there exist $\epsilon_0 > 0$ and a sequence of balls $\{B_j\}_{j=1}^{\infty} = \{B(x_j, r)\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} |x_j| = \infty$ such that, for every j,

$$\frac{\mathrm{MO}(b, B_j)}{\psi(B_j)} > \epsilon_0.$$

By $\lim_{j\to 0} |x_j| = \infty$ and the assumption (4.17) we may choose a subsequence $\{B_{j(k)}\}_{k=1}^{\infty}$ such that $\nu_2 B_{j(k)} \cap \nu_2 B_{j(k+1)} = \emptyset$ and

$$\varphi(B_{j(k+1)})^{1/p}\psi(B_{j(k+1)})|B_{j(k+1)}|^{1/q} \le \mu_0\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})|B_{j(k)}|^{1/q}.$$

Then, in a similar way to Step 1 we conclude that $[b,T]$ is not compact. □

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