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# Diophantine approximation in number fields and geometry of products of symmetric spaces

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**Abstract.** Dirichlet's theorem in Diophantine approximation is known to be closely related to geometry of the hyperbolic plane. In this paper we consider approximation in the setting of number fields and study relation between systems of linear forms and geometry of products of symmetric spaces.

## 1. Introduction.

It has been known for a long time that Dirichlet's theorem in Diophantine approximation is closely related to geometry of the hyperbolic plane. Let

$$\boldsymbol{H} = \left\{ x + \sqrt{-1}y \mid x, y \in \mathbf{R};\ y > 0 \right\}$$

be the upper half-plane equipped with the Poincaré metric $(dx^2 + dy^2)/y^2$. The group $SL(2, \mathbf{R})$ acts on $\boldsymbol{H}$ isometrically as a group of linear fractional transformations:

$$g \cdot z = \frac{\lambda z + \mu}{\nu z + \xi} \quad \text{for} \ \ z = x + \sqrt{-1}y \in \boldsymbol{H}, \ \ g = \begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix} \in SL(2, \mathbf{R}).$$

A horoball in $\boldsymbol{H}$ is an open ball tangent to the boundary at infinity of $\boldsymbol{H}$: it is a subset of the form

$$HB(C) = \left\{ x + \sqrt{-1}\, y \in \boldsymbol{H} \mid y > C \right\}$$

for a positive number $C$ or its translate by an element of $SL(2, \mathbf{R})$. For a given real number $a$, let $\gamma_a : [0, \infty) \longrightarrow \boldsymbol{H}$ be the geodesic ray defined by

$$\gamma_a(s) = a + e^{-s}\sqrt{-1} \quad \text{for} \ \ s \geq 0.$$

The image under

$$g = \begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix} \in SL(2, \mathbf{R})$$

of the horoball $HB(C)$ is the interior of the circle

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$$\left(x-\frac{\lambda}{\nu}\right)^2+\left(y-\frac{1}{2C\nu^2}\right)^2=\left(\frac{1}{2C\nu^2}\right)^2$$

tangent to the real line at $\lambda/\nu$ if $\nu \neq 0$, and the following holds.

Theorem 1.1 (Ford [**11**]). *Let $\lambda, \nu$ be coprime integers with $\nu > 0$ and let*

$$g=\begin{pmatrix}\lambda & \mu\\ \nu & \xi\end{pmatrix}\in SL(2,\mathbf{Z}).$$

*Then we have*

$$\left|a-\frac{\lambda}{\nu}\right|<\frac{C}{\nu^2} \tag{1.1}$$

*if and only if the geodesic ray $\gamma_a$ intersects the image under $g$ of the horoball $HB(1/(2C))$.*

Based on this correspondence Ford ([**11**]) gave a geometric proof of a theorem of Hurwitz without using the theory of continued fractions. Similar relation as Theorem 1.1 between approximation of a complex number by ratios of algebraic integers in an imaginary quadratic field and geometry of the 3-dimensional hyperbolic space was also studied in [**12**], [**13**], [**19**].

In [**15**] we studied such relation in the case of real quadratic fields and totally complex quartic fields. The problem is to find appropriate generalization of the inequality (1.1) and corresponding spaces. It seems to be the answer that the inequality is the one treated in [**10**], [**21**] and the spaces are products of 2-dimensional and 3-dimensional hyperbolic spaces, although we treated other inequalities in [**15**]. In this paper we study more general cases of linear forms. We consider the following situation.

Let $\boldsymbol{k}$ be a number field of degree $d=l+2m$ with $l$ real places and $m$ complex places. We denote by $\iota_1,\dots,\iota_l:\boldsymbol{k}\longrightarrow\mathbf{R}$ the real embeddings and $\iota_{l+1},\dots,\iota_{l+m}:\boldsymbol{k}\longrightarrow\mathbf{C}$ the complex embeddings which are not complex conjugate to each other. Let $\boldsymbol{k}_M=\mathbf{R}^l\times\mathbf{C}^m$ be the Minkowski space associated to $\boldsymbol{k}$. We denote by $\iota$ the embedding $\mathbf{R}\longrightarrow\boldsymbol{k}_M$ defined by

$$\iota(\lambda)=(\lambda,\dots,\lambda)\quad\text{for }\ \lambda\in\mathbf{R}.$$

The twisted diagonal embedding $\iota_{\boldsymbol{k}}:\boldsymbol{k}\longrightarrow\boldsymbol{k}_M$ is given by

$$\iota_{\boldsymbol{k}}(a)=(\iota_1(a),\dots,\iota_l(a),\iota_{l+1}(a),\dots,\iota_{l+m}(a))\quad\text{for }\ a\in\boldsymbol{k}. \tag{1.2}$$

For any positive integer $q$, this embedding $\iota_{\boldsymbol{k}}$ induces an embedding $\boldsymbol{k}^q\longrightarrow(\boldsymbol{k}_M)^q$, which we also denote by $\iota_{\boldsymbol{k}}$:

$$\iota_{\boldsymbol{k}}(\boldsymbol{a})=(\iota_{\boldsymbol{k}}(a_1),\dots,\iota_{\boldsymbol{k}}(a_q))\quad\text{for }\ \boldsymbol{a}=(a_1,\dots,a_q)\in\boldsymbol{k}^q.$$

For $\eta=(\eta^1,\dots,\eta^{l+m})$, $\rho=(\rho^1,\dots,\rho^{l+m})\in\boldsymbol{k}_M$ and $\lambda\in\mathbf{R}$, we put

$$\|\eta\|=\max_{1\le i\le l+m}|\eta^i|,\quad \lambda\cdot\eta=(\lambda\eta^1,\dots,\lambda\eta^{l+m}),$$

and

$$\eta + \rho = (\eta^1 + \rho^1, \ldots, \eta^{l+m} + \rho^{l+m}), \quad \eta \cdot \rho = (\eta^1 \rho^1, \ldots, \eta^{l+m} \rho^{l+m}),$$

where $| \ |$ is the usual Euclidean absolute value on $\mathbf{R}$ or $\mathbf{C}$.

Let $n$, $p$ be integers such that $n-1 \geq p \geq 1$ and $L = (L_{ij})$ a $p \times (n-p)$ matrix with entries in $\boldsymbol{k}_M$. For $i = 1, \ldots, n-p$, we denote by $L_i$ the $\mathbf{R}$-linear map $(\boldsymbol{k}_M)^p \longrightarrow \boldsymbol{k}_M$ determined by the $i$th column of $L$:

$$L_i(\boldsymbol{x}) = \sum_{k=1}^{p} L_{ki} x_k \quad \text{for } \boldsymbol{x} = (x_1, \ldots, x_p) \in (\boldsymbol{k}_M)^p.$$

We call such an $\mathbf{R}$-linear map a $\boldsymbol{k}_M$-form. Let

$$\boldsymbol{L}(\boldsymbol{x}) = (L_1(\boldsymbol{x}), \ldots, L_{n-p}(\boldsymbol{x})) \quad \text{and} \quad \|\boldsymbol{a}\| = \max_{1 \leq i \leq q} \|a_i\|$$

for $\boldsymbol{x} \in (\boldsymbol{k}_M)^p$ and $\boldsymbol{a} = (a_1, \ldots, a_q) \in (\boldsymbol{k}_M)^q$, $q \geq 1$.

Let $\mathcal{O}_{\boldsymbol{k}}$ be the ring of integers of $\boldsymbol{k}$ and consider the inequality

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{x})\|^p \, \|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{x})) - \iota_{\boldsymbol{k}}(\boldsymbol{y})\|^{n-p} < C \tag{1.3}$$

for $\boldsymbol{x} \in (\mathcal{O}_{\boldsymbol{k}})^p - \{\boldsymbol{0}\}$ and $\boldsymbol{y} \in (\mathcal{O}_{\boldsymbol{k}})^{n-p}$.

For any ring $R$ and positive integers $q, q'$ we denote by $M(q, q'; R)$ the set of all $q \times q'$ matrices with entries in $R$. Let $\Delta_{\boldsymbol{k}}$ be the discriminant of $\boldsymbol{k}$ and

$$C_{\boldsymbol{k}} = \left\{ \left( \frac{2}{\pi} \right)^m |\Delta_{\boldsymbol{k}}|^{1/2} \right\}^{1/d} \geq 1.$$

Then the following generalization of Dirichlet's theorem follows from Minkowski's convex body theorem.

Theorem 1.2. *Suppose that $C > (C_{\boldsymbol{k}})^n$. Then the following hold.*
(1) *For every $L \in M(p, n-p; \boldsymbol{k}_M)$ there exist $\boldsymbol{x} \in (\mathcal{O}_{\boldsymbol{k}})^p - \{\boldsymbol{0}\}$, $\boldsymbol{y} \in (\mathcal{O}_{\boldsymbol{k}})^{n-p}$ satisfying the inequality* (1.3).
(2) *If $\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a}))$ does not belong to $(\iota_{\boldsymbol{k}}(\mathcal{O}_{\boldsymbol{k}}))^{n-p}$ for any $\boldsymbol{a} \in (\mathcal{O}_{\boldsymbol{k}})^p - \{\boldsymbol{0}\}$, then there exist infinitely many distinct pairs $(\boldsymbol{x}, \boldsymbol{y}) \in (\mathcal{O}_{\boldsymbol{k}})^p \times (\mathcal{O}_{\boldsymbol{k}})^{n-p} - \{\boldsymbol{0}\} \times (\mathcal{O}_{\boldsymbol{k}})^{n-p}$ satisfying* (1.3).

The inequality (1.3) is the same as the one treated in [**22**] and is different from the one treated in [**5**] (see Section 2). It is also different from the one treated in [**26**, Theorem 1]. In the case $n = 2$, (1.3) becomes an inequality for a single $\boldsymbol{k}_M$-form in one variable, which coincides with the inequality treated in [**10**], [**21**] (see also [**24**]).

We say that the system $L_1, \ldots, L_{n-p}$ of $\boldsymbol{k}_M$-forms is badly approximable if there exists a positive constant $C$ such that

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{x})\|^p \, \|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{x})) - \iota_{\boldsymbol{k}}(\boldsymbol{y})\|^{n-p} \geq C \tag{1.4}$$

for any $\boldsymbol{x} \in (\mathcal{O}_{\boldsymbol{k}})^p - \{\boldsymbol{0}\}$ and $\boldsymbol{y} \in (\mathcal{O}_{\boldsymbol{k}})^{n-p}$. Let $\mathcal{B}_{n,p,\boldsymbol{k}}$ be the set of $L \in M(p, n-p; \boldsymbol{k}_M)$ such that the system of $\boldsymbol{k}_M$-forms induced from $L$ is badly approximable.

We show that one can treat the inequalities (1.3), (1.4) in the same geometric framework as in Theorem 1.1 by replacing the upper half-plane $\boldsymbol{H}$ with $\widetilde{V} = (SL(n,\mathbf{R})/SO(n))^l \times (SL(n,\mathbf{C})/SU(n))^m$, $SL(2,\mathbf{Z})$ with $SL(n,\mathcal{O}_{\boldsymbol{k}})$, $\gamma_a$ with a suitable geodesic ray $\gamma_L$ in $\widetilde{V}$, $\{HB(C)\}_{C\geq 1}$ with a suitable family of horoballs in $\widetilde{V}$, respectively.

Let $V = SL(n,\mathbf{R})/SO(n)$ and $\widehat{V} = SL(n,\mathbf{C})/SU(n)$. We equip $V$ with the left $SL(n,\mathbf{R})$-invariant Riemannian metric induced from the Killing form of the Lie algebra of $SL(n,\mathbf{R})$. Similarly we give $\widehat{V}$ the left $SL(n,\mathbf{C})$-invariant Riemannian metric induced from the Killing form of the Lie algebra of $SL(n,\mathbf{C})$.

For any positive integers $q, q'$, each embedding $\iota_k$ can be extended to an embedding $M(q,q';\boldsymbol{k}) \longrightarrow M(q,q';\mathbf{R})$ by

$$\iota_k(g) = (\iota_k(g_{ij})) \tag{1.5}$$

for $g = (g_{ij}) \in M(q,q';\boldsymbol{k})$, if $\iota_k$ is real. Similarly, in the case $\iota_k$ is complex, $\iota_k$ can be extended to an embedding $M(q,q';\boldsymbol{k}) \longrightarrow M(q,q';\mathbf{C})$ by the same formula (1.5). Then the twisted diagonal embedding $\iota_{\boldsymbol{k}}$ given by (1.2) can be extended to an embedding $SL(n,\boldsymbol{k}) \longrightarrow SL(n,\mathbf{R})^l \times SL(n,\mathbf{C})^m$ by

$$\iota_{\boldsymbol{k}}(g) = (\iota_1(g),\dots,\iota_l(g),\iota_{l+1}(g),\dots,\iota_{l+m}(g)) \quad \text{for } g \in SL(n,\boldsymbol{k}). \tag{1.6}$$

Thus the group $SL(n,\boldsymbol{k})$ acts isometrically on the Riemannian product $V^l \times \widehat{V}^m$ through the embedding $\iota_{\boldsymbol{k}}$:

$$\iota_{\boldsymbol{k}}(g)\cdot x = (\iota_1(g)\cdot x_1,\dots,\iota_{l+m}(g)\cdot x_{l+m}) \quad \text{for } x = (x_1,\dots,x_{l+m}) \in V^l \times \widehat{V}^m. \tag{1.7}$$

We study the relation between the inequalities (1.3), (1.4) and geometry of the symmetric space $V^l \times \widehat{V}^m$ through the action of the group $\Gamma = SL(n,\mathcal{O}_{\boldsymbol{k}})$ on $V^l \times \widehat{V}^m$. We define a geodesic ray $\gamma^*$ in $V^l \times \widehat{V}^m$ by the formula (3.13) in Section 3 and take a family of horoballs $\{B(\gamma^*,\tau)\}_{\tau\geq 0}$ determined by $\gamma^*$ according to Definition 3.1. We also define a geodesic ray $\gamma_L : [0,\infty) \longrightarrow V^l \times \widehat{V}^m$ for a given $L \in M(p,n-p;\boldsymbol{k}_M)$ by the formula (3.19) in Section 3. Then we consider when $\gamma_L$ intersects translates of $B(\gamma^*,\tau)$ by elements of $SL(n,\boldsymbol{k})$.

Let $\varepsilon_1,\dots,\varepsilon_{l+m-1}$ be a system of fundamental units of $\mathcal{O}_{\boldsymbol{k}}$ and let

$$\begin{aligned} C_1 = \max\big\{&|\iota_j(\varepsilon_i)|^{2(l+m-1)},\ |\iota_j(\varepsilon_i)|^{-2(l+m-1)} \\ &\big|\, i = 1,\dots,l+m-1;\ j = 1,\dots,l+m\big\}. \end{aligned}$$

We denote by $I_n$ the unit matrix of order $n$. The relation corresponding to Theorem 1.1 is as follows.

Theorem 1.3. *Let $D$ be a positive integer and let $g = (a_{ij}) \in SL(n,\boldsymbol{k})$ such that all the entries in the nth row of the matrix $(DI_n)g$ belong to $\mathcal{O}_{\boldsymbol{k}}$. Suppose that*

$$\tau \geq \frac{\sqrt{2d}\,n}{\sqrt{n-1}}\log D.$$

*Then we have the following, where* $\kappa = \exp\{-\sqrt{2(n-1)}\,\tau/(n\sqrt{d})\}$.
(1) *If* $\boldsymbol{a} = (a_{n1}, \ldots, a_{np}) \neq \boldsymbol{0}$ *and*

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^{p}\,\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) + \iota_{\boldsymbol{k}}(\boldsymbol{b})\|^{n-p} < \left(\frac{\kappa}{n}\right)^{n/2},$$

*where* $\boldsymbol{b} = (a_{n,p+1}, \ldots, a_{nn})$, *then* $\gamma_L([0,\infty))$ *intersects the horoball* $\iota_{\boldsymbol{k}}(g)^{-1} \cdot B(\gamma^*, \tau)$.

(2) *If* $\gamma_L([0,\infty))$ *intersects the horoball* $\iota_{\boldsymbol{k}}(g)^{-1} \cdot B(\gamma^*, \tau)$, *then* $\boldsymbol{a} \neq \boldsymbol{0}$ *and there exists a unit* $\omega \in \mathcal{O}_{\boldsymbol{k}}$ *such that*

$$\|\iota_{\boldsymbol{k}}(\omega\boldsymbol{a})\|^{p}\,\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\omega\boldsymbol{a})) + \iota_{\boldsymbol{k}}(\omega\boldsymbol{b})\|^{n-p} < (C_1\kappa)^{n/2}.$$

Using this correspondence, we study the set $\mathcal{B}_{n,p,\boldsymbol{k}}$. Let $B$ be the subgroup of $SL(n,\boldsymbol{k})$ consisting of all the upper triangular matrices in $SL(n,\boldsymbol{k})$ and let $g_1, \ldots, g_h$ be a complete representative system of the double coset classes $\Gamma\backslash SL(n,\boldsymbol{k})/B$.

THEOREM 1.4. *Let* $L \in M(p, n-p; \boldsymbol{k}_M)$. *The following two conditions are equivalent.*
(1) *There exists a non-negative number* $\tau$ *such that* $\gamma_L([0,\infty))$ *does not intersect*

$$\bigcup_{i=1}^{h}\bigcup_{g\in\Gamma} \iota_{\boldsymbol{k}}(g)\iota_{\boldsymbol{k}}(g_i)\cdot B(\gamma^*, \tau).$$

(2) *The system of* $\boldsymbol{k}_M$*-forms* $L_1, \ldots, L_{n-p}$ *induced from* $L$ *is badly approximable.*

Let $\Pi : V^l \times \widehat{V}^m \longrightarrow \iota_{\boldsymbol{k}}(\Gamma)\backslash(V^l \times \widehat{V}^m)$ be the natural projection. Since the first condition of Theorem 1.4 is equivalent to the condition that $\Pi \circ \gamma_L([0,\infty))$ is relatively compact in $\iota_{\boldsymbol{k}}(\Gamma)\backslash(V^l \times \widehat{V}^m)$ (cf. Lemma 6.1), we have the following.

THEOREM 1.5. *The system* $L_1, \ldots, L_{n-p}$ *of* $\boldsymbol{k}_M$*-forms is badly approximable if and only if* $\Pi \circ \gamma_L([0,\infty))$ *is relatively compact in* $\iota_{\boldsymbol{k}}(\Gamma)\backslash(V^l \times \widehat{V}^m)$.

By topological consideration (see Lemma 6.2) this is equivalent to the following. Let $\alpha_0$ be the diagonal matrix such that the first $p$ diagonal elements are equal to 1 and the last $n-p$ diagonal elements are equal to $-\lambda = -p/(n-p)$:

$$\alpha_0 = \mathrm{diag}(1, \ldots, 1, -\lambda, \ldots, -\lambda).$$

THEOREM 1.6. *Let*

$$g_s = \left(e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)}, \ldots, e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)}\right) \in \widetilde{G} = (SL(n,\mathbf{R}))^l \times (SL(n,\mathbf{C}))^m,$$

*where* $|\alpha_0| = \sqrt{2n^2p/(n-p)}$. *Then the following two conditions are equivalent.*
(1) *The trajectory* $\{\iota_{\boldsymbol{k}}(\Gamma)u_L g_s \mid s \geq 0\}$ *is relatively compact in* $\iota_{\boldsymbol{k}}(\Gamma)\backslash\widetilde{G}$.
(2) *The system of* $\boldsymbol{k}_M$*-forms induced from* $L$ *is badly approximable.*

Theorem 1.6 is a generalization of Dani's correspondence ([**6**, Theorem 2.20]), which coincides with the one proved in [**22**] by a different method. If $\boldsymbol{k} = \mathbf{Q}$ this is the original correspondence of Dani and if $n = 2$ this is Proposition 3.1 of [**10**]. Our geometric approach makes it clear how Theorem 1.1 relates to Dani's correspondence and its generalization.

Using this correspondence, Ly showed the following.

THEOREM 1.7 (cf. [**22**, Theorems 1.9, 1.10]). *The set $\mathcal{B}_{n,p,\boldsymbol{k}}$ has zero Lebesgue measure, when we identify $M(p, n-p; \boldsymbol{k}_M)$ with $\mathbf{R}^{dp(n-p)}$. Furthermore, $\mathcal{B}_{n,p,\boldsymbol{k}}$ is thick, and in particular has Hausdorff dimension $dp(n-p)$.*

We recall that the case $\boldsymbol{k} = \mathbf{Q}$ was established in [**20**], [**25**] and the case $n = 2$, $p = 1$ was proved in [**10**].

For $1 \leq j \leq p$, let $L'_j : (\boldsymbol{k}_M)^{n-p} \longrightarrow \boldsymbol{k}_M$ be the $\boldsymbol{k}_M$-form determined by the $j$th column of ${}^tL \in M(n-p, p; \boldsymbol{k}_M)$:

$$L'_j(\boldsymbol{y}) = \sum_{k=1}^{n-p} L_{jk} y_k \quad \text{for } \boldsymbol{y} = (y_1, \ldots, y_{n-p}) \in (\boldsymbol{k}_M)^{n-p}.$$

Let

$$\boldsymbol{L}'(\boldsymbol{y}) = (L'_1(\boldsymbol{y}), \ldots, L'_p(\boldsymbol{y})) \quad \text{for } \boldsymbol{y} \in (\boldsymbol{k}_M)^{n-p}.$$

THEOREM 1.8. *Let $n \geq 3$. The following two conditions are equivalent.*
(1) *The system of $\boldsymbol{k}_M$-forms $L_1, \ldots, L_{n-p}$ induced from the matrix $L \in M(p, n-p; \boldsymbol{k}_M)$ is badly approximable.*
(2) *The system of $\boldsymbol{k}_M$-forms $L'_1, \ldots, L'_p$ induced from the transpose ${}^tL \in M(n-p, p; \boldsymbol{k}_M)$ of $L$ is badly approximable.*

Theorem 1.8 is a new generalization of Khintchine's theorem (Theorem 5B of [**27**, Chapter IV]). This is shown by considering a family of horoballs $\{B(\gamma_*, \tau)\}_{\tau \geq 0}$ obtained by exchanging $\gamma^*$ with the geodesic ray $\gamma_*$ defined by the formula (7.1) in Section 7. In the case $\boldsymbol{k} = \mathbf{Q}$, a geometric interpretation of the transference principle is given in [**7**] by means of Busemann functions on the quotient space $SL(n, \mathbf{Z}) \backslash SL(n, \mathbf{R}) / SO(n)$ associated to two different geodesic rays.

So far we have seen that there are abundant badly approximable systems of $\boldsymbol{k}_M$-forms. However, it is another problem to find explicit examples of such systems. For this, it is already possible to use Theorem 6.5 of [**5**] in general (see Proposition 2.1), and results in [**18**] in the case $n = 2$. We present another geometric method to construct badly approximable systems of $\boldsymbol{k}_M$-forms by using Theorem 1.5.

Let $\boldsymbol{k}'$ be a number field of degree $d' = l' + 2m'$ with $l'$ real places and $m'$ complex places. We denote by $\iota'_1, \ldots, \iota'_{l'} : \boldsymbol{k}' \longrightarrow \mathbf{R}$ the real embeddings and $\iota'_{l'+1}, \ldots, \iota'_{l'+m'} : \boldsymbol{k}' \longrightarrow \mathbf{C}$ the complex embeddings which are not complex conjugate to each other. Let $\boldsymbol{k}'_M = \mathbf{R}^{l'} \times \mathbf{C}^{m'}$ be the Minkowski space associated to $\boldsymbol{k}'$. Suppose that $\boldsymbol{k}'$ is a subfield of $\boldsymbol{k}$. Then there exists a natural embedding of $\boldsymbol{k}'_M$ into $\boldsymbol{k}_M$, which can be extended to an embedding

$$\varphi_{\boldsymbol{k}', \boldsymbol{k}} : M(p, n-p; \boldsymbol{k}'_M) \longrightarrow M(p, n-p; \boldsymbol{k}_M)$$

(see Section 8 for precise definitions).

THEOREM 1.9. *Let $\boldsymbol{k} \neq \mathbf{Q}$. Suppose that the system of $\boldsymbol{k}'_M$-forms induced from a matrix $L' \in M(p, n-p; \boldsymbol{k}'_M)$ is badly approximable. Then the system of $\boldsymbol{k}_M$-forms induced from the matrix $\varphi_{\boldsymbol{k}', \boldsymbol{k}}(L') \in M(p, n-p; \boldsymbol{k}_M)$ is badly approximable.*

Let $\boldsymbol{k}' = \mathbf{Q}$. Then $\varphi_{\mathbf{Q}, \boldsymbol{k}}(L')$ induces a badly approximable system of $\boldsymbol{k}_M$-forms if $L' \in M(p, n-p; \mathbf{R})$ is a matrix which induces a badly approximable system of linear forms. Since $\mathcal{B}_{n,p,\mathbf{Q}}$ has the power of the continuum ([**25**]), we can construct uncountably many matrices in $\mathcal{B}_{n,p,\boldsymbol{k}}$. In particular, it is possible that we obtain a concrete example of badly approximable system of $\boldsymbol{k}_M$-forms when $L'$ is a matrix in $M(n, n-p; \mathbf{R})$ which induces one of Perron's examples ([**23**]) of badly approximable systems of linear forms (see Theorem 4B of [**27**, Chapter II]).

Finally we mention Diophantine approximation with weights in the setting of number fields. If $n = 2$, $l+m \geq 2$, it is possible to consider notion of weighted badly approximable vectors by using weighted norms on $\boldsymbol{k}_M$. In the case where $\boldsymbol{k}$ is a totally real number field, it was shown in [**1**] that the set of weighted badly approximable vectors with respect to a given weighted norm is thick, which is a generalization of the result in [**10**] for real quadratic fields.

This paper is organized as follows. In Section 2 we prove Theorem 1.2 and compare it with results of Burger ([**5**]). We collect necessary facts on symmetric spaces in Section 3. In Section 4 we prove Theorem 1.3 and find a basic correspondence. We prove Theorem 1.4 in Section 5. In Section 6 we describe a relatively compactness criterion and prove Theorem 1.5. We prove Theorem 1.8 in Section 7 and prove Theorem 1.9 in Section 8.

## 2. Convex body theorem.

In this section we prove Theorem 1.2 by using Minkowski's convex body theorem and compare it with results in Burger's paper [**5**].

We first remark that Theorem 7.1 of [**28**] and the standard argument as in the proof of Theorem 2A of [**27**, Chapter II] yield the following version of Minkowski's convex body theorem.

THEOREM 2.1. *Let $\Lambda$ be an $(nd)$-dimensional lattice in $\mathbf{R}^{nd}$ with fundamental domain $T$, and $\mathrm{vol}(T)$ the volume of $T$. Let $\mathcal{R}$ be a compact convex subset of $\mathbf{R}^{nd}$ with volume $\mathrm{vol}(\mathcal{R})$, which is symmetric about the origin $\mathbf{0} \in \mathbf{R}^{nd}$. If $\mathrm{vol}(\mathcal{R}) \geq 2^{nd}\mathrm{vol}(T)$, then $\mathcal{R}$ contains a non-zero point of $\Lambda$.*

PROOF OF THEOREM 1.2. Let $L = (L_{ij}) \in M(p, n-p; \boldsymbol{k}_M)$ and $L_{ij} = (L_{ij}^1, \ldots, L_{ij}^{l+m}) \in \boldsymbol{k}_M$ for each $i, j$. For $j = 1, \ldots, n-p$ and $q = 1, \ldots, l$, we define an $\mathbf{R}$-linear map $L_j^q : \mathbf{R}^p \longrightarrow \mathbf{R}$ by

$$L_j^q(\boldsymbol{x}) = \sum_{k=1}^{p} L_{kj}^q x_k \quad \text{for } \boldsymbol{x} = (x_1, \ldots, x_p) \in \mathbf{R}^p.$$

For $j = 1, \dots, n-p$ and $q = l+1, \dots, l+m$, we define an $\mathbf{R}$-linear map $L_j^q : \mathbf{C}^p \longrightarrow \mathbf{C}$ by

$$L_j^q(\boldsymbol{x}) = \sum_{k=1}^{p} L_{kj}^q x_k \quad \text{for } \boldsymbol{x} = (x_1, \dots, x_p) \in \mathbf{C}^p.$$

Let $Q > (C_{\boldsymbol{k}})^n$ and put

$$\delta = Q^{1/p}, \quad \varepsilon = \left\{ \frac{(C_{\boldsymbol{k}})^n}{Q} \right\}^{1/(n-p)} < 1.$$

Let

$$L^q = (L_{ij}^q) \in M(p, n-p; \mathbf{R})$$

for $q = 1, \dots, l$, and

$$L^q = (L_{ij}^q) \in M(p, n-p; \mathbf{C})$$

for $q = l+1, \dots, l+m$. For each $q$ we put

$$B_q = (b_{ij}^q) = \begin{pmatrix} \delta^{-1} I_p & O \\ \varepsilon^{-1}\, {}^t L^q & -\varepsilon^{-1} I_{n-p} \end{pmatrix},$$

where $I_k$ is the unit matrix of order $k$ for any positive integer $k$. Let

$$\mathcal{R}_q = \left\{ \boldsymbol{x} = (x_1, \dots, x_n) \in \mathbf{R}^n \;\middle|\; \left| \sum_{k=1}^{n} b_{ik}^q x_k \right| \le 1 \text{ for } i = 1, \dots, n \right\}$$

for $q = 1, \dots, l$, and

$$\mathcal{R}_q = \left\{ \boldsymbol{x} = (x_1, \dots, x_n) \in \mathbf{C}^n \;\middle|\; \left| \sum_{k=1}^{n} b_{ik}^q x_k \right| \le 1 \text{ for } i = 1, \dots, n \right\}$$

for $q = l+1, \dots, l+m$. Then the volume $\mathrm{vol}(\mathcal{R}_q)$ of $\mathcal{R}_q$ is given by

$$\mathrm{vol}(\mathcal{R}_q) = \begin{cases} 2^n \delta^p \varepsilon^{n-p} & \text{if } 1 \le q \le l, \\ \pi^n \delta^{2p} \varepsilon^{2(n-p)} & \text{if } l+1 \le q \le l+m. \end{cases}$$

We put

$$\mathcal{R} = \mathcal{R}_1 \times \cdots \times \mathcal{R}_{l+m} \subset (\mathbf{R}^n)^l \times (\mathbf{C}^n)^m.$$

Under the natural identification of $\mathbf{C}$ with $\mathbf{R}^2$, we regard $(\mathbf{R}^n)^l \times (\mathbf{C}^n)^m$ as $\mathbf{R}^{nl} \times \mathbf{R}^{2nm} = \mathbf{R}^{nd}$. Then $\mathcal{R}$ is a compact symmetric convex subset of $\mathbf{R}^{nd}$ and its volume is given by

$$\mathrm{vol}(\mathcal{R}) = \mathrm{vol}(\mathcal{R}_1) \times \cdots \times \mathrm{vol}(\mathcal{R}_{l+m}) = 2^{nl} \pi^{nm} (C_{\boldsymbol{k}})^{nd} = 2^{n(l+m)} |\Delta_{\boldsymbol{k}}|^{n/2}.$$

On the other hand, $\iota_{\boldsymbol{k}}((\mathcal{O}_{\boldsymbol{k}})^n)$ is a lattice in $(\boldsymbol{k}_M)^n = \mathbf{R}^{nd}$ because $\iota_{\boldsymbol{k}}(\mathcal{O}_{\boldsymbol{k}})$ is a lattice in $\boldsymbol{k}_M$. Let $\Lambda = \iota_{\boldsymbol{k}}((\mathcal{O}_{\boldsymbol{k}})^n)$ and $T$ a fundamental domain of $\Lambda$. Since the volume of a fundamental domain for $\iota_{\boldsymbol{k}}(\mathcal{O}_{\boldsymbol{k}})$ in $\boldsymbol{k}_M = \mathbf{R}^d$ is equal to $2^{-m}|\Delta_{\boldsymbol{k}}|^{1/2}$ (cf. [**28**, Theorem 9.4]), the volume of $T$ is given by

$$\mathrm{vol}(T) = \left\{2^{-m}|\Delta_{\boldsymbol{k}}|^{1/2}\right\}^n = 2^{-nm}|\Delta_{\boldsymbol{k}}|^{n/2}.$$

Hence $\mathrm{vol}(\mathcal{R}) = 2^{nd}\mathrm{vol}(T)$ and $\mathcal{R}$ contains a non-zero point $\iota_{\boldsymbol{k}}((\boldsymbol{a},\boldsymbol{b}))$ due to Theorem 2.1, where $\boldsymbol{a} = (a_1,\dots,a_p) \in (\mathcal{O}_{\boldsymbol{k}})^p$, $\boldsymbol{b} = (b_1,\dots,b_{n-p}) \in (\mathcal{O}_{\boldsymbol{k}})^{n-p}$. Then

$$(\iota_q(\boldsymbol{a}), \iota_q(\boldsymbol{b})) \in \mathcal{R}_q \quad \text{for } q = 1,\dots,l+m,$$

which means that

$$|\iota_q(a_i)| \le \delta = Q^{1/p}, \quad |L_j^q(\iota_q(\boldsymbol{a})) - \iota_q(b_j)| \le \varepsilon = \left\{\frac{(C_{\boldsymbol{k}})^n}{Q}\right\}^{1/(n-p)} \tag{2.1}$$

for $i = 1,\dots,p$; $j = 1,\dots,n-p$; and $q = 1,\dots,l+m$. Then we have

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\| = \max_{1\le i\le p}\|\iota_{\boldsymbol{k}}(a_i)\| = \max\bigl\{|\iota_q(a_i)| \bigm| 1\le i\le p,\ 1\le q\le l+m\bigr\} \le Q^{1/p} \tag{2.2}$$

and

$$\begin{aligned}\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) - \iota_{\boldsymbol{k}}(\boldsymbol{b})\| &= \max_{1\le j\le n-p}\|L_j(\iota_{\boldsymbol{k}}(\boldsymbol{a})) - \iota_{\boldsymbol{k}}(b_j)\| \\ &= \max\bigl\{|L_j^q(\iota_q(\boldsymbol{a})) - \iota_q(b_j)| \bigm| 1\le j\le n-p,\ 1\le q\le l+m\bigr\} \le \left\{\frac{(C_{\boldsymbol{k}})^n}{Q}\right\}^{1/(n-p)},\end{aligned} \tag{2.3}$$

which imply that

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^p\,\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) - \iota_{\boldsymbol{k}}(\boldsymbol{b})\|^{n-p} \le (C_{\boldsymbol{k}})^n < C.$$

If $\boldsymbol{a} = \mathbf{0}$, then it follows from (2.1) that $|\iota_q(b_j)| \le \varepsilon < 1$ for all $j$ and $q$. Then the norm $N_{\boldsymbol{k}}(b_j)$ of $b_j$ satisfies

$$|N_{\boldsymbol{k}}(b_j)| = \left|\prod_{q=1}^{l}\iota_q(b_j)\right|\left|\prod_{q=l+1}^{l+m}\iota_q(b_j)\right|^2 < 1,$$

which shows that $b_j = 0$ for $j = 1,\dots,n-p$, and $\boldsymbol{b} = \mathbf{0}$. This is a contradiction, and hence $\boldsymbol{a} \ne \mathbf{0}$. This proves (1).

Suppose that the assumption of (2) is satisfied. Since for fixed $Q$ there are only finitely many pairs $(\boldsymbol{a},\boldsymbol{b})$ with $\boldsymbol{a} \in (\mathcal{O}_{\boldsymbol{k}})^p - \{\mathbf{0}\}$, $\boldsymbol{b} \in (\mathcal{O}_{\boldsymbol{k}})^{n-p}$ satisfying (2.2) and (2.3), let $C'$ be the minimum of the numbers $\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) - \iota_{\boldsymbol{k}}(\boldsymbol{b})\|$ for such pairs. Then we have $C' > 0$. Let $Q'$ be a positive number such that

$$\left\{\frac{(C_{\boldsymbol{k}})^n}{Q'}\right\}^{1/(n-p)} < C'.$$

Then we can find a pair $(\boldsymbol{a}', \boldsymbol{b}')$ with $\boldsymbol{a}' \in (\mathcal{O}_{\boldsymbol{k}})^p - \{\boldsymbol{0}\}$, $\boldsymbol{b}' \in (\mathcal{O}_{\boldsymbol{k}})^{n-p}$ such that

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{a}')\| \leq (Q')^{1/p}, \quad \|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a}')) - \iota_{\boldsymbol{k}}(\boldsymbol{b}')\| \leq \left\{ \frac{(C_{\boldsymbol{k}})^n}{Q'} \right\}^{1/(n-p)}$$

in the same way. This pair is different from $(\boldsymbol{a}, \boldsymbol{b})$. Repeating this procedure, we obtain infinitely many distinct solutions $(\boldsymbol{x}, \boldsymbol{y}) \in (\mathcal{O}_{\boldsymbol{k}})^p \times (\mathcal{O}_{\boldsymbol{k}})^{n-p} - \{\boldsymbol{0}\} \times (\mathcal{O}_{\boldsymbol{k}})^{n-p}$ of the inequality (1.3). This proves (2). $\square$

We have also proved that the following holds.

THEOREM 2.2. *For every $L \in M(p, n-p; \boldsymbol{k}_M)$ and every real number $C > (C_{\boldsymbol{k}})^{n/p}$, there exist $\boldsymbol{x} \in (\mathcal{O}_{\boldsymbol{k}})^p - \{\boldsymbol{0}\}$, $\boldsymbol{y} \in (\mathcal{O}_{\boldsymbol{k}})^{n-p}$ such that*

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{x})\| \leq C, \quad \|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{x})) - \iota_{\boldsymbol{k}}(\boldsymbol{y})\| \leq (C_{\boldsymbol{k}})^{n/(n-p)} C^{-p/(n-p)}. \tag{2.4}$$

REMARK 2.1. In the proof of Theorem 1.2, the existence of $\boldsymbol{a} = (a_1, \ldots, a_p) \in (\mathcal{O}_{\boldsymbol{k}})^p - \{\boldsymbol{0}\}$ and $\boldsymbol{b} = (b_1, \ldots, b_{n-p}) \in (\mathcal{O}_{\boldsymbol{k}})^{n-p}$ satisfying (2.1) corresponds to Lemma 5.1 of [**5**] in the case where $S$ is the set $S_0$ of all places of $\boldsymbol{k}$ lying over infinity.

For any positive integer $r$, let

$$h_{S_0}(\boldsymbol{x}) = \left\{ \prod_{q=1}^{l} \max_{1 \leq i \leq r} |\iota_q(x_i)|^{1/d} \right\} \left\{ \prod_{q=l+1}^{l+m} \max_{1 \leq i \leq r} |\iota_q(x_i)|^{2/d} \right\}$$

be the $S_0$-height of $\boldsymbol{x}$ for $\boldsymbol{x} = (x_1, \ldots, x_r) \in \boldsymbol{k}^r$. For $\boldsymbol{a} = (a_1, \ldots, a_p) \in \boldsymbol{k}^p$ and $\boldsymbol{b} = (b_1, \ldots, b_{n-p}) \in \boldsymbol{k}^{n-p}$, we regard $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{k}^n$ and let

$$h_{S_0}(\boldsymbol{a}, \boldsymbol{b}) = h_{S_0}((\boldsymbol{a}, \boldsymbol{b}))$$

be the $S_0$-height of $\boldsymbol{a}$ and $\boldsymbol{b}$. We also put

$$\begin{aligned} &H(\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) - \iota_{\boldsymbol{k}}(\boldsymbol{b})) \\ &\quad = \left\{ \prod_{q=1}^{l} \max_{1 \leq j \leq n-p} |L_j^q(\iota_q(\boldsymbol{a})) - \iota_q(b_j)|^{1/d} \right\} \left\{ \prod_{q=l+1}^{l+m} \max_{1 \leq j \leq n-p} |L_j^q(\iota_q(\boldsymbol{a})) - \iota_q(b_j)|^{2/d} \right\}. \end{aligned}$$

Then it follows from (2.1) that the following holds, which is equivalent to [**5**, Theorem 5.2] in the case where $S$ is the set $S_0$ of all places of $\boldsymbol{k}$ lying over infinity, under the natural identification

$$M(p, n-p; \boldsymbol{k}_M) = M(p, n-p; \mathbf{R})^l \times M(p, n-p; \mathbf{C})^m. \tag{2.5}$$

THEOREM 2.3. *For every $L \in M(p, n-p; \boldsymbol{k}_M)$ and every real number $C > (C_{\boldsymbol{k}})^{n/p}$, there exist $\boldsymbol{x} \in (\mathcal{O}_{\boldsymbol{k}})^p - \{\boldsymbol{0}\}$, $\boldsymbol{y} \in (\mathcal{O}_{\boldsymbol{k}})^{n-p}$ such that*

$$h_{S_0}(\boldsymbol{x}) \leq C, \quad H(\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{x})) - \iota_{\boldsymbol{k}}(\boldsymbol{y})) \leq (C_{\boldsymbol{k}})^{n/(n-p)} C^{-p/(n-p)}. \tag{2.6}$$

By the inequality of arithmetic and geometric means we have

$$\begin{aligned} h_{S_0}(\boldsymbol{x}) &\le \frac{1}{d}\left\{\sum_{q=1}^{l}\max_{1\le i\le p}|\iota_q(x_i)| + 2\sum_{q=l+1}^{l+m}\max_{1\le i\le p}|\iota_q(x_i)|\right\} \\ &\le \max\bigl\{|\iota_q(x_i)| \bigm| 1\le i\le p,\ 1\le q\le l+m\bigr\} = \|\iota_{\boldsymbol{k}}(\boldsymbol{x})\| \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &H(\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{x})) - \iota_{\boldsymbol{k}}(\boldsymbol{y})) \\ &\le \frac{1}{d}\left\{\sum_{q=1}^{l}\max_{1\le j\le n-p}|L_j^q(\iota_q(\boldsymbol{x})) - \iota_q(y_j)| + 2\sum_{q=l+1}^{l+m}\max_{1\le j\le n-p}|L_j^q(\iota_q(\boldsymbol{x})) - \iota_q(y_j)|\right\} \\ &\le \max\bigl\{|L_j^q(\iota_q(\boldsymbol{x})) - \iota_q(y_j)| \bigm| 1\le j\le n-p,\ 1\le q\le l+m\bigr\} = \|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{x})) - \iota_{\boldsymbol{k}}(\boldsymbol{y})\| \end{aligned} \tag{2.8}$$

for $\boldsymbol{x} = (x_1,\dots,x_p) \in \boldsymbol{k}^p$, $\boldsymbol{y} = (y_1,\dots,y_{n-p}) \in \boldsymbol{k}^{n-p}$. Therefore Burger's inequality (2.6) follows from (2.4), although both of them follow from the same inequalities (2.1).

We remark that Burger's definition of 'badly approximable $S_0$-systems of linear forms' ([**5**, p.237]) is rephrased as follows under the identification (2.5):

DEFINITION 2.1. Let $L \in M(p, n-p; \boldsymbol{k}_M)$. Then we say ${}^tL \in M(n-p, p; \mathbf{R})^l \times M(n-p, p; \mathbf{C})^m$ is a badly approximable $S_0$-system of linear forms (of dimension $(n-p)\times p$) if there exists a constant $C(\boldsymbol{k}, L) > 0$ depending only on $\boldsymbol{k}$ and $L$ such that

$$h_{S_0}(\boldsymbol{x}, \boldsymbol{y})^p H(\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{x})) - \iota_{\boldsymbol{k}}(\boldsymbol{y}))^{n-p} > C(\boldsymbol{k}, L)$$

for every $\boldsymbol{x} \in (\mathcal{O}_{\boldsymbol{k}})^p - \{\mathbf{0}\}$ and $\boldsymbol{y} \in (\mathcal{O}_{\boldsymbol{k}})^{n-p}$.

PROPOSITION 2.1. *The system of $\boldsymbol{k}_M$-forms induced from $L \in M(p, n-p; \boldsymbol{k}_M)$ is badly approximable if ${}^tL$ is a badly approximable $S_0$-system of linear forms in the sense of Burger.*

PROOF. Let $L \in M(p, n-p; \boldsymbol{k}_M)$ and suppose that ${}^tL$ is a badly approximable $S_0$-system of linear forms. Then $\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{x}))$ does not belong to $(\iota_{\boldsymbol{k}}(\mathcal{O}_{\boldsymbol{k}}))^{n-p}$ for any $\boldsymbol{x} \in (\mathcal{O}_{\boldsymbol{k}})^p - \{\mathbf{0}\}$ and $\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{x})) - \iota_{\boldsymbol{k}}(\boldsymbol{y})\| > 0$ for such $\boldsymbol{x}$ and any $\boldsymbol{y} \in (\mathcal{O}_{\boldsymbol{k}})^{n-p}$.

Suppose that $\boldsymbol{a} = (a_1,\dots,a_p) \in (\mathcal{O}_{\boldsymbol{k}})^p - \{\mathbf{0}\}$ and $\boldsymbol{b} = (b_1,\dots,b_{n-p}) \in (\mathcal{O}_{\boldsymbol{k}})^{n-p}$.

If $\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) - \iota_{\boldsymbol{k}}(\boldsymbol{b})\| \ge 1$, then we have

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^p\,\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) - \iota_{\boldsymbol{k}}(\boldsymbol{b})\|^{n-p} \ge \|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^p \ge 1 > 0.$$

Suppose that $\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) - \iota_{\boldsymbol{k}}(\boldsymbol{b})\| < 1$. Then we have

$$|L_j^q(\iota_q(\boldsymbol{a})) - \iota_q(b_j)| < 1 \tag{2.9}$$

for $j = 1,\dots,n-p$ and $q = 1,\dots,l+m$. Let

$$C' = \max\bigl\{|L_{ij}^q| \bigm| 1\le i\le p;\ 1\le j\le n-p;\ 1\le q\le l+m\bigr\}$$

and $C'' = p\,C' + 1$. Since

$$|L_j^q(\iota_q(\boldsymbol{a}))| = \left|\sum_{k=1}^{p} L_{kj}^q \iota_q(a_k)\right| \leq \sum_{k=1}^{p} \left|L_{kj}^q\right| |\iota_q(a_k)| \leq \|\iota_{\boldsymbol{k}}(\boldsymbol{a})\| \sum_{k=1}^{p} \left|L_{kj}^q\right| \leq p\,C' \|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|,$$

it follows from (2.9) that

$$|\iota_q(b_j)| < 1 + |L_j^q(\iota_q(\boldsymbol{a}))| < 1 + p\,C' \|\iota_{\boldsymbol{k}}(\boldsymbol{a})\| \leq (p\,C' + 1)\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\| = C'' \|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|$$

for each $j$ and $q$. This and (2.7) show that

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{b})\| < C'' \|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|$$

and

$$h_{S_0}(\boldsymbol{a}, \boldsymbol{b}) = h_{S_0}((\boldsymbol{a}, \boldsymbol{b})) \leq \|\iota_{\boldsymbol{k}}((\boldsymbol{a}, \boldsymbol{b}))\| \leq \max\{\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|, \|\iota_{\boldsymbol{k}}(\boldsymbol{b})\|\} < C'' \|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|.$$

Since ${}^tL$ is a badly approximable $S_0$-system of linear forms, we have

$$h_{S_0}(\boldsymbol{a}, \boldsymbol{b})^p H(\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) - \iota_{\boldsymbol{k}}(\boldsymbol{b}))^{n-p} > C(\boldsymbol{k}, L) > 0$$

for a constant $C(\boldsymbol{k}, L)$ depending only on $\boldsymbol{k}$ and $L$. Then it follows from (2.8) that

$$(C'')^p \|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^p \|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) - \iota_{\boldsymbol{k}}(\boldsymbol{b})\|^{n-p} \geq h_{S_0}(\boldsymbol{a}, \boldsymbol{b})^p H(\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{x})) - \iota_{\boldsymbol{k}}(\boldsymbol{y}))^{n-p} > C(\boldsymbol{k}, L).$$

Let

$$C''' = \min\{1,\ C(\boldsymbol{k}, L)/(C'')^p\} > 0.$$

Then we have

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{x})\|^p \|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{x})) - \iota_{\boldsymbol{k}}(\boldsymbol{y})\|^{n-p} \geq C'''$$

for any $\boldsymbol{x} \in (\mathcal{O}_{\boldsymbol{k}})^p - \{\boldsymbol{0}\}$ and $\boldsymbol{y} \in (\mathcal{O}_{\boldsymbol{k}})^{n-p}$. □

## 3. Geometry of $(SL(n, \mathbf{R})/SO(n))^l \times (SL(n, \mathbf{C})/SU(n))^m$.

Let $n \geq 2$, $G = SL(n, \mathbf{R})$ and $K = SO(n)$. Let $V = G/K$ and denote by $x_0$ the coset of the identity element of $G$. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of $G$ and $K$, respectively, and by $\mathfrak{p}$ the set of all real symmetric matrices of order $n$ with trace 0. Then the direct sum decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition. The tangent space of $V$ at $x_0$ is naturally identified with $\mathfrak{p}$ through the differential at the identity element of the projection $G \longrightarrow V$. The inner product $\langle\ ,\ \rangle$ on $\mathfrak{p}$ defined by the Killing form of $\mathfrak{g}$ is

$$\langle X, Y\rangle = 2n \cdot \mathrm{trace}(XY) \quad \text{for} \ \ X, Y \in \mathfrak{p}. \tag{3.1}$$

It can be extended to a left $G$-invariant Riemannian metric on $V$. Then $V$ equipped with the resulting metric is a symmetric space of noncompact type (see [**8**], [**17**] for more details on symmetric spaces of noncompact type).

Similarly, let $\widehat{G} = SL(n, \mathbf{C})$ and $\widehat{K} = SU(n)$. Let $\widehat{V} = \widehat{G}/\widehat{K}$ and denote by $\widehat{x}_0$ the coset of the identity element of $\widehat{G}$. We denote by $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{k}}$ the Lie algebras of $\widehat{G}$ and $\widehat{K}$, respectively, and by $\widehat{\mathfrak{p}}$ the set of all Hermitian matrices of order $n$ with trace 0. Then the direct sum decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{k}} + \widehat{\mathfrak{p}}$ is a Cartan decomposition. The tangent space of $\widehat{V}$ at $\widehat{x}_0$ is naturally identified with $\widehat{\mathfrak{p}}$ through the differential at the identity element of the projection $\widehat{G} \longrightarrow \widehat{V}$. The inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\widehat{\mathfrak{p}}$ defined by the Killing form of $\widehat{\mathfrak{g}}$ is

$$\langle\!\langle X, Y \rangle\!\rangle = 4n \cdot \mathrm{trace}(XY) \quad \text{for} \quad X, Y \in \widehat{\mathfrak{p}}. \tag{3.2}$$

It can be extended to a left $\widehat{G}$-invariant Riemannian metric on $\widehat{V}$. Then $\widehat{V}$ equipped with the resulting metric is a symmetric space of noncompact type.

The Riemannian product $\widetilde{V} = V^l \times \widehat{V}^m$ as well as $V$ and $\widehat{V}$ are Hadamard manifolds, that is, simply connected, complete Riemannian manifolds of nonpositive sectional curvature. Let $d_{\widetilde{V}}$ be the distance on $\widetilde{V}$ induced from the product metric on $\widetilde{V}$. The group $\Gamma = SL(n, \mathcal{O}_{\boldsymbol{k}})$ acts isometrically on $\widetilde{V}$ by (1.7) and the volume of the quotient space $\iota_{\boldsymbol{k}}(\Gamma)\backslash\widetilde{V}$ is finite.

Let $W$ be a Hadamard manifold and $d_W$ the distance on $W$ induced from the Riemannian metric of $W$. A geodesic $\gamma : [0, \infty) \longrightarrow W$ is a geodesic ray if it realizes the distance between any two points on it: $d_W(\gamma(s), \gamma(s')) = |s - s'|$ for any $s, s' \geq 0$. Any unit speed geodesic $[0, \infty) \longrightarrow W$ is a geodesic ray.

Definition 3.1 (cf. [**8**], [**2**], [**9**]). (1) Let $\gamma : [0, \infty) \longrightarrow W$ be a geodesic ray. Then the function $b(\gamma)$ on $W$ defined by

$$b(\gamma)(x) = \lim_{s\to\infty} \{d_W(x, \gamma(s)) - s\} \quad \text{for} \quad x \in W$$

is called the Busemann function associated to $\gamma$.
(2) For any geodesic ray $\gamma$ and any real number $\tau$, the set $B(\gamma, \tau) = b(\gamma)^{-1}(-\infty, -\tau)$ is called a horoball in $W$.

For an isometry $g$ of $W$ and $x \in W$, we denote by $g \cdot x$ the image of $x$ under $g$. We use the similar notation for subsets of $W$ and geodesics in $W$. Then we have the following for any isometry $g$ of $W$ and $\tau \in \mathbf{R}$.

$$b(g \cdot \gamma)(x) = b(\gamma)(g^{-1} \cdot x) \quad \text{for} \quad x \in W, \tag{3.3}$$

$$g \cdot B(\gamma, \tau) = B(g \cdot \gamma, \tau). \tag{3.4}$$

Proposition 3.1 (Proposition 1.10.2.(4) of [**8**, Chapter I]). *Let $\gamma$ be a geodesic ray in $W$. Then we have*

$$|b(\gamma)(x) - b(\gamma)(y)| \leq d_W(x, y) \quad \textit{for any} \quad x, y \in W.$$

Let

$$A = \left\{a = \mathrm{diag}(a_1, \ldots, a_n) \in G \;\middle|\; a_1, \ldots, a_n > 0\right\} \tag{3.5}$$

and let $N$ be the subgroup of $G$ consisting of all the upper triangular matrices with diagonal elements 1. Then $G = NAK$ is an Iwasawa decomposition of $G$ and

$$V = NA \cdot x_0 = \{va \cdot x_0 \mid v \in N,\, a \in A\}.$$

For any $g \in G$, $gA \cdot x_0$ is a totally geodesically embedded Euclidean space in $V$.

Let $\widehat{A} = A$ and let $\widehat{N}$ be the subgroup of $\widehat{G}$ consisting of all the upper triangular matrices with diagonal elements 1. Then $\widehat{G} = \widehat{N}\widehat{A}\widehat{K}$ is an Iwasawa decomposition of $\widehat{G}$ and

$$\widehat{V} = \widehat{N}\widehat{A} \cdot \widehat{x}_0 = \left\{va \cdot \widehat{x}_0 \,\middle|\, v \in \widehat{N},\, a \in \widehat{A}\right\}.$$

For any $g \in \widehat{G}$, $g\widehat{A} \cdot \widehat{x}_0$ is a totally geodesically embedded Euclidean space in $\widehat{V}$.

Let

$$\mathfrak{a} = \widehat{\mathfrak{a}} = \{\alpha = \mathrm{diag}(\alpha_1, \ldots, \alpha_n) \mid \alpha_1, \ldots, \alpha_n \in \mathbf{R};\; \alpha_1 + \cdots + \alpha_n = 0\}.$$

The Lie algebra $\mathfrak{a}$ of $A$ is a maximal abelian subspace of $\mathfrak{p}$. For any $\alpha \in \mathfrak{a} - \{0\}$ the map $\gamma_\alpha : [0, \infty) \longrightarrow V$ defined by

$$\gamma_\alpha(s) = \left(\exp s\frac{\alpha}{|\alpha|}\right) \cdot x_0 = e^{s\alpha/|\alpha|} \cdot x_0 = e^{s\alpha/|\alpha|}K$$

is a geodesic ray in $V$, where $\exp : \mathfrak{g} \longrightarrow G$ is the exponential mapping and $|\alpha| = \sqrt{\langle\alpha,\, \alpha\rangle}$. Any geodesic ray in $V$ is of the form $g \cdot \gamma_\alpha$ for some $g \in G$ and $\alpha \in \mathfrak{a} - \{0\}$ (see Theorem 6.2 of [**17**, Chapter V]).

Similarly, for any $\alpha \in \widehat{\mathfrak{a}} - \{0\}$ the map $\widehat{\gamma}_\alpha : [0, \infty) \longrightarrow \widehat{V}$ defined by

$$\widehat{\gamma}_\alpha(s) = \left(\exp s\frac{\alpha}{\|\alpha\|}\right) \cdot \widehat{x}_0 = e^{s\alpha/\|\alpha\|} \cdot \widehat{x}_0 = e^{s\alpha/\|\alpha\|}\widehat{K}$$

is a geodesic ray in $\widehat{V}$, where $\exp : \widehat{\mathfrak{g}} \longrightarrow \widehat{G}$ is the exponential mapping and $\|\alpha\| = \sqrt{\langle\!\langle\alpha,\, \alpha\rangle\!\rangle}$. Any geodesic ray in $\widehat{V}$ is of the form $g \cdot \widehat{\gamma}_\alpha$ for some $g \in \widehat{G}$ and $\alpha \in \widehat{\mathfrak{a}} - \{0\}$.

For any $\alpha \in \mathfrak{a} - \{0\} = \widehat{\mathfrak{a}} - \{0\}$, $\gamma_\alpha([0, \infty)) \subset A \cdot x_0$ and $A \cdot x_0$ is isometric to the Euclidean space $\mathfrak{a}$ equipped with the metric $\langle\ ,\ \rangle$ induced from the Killing form of the Lie algebra of $G$. Similarly, $\widehat{\gamma}_\alpha([0, \infty)) \subset \widehat{A} \cdot \widehat{x}_0$ and $\widehat{A} \cdot \widehat{x}_0$ is isometric to the Euclidean space $\widehat{\mathfrak{a}}$ equipped with the metric $\langle\!\langle\ ,\ \rangle\!\rangle$ induced from the Killing form of the Lie algebra of $\widehat{G}$. Hence a direct computation in $\mathbf{R}^{n-1}$ gives the following.

LEMMA 3.1. *Let $\alpha \in \mathfrak{a} - \{0\} = \widehat{\mathfrak{a}} - \{0\}$. Then*

$$b(\gamma_\alpha)(e^\beta \cdot x_0) = \left\langle \frac{\alpha}{|\alpha|},\, \beta \right\rangle \quad \textit{for}\ \ \beta \in \mathfrak{a} \tag{3.6}$$

*and*

$$b(\widehat{\gamma}_\alpha)(e^\beta \cdot \widehat{x}_0) = \left\langle\!\!\left\langle \frac{\alpha}{\|\alpha\|},\, \beta \right\rangle\!\!\right\rangle \quad \textit{for}\ \ \beta \in \widehat{\mathfrak{a}}. \tag{3.7}$$

For each $i = 1, \dots, n-1$, we define a linear function $\theta_i$ on $\mathfrak{a} = \widehat{\mathfrak{a}}$ by

$$\theta_i(\alpha) = \alpha_i - \alpha_{i+1} \quad \text{for} \;\; \alpha = \mathrm{diag}(\alpha_1, \dots, \alpha_n) \in \mathfrak{a} = \widehat{\mathfrak{a}}.$$

Let

$$\begin{aligned} \mathfrak{a}^+ &= \{\alpha \in \mathfrak{a} \,|\, \theta_i(\alpha) \geq 0 \quad \text{for} \;\; i = 1, \dots, n-1\} \\ &= \{\mathrm{diag}(\alpha_1, \dots, \alpha_n) \in \mathfrak{a} \,|\, \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n\} \end{aligned}$$

and $\widehat{\mathfrak{a}}^+ = \mathfrak{a}^+$. Then $\mathfrak{a}^+$ is a closed Weyl chamber of $\mathfrak{a}$ and $\widehat{\mathfrak{a}}^+$ is a closed Weyl chamber of $\widehat{\mathfrak{a}}$.

LEMMA 3.2 (cf. [**14**, Lemma 2-3] and [**16**, Lemma 5.1]). *Let $\alpha \in \mathfrak{a}^+ - \{0\} = \widehat{\mathfrak{a}}^+ - \{0\}$. Then the Busemann function $b(\gamma_\alpha)$ is invariant under the action of the group $N$ on $V$ and the Busemann function $b(\widehat{\gamma}_\alpha)$ is invariant under the action of the group $\widehat{N}$ on $\widehat{V}$.*

Let $P(n, \mathbf{R})$ be the set of all positive definite, real symmetric matrices contained in $G$. The group $G$ acts transitively on $P(n, \mathbf{R})$ as follows.

$$g \cdot x = gx\,{}^t g \quad \text{for} \;\; x \in P(n, \mathbf{R}), \;\; g \in G.$$

Since the isotropy group at the unit matrix is $K$, $P(n, \mathbf{R})$ is diffeomorphic to $V = G/K$. From now on we identify $V$ with $P(n, \mathbf{R})$. Under this identification, $x_0$ is the unit matrix $I_n$. Similarly, let $P(n, \mathbf{C})$ be the set of all positive definite, Hermitian matrices contained in $\widehat{G}$. The group $\widehat{G}$ acts transitively on $P(n, \mathbf{C})$ as follows.

$$g \cdot x = gx\,{}^t \overline{g} \quad \text{for} \;\; x \in P(n, \mathbf{C}), \;\; g \in \widehat{G}.$$

Since the isotropy group at the unit matrix is $\widehat{K}$, $P(n, \mathbf{C})$ is diffeomorphic to $\widehat{V} = \widehat{G}/\widehat{K}$. From now on we identify $\widehat{V}$ with $P(n, \mathbf{C})$. Under this identification, $\widehat{x}_0$ is the unit matrix $I_n$.

We remark that $P(n, \mathbf{R}) \subset P(n, \mathbf{C})$. For $z \in P(n, \mathbf{C})$ we denote by $\square_j(z)$ the $(j \times j)$-minor determinant in the lower right corner of $z$. For a permutation $\sigma$ of $n$ letters and $z = (z_{ij}) \in P(n, \mathbf{C})$, let

$$\sigma \cdot z = (z_{\sigma(i)\sigma(j)}) = \begin{pmatrix} z_{\sigma(1)\sigma(1)} & z_{\sigma(1)\sigma(2)} & \cdots & z_{\sigma(1)\sigma(n)} \\ z_{\sigma(2)\sigma(1)} & z_{\sigma(2)\sigma(2)} & \cdots & z_{\sigma(2)\sigma(n)} \\ \vdots & \vdots & & \vdots \\ z_{\sigma(n)\sigma(1)} & z_{\sigma(n)\sigma(2)} & \cdots & z_{\sigma(n)\sigma(n)} \end{pmatrix}.$$

LEMMA 3.3 (cf. [**14**, Lemma 2-5] and [**16**, Lemma 5.3]). *Let $\alpha = \mathrm{diag}(\alpha_1, \dots, \alpha_n) \in \mathfrak{a} - \{0\} = \widehat{\mathfrak{a}} - \{0\}$. If we take a permutation $\sigma$ such that*

$$\alpha_{\sigma(1)} \geq \alpha_{\sigma(2)} \geq \cdots \geq \alpha_{\sigma(n)},$$

*then we have*

$$b(\gamma_\alpha)(z) = \frac{n}{|\alpha|} \log \left\{ \prod_{k=1}^{n-1} \square_k(\sigma \cdot z)^{\alpha_{\sigma(n-k)} - \alpha_{\sigma(n-k+1)}} \right\} \tag{3.8}$$

*for* $z \in P(n, \mathbf{R})$ *and*

$$b(\widehat{\gamma}_\alpha)(z) = \frac{2n}{\|\alpha\|} \log \left\{ \prod_{k=1}^{n-1} \square_k(\sigma \cdot z)^{\alpha_{\sigma(n-k)} - \alpha_{\sigma(n-k+1)}} \right\} \tag{3.9}$$

*for* $z \in P(n, \mathbf{C})$.

Let $\alpha \in \mathfrak{a}^+$ and define a geodesic $\widetilde{\gamma}_\alpha : [0, \infty) \longrightarrow \widetilde{V} = V^l \times \widehat{V}^m$ by

$$\widetilde{\gamma}_\alpha(s) = \left( \gamma_\alpha \left( \frac{s}{\sqrt{d}} \right), \ldots, \gamma_\alpha \left( \frac{s}{\sqrt{d}} \right), \widehat{\gamma}_\alpha \left( \frac{\sqrt{2} s}{\sqrt{d}} \right), \ldots, \widehat{\gamma}_\alpha \left( \frac{\sqrt{2} s}{\sqrt{d}} \right) \right) \tag{3.10}$$

for $s \geq 0$. In the right hand side of (3.10), the first $l$ entries are equal to

$$\gamma_\alpha \left( \frac{s}{\sqrt{d}} \right) = e^{s\alpha/(\sqrt{d}\,|\alpha|)} \cdot x_0$$

and the last $m$ entries are equal to

$$\widehat{\gamma}_\alpha \left( \frac{\sqrt{2}\, s}{\sqrt{d}} \right) = e^{\sqrt{2}\, s\alpha/(\sqrt{d}\,\|\alpha\|)} \cdot \widehat{x}_0 = e^{s\alpha/(\sqrt{d}\,|\alpha|)} \cdot \widehat{x}_0.$$

Since

$$l \cdot \left( \frac{1}{\sqrt{d}} \right)^2 + m \cdot \left( \frac{\sqrt{2}}{\sqrt{d}} \right)^2 = 1,$$

$\widetilde{\gamma}_\alpha$ is a unit speed geodesic. From [**2**, Section 3.8] we have

$$b(\widetilde{\gamma}_\alpha)(z_1, \ldots, z_{l+m}) = \frac{1}{\sqrt{d}} \left\{ \sum_{j=1}^{l} b(\gamma_\alpha)(z_j) \right\} + \frac{\sqrt{2}}{\sqrt{d}} \left\{ \sum_{j=l+1}^{l+m} b(\widehat{\gamma}_\alpha)(z_j) \right\} \tag{3.11}$$

for $z_1, \ldots, z_l \in P(n, \mathbf{R})$ and $z_{l+1}, \ldots, z_{l+m} \in P(n, \mathbf{C})$.

Let $\widetilde{G} = G^l \times \widehat{G}^m$, $\widetilde{K} = K^l \times \widehat{K}^m$,

$$\widetilde{A} = A^l \times A^m = A^l \times \widehat{A}^m \subset \widetilde{G}$$

and $\widetilde{N} = N^l \times \widehat{N}^m$. Then we have $V^l \times \widehat{V}^m = \widetilde{N}\widetilde{A} \cdot z_0$, where

$$z_0 = (x_0, \ldots, x_0, \widehat{x}_0, \ldots, \widehat{x}_0) \in V^l \times \widehat{V}^m.$$

Let

$$\widetilde{A}_{\mathbf{Q}} = \left\{ (a, \ldots, a) \in \widetilde{A} \,\middle|\, a \in A = \widehat{A} \right\}$$

and

$$\begin{aligned} \widetilde{A}' = \Big\{ \big(\operatorname{diag}\big(b_1^1, \ldots, b_n^1\big), \ldots, \operatorname{diag}\big(b_1^{l+m}, \ldots, b_n^{l+m}\big)\big) \in \widetilde{A} \\ \Big|\, \big(b_j^1 \cdots b_j^l\big)\big(b_j^{l+1} \cdots b_j^{l+m}\big)^2 = 1 \quad \text{for } j = 1, \ldots, n \Big\}. \end{aligned}$$

Then $\widetilde{A} = \widetilde{A}'\widetilde{A}_{\mathbf{Q}}$. Using the method 'restriction of scalars' (cf. Proposition 6.1.3 and Corollary 6.1.4 of [**29**]), one can find an algebraic group $\boldsymbol{G}$ defined over $\mathbf{Q}$ such that the group $\boldsymbol{G}_{\mathbf{R}}$ of real points of $\boldsymbol{G}$ is isomorphic to $\widetilde{G}$ and the group $\boldsymbol{G}_{\mathbf{Z}}$ of integral points of $\boldsymbol{G}$ is isomorphic to $\varGamma = SL(n, \mathcal{O}_{\boldsymbol{k}})$. Then $\widetilde{A}_{\mathbf{Q}}$ is isomorphic to the topological identity component of the group of real points of a maximal $\mathbf{Q}$-split torus of $\boldsymbol{G}$ and $\widetilde{A}$ is isomorphic to the topological identity component of the group of real points of a maximal $\mathbf{R}$-split torus of $\boldsymbol{G}$. The Lie algebra $\widetilde{\mathfrak{a}}_{\mathbf{Q}}$ of $\widetilde{A}_{\mathbf{Q}}$ and the Lie algebra $\widetilde{\mathfrak{a}}'$ of $\widetilde{A}'$ are orthogonal to each other with respect to the product metric of $(\mathfrak{a})^l \times (\widehat{\mathfrak{a}})^m$.

If $\alpha \in \mathfrak{a} - \{0\}$ and $\widetilde{\beta} = (\beta_1, \ldots, \beta_{l+m}) \in \widetilde{\mathfrak{a}}'$, then it follows from (3.11) that

$$\begin{aligned} &b(\widetilde{\gamma}_\alpha)((e^{\beta_1} \cdot x_0, \ldots, e^{\beta_l} \cdot x_0, e^{\beta_{l+1}} \cdot \widehat{x}_0, \ldots, e^{\beta_{l+m}} \cdot \widehat{x}_0)) \\ &\quad = \frac{1}{\sqrt{d}} \left\{ -\sum_{j=1}^{l} \left\langle \frac{\alpha}{|\alpha|}, \beta_j \right\rangle \right\} + \frac{\sqrt{2}}{\sqrt{d}} \left\{ -\sum_{j=l+1}^{l+m} \left\langle\!\!\left\langle \frac{\alpha}{\|\alpha\|}, \beta_j \right\rangle\!\!\right\rangle \right\} \\ &\quad = -\frac{1}{\sqrt{d}} \left\langle \frac{\alpha}{|\alpha|}, \sum_{j=1}^{l} \beta_j + 2 \sum_{j=l+1}^{l+m} \beta_j \right\rangle \end{aligned}$$

From this and Lemma 3.2 we have the following, because $\widetilde{V} = \widetilde{N}\widetilde{A}'\widetilde{A}_{\mathbf{Q}} \cdot z_0$ and $\widetilde{A}'$ normalizes $\widetilde{N}$.

Lemma 3.4. *If $\alpha \in \mathfrak{a}^+ - \{0\} = \widehat{\mathfrak{a}}^+ - \{0\}$, then*

$$b(\widetilde{\gamma}_\alpha)(ua' \cdot z) = b(\widetilde{\gamma}_\alpha)(z) \quad \text{for } z \in \widetilde{V},\, u \in \widetilde{N} \text{ and } a' \in \widetilde{A}'. \tag{3.12}$$

Let $\alpha^*$ be the diagonal matrix in $\mathfrak{a}^+ = \widehat{\mathfrak{a}}^+$ such that the first $n-1$ diagonal elements are equal to 1:

$$\alpha^* = \operatorname{diag}(1, \ldots, 1, -(n-1)).$$

Then we have

$$|\alpha^*| = \sqrt{2n^2(n-1)}, \quad \|\alpha^*\| = \sqrt{4n^2(n-1)}.$$

We define a geodesic ray $\gamma^* : [0, \infty) \longrightarrow V^l \times \widehat{V}^m$ by

$$\gamma^*(s) = \widetilde{\gamma}_{\alpha^*}(s) = \left( \gamma_{\alpha^*}\left(\frac{s}{\sqrt{d}}\right), \ldots, \gamma_{\alpha^*}\left(\frac{s}{\sqrt{d}}\right),\, \widehat{\gamma}_{\alpha^*}\left(\frac{\sqrt{2}s}{\sqrt{d}}\right), \ldots, \widehat{\gamma}_{\alpha^*}\left(\frac{\sqrt{2}s}{\sqrt{d}}\right) \right) \tag{3.13}$$

for $s \geq 0$.

It follows from Lemma 3.3 that

$$b(\gamma_{\alpha^*})(z) = \frac{n}{|\alpha^*|}\log\{\Box_1(z)\}^n = \frac{n}{\sqrt{2(n-1)}}\log\Box_1(z) \tag{3.14}$$

for $z=(z_{ij})\in P(n,\mathbf{R})$ and

$$b(\widehat{\gamma}_{\alpha^*})(z) = \frac{2n}{\|\alpha^*\|}\log\{\Box_1(z)\}^n = \frac{n}{\sqrt{n-1}}\log\Box_1(z) \tag{3.15}$$

for $z=(z_{ij})\in P(n,\mathbf{C})$. From (3.11), (3.14) and (3.15) we have

$$\begin{aligned} b(\gamma^*)(z_1,\dots,z_{l+m}) &= \frac{1}{\sqrt{d}}\left\{\sum_{j=1}^{l} b(\gamma_{\alpha^*})(z_j)\right\} + \frac{\sqrt{2}}{\sqrt{d}}\left\{\sum_{j=l+1}^{l+m} b(\widehat{\gamma}_{\alpha^*})(z_j)\right\} \\ &= \frac{n}{\sqrt{2d(n-1)}}\left\{\sum_{j=1}^{l}\log\Box_1(z_j)\right\} + \frac{\sqrt{2}\,n}{\sqrt{d(n-1)}}\left\{\sum_{j=l+1}^{l+m}\log\Box_1(z_j)\right\} \\ &= \frac{n}{\sqrt{2d(n-1)}}\log\left\{\left(\prod_{j=1}^{l}\Box_1(z_j)\right)\left(\prod_{j=l+1}^{l+m}\Box_1(z_j)\right)^2\right\} \end{aligned} \tag{3.16}$$

for $z_1,\dots,z_l\in P(n,\mathbf{R})$ and $z_{l+1},\dots,z_{l+m}\in P(n,\mathbf{C})$.

Let $\alpha_0$ be the diagonal matrix in $\mathfrak{a}^+ = \widehat{\mathfrak{a}}^+$ such that the first $p$ diagonal elements are equal to 1 and the last $n-p$ diagonal elements are equal to $-\lambda = -p/(n-p)$:

$$\alpha_0 = \operatorname{diag}(1,\dots,1,-\lambda,\dots,-\lambda). \tag{3.17}$$

Then we have

$$|\alpha_0| = \sqrt{\langle\alpha_0,\,\alpha_0\rangle} = \sqrt{2n^2p/(n-p)},\quad \|\alpha_0\| = \sqrt{\langle\!\langle\alpha_0,\,\alpha_0\rangle\!\rangle} = \sqrt{4n^2p/(n-p)}.$$

We define a geodesic ray $\gamma_0 : [0,\infty) \longrightarrow V^l\times\widehat{V}^m$ by

$$\begin{aligned} \gamma_0(s) = \widetilde{\gamma}_{-\alpha_0}(s) = \Big(& e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)}\cdot x_0,\dots,e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)}\cdot x_0, \\ & e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)}\cdot \hat{x}_0,\dots,e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)}\cdot \hat{x}_0\Big) \end{aligned} \tag{3.18}$$

for $s\geq 0$.

Let $L=(L_{ij})$ be a $p\times(n-p)$ matrix with entries in $\boldsymbol{k}_M$. Let

$$L_{ij} = (L_{ij}^1,\dots,L_{ij}^l,L_{ij}^{l+1},\dots,L_{ij}^{l+m}) \in \boldsymbol{k}_M = \mathbf{R}^l\times\mathbf{C}^m$$

for each $i$, $j$. We put

$$L^q = (L_{ij}^q) \in M(p,n-p;\mathbf{R})$$

for $q=1,\dots,l$, and

$$L^q = (L^q_{ij}) \in M(p, n-p; \mathbf{C})$$

for $q = l+1, \ldots, l+m$. Let

$$u_L^q = \begin{pmatrix} I_p & L^q \\ O & I_{n-p} \end{pmatrix} \in SL(n, \mathbf{R})$$

for $q = 1, \ldots, l$,

$$u_L^q = \begin{pmatrix} I_p & L^q \\ O & I_{n-p} \end{pmatrix} \in SL(n, \mathbf{C})$$

for $q = l+1, \ldots, l+m$, and

$$u_L = (u_L^1, \ldots, u_L^l, u_L^{l+1}, \ldots, u_L^{l+m}) \in \widetilde{G} = G^l \times \widehat{G}^m,$$

where $I_k$ is the unit matrix of order $k$ for any positive integer $k$.

We define a geodesic ray $\gamma_L : [0, \infty) \longrightarrow V^l \times \widehat{V}^m$ by

$$\gamma_L(s) = u_L \cdot \gamma_0(s) \quad \text{for} \;\; s \geq 0. \tag{3.19}$$

## 4. Basic correspondence.

In this section we prove Theorem 1.3.

Proof of Theorem 1.3. (1) It suffices to show the following: if $\gamma_L([0,\infty))$ does not intersect $\iota_{\boldsymbol{k}}(g)^{-1} \cdot B(\gamma^*, \tau)$, then $\boldsymbol{a} = \boldsymbol{0}$ or

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^p \, \|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) + \iota_{\boldsymbol{k}}(\boldsymbol{b})\|^{n-p} \geq \left(\frac{\kappa}{n}\right)^{n/2}.$$

First we compute the value of $b(\gamma^*)(\iota_{\boldsymbol{k}}(g) \cdot \gamma_L(s))$. Let

$$\iota_{\boldsymbol{k}}(g) \cdot \gamma_L(s) = (z_1, \ldots, z_{l+m}) \in V^l \times \hat{V}^m.$$

The $q$th entry $z_q$ of $\iota_{\boldsymbol{k}}(g) \cdot \gamma_L(s)$ is

$$z_q = \begin{cases} \iota_q(g) \cdot u_L^q e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)} \cdot x_0 & \text{if } \; 1 \leq q \leq l, \\ \iota_q(g) \cdot u_L^q e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)} \cdot \hat{x}_0 & \text{if } \; l+1 \leq q \leq l+m. \end{cases}$$

Let $(c_1^q \cdots c_n^q)$ be the $n$th row of the matrix

$$\iota_q(g) u_L^q e^{-s\alpha_0/(\sqrt{d}|\alpha_0|)} = \iota_q(g) \begin{pmatrix} e^{-s/(\sqrt{d}\,|\alpha_0|)} I_p & e^{\lambda s/(\sqrt{d}\,|\alpha_0|)} L^q \\ O & e^{\lambda s/(\sqrt{d}\,|\alpha_0|)} I_{n-p} \end{pmatrix}.$$

In the sequel, we write $y^{(q)}$ instead of $\iota_q(y)$ for $y \in \boldsymbol{k}$, $\boldsymbol{y}^{(q)}$ instead of $\iota_q(\boldsymbol{y})$ for $\boldsymbol{y} \in \boldsymbol{k}^j$ with $j \geq 2$:

$$y^{(q)} = \iota_q(y), \quad \boldsymbol{y}^{(q)} = \iota_q(\boldsymbol{y}) \quad \text{for} \;\; y \in \boldsymbol{k}, \, \boldsymbol{y} \in \boldsymbol{k}^j. \tag{4.1}$$

For $1 \leq j \leq n-p$, we denote by $L_j^q$ the linear form determined by the $j$th column of $L^q$. If $1 \leq q \leq l$, $L_j^q : \mathbf{R}^p \longrightarrow \mathbf{R}$ is given by

$$L_j^q(\boldsymbol{y}) = \sum_{k=1}^{p} L_{kj}^q y_k \quad \text{for } \ \boldsymbol{y} = (y_1, \ldots, y_p) \in \mathbf{R}^p$$

and if $l+1 \leq q \leq l+m$, $L_j^q : \mathbf{C}^p \longrightarrow \mathbf{C}$ is given by

$$L_j^q(\boldsymbol{y}) = \sum_{k=1}^{p} L_{kj}^q y_k \quad \text{for } \ \boldsymbol{y} = (y_1, \ldots, y_p) \in \mathbf{C}^p.$$

Then we have

$$\begin{cases} c_i^q = e^{-s/(\sqrt{d}\,|\alpha_0|)} a_{ni}^{(q)} & \text{for } \ 1 \leq i \leq p, \\ c_{p+j}^q = (L_j^q(\boldsymbol{a}^{(q)}) + a_{n,p+j}^{(q)}) e^{\lambda s/(\sqrt{d}\,|\alpha_0|)} & \text{for } \ 1 \leq j \leq n-p. \end{cases}$$

From (3.16) we have

$$\begin{aligned} b(\gamma^*)(\iota_{\boldsymbol{k}}(g) \cdot \gamma_L(s)) &= b(\gamma^*)(z_1, \ldots, z_{l+m}) \\ &= \frac{n}{\sqrt{2d(n-1)}} \log \left\{ \left( \prod_{q=1}^{l} \Box_1(z_q) \right) \left( \prod_{q=l+1}^{l+m} \Box_1(z_q) \right)^2 \right\} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \Box_1(z_q) &= |c_1^q|^2 + |c_2^q|^2 + \cdots + |c_n^q|^2 \\ &= |\boldsymbol{a}^{(q)}|^2 e^{-2s/(\sqrt{d}\,|\alpha_0|)} + \left( \sum_{j=1}^{n-p} |L_j^q(\boldsymbol{a}^{(q)}) + a_{n,p+j}^{(q)}|^2 \right) e^{2\lambda s/(\sqrt{d}\,|\alpha_0|)}, \end{aligned} \tag{4.3}$$

where $|\boldsymbol{a}^{(q)}|$ is the Euclidean norm on $\mathbf{R}^p$ or $\mathbf{C}^p$: $|\boldsymbol{a}^{(q)}|^2 = |a_{n1}^{(q)}|^2 + \cdots + |a_{np}^{(q)}|^2$.

Suppose that $\gamma_L([0, \infty))$ does not intersect $\iota_{\boldsymbol{k}}(g)^{-1} \cdot B(\gamma^*, \tau)$. Then we have

$$b(\gamma^*)(\iota_{\boldsymbol{k}}(g) \cdot \gamma_L(s)) \geq -\tau \quad \text{for all } \ s \geq 0.$$

Since

$$|\boldsymbol{a}^{(q)}|^2 = |a_{n1}^{(q)}|^2 + \cdots + |a_{np}^{(q)}|^2 \leq \|\iota_{\boldsymbol{k}}(a_{n1})\|^2 + \cdots + \|\iota_{\boldsymbol{k}}(a_{np})\|^2 \leq p \|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^2$$

and

$$\begin{aligned} &\sum_{j=1}^{n-p} |L_j^q(\boldsymbol{a}^{(q)}) + a_{n,p+j}^{(q)}|^2 \\ &\quad \leq \|L_1(\iota_{\boldsymbol{k}}(\boldsymbol{a})) + \iota_{\boldsymbol{k}}(a_{n,p+1})\|^2 + \cdots + \|L_{n-p}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) + \iota_{\boldsymbol{k}}(a_{nn})\|^2 \\ &\quad \leq (n-p) \|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) + \iota_{\boldsymbol{k}}(\boldsymbol{b})\|^2, \end{aligned}$$

it follows from (4.3) that

$$\square_1(z_q) \leq p\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^2 e^{-2s/(\sqrt{d}\,|\alpha_0|)} + (n-p)\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) + \iota_{\boldsymbol{k}}(\boldsymbol{b})\|^2 e^{2\lambda s/(\sqrt{d}\,|\alpha_0|)}.$$

This and (4.2) imply that

$$\begin{aligned} \frac{n}{\sqrt{2d(n-1)}} \cdot d \log \Big\{ p\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^2 e^{-2s/(\sqrt{d}\,|\alpha_0|)} \\ + (n-p)\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) + \iota_{\boldsymbol{k}}(\boldsymbol{b})\|^2 e^{2\lambda s/(\sqrt{d}\,|\alpha_0|)} \Big\} \\ \geq b(\gamma^*)(\iota_{\boldsymbol{k}}(g) \cdot \gamma_L(s)) \geq -\tau \end{aligned} \tag{4.4}$$

for all $s \geq 0$. We recall that

$$\lambda = \frac{p}{n-p}, \qquad \kappa = \exp\left(-\tau\sqrt{2(n-1)}/(\sqrt{d}\,n)\right).$$

Let

$$X = \left\{ e^{2s/(\sqrt{d}\,|\alpha_0|)} \right\}^{1/(n-p)}, \qquad \xi = \|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) + \iota_{\boldsymbol{k}}(\boldsymbol{b})\|^2$$

and

$$f(X) = (n-p)\xi X^n - \kappa X^{n-p} + p\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^2.$$

Then (4.4) implies that $f(X) \geq 0$ for any $X \geq 1$.

Let $\boldsymbol{a} = \boldsymbol{0}$. Then $\boldsymbol{b} \neq \boldsymbol{0}$ and

$$f(X) = (n-p)\|\iota_{\boldsymbol{k}}(\boldsymbol{b})\|^2 X^{n-p} \left\{ X^p - \frac{\kappa}{(n-p)\|\iota_{\boldsymbol{k}}(\boldsymbol{b})\|^2} \right\}.$$

Since $0 < \kappa \leq 1/D^2$ and $(n-p)\|\iota_{\boldsymbol{k}}(\boldsymbol{b})\|^2 \geq 1/D^2$, we have $f(X) \geq 0$ for any $X \geq 1$.

Let $\boldsymbol{a} \neq \boldsymbol{0}$. If $\xi = 0$, then

$$f(X) = -\kappa X^{n-p} + p\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^2 < 0$$

for sufficiently large $X$, which is a contradiction. Hence $\xi \neq 0$. Then

$$f'(X) = n(n-p)\xi X^{n-p-1}\left(X^p - \frac{\kappa}{n\xi}\right).$$

Let $X_0 = \{\kappa/(n\xi)\}^{1/p}$. Then $f(X)$ is monotone decreasing in the interval $(0, X_0)$ and monotone increasing in $(X_0, \infty)$. If

$$f(X_0) = -p\left(\frac{\kappa}{n}\right)^{n/p} \cdot \left(\frac{1}{\xi}\right)^{(n-p)/p} + p\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^2 < 0,$$

then

$$\left(\frac{1}{\xi}\right)^{(n-p)/p} > \left(\frac{n}{\kappa}\right)^{n/p} \|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^2$$

and

$$(X_0)^{n-p} = \left(\frac{\kappa}{n}\right)^{(n-p)/p} \cdot \left(\frac{1}{\xi}\right)^{(n-p)/p} > \frac{n}{\kappa} \|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^2 \geq nD^2 \cdot \frac{1}{D^2} = n > 1,$$

which means that the inequality $f(X) < 0$ has a solution in the interval $[1, \infty)$. Therefore $f(X_0) \geq 0$, which implies that

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^{2p} \geq \left(\frac{\kappa}{n}\right)^n \left(\frac{1}{\xi}\right)^{n-p}.$$

The last inequality is equivalent to

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^p \, \|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a})) + \iota_{\boldsymbol{k}}(\boldsymbol{b})\|^{n-p} \geq \left(\frac{\kappa}{n}\right)^{n/2}.$$

(2) Suppose that $\gamma_L([0, \infty))$ intersects $\iota_{\boldsymbol{k}}(g)^{-1} \cdot B(\gamma^*, \tau)$. Then there exists $s_0 \geq 0$ such that

$$b(\gamma^*)(\iota_{\boldsymbol{k}}(g) \cdot \gamma_L(s_0)) < -\tau.$$

Let $\iota_{\boldsymbol{k}}(g) \cdot \gamma_L(s_0) = ua'a \cdot z_0$;

$$\begin{aligned} u &= \left( \begin{pmatrix} I_p & \Xi^1 \\ O & I_{n-p} \end{pmatrix}, \dots, \begin{pmatrix} I_p & \Xi^{l+m} \\ O & I_{n-p} \end{pmatrix} \right) \in \widetilde{N}, \\ a' &= \left(a^1, \dots, a^{l+m}\right) \in \widetilde{A}', \quad a = (b, \dots, b) \in \widetilde{A}_{\mathbf{Q}}, \end{aligned}$$

where

$$\Xi^k \in M(p, n-p; \mathbf{R}) \ \text{ for } \ k = 1, \dots, l, \quad \Xi^k \in M(p, n-p; \mathbf{C}) \ \text{ for } \ k = l+1, \dots, l+m,$$
$$a^k = \operatorname{diag}(a_1^k, \dots, a_n^k) \ \text{ for } \ k = 1, \dots, l+m$$

and

$$b = \operatorname{diag}(b_1, \dots, b_n) \in A = \widehat{A}.$$

For any units $\upsilon_1, \dots, \upsilon_n \in \mathcal{O}_{\boldsymbol{k}}$ such that $\upsilon_1 \cdots \upsilon_n = 1$, let

$$\Upsilon = (\upsilon_1, \dots, \upsilon_n), \quad k_\Upsilon = \operatorname{diag}(\upsilon_1, \dots, \upsilon_n) \in \mathit{\Gamma}$$

and

$$a_\Upsilon = \iota_{\boldsymbol{k}}(k_\Upsilon) = \left( \operatorname{diag}\left(\upsilon_1^{(1)}, \dots, \upsilon_n^{(1)}\right), \dots, \operatorname{diag}\left(\upsilon_1^{(l+m)}, \dots, \upsilon_n^{(l+m)}\right) \right).$$

Recall that the norm $N_{\boldsymbol{k}}(\mu)$ of an element $\mu$ of $\boldsymbol{k}$ is given by

$$N(\mu) = \prod_{j=1}^{l} \iota_j(\mu) \prod_{j=l+1}^{l+m} |\iota_j(\mu)|^2.$$

Let

$$b'_\Upsilon = \left(\operatorname{diag}\left(|\upsilon_1^{(1)}|, \ldots, |\upsilon_n^{(1)}|\right), \ldots, \operatorname{diag}\left(|\upsilon_1^{(l+m)}|, \ldots, |\upsilon_n^{(l+m)}|\right)\right).$$

Since

$$\left(\prod_{q=1}^{l} |\upsilon_j^{(q)}|\right)\left(\prod_{q=l+1}^{l+m} |\upsilon_j^{(q)}|^2\right) = |N_{\boldsymbol{k}}(\upsilon_j)| = 1$$

for each $j$, we have $b'_\Upsilon \in \widetilde{A}'$. Let

$$b_\Upsilon = \left(\rho_\Upsilon^1, \ldots, \rho_\Upsilon^{l+m}\right),$$

where

$$\rho_\Upsilon^j = \operatorname{diag}\left(\rho_{\upsilon_1}^j, \ldots, \rho_{\upsilon_n}^j\right), \quad \rho_{\upsilon_i}^j = |\upsilon_i^{(j)}| a_i^j b_i \quad \text{for } j = 1, \ldots, l+m;\ i = 1, \ldots, n.$$

Since

$$\left(\upsilon_i^{(j)} a_i^j b_i\right) \overline{\left(\upsilon_i^{(j)} a_i^j b_i\right)} = \left|\upsilon_i^{(j)} a_i^j b_i\right|^2 = \left(|\upsilon_i^{(j)}| a_i^j b_i\right) \overline{\left(|\upsilon_i^{(j)}| a_i^j b_i\right)}$$

we have

$$a_\Upsilon a' a \cdot z_0 = b'_\Upsilon a' a \cdot z_0 = b_\Upsilon \cdot z_0. \tag{4.5}$$

Let $u' = a_\Upsilon u (a_\Upsilon)^{-1} \in \widetilde{N}$. Then it follows from Lemma 3.4 and (4.5) that

$$\iota_{\boldsymbol{k}}(k_\Upsilon)\iota_{\boldsymbol{k}}(g) \cdot \gamma_L(s_0) = a_\Upsilon u a' a \cdot z_0 = u' a_\Upsilon a' a \cdot z_0 = u' b'_\Upsilon a' a \cdot z_0$$

and

$$\begin{aligned} b(\gamma^*)(\iota_{\boldsymbol{k}}(k_\Upsilon)\iota_{\boldsymbol{k}}(g) \cdot \gamma_L(s_0)) &= b(\gamma^*)(u' b'_\Upsilon a' a \cdot z_0) = b(\gamma^*)(a \cdot z_0) \\ &= b(\gamma^*)(u a' a \cdot z_0) = b(\gamma^*)(\iota_{\boldsymbol{k}}(g) \cdot \gamma_L(s_0)). \end{aligned}$$

Hence we may consider $\iota_{\boldsymbol{k}}(k_\Upsilon)\iota_{\boldsymbol{k}}(g)\gamma_L(s_0)$ instead of $\iota_{\boldsymbol{k}}(g)\gamma_L(s_0)$ by taking suitable units $\upsilon_1, \ldots, \upsilon_n$.

Let

$$\iota_{\boldsymbol{k}}(k_\Upsilon)\iota_{\boldsymbol{k}}(g)\gamma_L(s_0) = (z_1, \ldots, z_{l+m}).$$

For $1 \le q \le l$ we have

$$z_q = \begin{pmatrix} \upsilon_1^{(q)} & & \\ & \ddots & \\ & & \upsilon_n^{(q)} \end{pmatrix} \begin{pmatrix} I_p & \Xi^q \\ O & I_{n-p} \end{pmatrix} \begin{pmatrix} (\upsilon_1^{(q)})^{-1} & & \\ & \ddots & \\ & & (\upsilon_n^{(q)})^{-1} \end{pmatrix} \begin{pmatrix} \rho^q_{\upsilon_1} & & \\ & \ddots & \\ & & \rho^q_{\upsilon_n} \end{pmatrix} \cdot x_0$$

and

$$\Box_1(z_q) = \left(\rho^q_{\upsilon_n}\right)^2 = |\upsilon_n^{(q)}|^2 (a_n^q)^2 (b_n)^2.$$

Similarly, for $l+1 \le q \le l+m$ we have

$$\Box_1(z_q) = \left|\rho^q_{\upsilon_n}\right|^2 = |\upsilon_n^{(q)}|^2 (a_n^q)^2 (b_n)^2.$$

Let $U_{\boldsymbol{k}}$ be the group of units of $\mathcal{O}_{\boldsymbol{k}}$ and let

$$\varphi : U_{\boldsymbol{k}} \longrightarrow W_{\boldsymbol{k}} = \left\{(y_1, \ldots, y_{l+m}) \in \mathbf{R}^{l+m} \;\middle|\; y_1 + \cdots + y_{l+m} = 0\right\}$$

be a homomorphism defined by

$$\varphi(\upsilon) = \left(\log|\iota_1(\upsilon)|, \ldots, \log|\iota_l(\upsilon)|, \log|\iota_{l+1}(\upsilon)|^2, \ldots, \log|\iota_{l+m}(\upsilon)|^2\right)$$

for $\upsilon \in U_{\boldsymbol{k}}$. Let

$$\begin{aligned} \mathcal{D} = \big\{ &\lambda_1 \varphi(\varepsilon_1) + \cdots + \lambda_{l+m-1}\varphi(\varepsilon_{l+m-1}) \,|\, \\ &- 1/2 \le \lambda_i < 1/2 \quad \text{for} \quad i = 1, \ldots, l+m-1 \big\}. \end{aligned}$$

Then the image $\varphi(U_{\boldsymbol{k}})$ is a cocompact lattice of $W_{\boldsymbol{k}}$ with a fundamental domain $\mathcal{D}$ due to Dirichlet's unit theorem. Accordingly, there exists a unit $\omega$ such that

$$(\log a_n^1, \ldots, \log a_n^l, 2\log a_n^{l+1}, \ldots, 2\log a_n^{l+m}) + \varphi(\omega) \in \mathcal{D}.$$

Let

$$C_2 = \max\left\{\log|\iota_j(\varepsilon_i)|,\ -\log|\iota_j(\varepsilon_i)| \;\middle|\; i = 1, \ldots, l+m-1;\ j = 1, \ldots, l+m\right\}.$$

Then $C_1 = e^{2(l+m-1)C_2}$ and there exist $\lambda_1, \ldots, \lambda_{l+m-1} \in [-1/2,\ 1/2)$ such that

$$\log a_n^j + \log|\iota_j(\omega)| = \sum_{i=1}^{l+m-1} \lambda_i \log|\iota_j(\varepsilon_i)|,$$

for $j = 1, \ldots, l+m$. This shows that

$$\left|\log(|\iota_j(\omega)| a_n^j)\right| \le \sum_{i=1}^{l+m-1} |\lambda_i| \left|\log|\iota_j(\varepsilon_i)|\right| \le \frac{(l+m-1)C_2}{2}.$$

Hence we have

$$-(l+m-1)C_2 \le \log|\iota_j(\omega)| a_n^j - \log|\iota_{j'}(\omega)| a_n^{j'} \le (l+m-1)C_2$$

and

$$e^{-(l+m-1)C_2} \leq \frac{|\iota_j(\omega)|a_n^j}{|\iota_{j'}(\omega)|a_n^{j'}} \leq e^{(l+m-1)C_2} \quad \text{for } \; j, j' \in \{1, \ldots, l+m\}.$$

Let

$$\upsilon_1 = \cdots = \upsilon_{n-2} = 1, \quad \upsilon_{n-1} = \omega^{-1}, \quad \upsilon_n = \omega.$$

Then we have

$$\frac{\square_1(z_j)}{\square_1(z_{j'})} = \left( \frac{|\iota_j(\omega)|a_n^j b_n}{|\iota_{j'}(\omega)|a_n^{j'} b_n} \right)^2 = \left( \frac{|\iota_j(\omega)|a_n^j}{|\iota_{j'}(\omega)|a_n^{j'}} \right)^2$$

and

$$\frac{1}{C_1} \leq \frac{\square_1(z_j)}{\square_1(z_{j'})} \leq C_1.$$

In particular,

$$\square_1(z_j) \geq \frac{1}{C_1} \left\{ \max_{1 \leq q \leq l+m} \square_1(z_q) \right\} \tag{4.6}$$

for any $j$. From (3.16) we obtain

$$d \cdot \frac{n}{\sqrt{2d(n-1)}} \log \left[ \frac{1}{C_1} \max_{1 \leq q \leq l+m} \square_1(z_q) \right] \leq b(\gamma^*)(z_1, \ldots, z_{l+m}) < -\tau.$$

This means that

$$\max_{1 \leq q \leq l+m} \square_1(z_q) < C_1 \exp \left\{ -\tau \sqrt{2(n-1)}/(n\sqrt{d}) \right\} = C_1 \kappa. \tag{4.7}$$

On the other hand, we have

$$\iota_q(k_\Upsilon)\iota_q(g) = \operatorname{diag}\left(1, \ldots, 1, (\omega^{(q)})^{-1}, \omega^{(q)}\right) \left(a_{ij}^{(q)}\right)$$

and

$$\begin{aligned} \square_1(z_q) &= |(\omega \boldsymbol{a})^{(q)}|^2 e^{-2s_0/(\sqrt{d}\,|\alpha_0|)} \\ &\quad + \left( \sum_{j=1}^{n-p} |L_j^q((\omega \boldsymbol{a})^{(q)}) + (\omega a_{n,p+j})^{(q)}|^2 \right) e^{2\lambda s_0/(\sqrt{d}\,|\alpha_0|)}. \end{aligned} \tag{4.8}$$

Hence it follows from (4.7) that

$$|(\omega \boldsymbol{a})^{(q)}|^2 e^{-2s_0/(\sqrt{d}\,|\alpha_0|)} < C_1 \kappa$$

and

$$\left(\sum_{j=1}^{n-p} |L_j^q((\omega \boldsymbol{a})^{(q)}) + (\omega a_{n,p+j})^{(q)}|^2\right) e^{2\lambda s_0/(\sqrt{d}\,|\alpha_0|)} < C_1\kappa$$

for each $q$. Then we have

$$\|\iota_{\boldsymbol{k}}(\omega a_{ni})\|^2 = \max_{1\le q\le l+m} |(\omega a_{ni})^{(q)}|^2 \le \max_{1\le q\le l+m} |(\omega \boldsymbol{a})^{(q)}|^2 < C_1\kappa e^{2s_0/(\sqrt{d}\,|\alpha_0|)}$$

for $1 \le i \le p$, and

$$\begin{aligned}\|L_j(\iota_{\boldsymbol{k}}(\omega \boldsymbol{a})) + \iota_{\boldsymbol{k}}(\omega a_{n,p+j})\|^2 &= \max_{1\le q\le l+m} |L_j^q((\omega \boldsymbol{a})^{(q)}) + (\omega a_{n,p+j})^{(q)}|^2 \\ &< C_1\kappa e^{-2\lambda s_0/(\sqrt{d}\,|\alpha_0|)}\end{aligned}$$

for $1 \le j \le n-p$, which imply that

$$\|\iota_{\boldsymbol{k}}(\omega \boldsymbol{a})\|^2 = \max_{1\le i\le p} \|\iota_{\boldsymbol{k}}(\omega a_{ni})\|^2 < C_1\kappa e^{2s_0/(\sqrt{d}\,|\alpha_0|)}$$

and

$$\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\omega \boldsymbol{a})) + \iota_{\boldsymbol{k}}(\omega \boldsymbol{b})\|^2 = \max_{1\le j\le n-p} \|L_j(\iota_{\boldsymbol{k}}(\omega \boldsymbol{a})) + \iota_{\boldsymbol{k}}(\omega a_{n,p+j})\|^2 < C_1\kappa e^{-2\lambda s_0/(\sqrt{d}\,|\alpha_0|)}.$$

We recall that $\lambda = p/(n-p)$. Then we have

$$\|\iota_{\boldsymbol{k}}(\omega \boldsymbol{a})\|^p < (C_1\kappa)^{p/2} e^{ps_0/(\sqrt{d}\,|\alpha_0|)}$$

and

$$\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\omega \boldsymbol{a})) + \iota_{\boldsymbol{k}}(\omega \boldsymbol{b})\|^{n-p} < (C_1\kappa)^{(n-p)/2} e^{-ps_0/(\sqrt{d}\,|\alpha_0|)}.$$

We conclude that

$$\|\iota_{\boldsymbol{k}}(\omega \boldsymbol{a})\|^p \, \|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\omega \boldsymbol{a})) + \iota_{\boldsymbol{k}}(\omega \boldsymbol{b})\|^{n-p} < (C_1\kappa)^{n/2}.$$

If $\boldsymbol{a} = \boldsymbol{0}$, then $\boldsymbol{b} \ne \boldsymbol{0}$ and it follows from (4.8) that

$$\Box_1(z_q) = |(\omega \boldsymbol{b})^{(q)}|^2 e^{2\lambda s_0/(\sqrt{d}\,|\alpha_0|)} \quad \text{for } q = 1, \ldots, l+m.$$

This shows that

$$\left(\prod_{j=1}^{l} \Box_1(z_j)\right)\left(\prod_{j=l+1}^{l+m} \Box_1(z_j)\right)^2 \ge \left\{\sum_{j=1}^{n-p} \frac{|N_{\boldsymbol{k}}(D\omega a_{n,p+j})|^2}{D^{2d}}\right\} e^{2\sqrt{d}\,\lambda s_0/|\alpha_0|} \ge \frac{1}{D^{2d}},$$

which implies that

$$-\tau > b(\gamma^*)(\iota_{\boldsymbol{k}}(k_\Upsilon)\iota_{\boldsymbol{k}}(g) \cdot \gamma_L(s_0))$$

$$= \frac{n}{\sqrt{2d(n-1)}} \log \left\{ \left( \prod_{j=1}^{l} \square_1(z_j) \right) \left( \prod_{j=l+1}^{l+m} \square_1(z_j) \right)^2 \right\} \geq \frac{-\sqrt{2d}\, n}{\sqrt{n-1}} \log D$$

and

$$\tau < \frac{\sqrt{2d}\, n}{\sqrt{n-1}} \log D.$$

This is a contradiction. Hence $\boldsymbol{a} \neq \mathbf{0}$. $\square$

## 5. Badly approximable systems and horoballs.

In this section we prove Theorem 1.4.

We recall that $B$ is the subgroup of $SL(n, \boldsymbol{k})$ consisting of all the upper triangular matrices in $SL(n, \boldsymbol{k})$ and $\Gamma = SL(n, \mathcal{O}_{\boldsymbol{k}})$. Let $g_1, \ldots, g_h$ be a complete representative system of the double coset classes $\Gamma \backslash SL(n, \boldsymbol{k})/B$. Let $D$ be a positive integer such that

$$(DI_n) g_j^{-1} \in M(n, n; \mathcal{O}_{\boldsymbol{k}}) \quad \text{for} \;\; j = 1, \ldots, h. \tag{5.1}$$

Proof of Theorem 1.4. Suppose that (1) is not satisfied. Then there exist a number $i_0 \in \{1, \ldots, h\}$ and sequences $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbf{R}$, $\{s_j\}_{j=1}^{\infty} \subset [0, \infty)$,

$$\{k_j\}_{j=1}^{\infty} \subset \Gamma g_{i_0}$$

such that

$$\frac{\sqrt{2d}\, n}{\sqrt{n-1}} \log D < \lambda_1 < \lambda_2 < \lambda_3 < \cdots, \quad \lim_{j \to \infty} \lambda_j = \infty$$

and

$$b(\gamma^*)(\iota_{\boldsymbol{k}}(k_j^{-1}) \cdot \gamma_L(s_j)) < -\lambda_j.$$

Let $(a_{j1} \cdots a_{jn})$ be the $n$th row of $k_j^{-1}$ and let

$$\boldsymbol{a}_j = (a_{j1}, \ldots, a_{jp}) \in \left( \frac{1}{D} \mathcal{O}_{\boldsymbol{k}} \right)^p, \quad \boldsymbol{b}_j = (a_{j,p+1}, \ldots, a_{jn}) \in \left( \frac{1}{D} \mathcal{O}_{\boldsymbol{k}} \right)^{n-p}.$$

From Theorem 1.3 (2), there exists a unit $\omega_j \in \mathcal{O}_{\boldsymbol{k}}$ such that

$$\|\iota_{\boldsymbol{k}}(\omega_j \boldsymbol{a}_j)\|^p \, \|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\omega_j \boldsymbol{a}_j)) + \iota_{\boldsymbol{k}}(\omega_j \boldsymbol{b}_j)\|^{n-p} < (C_1)^{n/2} \exp\left\{ -\sqrt{n-1}\, \lambda_j / \sqrt{2d} \right\}. \tag{5.2}$$

and $\boldsymbol{a}_j \neq 0$ for each $j$. Since

$$\lim_{j \to \infty} \exp\left\{ -\sqrt{n-1}\, \lambda_j / \sqrt{2d} \right\} = 0,$$

the inequality (5.2) shows that the system of $\boldsymbol{k}_M$-forms induced from $L$ is not badly approximable and the condition (2) is not satisfied.

Suppose that the condition (1) is satisfied. Let

$$\boldsymbol{a} = (a_1, \ldots, a_p) \in (\mathcal{O}_{\boldsymbol{k}})^p - \{\boldsymbol{0}\}, \quad \boldsymbol{b} = (b_1, \ldots, b_{n-p}) \in (\mathcal{O}_{\boldsymbol{k}})^{n-p}.$$

Suppose that $a_{q_0} \neq 0$ with $1 \leq q_0 \leq p$. We define a matrix $g' = (g'_{ij}) \in SL(n, \boldsymbol{k})$ by

$$g'_{ij} = \begin{cases} \varepsilon / a_{q_0} & \text{if } i = 1 \text{ and } j = n, \\ 1 & \text{if } i + j = n+1 \text{ and } 2 \leq i \leq n-1, \\ -a_{q_0} & \text{if } i = n \text{ and } j = 1, \\ a_1 & \text{if } i = n \text{ and } j = q_0, \\ a_j & \text{if } i = n,\ 2 \leq j \leq p \text{ and } j \neq q_0, \\ b_{j-p} & \text{if } i = n \text{ and } p+1 \leq j, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varepsilon$ is equal to 1 or $-1$. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be the standard basis of $\mathbf{R}^n$ and put

$$w = \left({}^t\boldsymbol{e}_{q_0}\ {}^t\boldsymbol{e}_2 \cdots {}^t\boldsymbol{e}_{q_0-1} \quad -{}^t\boldsymbol{e}_1 \quad {}^t\boldsymbol{e}_{q_0+1} \cdots {}^t\boldsymbol{e}_n\right) \in SL(n, \mathbf{Z}).$$

Let $g^{-1} = g'w$. Then $g^{-1} \in SL(n, \boldsymbol{k})$ and the $n$th row of $g^{-1}$ is $(a_1 \cdots a_p\ b_1 \cdots b_{n-p})$.

There exist $i_0 \in \{1, \ldots, h\}$, $u \in B$ and $k \in \Gamma$ such that $g^{-1} = u g_{i_0}^{-1} k$. Let $(a_{i_0,1} \cdots a_{i_0,n})$ be the $n$th row of the matrix $(DI_n) g_{i_0}^{-1}$. The ideal generated by the set of all the elements in the $n$th row of the matrix $(DI_n) g_{i_0}^{-1}$ coincides with the ideal generated by the set of all the elements in the $n$th row of the matrix $(DI_n) g_{i_0}^{-1} k$, which we denote by $\mathfrak{c}_{i_0}$. Let $\nu$ be the $(n,n)$-entry of the matrix $u^{-1}$. Since

$$(DI_n) u^{-1} g^{-1} = (DI_n) g_{i_0}^{-1} k,$$

we have

$$(D)(a_1, \ldots, a_p,\ b_1, \ldots, b_{n-p})(\nu) = \mathfrak{c}_{i_0}$$

and

$$D^d |N_{\boldsymbol{k}}(\nu)| N((a_1, \ldots, a_p,\ b_1, \ldots, b_{n-p})) = N(\mathfrak{c}_{i_0}).$$

This implies that

$$|N_{\boldsymbol{k}}(\nu)| \leq \frac{N(\mathfrak{c}_{i_0})}{D^d}. \tag{5.3}$$

From Dirichlet's unit theorem, for any $\mu \in \boldsymbol{k}$ with $|N_{\boldsymbol{k}}(\mu)| = 1$ there exists a unit $\omega$ of $\boldsymbol{k}$ with

$$\|\iota_{\boldsymbol{k}}(\omega\mu)\| \leq C_3,$$

where

$$C_3 = e^{(l+m-1)C_2/2} = \sqrt{C_1}.$$

Let $\mu' \in \boldsymbol{k} - \{0\}$. Applying the above to $\mu'/|N_{\boldsymbol{k}}(\mu')|^{1/d}$, one can find a unit $\omega'$ such that

$$\|\iota_{\boldsymbol{k}}(\omega'\mu')\| \leq C_3 |N_{\boldsymbol{k}}(\mu')|^{1/d}$$

(cf. Lemma 2.4 of [**10**]). We take a unit $\omega''$ such that

$$\|\iota_{\boldsymbol{k}}(\omega''\nu)\| \leq C_3 |N_{\boldsymbol{k}}(\nu)|^{1/d} \tag{5.4}$$

and put

$$\upsilon = \operatorname{diag}\big((\omega'')^{-1}, 1, \ldots, 1, \omega''\big).$$

From (3.16) we have

$$b(\gamma^*)(\iota_{\boldsymbol{k}}(\upsilon) \cdot z) = b(\gamma^*)(z) \quad \text{for} \quad z \in \widetilde{V}.$$

Let $g'' = u^{-1}g^{-1} \in g_{i_0}^{-1}\Gamma$. From the assumption we have

$$b(\gamma^*)(\iota_{\boldsymbol{k}}(\upsilon)\iota_{\boldsymbol{k}}(g'') \cdot \gamma_L(s)) = b(\gamma^*)(\iota_{\boldsymbol{k}}(g'') \cdot \gamma_L(s)) \geq -\tau$$

for $s \geq 0$. Hence $\gamma_L([0,\infty))$ does not intersect $\iota_{\boldsymbol{k}}((\upsilon g''))^{-1} \cdot B(\gamma^*, \tau)$. Since the $n$th row of the matrix $\upsilon g''$ is $(\omega''\nu\boldsymbol{a}\ \omega''\nu\boldsymbol{b})$, it follows from Theorem 1.3 (1) that

$$\|\iota_{\boldsymbol{k}}(\omega''\nu\boldsymbol{a})\|^p \, \|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\omega''\nu\boldsymbol{a})) + \iota_{\boldsymbol{k}}(\omega''\nu\boldsymbol{b})\|^{n-p} \geq \left(\frac{\kappa}{n}\right)^{n/2}.$$

From (5.4) we have

$$\begin{aligned}
\|\iota_{\boldsymbol{k}}(\omega''\nu a_j)\| &= \max_{1\leq i\leq l+m} |(\omega''\nu)^{(i)} a_j^{(i)}| \\
&\leq \|\iota_{\boldsymbol{k}}(\omega''\nu)\| \max_{1\leq i\leq l+m} |a_j^{(i)}| = \|\iota_{\boldsymbol{k}}(\omega''\nu)\| \|\iota_{\boldsymbol{k}}(a_j)\| \leq C_3 |N_{\boldsymbol{k}}(\nu)|^{1/d} \|\iota_{\boldsymbol{k}}(a_j)\|
\end{aligned}$$

for each $j = 1, \ldots, p$. Hence we obtain

$$\|\iota_{\boldsymbol{k}}(\omega''\nu\boldsymbol{a})\| = \max_{1\leq j\leq p} \|\iota_{\boldsymbol{k}}(\omega''\nu a_j)\| \leq C_3 |N_{\boldsymbol{k}}(\nu)|^{1/d} \|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|.$$

Let $L = (L_{ij})$ and

$$L_{ij} = (L_{ij}^1, \ldots, L_{ij}^l, L_{ij}^{l+1}, \ldots, L_{ij}^{l+m}) \in \boldsymbol{k}_M = \mathbf{R}^l \times \mathbf{C}^m.$$

From (5.4) we have

$$\|L_j(\iota_{\boldsymbol{k}}(\omega''\nu\boldsymbol{a})) + \iota_{\boldsymbol{k}}(\omega''\nu b_j)\| = \max_{1\leq i\leq l+m} \left| \left( \sum_{n'=1}^{p} L_{n'j}^i (\omega''\nu)^{(i)} a_{n'}^{(i)} \right) + (\omega''\nu)^{(i)} b_j^{(i)} \right|$$

$$= \max_{1\le i\le l+m}\left|(\omega''\nu)^{(i)}\left(\left(\sum_{n'=1}^{p} L^{i}_{n'j}a^{(i)}_{n'}\right)+b^{(i)}_{j}\right)\right|$$
$$\le \|\iota_{\boldsymbol{k}}(\omega''\nu)\| \max_{1\le i\le l+m}\left|\left(\sum_{n'=1}^{p} L^{i}_{n'j}a^{(i)}_{n'}\right)+b^{(i)}_{j}\right| = \|\iota_{\boldsymbol{k}}(\omega''\nu)\|\ \|L_j(\iota_{\boldsymbol{k}}(\boldsymbol{a}))+\iota_{\boldsymbol{k}}(b_j)\|$$

for each $j=1,\dots,n-p$. Hence we obtain

$$\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\omega''\nu\boldsymbol{a}))+\iota_{\boldsymbol{k}}(\omega''\nu\boldsymbol{b})\| = \max_{1\le j\le n-p}\|L_j(\iota_{\boldsymbol{k}}(\omega''\nu\boldsymbol{a}))+\iota_{\boldsymbol{k}}(\omega''\nu b_j)\|$$
$$\le \|\iota_{\boldsymbol{k}}(\omega''\nu)\|\ \|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a}))+\iota_{\boldsymbol{k}}(\boldsymbol{b})\| \le C_3|N_{\boldsymbol{k}}(\nu)|^{1/d}\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a}))+\iota_{\boldsymbol{k}}(\boldsymbol{b})\|$$

from (5.4). Therefore

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\| \ge \frac{1}{C_3|N_{\boldsymbol{k}}(\nu)|^{1/d}}\|\iota_{\boldsymbol{k}}(\omega''\nu\boldsymbol{a})\|,$$

$$\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a}))+\iota_{\boldsymbol{k}}(\boldsymbol{b})\| \ge \frac{1}{C_3|N_{\boldsymbol{k}}(\nu)|^{1/d}}\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\omega''\nu\boldsymbol{a}))+\iota_{\boldsymbol{k}}(\omega''\nu\boldsymbol{b})\|,$$

and

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^p\,\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a}))+\iota_{\boldsymbol{k}}(\boldsymbol{b})\|^{n-p} \ge \frac{1}{(C_3)^n|N_{\boldsymbol{k}}(\nu)|^{n/d}}\left(\frac{\kappa}{n}\right)^{n/2}. \tag{5.5}$$

Let $\mathfrak{c}_i$ be the ideal generated by the set of all entries in the $n$th row of the matrix $(DI_n)g_i^{-1}$ for each $i=1,\dots,h$ and let

$$C_4 = \max_{1\le i\le h} N(\mathfrak{c}_i).$$

Then it follows from (5.3) that

$$|N_{\boldsymbol{k}}(\nu)| \le \frac{C_4}{D^d}.$$

From this and (5.5), we obtain

$$\|\iota_{\boldsymbol{k}}(\boldsymbol{a})\|^p\,\|\boldsymbol{L}(\iota_{\boldsymbol{k}}(\boldsymbol{a}))+\iota_{\boldsymbol{k}}(\boldsymbol{b})\|^{n-p} \ge \frac{D^n}{(C_3)^n(C_4)^{n/d}}\left(\frac{\kappa}{n}\right)^{n/2} > 0.$$

Therefore the system of $\boldsymbol{k}_M$-forms induced from $L$ is badly approximable and the condition (2) is satisfied. □

## 6. Relatively compactness criterion.

We recall that $D$ is a positive integer satisfying (5.1) and $\Pi : \widetilde{V} \longrightarrow \iota_{\boldsymbol{k}}(\Gamma)\backslash\widetilde{V}$ is the natural projection.

Proposition 6.1. *For any $q \in \{1,\dots,h\}$, we have*

$$d_{\widetilde{V}}(z_0,\, \iota_{\boldsymbol{k}}(gg_q)b\cdot z_0) \geq d_{\widetilde{V}}(z_0,\, b\cdot z_0) - 2n\sqrt{2dn}\,\log D \quad \textit{for any} \;\; b\in \widetilde{A}_{\mathbf{Q}} \;\; \textit{and} \;\; g\in \Gamma.$$

Proof. Suppose that $b \neq e$. Let

$$\beta = \mathrm{diag}(\beta_1, \ldots, \beta_n) \in \log A$$

such that

$$|\beta| = \sqrt{\langle \beta,\, \beta\rangle} = 1/\sqrt{d}$$

and

$$b = (e^{s_0\beta}, \ldots, e^{s_0\beta}).$$

We consider the geodesic ray $\widetilde{\gamma}_\beta : [0,\infty) \longrightarrow \widetilde{V}$ defined by

$$\widetilde{\gamma}_\beta(s) = (e^{s\beta}\cdot x_0, \ldots, e^{s\beta}\cdot x_0,\, e^{s\beta}\cdot \widehat{x}_0, \ldots, e^{s\beta}\cdot \widehat{x}_0) \quad \text{for} \;\; s\geq 0.$$

Let $(gg_q)^{-1} = (g_{ij})$ and let $\sigma$ be a permutation such that

$$\beta_{\sigma(1)} \geq \beta_{\sigma(2)} \geq \cdots \geq \beta_{\sigma(n)}.$$

We define an element $g' = (g'_{ij})$ of $SL(n,\boldsymbol{k})$ by

$$g'_{ij} = g_{\sigma(i)\,\sigma(j)}.$$

Let

$$\iota_{\boldsymbol{k}}(g')\cdot z_0 = \sigma\cdot(\iota_{\boldsymbol{k}}(gg_q)^{-1}\cdot z_0) = (\xi_1,\ldots,\xi_n).$$

For each $r\in\{1,\ldots,n-1\}$, there exists $r$ integers $n'_1,\ldots,n'_r$ such that

$$\begin{vmatrix} g'_{n+1-r,\,n'_1} & \cdots & g'_{n+1-r,\,n'_r} \\ \cdots & \cdots & \cdots \\ g'_{n,\,n'_1} & \cdots & g'_{n,\,n'_r} \end{vmatrix} \neq 0.$$

We denote by $\triangle_r$ the value of this minor determinant. Then

$$\Box_r(\xi_{i'}) \geq \left\{(\triangle_r)^{(i')}\right\}^2, \quad \Box_r(\xi_{j'}) \geq |(\triangle_r)^{(j')}|^2$$

for $1\leq i'\leq l$ and $l+1\leq j'\leq l+m$. Hence we have

$$\left(\prod_{i'=1}^{l} \Box_r(\xi_{i'})\right)\left(\prod_{j'=l+1}^{l+m} \Box_r(\xi_{j'})\right)^2 \geq |N_{\boldsymbol{k}}(\triangle_r)|^2.$$

Since $D^r\triangle_r \in \mathcal{O}_{\boldsymbol{k}}$, we also have

$$|N_{\boldsymbol{k}}(\triangle_r)|^2 \geq \frac{1}{D^{2dr}} \geq \frac{1}{D^{2dn}}.$$

We remark that $|\beta_{\sigma(1)}| \leq |\beta|/\sqrt{2n}$, $|\beta_{\sigma(n)}| \leq |\beta|/\sqrt{2n}$, and hence

$$\frac{\beta_{\sigma(1)} - \beta_{\sigma(n)}}{|\beta|} = \frac{|\beta_{\sigma(1)} - \beta_{\sigma(n)}|}{|\beta|} \leq \frac{|\beta_{\sigma(1)}|}{|\beta|} + \frac{|\beta_{\sigma(n)}|}{|\beta|} \leq \sqrt{\frac{2}{n}}.$$

Then it follows from Lemma 3.3 and (3.11), (3.8), (3.9) that

$$\begin{aligned}
&b(\widetilde{\gamma}_\beta)(\iota_{\boldsymbol{k}}(gg_q)^{-1} \cdot z_0) = \frac{1}{\sqrt{d}} \left\{ \sum_{j=1}^{l} b(\gamma_\beta)(\xi_j) \right\} + \frac{\sqrt{2}}{\sqrt{d}} \left\{ \sum_{j=l+1}^{l+m} b(\widehat{\gamma}_\beta)(\xi_j) \right\} \\
&= \frac{n}{\sqrt{d}\ |\beta|} \log \left[ \prod_{r=1}^{n-1} \left\{ \left( \prod_{j=1}^{l} \square_r(\xi_j) \right) \left( \prod_{j=l+1}^{l+m} \square_r(\xi_j) \right)^2 \right\}^{\beta_{\sigma(n-r)} - \beta_{\sigma(n-r+1)}} \right] \\
&\geq \frac{n}{\sqrt{d}\ |\beta|} \log \left\{ \prod_{r=1}^{n-1} \left( \frac{1}{D^{2dn}} \right)^{\beta_{\sigma(n-r)} - \beta_{\sigma(n-r+1)}} \right\} \\
&= \frac{n}{\sqrt{d}\ |\beta|} \log \left( \frac{1}{D^{2dn}} \right)^{\beta_{\sigma(1)} - \beta_{\sigma(n)}} = -2n^2 \sqrt{d} \cdot \frac{\beta_{\sigma(1)} - \beta_{\sigma(n)}}{|\beta|} \log D \geq -2n\sqrt{2dn} \log D.
\end{aligned}$$

Suppose that

$$d_{\widetilde{V}}(z_0,\, \iota_{\boldsymbol{k}}(gg_q)b \cdot z_0) < d_{\widetilde{V}}(z_0,\, b \cdot z_0) - 2n\sqrt{2dn} \log D.$$

Then we have

$$\begin{aligned}
d_{\widetilde{V}}(\iota_{\boldsymbol{k}}(gg_q)^{-1} \cdot z_0,\, \widetilde{\gamma}_\beta(s_0)) &= d_{\widetilde{V}}(z_0,\, \iota_{\boldsymbol{k}}(gg_q) \cdot \widetilde{\gamma}_\beta(s_0)) = d_{\widetilde{V}}(z_0,\, \iota_{\boldsymbol{k}}(gg_q)b \cdot z_0) \\
&< d_{\widetilde{V}}(z_0,\, b \cdot z_0) - 2n\sqrt{2dn} \log D = s_0 - 2n\sqrt{2dn} \log D.
\end{aligned}$$

Since the function

$$[0, \infty) \ni s \longmapsto d_{\widetilde{V}}(\iota_{\boldsymbol{k}}(gg_q)^{-1} \cdot z_0,\, \widetilde{\gamma}_\beta(s)) - s$$

is monotone decreasing by the triangle inequality (cf. [**2**, Section 3]), we obtain

$$\begin{aligned}
b(\widetilde{\gamma}_\beta)(\iota_{\boldsymbol{k}}(gg_q)^{-1} \cdot z_0) &= \lim_{s \to \infty} \left\{ d_{\widetilde{V}}(\iota_{\boldsymbol{k}}(gg_q)^{-1} \cdot z_0,\, \widetilde{\gamma}_\beta(s)) - s \right\} \\
&\leq d_{\widetilde{V}}(\iota_{\boldsymbol{k}}(gg_q)^{-1} \cdot z_0,\, \widetilde{\gamma}_\beta(s_0)) - s_0 < -2n\sqrt{2dn} \log D,
\end{aligned}$$

which is a contradiction. □

LEMMA 6.1. *Let $\gamma : [0, \infty) \longrightarrow \widetilde{V} = V^l \times \widehat{V}^m$ be an arbitrary geodesic ray and*

$$\alpha = \operatorname{diag}(\alpha_1, \ldots, \alpha_n) \in \mathfrak{a}^+ - \{0\} = \widehat{\mathfrak{a}}^+ - \{0\}.$$

*Then the following two conditions are equivalent.*

(1) $\gamma([0,\infty))$ *does not intersect*

$$\bigcup_{i=1}^{h} \bigcup_{g\in\Gamma} \iota_{\boldsymbol{k}}(g)\iota_{\boldsymbol{k}}(g_i)\cdot B(\widetilde{\gamma}_\alpha,\, \tau)$$

*for some non-negative constant* $\tau$.

(2) $\Pi\circ\gamma([0,\infty))$ *is relatively compact in* $\iota_{\boldsymbol{k}}(\Gamma)\backslash\widetilde{V}$.

For positive numbers $c$ let

$$A_c = \{a=\mathrm{diag}(a_1,\dots,a_n)\in A \,|\, a_i/a_{i+1}\ge c \ \text{ for } \ i=1,\dots,n-1\}, \tag{6.1}$$

$$(\widetilde{A}_{\mathbf{Q}})_c = \left\{(a,\dots,a)\in\widetilde{A}_{\mathbf{Q}} \,\middle|\, a\in A_c\right\} \tag{6.2}$$

and let $\eta$ be a compact subset of $\widetilde{N}\widetilde{A}'$ containing the identity element. Then a Siegel set in $\widetilde{V}$ is a set of the form $\mathfrak{S}_{c,\eta}=\eta(\widetilde{A}_{\mathbf{Q}})_c\cdot z_0$ for some $c>0$ and a compact subset $\eta$ of $\widetilde{N}\widetilde{A}'$ containing the identity element.

Theorem 6.1 (cf. [**3**], [**4**]). *There exist a positive number* $C_5<1$ *and a compact subset* $\eta_1$ *of* $\widetilde{N}\widetilde{A}'$ *containing the identity element such that the following holds*:

$$\iota_{\boldsymbol{k}}(\Gamma)\left(\bigcup_{i=1}^{h}\iota_{\boldsymbol{k}}(g_i)\,\mathfrak{S}_{c,\eta}\right)=\widetilde{V}$$

*if* $0<c\le C_5$ *and* $\eta_1\subset\eta\subset\widetilde{N}\widetilde{A}'$.

Let

$$\mu=\log C_5,\quad \mathcal{S}=\mathfrak{S}_{C_5,\eta_1}=\eta_1(\widetilde{A}_{\mathbf{Q}})_{C_5}\cdot z_0 \tag{6.3}$$

and

$$\mathfrak{S}=\bigcup_{i=1}^{h}\iota_{\boldsymbol{k}}(g_i)\cdot\mathcal{S}. \tag{6.4}$$

We remark that

$$\log(\widetilde{A}_{\mathbf{Q}})_1=\left\{\log b\,\middle|\, b\in(\widetilde{A}_{\mathbf{Q}})_1\right\}=\{(\beta,\dots,\beta)\in(\mathfrak{a}^+)^l\times(\widehat{\mathfrak{a}}^+)^m\}\subset\log(\widetilde{A}_{\mathbf{Q}})_{C_5}$$

and

$$\log(\widetilde{A}_{\mathbf{Q}})_{C_5}=\{(\beta,\dots,\beta)\in\mathfrak{a}^l\times\widehat{\mathfrak{a}}^m\,|\,\theta_i(\beta)\ge\mu=\log C_5\quad\text{for }\ i=1,\dots,n-1\}.$$

We first consider in the Lie algebra $\mathfrak{a}$. Let

$$\beta_0=\mathrm{diag}\left(\frac{n-1}{2}\,\mu,\ \frac{n-3}{2}\,\mu,\dots,\frac{n-(2n-1)}{2}\,\mu\right)\in\mathfrak{a}, \tag{6.5}$$

where the $i$th diagonal element is equal to $\{n-(2i-1)\}\,\mu/2$. For each $i=1,\dots,n-1$, let $v_i$ be the diagonal matrix such that the first $i$ diagonal elements are equal to $1/i$ and the last $n-i$ elements are equal to $-1/(n-i)$:

$$v_i = \mathrm{diag}\left(\frac{1}{i},\dots,\frac{1}{i},-\frac{1}{n-i},\dots,-\frac{1}{n-i}\right) \in \mathfrak{a}. \tag{6.6}$$

We denote by $l_i$ the half-line

$$[0,\infty) \ni s \longmapsto \beta_0 + s v_i \tag{6.7}$$

in $\mathfrak{a}$. Then $\log A_{C_5}$ is the smallest convex subset of $\mathfrak{a}$ containing the $n-1$ half-lines $l_1,\dots,l_{n-1}$ and it is an infinite cone in $\mathfrak{a}$ with apex $\beta_0$.

Let $\alpha \in \mathfrak{a}^+ - \{0\}$ and let $l$ be the line

$$(-\infty,\infty) \ni s \longmapsto s\cdot\frac{\alpha}{|\alpha|}.$$

For $\nu \geq 0$, let $H(\nu)$ be the hyperplane in $\mathfrak{a}$ through $\nu\alpha/|\alpha|$ which is perpendicular to $l$.

Let $\delta_{ij}$ be the angle between $v_i$ and $v_j$. For $i<j$ we have

$$0 < \cos\delta_{ij} = \sqrt{\frac{i(n-j)}{j(n-i)}} < 1 \tag{6.8}$$

and $0 < \delta_{ij} < \pi/2$. Since $\alpha \in \mathfrak{a}^+ - \{0\}$, we may write $\alpha = \sum_{k=1}^{n-1}\lambda_k v_k$ by non-negative numbers $\lambda_1,\dots,\lambda_{n-1}$. Then, for each $i=1,\dots,n-1$, the angle between $\alpha$ and $v_i$ is smaller than $\pi/2$ and hence $l_i$ intersects $H(\nu)$ at one point, say $P_i(\nu)$. The intersection $H(\nu)\cap\log A_{C_5}$ is the smallest convex subset of $H(\nu)$ containing the $n-1$ points $P_1(\nu),\dots,P_{n-1}(\nu)$, and is compact.

We recall that $d_{\widetilde{V}}$ is the distance on $\widetilde{V}$ induced from the product metric on $\widetilde{V} = V^l \times \widehat{V}^m$. We define a distance $\overline{d}$ on $\iota_{\boldsymbol{k}}(\Gamma)\backslash\widetilde{V}$ by

$$\overline{d}(\Pi(z),\,\Pi(z')) = \inf_{g\in\Gamma} d_{\widetilde{V}}(z,\,\iota_{\boldsymbol{k}}(g)\cdot z') \quad \text{for } \ z,z'\in\widetilde{V}.$$

Proof of Lemma 6.1. Suppose that the condition (1) is satisfied. Then the subset

$$\begin{aligned}&\Big\{\widetilde{\beta} = (\beta,\dots,\beta) \in \log(\widetilde{A}_{\mathbf{Q}})_{C_5} \\ &\qquad\qquad \Big|\, b(\widetilde{\gamma}_\alpha)\big((e^\beta\cdot x_0,\dots,e^\beta\cdot x_0,e^\beta\cdot\widehat{x}_0,\dots,e^\beta\cdot\widehat{x}_0)\big) \geq -\tau\Big\} \\ &= \left\{\widetilde{\beta} = (\beta,\dots,\beta)\in\log(\widetilde{A}_{\mathbf{Q}})_{C_5} \,\middle|\, \beta\in\log A_{C_5},\ \left\langle\frac{\alpha}{|\alpha|},\,\beta\right\rangle \leq \frac{\tau}{\sqrt{d}}\right\}\end{aligned}$$

of $\mathfrak{a}^l \times \widehat{\mathfrak{a}}^m$ is homeomorphic to a cone in $\mathfrak{a}$ over $H(\tau/\sqrt{d})\cap\log A_{C_5}$ with apex $\beta_0$, and is compact. Let

$$\mathcal{C}_1 = \mathcal{S} \cap b(\widetilde{\gamma}_\alpha)^{-1}([-\tau, \infty)).$$

Since $b(\widetilde{\gamma}_\alpha)$ is $\widetilde{N}\widetilde{A}'$-invariant (cf. Lemma 3.4), we have

$$\mathcal{C}_1 = \Bigg\{ v \cdot (e^\beta \cdot x_0, \dots, e^\beta \cdot x_0,\, e^\beta \cdot \widehat{x}_0, \dots, e^\beta \cdot \widehat{x}_0) \\ \Bigg|\, \beta \in \log A_{C_5},\, \left\langle \frac{\alpha}{|\alpha|},\, \beta \right\rangle \le \tau/\sqrt{d},\, v \in \eta_1 \Bigg\}$$

and $\mathcal{C}_1$ is compact. Let

$$\mathcal{C}_j = \iota_{\boldsymbol{k}}(g_j) \cdot \mathcal{C}_1 = \iota_{\boldsymbol{k}}(g_j) \cdot \mathcal{S} \cap b(\iota_{\boldsymbol{k}}(g_j) \cdot \widetilde{\gamma}_\alpha)^{-1}([-\tau, \infty))$$

for $j = 1, \dots, h$ and

$$\mathcal{C} = \bigcup_{j=1}^{h} \mathcal{C}_j.$$

Then $\mathcal{C}$ is also compact.

For each $s \ge 0$, there exist $g \in \mathit{\Gamma}$ and $i_0 \in \{1, \dots, h\}$ such that $\iota_{\boldsymbol{k}}(g) \cdot \gamma(s) \in \iota_{\boldsymbol{k}}(g_{i_0}) \cdot \mathcal{S}$, due to Theorem 6.1. From the assumption, $\gamma(s)$ does not belong to

$$\iota_{\boldsymbol{k}}(g)^{-1} \cdot \iota_{\boldsymbol{k}}(g_{i_0}) \cdot B(\widetilde{\gamma}_\alpha,\, \tau) = \iota_{\boldsymbol{k}}(g)^{-1} \cdot B(\iota_{\boldsymbol{k}}(g_{i_0}) \cdot \widetilde{\gamma}_\alpha,\, \tau)$$

and

$$b(\iota_{\boldsymbol{k}}(g_{i_0}) \cdot \widetilde{\gamma}_\alpha)(\iota_{\boldsymbol{k}}(g) \cdot \gamma(s)) > -\tau.$$

This means that

$$\iota_{\boldsymbol{k}}(g) \cdot \gamma(s) \in \iota_{\boldsymbol{k}}(g_{i_0}) \cdot \mathcal{C}_1 \subset \mathcal{C}$$

and $\Pi(\gamma(s)) \subset \Pi(\mathcal{C})$. Since $s \ge 0$ is arbitrary and $\Pi(\mathcal{C})$ is compact, we have $\Pi(\gamma([0, \infty))) \subset \Pi(\mathcal{C})$ and $\Pi(\gamma([0, \infty)))$ is relatively compact.

Suppose that the condition (2) is satisfied.

Claim. *There exists a positive number $C_6$ such that the following holds: if $g \in \mathit{\Gamma}$, $s \ge 0$ and $\iota_{\boldsymbol{k}}(g) \cdot \gamma(s) \in \iota_{\boldsymbol{k}}(g_i) \cdot \mathcal{S}$, then $b(\iota_{\boldsymbol{k}}(g_i) \cdot \widetilde{\gamma}_\alpha)(\iota_{\boldsymbol{k}}(g) \cdot \gamma(s)) \ge -C_6$.*

Proof of Claim. If this is not true, then there exist a number $i_0 \in \{1, \dots, h\}$ and sequences $\{\lambda_j\}_{j=1}^\infty \subset \mathbf{R}$, $\{k_j\}_{j=1}^\infty \subset \mathit{\Gamma}$, $\{s_j\}_{j=1}^\infty \subset [0, \infty)$ such that

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots, \quad \lim_{j \to \infty} \lambda_j = \infty,$$

$\iota_{\boldsymbol{k}}(k_j) \cdot \gamma(s_j) \in \iota_{\boldsymbol{k}}(g_{i_0}) \cdot \mathcal{S}$ and $b(\iota_{\boldsymbol{k}}(g_{i_0}) \cdot \widetilde{\gamma}_\alpha)(\iota_{\boldsymbol{k}}(k_j) \cdot \gamma(s_j)) < -\lambda_j$.

Let

$$\iota_{\boldsymbol{k}}(k_j) \cdot \gamma(s_j) = \iota_{\boldsymbol{k}}(g_{i_0}) u_j a_j \cdot z_0; \;\; u_j \in \eta_1, \, a_j \in (\widetilde{A}_{\mathbf{Q}})_{C_5}.$$

It follows from Lemma 3.4 that

$$\begin{aligned} b(\widetilde{\gamma}_\alpha)(a_j \cdot z_0) &= b(\widetilde{\gamma}_\alpha)(u_j a_j \cdot z_0) \\ &= b(\iota_{\boldsymbol{k}}(g_{i_0}) \cdot \widetilde{\gamma}_\alpha)(\iota_{\boldsymbol{k}}(g_{i_0}) u_j a_j \cdot z_0) = b(\iota_{\boldsymbol{k}}(g_{i_0}) \cdot \widetilde{\gamma}_\alpha)(\iota_{\boldsymbol{k}}(k_j) \cdot \gamma(s_j)) < -\lambda_j. \end{aligned}$$

Since

$$|b(\widetilde{\gamma}_\alpha)(z) - b(\widetilde{\gamma}_\alpha)(z')| \leq d_{\widetilde{V}}(z, \, z') \quad \text{for} \;\; z, z' \in \widetilde{V}$$

from Proposition 3.1, we obtain

$$d_{\widetilde{V}}(z_0, \, a_j \cdot z_0) \geq |b(\widetilde{\gamma}_\alpha)(z_0) - b(\widetilde{\gamma}_\alpha)(a_j \cdot z_0)| = |b(\widetilde{\gamma}_\alpha)(a_j \cdot z_0)| > \lambda_j. \tag{6.9}$$

Remark that the set $\{a^{-1} v a \,|\, a \in (\widetilde{A}_{\mathbf{Q}})_{C_5}, \, v \in \eta_1\}$ is compact. We put

$$C_7 = \max \left\{ d_{\widetilde{V}}(va \cdot z_0, a \cdot z_0) \,\middle|\, a \in (\widetilde{A}_{\mathbf{Q}})_{C_5}, \, v \in \eta_1 \right\}.$$

For any $g \in \Gamma$ we have

$$\begin{aligned} d_{\widetilde{V}}&(\iota_{\boldsymbol{k}}(g k_j) \cdot \gamma(s_j), \, z_0) = d_{\widetilde{V}}(\iota_{\boldsymbol{k}}(g) \iota_{\boldsymbol{k}}(g_{i_0}) u_j a_j \cdot z_0, \, z_0) \\ &\geq d_{\widetilde{V}}(z_0, \, \iota_{\boldsymbol{k}}(g) \iota_{\boldsymbol{k}}(g_{i_0}) a_j \cdot z_0) - d_{\widetilde{V}}(\iota_{\boldsymbol{k}}(g) \iota_{\boldsymbol{k}}(g_{i_0}) a_j \cdot z_0, \, \iota_{\boldsymbol{k}}(g) \iota_{\boldsymbol{k}}(g_{i_0}) u_j a_j \cdot z_0) \\ &= d_{\widetilde{V}}(z_0, \, \iota_{\boldsymbol{k}}(g) \iota_{\boldsymbol{k}}(g_{i_0}) a_j \cdot z_0) - d_{\widetilde{V}}(a_j \cdot z_0, \, u_j a_j \cdot z_0) \\ &\geq d_{\widetilde{V}}(z_0, \, \iota_{\boldsymbol{k}}(g) \iota_{\boldsymbol{k}}(g_{i_0}) a_j \cdot z_0) - C_7. \end{aligned}$$

Then, it follows from Proposition 6.1 and (6.9) that

$$d_{\widetilde{V}}(\iota_{\boldsymbol{k}}(g k_j) \cdot \gamma(s_j), \, z_0) \geq d_{\widetilde{V}}(z_0, \, a_j \cdot z_0) - 2n\sqrt{2dn} \, \log D - C_7 > \lambda_j - 2n\sqrt{2dn} \, \log D - C_7.$$

Since $g$ is an arbitrary element of $\Gamma$, we obtain

$$\overline{d}(\Pi(\gamma(s_j)), \, \Pi(z_0)) \geq \lambda_j - 2n\sqrt{2dn} \, \log D - C_7.$$

Hence $\Pi \circ \gamma([0, \infty))$ is not bounded because $\lim_{j \to \infty} \lambda_j = \infty$. This is a contradiction. $\square$

Let $C_8$ be the diameter of the compact set $b(\widetilde{\gamma}_\alpha)^{-1}([-C_6, \infty)) \cap \mathcal{S}$. For any $s \geq 0$, there exist a number $i_0 \in \{1, \ldots, h\}$ and $g \in \Gamma$ such that $\iota_{\boldsymbol{k}}(g) \cdot \gamma(s) \in \iota_{\boldsymbol{k}}(g_{i_0}) \cdot \mathcal{S}$. From Claim we have

$$\iota_{\boldsymbol{k}}(g) \cdot \gamma(s) \in \iota_{\boldsymbol{k}}(g_{i_0}) \cdot \left\{ b(\widetilde{\gamma}_\alpha)^{-1}([-C_6, \, \infty)) \cap \mathcal{S} \right\}$$

and

$$d_{\widetilde{V}}(z_0, \, \iota_{\boldsymbol{k}}(g_{i_0}^{-1} g) \cdot \gamma(s)) \leq C_8.$$

Then, for any $j \in \{1, \ldots, h\}$ and $g' \in \Gamma$, we obtain

$$
\begin{aligned}
&b(\widetilde{\gamma}_\alpha)(\iota_{\boldsymbol{k}}(g_j^{-1}{g'}^{-1})\cdot\gamma(s)) = b(\widetilde{\gamma}_\alpha)(\iota_{\boldsymbol{k}}((g_j^{-1}{g'}^{-1}g^{-1}g_{i_0})g_{i_0}^{-1}g^{-1})\cdot\gamma(s))\\
&\quad\geq b(\widetilde{\gamma}_\alpha)(\iota_{\boldsymbol{k}}(g_j^{-1}{g'}^{-1}g^{-1}g_{i_0})\cdot z_0)\\
&\qquad\qquad - d_{\widetilde{V}}(\iota_{\boldsymbol{k}}((g_j^{-1}{g'}^{-1}g^{-1}g_{i_0})g_{i_0}^{-1}g)\cdot\gamma(s),\ \iota_{\boldsymbol{k}}(g_j^{-1}{g'}^{-1}g^{-1}g_{i_0})\cdot z_0)\\
&\quad= b(\widetilde{\gamma}_\alpha)(\iota_{\boldsymbol{k}}(g_j^{-1}{g'}^{-1}g^{-1}g_{i_0})\cdot z_0) - d_{\widetilde{V}}(\iota_{\boldsymbol{k}}(g_{i_0}^{-1}g)\cdot\gamma(s),\ z_0)\\
&\quad\geq b(\widetilde{\gamma}_\alpha)(\iota_{\boldsymbol{k}}(g_j^{-1}({g'}^{-1}g^{-1})g_{i_0})\cdot z_0) - C_8.
\end{aligned}
$$

Since

$$
gg',\ (D^{n-1}I_n)g_j,\ (DI_n)g_{i_0}^{-1} \in M(n,n;\mathcal{O}_{\boldsymbol{k}}),
$$

we have

$$
(D^n I_n)\left\{g_j^{-1}({g'}^{-1}g^{-1})g_{i_0}\right\}^{-1} \in M(n,n;\mathcal{O}_{\boldsymbol{k}}).
$$

By the same argument as that in the proof of Proposition 6.1, we have

$$
b(\widetilde{\gamma}_\alpha)(\iota_{\boldsymbol{k}}(g_j^{-1}({g'}^{-1}g^{-1})g_{i_0})\cdot z_0) \geq -2n^2\sqrt{2dn}\,\log D.
$$

Let $C_9 = 2n^2\sqrt{2dn}\,\log D$. Then we have

$$
b(\iota_{\boldsymbol{k}}(g'g_j)\cdot\widetilde{\gamma}_\alpha)(\gamma(s)) = b(\widetilde{\gamma}_\alpha)(\iota_{\boldsymbol{k}}(g_j^{-1}{g'}^{-1})\cdot\gamma(s)) \geq -(C_8+C_9).
$$

We conclude

$$
\gamma(s)\notin\bigcup_{j=1}^{h} B(\iota_{\boldsymbol{k}}(g'g_j)\cdot\widetilde{\gamma}_\alpha,\ C_8+C_9) = \bigcup_{j=1}^{h}\iota_{\boldsymbol{k}}(g'g_j)\cdot B(\widetilde{\gamma}_\alpha,\ C_8+C_9)
$$

for all $g'\in\mathit{\Gamma}$, $s\in[0,\infty)$, which shows that the condition (1) is satisfied. □

Applying Lemma 6.1 to the geodesic ray $\gamma^*$, we obtain Theorem 1.5 from Theorem 1.4.

We consider three natural projections $f_1:\widetilde{G}\longrightarrow\widetilde{G}/\widetilde{K}$, $f_2:\widetilde{G}\longrightarrow\iota_{\boldsymbol{k}}(\mathit{\Gamma})\backslash\widetilde{G}$ and $\pi:\iota_{\boldsymbol{k}}(\mathit{\Gamma})\backslash\widetilde{G}\longrightarrow\iota_{\boldsymbol{k}}(\mathit{\Gamma})\backslash\widetilde{G}/\widetilde{K}$.

We remark that $\widetilde{V}=\widetilde{G}/\widetilde{K}$ is equipped with the quotient topology induced from $f_1$ and $\iota_{\boldsymbol{k}}(\mathit{\Gamma})\backslash\widetilde{G}$ is equipped with the quotient topology from $f_2$. Then the quotient topology on $\iota_{\boldsymbol{k}}(\mathit{\Gamma})\backslash\widetilde{G}/\widetilde{K}$ induced from $\Pi$ coincides with the quotient topology induced from $\pi$ because $\Pi\circ f_1=\pi\circ f_2$.

Lemma 6.2. *Let $E$ be a subset of $\widetilde{G}$. Then $\Pi\circ f_1(E)$ is relatively compact in $\iota_{\boldsymbol{k}}(\mathit{\Gamma})\backslash\widetilde{G}/\widetilde{K}$ if and only if $f_2(E)$ is relatively compact in $\iota_{\boldsymbol{k}}(\mathit{\Gamma})\backslash\widetilde{G}$.*

Proof. Suppose that $f_2(E)$ is relatively compact. Then there exists a compact subset $\mathcal{V}$ of $\iota_{\boldsymbol{k}}(\mathit{\Gamma})\backslash\widetilde{G}$ such that $f_2(E)\subset\mathcal{V}$. Since $\Pi\circ f_1(E)=\pi\circ f_2(E)\subset\pi(\mathcal{V})$ and $\pi(\mathcal{V})$ is compact, $\Pi\circ f_1(E)$ is relatively compact.

Suppose that $\Pi \circ f_1(E)$ is relatively compact. Then there exists a compact subset $\mathcal{W}$ of $\iota_{\boldsymbol{k}}(\Gamma)\backslash\widetilde{G}/\widetilde{K}$ such that $\Pi \circ f_1(E) \subset \mathcal{W}$.

Let $w \in \mathcal{W}$. We choose a point $\widetilde{w}$ of $\widetilde{V} = \widetilde{G}/\widetilde{K}$ such that $\Pi(\widetilde{w}) = w$. Since the action of $\iota_{\boldsymbol{k}}(\Gamma)$ on $\widetilde{V}$ is properly discontinuous, there exists an open neighborhood $\mathcal{U}$ of $\widetilde{w}$ such that

$$\#\bigl\{g \in \Gamma \,|\, \iota_{\boldsymbol{k}}(g) \cdot \widetilde{w} \in \mathcal{U}\bigr\} < \infty.$$

For a positive number $r$ we consider an open ball

$$B_{\widetilde{w}}(2r) = \bigl\{z \in \widetilde{V} \,\big|\, d_{\widetilde{V}}(z, \widetilde{w}) < 2r\bigr\}$$

of radius $2r$ in $\widetilde{V}$. If $r$ is sufficiently small, then the following holds: if $g \in \Gamma$, then $\iota_{\boldsymbol{k}}(g) \cdot B_{\widetilde{w}}(2r) = B_{\widetilde{w}}(2r)$ or $\iota_{\boldsymbol{k}}(g) \cdot B_{\widetilde{w}}(2r) \cap B_{\widetilde{w}}(2r) = \emptyset$. By replacing $r$ with a smaller positive number if necessary, we may suppose that the closure of $B_{\widetilde{w}}(r)$ is compact and contractible. Let

$$\widetilde{U}_w = B_{\widetilde{w}}(r), \quad U_w = \Pi(B_{\widetilde{w}}(r)).$$

Then

$$\Pi^{-1}(U_w) = \bigcup_{g \in \Gamma} \iota_{\boldsymbol{k}}(g) \cdot \widetilde{U}_w$$

and $U_w$ is an open subset of $\iota_{\boldsymbol{k}}(\Gamma)\backslash\widetilde{G}/\widetilde{K}$ containing $w$.

For each $w \in \mathcal{W}$ we take such an open subset $U_w$ of $\iota_{\boldsymbol{k}}(\Gamma)\backslash\widetilde{G}/\widetilde{K}$. Then $\{U_w\}_{w \in \mathcal{W}}$ is an open covering of $\mathcal{W}$. Since $\mathcal{W}$ is compact we can choose a finite subcovering, which we denote by $U_1, \dots, U_q$. We also denote by $\widetilde{U}_1, \dots, \widetilde{U}_q$ the corresponding open subsets $\widetilde{U}_w$ of $\widetilde{V}$:

$$U_i = \Pi(\widetilde{U}_i) \quad \text{for all } \ i = 1, \dots, q.$$

Let $F_i$ be the closure of $\widetilde{U}_i$. Then $F_i$ is a compact subset of $\widetilde{V}$ and

$$\bigcup_{i=1}^{q} \Pi(F_i) \supset \bigcup_{i=1}^{q} U_i \supset \mathcal{W} \supset \Pi \circ f_1(E).$$

The map $f_1 : \widetilde{G} \longrightarrow \widetilde{G}/\widetilde{K}$ is a fiber bundle with fiber $\widetilde{K}$. Since each $F_i$ is contractible, the restriction of $f_1$ to $f_1^{-1}(F_i)$ is a trivial bundle. Hence $f_1^{-1}(F_i)$ is diffeomorphic to the product $F_i \times \widetilde{K}$ and in particular compact.

Let

$$Z = \bigcup_{i=1}^{q} f_1^{-1}(F_i).$$

Then $Z$ is compact and

$$f_1(Z) = \bigcup_{i=1}^{q} F_i, \quad \Pi \circ f_1(Z) = \bigcup_{i=1}^{q} \Pi(F_i) \supset \Pi \circ f_1(E).$$

This means that for each $a \in E$ there exist $g \in \Gamma$ and $b \in Z$ such that $f_1(a) = \iota_{\boldsymbol{k}}(g) \cdot f_1(b)$. In other words, we have $a\widetilde{K} = \iota_{\boldsymbol{k}}(g) b\widetilde{K}$ and

$$a \in \iota_{\boldsymbol{k}}(\Gamma) \cdot (b\widetilde{K}) \subset \iota_{\boldsymbol{k}}(\Gamma) \cdot (Z\widetilde{K}).$$

Since $a$ is arbitrary, we have $E \subset \iota_{\boldsymbol{k}}(\Gamma) \cdot (Z\widetilde{K})$ and $f_2(E) \subset f_2(Z\widetilde{K})$. This shows that $f_2(E)$ is relatively compact because $Z\widetilde{K}$, and hence $f_2(Z\widetilde{K})$ is compact. $\square$

From Lemma 6.2 it follows that Theorem 1.5 is equivalent to Theorem 1.6.

## 7. Dual systems of $\boldsymbol{k}_M$-forms.

In this section we prove Theorem 1.8.

From Lemma 6.1 it is natural to ask if analogous results to Theorem 1.3, Theorem 1.4 (and hence Theorems 1.5, 1.6) hold for geodesic rays $\widetilde{\gamma}_\alpha$ in $\widetilde{V}$, with $\alpha \in \mathfrak{a}^+ - \{0\} = \widehat{\mathfrak{a}}^+ - \{0\}$, different from $\gamma^*$.

We show that the following geodesic ray $\gamma_*$ is related to the system of $\boldsymbol{k}_M$-forms $L'_1, \ldots, L'_p$ induced from the transpose ${}^tL \in M(n-p, p; \boldsymbol{k}_M)$ of $L$.

Let $\alpha_*$ be the diagonal matrix in $\mathfrak{a}^+ = \widehat{\mathfrak{a}}^+$ such that the last $n-1$ diagonal elements are equal to $-1$:

$$\alpha_* = \mathrm{diag}(n-1,\ -1, \ldots, -1).$$

Then we have

$$|\alpha_*| = \sqrt{2n^2(n-1)}, \quad \|\alpha_*\| = \sqrt{4n^2(n-1)}.$$

We define a geodesic ray $\gamma_* : [0, \infty) \longrightarrow \widetilde{V} = V^l \times \widehat{V}^m$ by

$$\gamma_*(s) = \widetilde{\gamma}_{\alpha_*}(s) = \left( \gamma_{\alpha_*}\left(\frac{s}{\sqrt{d}}\right), \ldots, \gamma_{\alpha_*}\left(\frac{s}{\sqrt{d}}\right),\ \widehat{\gamma}_{\alpha_*}\left(\frac{\sqrt{2}\, s}{\sqrt{d}}\right), \ldots, \widehat{\gamma}_{\alpha_*}\left(\frac{\sqrt{2}\, s}{\sqrt{d}}\right) \right) \tag{7.1}$$

for $s \geq 0$.

Let $g = (g_{ij}) \in SL(n, \boldsymbol{k})$. We calculate the value of $b(\gamma_*)(\iota_{\boldsymbol{k}}(g) \cdot \gamma_L(s))$. For this, we recall that $L = (L_{ij}) \in M(p, n-p; \boldsymbol{k}_M)$,

$$L_{ij} = (L_{ij}^1, \ldots, L_{ij}^l,\ L_{ij}^{l+1}, \ldots, L_{ij}^{l+m}) \in \boldsymbol{k}_M$$

and

$$\begin{aligned} L^q &= (L_{ij}^q) \in M(p, n-p; \mathbf{R}) \quad \text{if } \ 1 \leq q \leq l, \\ L^q &= (L_{ij}^q) \in M(p, n-p; \mathbf{C}) \quad \text{if } \ l+1 \leq q \leq l+m. \end{aligned}$$

If we write

$$\gamma_L(s) = (\gamma_L^1(s), \ldots, \gamma_L^{l+m}(s)),$$

then

$$\gamma_L^q(s) = \begin{cases} u_L^q e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)} \cdot x_0 & \text{if } 1 \leq q \leq l, \\ u_L^q e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)} \cdot \hat{x}_0 & \text{if } l+1 \leq q \leq l+m, \end{cases} \tag{7.2}$$

where

$$u_L^q = \begin{pmatrix} I_p & L^q \\ O & I_{n-p} \end{pmatrix}.$$

Let $(L_j^q)'$ be the linear form determined by the $j$th column of ${}^t(L^q)$. If $1 \leq q \leq l$, then $(L_j^q)' : \mathbf{R}^{n-p} \longrightarrow \mathbf{R}$ is given by

$$(L_j^q)'(\boldsymbol{y}) = \sum_{k=1}^{n-p} L_{jk}^q y_k \quad \text{for } \boldsymbol{y} = (y_1, \ldots, y_{n-p}) \in \mathbf{R}^{n-p}.$$

If $l+1 \leq q \leq l+m$, then $(L_j^q)' : \mathbf{C}^{n-p} \longrightarrow \mathbf{C}$ is given by

$$(L_j^q)'(\boldsymbol{y}) = \sum_{k=1}^{n-p} L_{jk}^q y_k \quad \text{for } \boldsymbol{y} = (y_1, \ldots, y_{n-p}) \in \mathbf{C}^{n-p}.$$

Let

$$\xi_i = (\xi_i^1, \ldots, \xi_i^{l+m}) \in \boldsymbol{k}_M \quad \text{for } i = 1, \ldots, n-p,$$

and let

$$\xi^q = (\xi_1^q, \ldots, \xi_{n-p}^q) \quad \text{for } q = 1, \ldots, l+m.$$

If $1 \leq q \leq l$, then $\xi^q \in \mathbf{R}^{n-p}$ and if $l+1 \leq q \leq l+m$, then $\xi^q \in \mathbf{C}^{n-p}$. Let

$$\xi = (\xi_1, \ldots, \xi_{n-p}) \in (\boldsymbol{k}_M)^{n-p}.$$

Then we have

$$\boldsymbol{L}'(\xi) = (L_1'(\xi), \ldots, L_p'(\xi)) \in (\boldsymbol{k}_M)^p$$

and

$$L_j'(\xi) = ((L_j^1)'(\xi^1), \ldots, (L_j^{l+m})'(\xi^{l+m})) \in \boldsymbol{k}_M \quad \text{for } j = 1, \ldots, p.$$

From Lemma 3.3 we have

$$b(\gamma_{\alpha_*})(z) = \frac{n}{|\alpha_*|} \log\{\square_{n-1}(z)\}^n = \frac{n}{\sqrt{2(n-1)}} \log \square_{n-1}(z) \quad \text{for } z \in P(n, \mathbf{R}) \tag{7.3}$$

and

$$b(\widehat{\gamma}_{\alpha_*})(z) = \frac{2n}{\|\alpha_*\|} \log\{\Box_{n-1}(z)\}^n = \frac{n}{\sqrt{n-1}} \log \Box_{n-1}(z) \quad \text{for} \quad z \in P(n, \mathbf{C}). \tag{7.4}$$

From (3.11), (7.3) and (7.4) we have

$$\begin{aligned}
b(\gamma_*)(z_1, \dots, z_{l+m}) &= \frac{1}{\sqrt{d}} \left\{ \sum_{j=1}^{l} b(\gamma_{\alpha_*})(z_j) \right\} + \frac{\sqrt{2}}{\sqrt{d}} \left\{ \sum_{j=l+1}^{l+m} b(\widehat{\gamma}_{\alpha_*})(z_j) \right\} \\
&= \frac{n}{\sqrt{2d(n-1)}} \left\{ \sum_{j=1}^{l} \log \Box_{n-1}(z_j) \right\} + \frac{\sqrt{2}\, n}{\sqrt{d(n-1)}} \left\{ \sum_{j=l+1}^{l+m} \log \Box_{n-1}(z_j) \right\} \\
&= \frac{n}{\sqrt{2d(n-1)}} \log \left\{ \left( \prod_{j=1}^{l} \Box_{n-1}(z_j) \right) \left( \prod_{j=l+1}^{l+m} \Box_{n-1}(z_j) \right)^2 \right\}
\end{aligned} \tag{7.5}$$

for $z_1, \dots, z_l \in P(n, \mathbf{R})$ and $z_{l+1}, \dots, z_{l+m} \in P(n, \mathbf{C})$.

For $z \in P(n, \mathbf{C})$, the minor determinant $\Box_{n-1}(z)$ is the $(1,1)$-cofactor of $z$ and is equal to the $(1,1)$-entry of $z^{-1}$. Let

$$z = (z_1, \dots, z_{l+m}) = \iota_{\boldsymbol{k}}(g) \cdot \gamma_L(s).$$

From (7.2), we have

$$z_q = g^{(q)} u_L^q e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)} \cdot x_0 = g^{(q)} u_L^q e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)} e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)}\, {}^t u_L^q\, {}^t g^{(q)}$$

if $1 \le q \le l$, and

$$z_q = g^{(q)} u_L^q e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)} \cdot \hat{x}_0 = g^{(q)} u_L^q e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)} e^{-s\alpha_0/(\sqrt{d}\,|\alpha_0|)}\, {}^t \overline{u_L^q}\, {}^t \overline{g^{(q)}}$$

if $l+1 \le q \le l+m$. Then we have

$$\begin{aligned}
z_q^{-1} &= \left\{ {}^t g^{(q)} \right\}^{-1} {}^t\!\left( u_L^q \right)^{-1} e^{s\alpha_0/(\sqrt{d}\,|\alpha_0|)} e^{s\alpha_0/(\sqrt{d}\,|\alpha_0|)} \left( u_L^q \right)^{-1} \left\{ g^{(q)} \right\}^{-1} \\
&= {}^t\!\left\{ g^{(q)} \right\}^{-1} {}^t\!\left( u_L^q \right)^{-1} e^{s\alpha_0/(\sqrt{d}\,|\alpha_0|)} \cdot x_0
\end{aligned}$$

if $1 \le q \le l$, and

$$\begin{aligned}
z_q^{-1} &= \left\{ {}^t \overline{g^{(q)}} \right\}^{-1} {}^t\!\left( \overline{u_L^q} \right)^{-1} e^{s\alpha_0/(\sqrt{d}\,|\alpha_0|)} e^{s\alpha_0/(\sqrt{d}\,|\alpha_0|)} \left( u_L^q \right)^{-1} \left\{ g^{(q)} \right\}^{-1} \\
&= {}^t\!\left\{ \overline{g^{(q)}} \right\}^{-1} {}^t\!\left( \overline{u_L^q} \right)^{-1} e^{s\alpha_0/(\sqrt{d}\,|\alpha_0|)} \cdot \hat{x}_0
\end{aligned}$$

if $l+1 \le q \le l+m$.

Remark that

$${}^t\!\left( \overline{u_L^q} \right)^{-1} = \begin{pmatrix} I_p & O \\ -{}^t\overline{L^q} & I_{n-p} \end{pmatrix}.$$

Let $(c_1^q \cdots c_n^q)$ be the 1st row of the matrix

$$
{}^t\Bigl\{\overline{g^{(q)}}\Bigr\}^{-1}\,{}^t\bigl(\overline{u_L^q}\bigr)^{-1} e^{s\alpha_0/(\sqrt{d}\,|\alpha_0|)} = {}^t\Bigl\{\overline{g^{(q)}}\Bigr\}^{-1}\begin{pmatrix} e^{s/(\sqrt{d}\,|\alpha_0|)} I_p & O \\ -e^{s/(\sqrt{d}\,|\alpha_0|)}\,{}^t\overline{L^q} & e^{-\lambda s/(\sqrt{d}\,|\alpha_0|)} I_{n-p} \end{pmatrix}.
$$

Let $\widetilde{a}_{ij}$ be the $(i,j)$-cofactor of $g$ and $\widetilde{\boldsymbol{a}} = (\widetilde{a}_{1,p+1}, \ldots, \widetilde{a}_{1n})$. Then we have

$$
\begin{aligned}
c_i^q &= \overline{\Bigl\{\widetilde{a}_{1i}^{(q)} - (L_i^q)'(\widetilde{\boldsymbol{a}}^{(q)})\Bigr\}}\, e^{s/(\sqrt{d}\,|\alpha_0|)} && \text{for } i = 1, \ldots, p, \\
c_{p+j}^q &= \overline{\widetilde{a}_{1,p+j}^{(q)}}\, e^{-\lambda s/(\sqrt{d}\,|\alpha_0|)} && \text{for } j = 1, \ldots, n-p.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
b(\gamma_*)(\iota_{\boldsymbol{k}}(g)\cdot\gamma_L(s)) &= b(\gamma_*)(z_1, \ldots, z_{l+m}) \\
&= \frac{n}{\sqrt{2d(n-1)}} \log\left\{\left(\prod_{q=1}^{l} \square_{n-1}(z_q)\right)\left(\prod_{q=l+1}^{l+m} \square_{n-1}(z_q)\right)^2\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\square_{n-1}(z_q) &= |c_1^q|^2 + |c_2^q|^2 + \cdots + |c_n^q|^2 \\
&= |\widetilde{\boldsymbol{a}}^{(q)}|^2 e^{-2\lambda s/(\sqrt{d}\,|\alpha_0|)} + \left(\sum_{i=1}^{p} |(L_i^q)'(\widetilde{\boldsymbol{a}}^{(q)}) - \widetilde{a}_{1i}^{(q)}|^2\right) e^{2s/(\sqrt{d}\,|\alpha_0|)}.
\end{aligned}
$$

By the same argument as that in the proof of Theorem 1.3 we obtain the following.

THEOREM 7.1. *Let $D'$ be a positive integer, $g = (a_{ij}) \in SL(n, \boldsymbol{k})$, and $\widetilde{a}_{ij}$ the $(i,j)$-cofactor of $g$. Suppose that all the entries in the 1st row of the matrix $(D' I_n)\,{}^t g^{-1}$ belong to $\mathcal{O}_{\boldsymbol{k}}$ and*

$$
\tau \geq \frac{\sqrt{2d}\,n}{\sqrt{n-1}} \log D'.
$$

*Then we have the following, where $\kappa = \exp\{-\sqrt{2(n-1)}\,\tau/(n\sqrt{d})\}$.*

(1) *If $\widetilde{\boldsymbol{a}} = (\widetilde{a}_{1,p+1}, \ldots, \widetilde{a}_{1n}) \neq \boldsymbol{0}$ and*

$$
\|\iota_{\boldsymbol{k}}(\widetilde{\boldsymbol{a}})\|^{n-p} \|\boldsymbol{L}'(\iota_{\boldsymbol{k}}(\widetilde{\boldsymbol{a}})) - \iota_{\boldsymbol{k}}(\widetilde{\boldsymbol{b}})\|^p < \left(\frac{\kappa}{n}\right)^{n/2},
$$

*where $\widetilde{\boldsymbol{b}} = (\widetilde{a}_{11}, \ldots, \widetilde{a}_{1p})$, then $\gamma_L([0,\infty))$ intersects $\iota_{\boldsymbol{k}}(g)^{-1}\cdot B(\gamma_*, \tau)$.*

(2) *If $\gamma_L([0,\infty))$ intersects $\iota_{\boldsymbol{k}}(g)^{-1}\cdot B(\gamma_*, \tau)$, then $\widetilde{\boldsymbol{a}} \neq \boldsymbol{0}$ and there exists a unit $\omega \in \mathcal{O}_{\boldsymbol{k}}$ such that*

$$
\|\iota_{\boldsymbol{k}}(\omega\widetilde{\boldsymbol{a}})\|^{n-p} \|\boldsymbol{L}'(\iota_{\boldsymbol{k}}(\omega\widetilde{\boldsymbol{a}})) - \iota_{\boldsymbol{k}}(\omega\widetilde{\boldsymbol{b}})\|^p < (C_1\kappa)^{n/2}.
$$

From this we obtain the following by the same argument as that in the proof of

Theorem 1.4.

THEOREM 7.2. *Let $L \in M(p, n-p; \boldsymbol{k}_M)$. The following two conditions are equivalent.*

(1) *There exists a non-negative number $\tau$ such that $\gamma_L([0,\infty))$ does not intersect*

$$\bigcup_{i=1}^{h} \bigcup_{g\in\Gamma} \iota_{\boldsymbol{k}}(g)\iota_{\boldsymbol{k}}(g_i) \cdot B(\gamma_*, \tau).$$

(2) *The system of $\boldsymbol{k}_M$-forms $L'_1, \ldots, L'_p$ induced from the transpose ${}^tL \in M(n-p, p; \boldsymbol{k}_M)$ of $L$ is badly approximable.*

Combining Theorem 7.2 with Lemma 6.1, we obtain

THEOREM 7.3. *The following two conditions are equivalent.*

(1) *The system of $\boldsymbol{k}_M$-forms induced from the transpose ${}^tL$ of $L$ is badly approximable.*
(2) $\Pi \circ \gamma_L([0,\infty))$ *is relatively compact in $\iota_{\boldsymbol{k}}(\Gamma)\backslash\widetilde{V}$.*

Theorem 1.8 now follows from Theorem 1.5 and Theorem 7.3.

## 8. Constructing badly approximable systems of $\boldsymbol{k}_M$-forms.

Let $\boldsymbol{k}'$ be a number field of degree $d' = l' + 2m'$ with $l'$ real places and $m'$ complex places. Suppose that $\boldsymbol{k}'$ is a subfield of $\boldsymbol{k}$. We denote by $\iota'_1, \ldots, \iota'_{l'} : \boldsymbol{k}' \longrightarrow \mathbf{R}$ the real embeddings and $\iota'_{l'+1}, \ldots, \iota'_{l'+m'} : \boldsymbol{k}' \longrightarrow \mathbf{C}$ the complex embeddings which are not complex conjugate to each other.

For each $j \in \{1, \ldots, l+m\}$, we consider the restriction $\iota_j|_{\boldsymbol{k}'}$ of $\iota_j$ to $\boldsymbol{k}'$. If the image of this monomorphism $\iota_j|_{\boldsymbol{k}'}$ is contained in $\mathbf{R}$, there exists a unique number $n_j \in \{1, \ldots, l'\}$ such that $\iota_j|_{\boldsymbol{k}'} = \iota'_{n_j}$. If the image of $\iota_j|_{\boldsymbol{k}'}$ is not contained in $\mathbf{R}$, there exists a unique number $n_j \in \{l'+1, \ldots, l'+m'\}$ such that $\iota_j|_{\boldsymbol{k}'} = \iota'_{n_j}$ or $\iota_j|_{\boldsymbol{k}'}$ coincides with the complex conjugate of $\iota'_{n_j}$.

Let $\boldsymbol{k}'_M = \mathbf{R}^{l'} \times \mathbf{C}^{n'}$ be the Minkowski space associated to $\boldsymbol{k}'$. Then there exists a natural embedding $\varphi_{\boldsymbol{k}',\boldsymbol{k}} : \boldsymbol{k}'_M \longrightarrow \boldsymbol{k}_M$ defined by

$$\varphi_{\boldsymbol{k}',\boldsymbol{k}}(\xi_1, \ldots, \xi_{l'+m'}) = (\eta_1, \ldots, \eta_{l+m}),$$

where $\eta_j = \xi_{n_j}$ if $\iota_j|_{\boldsymbol{k}'} = \iota'_{n_j}$, $\eta_j = \overline{\xi_{n_j}}$ if the image of $\iota_j|_{\boldsymbol{k}'}$ is not contained in $\mathbf{R}$ and $\iota_j|_{\boldsymbol{k}'}$ coincides with the complex conjugate of $\iota'_{n_j}$. This map can be extended to an embedding $\varphi_{\boldsymbol{k}',\boldsymbol{k}} : M(p, n-p; \boldsymbol{k}'_M) \longrightarrow M(p, n-p; \boldsymbol{k}_M)$ by

$$\varphi_{\boldsymbol{k}',\boldsymbol{k}}(L') = \big(\varphi_{\boldsymbol{k}',\boldsymbol{k}}(L'_{ij})\big) \quad \text{for} \;\; L' = (L'_{ij}) \in M(p, n-p; \boldsymbol{k}'_M).$$

We define the twisted diagonal embedding $\iota_{\boldsymbol{k}'} : \boldsymbol{k}' \longrightarrow \boldsymbol{k}'_M$ by

$$\iota_{\boldsymbol{k}'}(a) = (\iota'_1(a), \ldots, \iota'_{l'}(a), \iota'_{l'+1}(a), \ldots, \iota'_{l'+m'}(a)) \quad \text{for} \;\; a \in \boldsymbol{k}',$$

and let $\mathcal{O}_{\boldsymbol{k}'}$ be the ring of integers of $\boldsymbol{k}'$.

We prove Theorem 1.9 in the same manner as in the proof of Proposition 8.5 in [**15**].

Proof of Theorem 1.9. For each $j \in \{l+1, \ldots, l+m\}$ there exists a unique number $n_j \in \{1, \ldots, l'+m'\}$ such that $\iota_j|_{\boldsymbol{k}'} = \iota'_{n_j}$ or $\iota_j|_{\boldsymbol{k}'}$ coincides with the complex conjugate of $\iota'_{n_j}$. In the latter case, we exchange $\iota_j$ with its complex conjugate. After this procedure, we may suppose that for any $j \in \{1, \ldots, l+m\}$ there exists a unique number $n_j \in \{1, \ldots, l'+m'\}$ such that $\iota_j|_{\boldsymbol{k}'} = \iota'_{n_j}$.

When we replace complex embeddings $\iota_{l+k_1}, \ldots, \iota_{l+k_q}$ with their complex conjugates in this way, the twisted embedding $\iota_{\boldsymbol{k}}$ should be changed. Let $\iota''_{\boldsymbol{k}}$ be the resultant embedding and let $\Pi'' : V^l \times \widehat{V}^m \longrightarrow \iota''_{\boldsymbol{k}}(\Gamma)\backslash(V^l \times \widehat{V}^m)$ be the natural projection. For

$$L = (L_{ij}) \in M(p, n-p; \boldsymbol{k}_M);\ L_{ij} = (L^1_{ij}, \ldots, L^l_{ij}, L^{l+1}_{ij}, \ldots, L^{l+m}_{ij}) \in \boldsymbol{k}_M,$$

let

$$L'' = (L''_{ij}) \in M(p, n-p; \boldsymbol{k}_M);\ L''_{ij} = (L^1_{ij}, \ldots, L^l_{ij}, (L''_{ij})^{l+1}, \ldots, (L''_{ij})^{l+m}) \in \boldsymbol{k}_M,$$

where

$$(L''_{ij})^{l+k_1} = \overline{L^{l+k_1}_{ij}}, \ldots, (L''_{ij})^{l+k_q} = \overline{L^{l+k_q}_{ij}}$$

and $(L''_{ij})^{l+k} = L^{l+k}_{ij}$ for other $k$. The diffeomorphism of $\widetilde{G} = (SL(n, \mathbf{R}))^l \times (SL(n, \mathbf{C}))^m$ which sends $(g_1, \ldots, g_{l+m}) \in \widetilde{G}$ to $(g_1, \ldots, g_l, g''_{l+1}, \ldots, g''_{l+m}) \in \widetilde{G}$, where

$$g''_{l+k_1} = \overline{g}_{l+k_1}, \ldots, g''_{l+k_q} = \overline{g}_{l+k_q}$$

and $g''_{l+k} = g_{l+k}$ for other $k$, induces a diffeomorphism of $V^l \times \widehat{V}^m$ and a homeomorphism $\iota_{\boldsymbol{k}}(\Gamma)\backslash(V^l \times \widehat{V}^m) \longrightarrow \iota''_{\boldsymbol{k}}(\Gamma)\backslash(V^l \times \widehat{V}^m)$. Since $\Pi \circ \gamma_L([0, \infty))$ is sent to $\Pi'' \circ \gamma_{L''}([0, \infty))$ by the last homeomorphism, it follows from Theorem 1.5 that $L$ induces a badly approximable system of $\boldsymbol{k}_M$-forms with respect to the embedding $\iota_{\boldsymbol{k}}$ if and only if $L''$ induces a badly approximable system of $\boldsymbol{k}_M$-forms with respect to $\iota''_{\boldsymbol{k}}$.

From this observation, it suffices to prove the assertion of this theorem under the following conditions (A) and (B).

(A) For any $j \in \{1, \ldots, l+m\}$, there exists a unique number $n_j \in \{1, \ldots, l'+m'\}$ such that $\iota_j|_{\boldsymbol{k}'} = \iota'_{n_j}$.

(B) The embedding $\varphi_{\boldsymbol{k}', \boldsymbol{k}}$ is given by

$$\varphi_{\boldsymbol{k}', \boldsymbol{k}}(\xi_1, \ldots, \xi_{l'+m'}) = (\eta_1, \ldots, \eta_{l+m});\ \eta_j = \xi_{n_j} \quad \text{for } j = 1, \ldots, l+m.$$

We define a geodesic ray $\gamma'_0 : [0, \infty) \longrightarrow \widetilde{V}' = V^{l'} \times \widehat{V}^{m'}$ by

$$\gamma'_0(s) = \Big(e^{-s\alpha_0/(\sqrt{d'}\,|\alpha_0|)} \cdot x_0, \ldots, e^{-s\alpha_0/(\sqrt{d'}\,|\alpha_0|)} \cdot x_0,$$
$$e^{-s\alpha_0/(\sqrt{d'}\,|\alpha_0|)} \cdot \widehat{x}_0, \ldots, e^{-s\alpha_0/(\sqrt{d'}\,|\alpha_0|)} \cdot \widehat{x}_0\Big)$$

for $s \geq 0$. Let $L' = (L'_{ij}) \in M(p, n-p; \boldsymbol{k}'_M)$ and

$$L'_{ij} = \Big((L'_{ij})^1, \ldots, (L'_{ij})^{l'}, (L'_{ij})^{l'+1}, \ldots, (L'_{ij})^{l'+m'}\Big) \in \boldsymbol{k}'_M = \mathbf{R}^{l'} \times \mathbf{C}^{m'}$$

for each $i$, $j$. We put

$$(L')^q = \big((L'_{ij})^q\big) \in M(p, n-p; \mathbf{R})$$

for $q = 1, \ldots, l'$, and

$$(L')^q = \big((L'_{ij})^q\big) \in M(p, n-p; \mathbf{C})$$

for $q = l'+1, \ldots, l'+m'$. Let

$$u^q_{L'} = \begin{pmatrix} I_p & (L')^q \\ O & I_{n-p} \end{pmatrix} \in SL(n, \mathbf{R})$$

for $q = 1, \ldots, l'$,

$$u^q_{L'} = \begin{pmatrix} I_p & (L')^q \\ O & I_{n-p} \end{pmatrix} \in SL(n, \mathbf{C})$$

for $q = l'+1, \ldots, l'+m'$, and

$$u_{L'} = (u^1_{L'}, \ldots, u^{l'}_{L'}, u^{l'+1}_{L'}, \ldots, u^{l'+m'}_{L'}) \in G^{l'} \times \widehat{G}^{m'}.$$

We define a geodesic ray $\gamma_{L'} : [0, \infty) \longrightarrow V^{l'} \times \widehat{V}^{m'}$ by

$$\gamma_{L'}(s) = u_{L'} \cdot \gamma'_0(s) \quad \text{for} \;\; s \geq 0. \tag{8.1}$$

There is a natural embedding $\iota_{V,\widehat{V}} : V \longrightarrow \widehat{V}$ defined by

$$\iota_{V,\widehat{V}}(g \cdot x_0) = g \cdot \hat{x}_0 \quad \text{for} \;\; g \in SL(n, \mathbf{R}).$$

We identify each point $z \in V$ with $\iota_{V,\widehat{V}}(z) \in \widehat{V}$. Let $\Delta : V^{l'} \times \widehat{V}^{m'} \longrightarrow V^l \times \widehat{V}^m$ be an embedding defined by

$$\begin{aligned} \Delta(z_1, \ldots, z_{l'+m'}) &= (w_1, \ldots, w_{l+m}); \\ w_j = z_{n_j} &\quad \text{for} \;\; j = 1, \ldots, l+m. \end{aligned}$$

We also define an embedding $\iota_0 : G^{l'} \times \widehat{G}^{m'} \longrightarrow G^l \times \widehat{G}^m$ by

$$\begin{aligned} \iota_0(g'_1, \ldots, g'_{l'+m'}) &= (g_1, \ldots, g_{l+m}); \\ g_j = g'_{n_j} &\quad \text{for} \;\; j = 1, \ldots, l+m. \end{aligned}$$

Then we have

$$\Delta(g' \cdot z) = \iota_0(g') \cdot \Delta(z) \quad \text{for} \;\; g' \in G^{l'} \times \widehat{G}^{m'}, \, z \in V^{l'} \times \widehat{V}^{m'}, \tag{8.2}$$

and

$$\Delta\left(\gamma_{L'}\left(\frac{\sqrt{d'}\,s}{\sqrt{d}}\right)\right) = \gamma_L(s) \quad \text{for } s \geq 0, \tag{8.3}$$

where $L = \varphi_{\boldsymbol{k}',\boldsymbol{k}}(L')$.

Let $\Gamma' = SL(n, \mathcal{O}_{\boldsymbol{k}'})$ and let $\Pi_1 : V^l \times \widehat{V}^m \longrightarrow \iota_0(\iota_{\boldsymbol{k}'}(\Gamma'))\backslash(V^l \times \widehat{V}^m)$ be the natural projection to the quotient space. Since $\iota_0(\iota_{\boldsymbol{k}'}(\Gamma'))$ is contained in $\iota_{\boldsymbol{k}}(\Gamma)$, we also have the projection $\Pi_2 : \iota_0(\iota_{\boldsymbol{k}'}(\Gamma'))\backslash(V^l \times \widehat{V}^m) \longrightarrow \iota_{\boldsymbol{k}}(\Gamma)\backslash(V^l \times \widehat{V}^m)$ as in the following diagram.

$$\begin{tikzcd}
V^{l'} \times \widehat{V}^{m'} \arrow[r, "\Delta"] & V^l \times \widehat{V}^m \arrow[r, "\Pi_1"] \arrow[dr, "\Pi"'] & \iota_0(\iota_{\boldsymbol{k}'}(\Gamma'))\backslash(V^l \times \widehat{V}^m) \arrow[d, "\Pi_2"] \\
& & \iota_{\boldsymbol{k}}(\Gamma)\backslash(V^l \times \widehat{V}^m)
\end{tikzcd}$$

Then $\Pi_2$ is continuous and $\Pi = \Pi_2 \circ \Pi_1$.

Let $\Pi' : V^{l'} \times \widehat{V}^{m'} \longrightarrow \iota_{\boldsymbol{k}'}(\Gamma')\backslash(V^{l'} \times \widehat{V}^{m'})$ be the natural projection. We remark that $\iota_0(\iota_{\boldsymbol{k}'}(\Gamma')))$ acts on the image $\operatorname{Im}\Delta$ of the embedding $\Delta$. Let $\Pi_0 : \operatorname{Im}\Delta \longrightarrow \iota_0(\iota_{\boldsymbol{k}'}(\Gamma'))\backslash\operatorname{Im}\Delta$ be the natural projection. It follows from (8.2) that $\Delta$ induces a homeomorphism $\overline{\Delta} : \iota_{\boldsymbol{k}'}(\Gamma')\backslash(V^{l'} \times \widehat{V}^{m'}) \longrightarrow \iota_0(\iota_{\boldsymbol{k}'}(\Gamma'))\backslash\operatorname{Im}\Delta$ defined by

$$\overline{\Delta}(\Pi'(z)) = \Pi_0(\Delta(z)) \quad \text{for } \ z \in V^{l'} \times \widehat{V}^{m'}.$$

Since the system of $\boldsymbol{k}'_M$-forms induced from $L'$ is badly approximable, it follows from Theorem 1.5 that there exists a compact subset $Z$ of $\iota_{\boldsymbol{k}'}(\Gamma')\backslash(V^{l'} \times \widehat{V}^{m'})$ such that

$$\Pi'\left(\gamma_{L'}([0,\infty))\right) \subset Z.$$

From (8.3) we have

$$\Pi_0(\gamma_L([0,\infty))) = \overline{\Delta}\left(\Pi'\big(\gamma_{L'}([0,\infty))\big)\right) \subset \overline{\Delta}(Z).$$

Let $\iota_1 : \iota_0(\iota_{\boldsymbol{k}'}(\Gamma'))\backslash\operatorname{Im}\Delta \longrightarrow \iota_0(\iota_{\boldsymbol{k}'}(\Gamma'))\backslash(V^l \times \widehat{V}^m)$ be the natural inclusion and let $Z' = \iota_1(\overline{\Delta}(Z))$. Then $Z'$ is compact and

$$\Pi_1(\gamma_L([0,\infty))) = \iota_1\left(\Pi_0\big(\gamma_L([0,\infty))\big)\right) \subset \iota_1(\overline{\Delta}(Z)) = Z'.$$

Therefore

$$\Pi \circ \gamma_L([0,\infty)) = \Pi_2\left(\Pi_1\big(\gamma_L([0,\infty))\big)\right) \subset \Pi_2(Z').$$

Since $\Pi_2(Z')$ is compact, $\Pi \circ \gamma_L([0,\infty))$ is relatively compact. From Theorem 1.5 we conclude that the system of $\boldsymbol{k}_M$-forms induced from $L = \varphi_{\boldsymbol{k}',\boldsymbol{k}}(L')$ is badly approximable. $\square$

If $\boldsymbol{k}' = \mathbf{Q}$, then $\boldsymbol{k}'_M = \mathbf{R}$ and badly approximable systems of $\boldsymbol{k}'_M$-forms are the usual badly approximable systems of linear forms. Since the set $\mathcal{B}_{n,p,\mathbf{Q}}$ has the power of the continuum ([**25**, Theorem 1]), we take a matrix $L' = (L'_{ij}) \in M(p, n-p; \mathbf{R})$ which induces a badly approximable system of linear forms. Let $L_{ij} = (L'_{ij}, \ldots, L'_{ij}) \in \boldsymbol{k}_M$ for each $i, j$ and let $L = (L_{ij}) \in M(p, n-p; \boldsymbol{k}_M)$. Then we have $L = \varphi_{\mathbf{Q}, \boldsymbol{k}}(L')$ and $L$ induces a badly approximable system of $\boldsymbol{k}_M$-forms.

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**Note Added in Proof.** After this paper was published online in Advance Publication, the author found that the description of the point $\iota_{\boldsymbol{k}}(g)\cdot\gamma_L(s_0)$ in the lines 12–19 on page 906 was not correct. It should be as follows.

Let $\iota_{\boldsymbol{k}}(g)\cdot\gamma_L(s_0) = ua'a\cdot z_0$;

$$\begin{aligned} u &= \left(u^1,\dots,u^{l+m}\right) \in \widetilde{N} = N^l \times \widehat{N}^m, \\ a' &= \left(a^1,\dots,a^{l+m}\right) \in \widetilde{A}', \quad a = (b,\dots,b) \in \widetilde{A}_{\mathbf{Q}}, \end{aligned}$$

where

$$a^k = \mathrm{diag}(a^k_1,\dots,a^k_n) \ \text{ for } \ k = 1,\dots,l+m$$

and

$$b = \mathrm{diag}(b_1,\dots,b_n) \in A = \widehat{A}.$$

The formula in the first line on page 908 should be replaced with

$$z_q = \begin{pmatrix} \upsilon_1^{(q)} & & \\ & \ddots & \\ & & \upsilon_n^{(q)} \end{pmatrix} u^q \begin{pmatrix} (\upsilon_1^{(q)})^{-1} & & \\ & \ddots & \\ & & (\upsilon_n^{(q)})^{-1} \end{pmatrix} \begin{pmatrix} \rho^q_{\upsilon_1} & & \\ & \ddots & \\ & & \rho^q_{\upsilon_n} \end{pmatrix} \cdot x_0$$

due to this change.

**Editorial Comment.** A few typos were also corrected on this occasion.