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Joint denseness of Hurwitz zeta functions with algebraic irrational parameters

By Yoonbok Lee and Hidehiko Mishou

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Abstract. In this paper, we study the joint denseness of the Riemann zeta function and Hurwitz zeta functions with certain algebraic irrational and transcendental parameters on $\Re s > 1$. We also provide evidence for the denseness of the Hurwitz zeta function with an algebraic irrational parameter on $1/2 < \Re s < 1$.

1. Introduction.

Let α be a real number with $0 < \alpha \leq 1$. The Hurwitz zeta function associated with α is defined by

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s} \tag{1.1}$$

for $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$. The Riemann zeta function $\zeta(s)$ is a special case of the Hurwitz zeta function since $\zeta(s, 1) = \zeta(s)$ and $\zeta(s, 1/2) = (2^s - 1)\zeta(s)$. It is well-known that the Hurwitz zeta function shares some of the properties of the Riemann zeta function such as an analytic continuation and a functional equation. Another interesting property of the Riemann zeta function is universality, which roughly states that vertical shifts of the Riemann zeta function on a closed disc in the right half of the critical strip can approximate every nonvanishing analytic function defined on the disc. See [**1**], [**4**] or [**9**] for details. We expect that universality holds for every Hurwitz zeta function $\zeta(s, \alpha)$ as well.

Indeed, the universality of $\zeta(s, \alpha)$ has been proved for both transcendental and rational α's (by different methods), but not for algebraic irrational α's. For a transcendental α, we apply Kronecker's theorem to the set

$$Z_\alpha := \{\log(n + \alpha) : n = 0, 1, 2, \dots\},$$

which is linearly independent over $\mathbb{Q}$. For a rational $\alpha = a/q$ with $\gcd(a, q) = 1$ and $q > 2$, we use the joint universality of Dirichlet L-functions $L(s, \chi)$ and the identity

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$$\zeta\left(s, \frac{a}{q}\right) = \frac{q^s}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) L(s, \chi). \tag{1.2}$$

Here, Kronecker's theorem is applied to the set

$$\{\log p : p \text{ prime number}\}.$$

It is not clear whether the universality of $\zeta(s, \alpha)$ with algebraic irrational α holds or not.

The linear independence of the set Z_α is important to study the value distribution of $\zeta(s, \alpha)$ for algebraic irrational α's. It is known that the set Z_α for an algebraic irrational α is linearly dependent over $\mathbb{Q}$, unlike the transcendental case. However, it seems that there is no pattern to the dependence, and this makes it hard to apply Kronecker's theorem. To get around this difficulty, we instead consider a maximal linearly independent subset of Z_α. How big is it? It is known to be infinite. Cassels [**2**] proved that at least 51 percent (in the sense of density) of Z_α is linearly independent over $\mathbb{Q}$. For large degree α's, Worley [**10**] improved this by providing bigger lower bounds depending on the degree. These results imply that $\zeta(s, \alpha)$ has infinitely many zeros in the strip $1 < \Re s < 1 + \delta$ for any fixed $\delta > 0$. (See [**2**] for the details.) This extends a theorem of Davenport and Heilbronn [**3**] for transcendental and rational α's with $0 < \alpha < 1$ and $\alpha \neq 1/2$. Moreover, it is easy to see from the proof in [**2**] that the set

$$\{\zeta(s, \alpha) : \Re s > 1\}$$

is dense in $\mathbb{C}$ for each algebraic irrational α. With more effort, the second author [**6**] proved the joint denseness of the Riemann zeta function and a Hurwitz zeta function with an algebraic irrational α. We extend these results further as follows.

Theorem 1. *Let $\alpha_1, \ldots, \alpha_r$ be algebraic irrational numbers in $(0, 1)$ such that*

$$\mathbb{Q}(\alpha_j) \cap \mathbb{Q}(\hat{\alpha}_j) = \mathbb{Q} \tag{1.3}$$

holds for each $j \leq r$, where $\hat{\alpha}_j = \{\alpha_i : i \leq r,\ i \neq j\}$. Let $\alpha_{r+1}, \ldots, \alpha_{r+\ell}$ be real transcendental numbers in $(0, 1)$ which are algebraically independent over $\mathbb{Q}$. Then the set

$$\left\{(\zeta(\sigma + it), \zeta(\sigma + it, \alpha_1), \ldots, \zeta(\sigma + it, \alpha_{r+\ell})) \in \mathbb{C}^{r+\ell+1} : \sigma > 1,\ t \in \mathbb{R}\right\}$$

is dense in $\mathbb{C}^{r+\ell+1}$.

Note that the assumption (1.3) and the algebraic independence imply the linear independence between the sets of Z_{α_j}'s. We will provide a stronger version of Cassels's lemma in Subsection 2.1, which combines ideas from [**2**] and [**10**]. It also explains a gap in the proof of Lemma 2 of [**6**]. Then we will prove Theorem 1 in Section 2.

At this point it is natural to ask about the joint denseness or universality of Hurwitz zeta functions with rational parameters. The joint denseness might be proved by applying the joint denseness of Dirichlet L-functions on $\Re s > 1$ to (1.2). However, the joint universality might not be true. A dual identity to (1.2) gives a representation for a

Dirichlet L-function in terms of Hurwitz zeta functions, so if the joint universality of the Hurwitz zeta functions is true, then by Rouché's theorem one can find an off-line zero of the Dirichlet L-function which would disprove the Riemann hypothesis. This makes it very interesting to study the joint denseness or universality of Hurwitz zeta functions with rational parameters.

As an application of the lemma in 2.1, we also study the denseness of the Hurwitz zeta function $\zeta(s,\alpha)$ for an algebraic irrational number α inside the critical strip. We believe the following conjecture is true.

Conjecture 1. *Let α be an algebraic irrational number with $0<\alpha<1$ and let $1/2<\sigma_0<1$ be fixed. Then the set*

$$\{\zeta(\sigma_0+it,\alpha) : t\in\mathbb{R}\}$$

is dense in $\mathbb{C}$.

Conjecture 1 was also stated in Gonek's thesis [**4**, p.122]. To provide evidence for Conjecture 1, in Section 3 we will study the denseness of the sum

$$\sum_n \frac{\chi(n+\alpha)}{(n+\alpha)^{\sigma_0}}$$

with a suitable sense of convergence, where χ is a character defined on $\{n+\alpha\}_{n\geq 0}$.

Recently, Sourmelidis and Steuding [**7**] announced surprising results including a substantial part of Conjecture 1. Theorem 1 in [**7**] provides explicit simultaneous estimations of $\zeta(s,\alpha)$ and its derivatives in the strip $1/2<\Re s\leq 1$ for algebraic irrational α of large degree. Theorem 2 in [**7**] is the universality theorem of $\zeta(s,\alpha)$ in the strip $1-(2\cdot 3^6)^{-1}\leq\sigma_0<\Re s<1$ for all but finitely many algebraic irrational α in $[A,1]$ of degree at most $(162(1-\sigma_0))^{-1/2}-1$, where $0<A<1$ is given.

2. Proof of Theorem 1.

Let $\alpha_1,\ldots,\alpha_r$ be algebraic irrational numbers and let $\alpha_{r+1},\ldots,\alpha_{r+\ell}$ be transcendental numbers satisfying the conditions in Theorem 1. Then we want to prove that for any $\varepsilon>0$ and any complex numbers $z_0,\ldots,z_{r+\ell}$, there exists a complex number s with $\Re s>1$ such that

$$|\log\zeta(s)-z_0|<\varepsilon$$

and

$$|\zeta(s,\alpha_j)-z_j|<\varepsilon$$

both hold for $j=1,\ldots,r+\ell$. Here is the outline of the proof of Theorem 1. In Subsection 2.1 we first prove an essential lemma. Then, in Subsection 2.2, we define a totally multiplicative function χ from

$$\mathcal{D}_{r+\ell}:=\mathbb{N}\cup\{n+\alpha_j : n\geq 0, j=1,\ldots,r+\ell\}$$

to

$$S^1 := \{z \in \mathbb{C} : |z| = 1\}$$

such that

$$\log \zeta(s : \chi) := -\sum_p \log\left(1 - \frac{\chi(p)}{p^s}\right) = z_0$$

and

$$\zeta(s, \alpha_j : \chi) := \sum_{n \geq 0} \frac{\chi(n + \alpha_j)}{(n + \alpha_j)^s} = z_j$$

for some s in $\Re s > 1$ and all $j = 1, \ldots, r + \ell$. In Subsection 2.3, we apply Kronecker's theorem to find numbers t satisfying

$$\zeta(s + it) \approx \zeta(s : \chi)$$

and

$$\zeta(s + it, \alpha_j) \approx \zeta(s, \alpha_j : \chi)$$

for $j = 1, \ldots, r + \ell$ and $\Re s > 1$, and then apply Rouché's theorem to complete the proof.

2.1. Lemma.

Let α be an algebraic irrational number of degree k and let $\mathfrak{a}$ denote the ideal denominator of α. We define $\mathcal{N}(\mathfrak{b}) = \mathcal{N}_{K/\mathbb{Q}}(\mathfrak{b})$ to be the norm of an ideal $\mathfrak{b}$ of the field $K = \mathbb{Q}(\alpha)$. Let I_α be the set of all prime ideals $\mathfrak{p} \nmid \mathfrak{a}$ in $K = \mathbb{Q}(\alpha)$ satisfying the following three properties:

(1) $\mathfrak{p}$ is of the first degree, that is, $\mathcal{N}(\mathfrak{p}) = p$ is a rational prime,

(2) $\mathfrak{p}$ is unambiguous, that is, $\mathfrak{p}^2 \nmid p$,

(3) if $\mathfrak{p} | (n + \alpha)\mathfrak{a}$ for a nonnegative integer n, then $\mathfrak{p}' \nmid (n + \alpha)\mathfrak{a}$ for any prime ideal $\mathfrak{p}' \neq \mathfrak{p}$ with $\mathcal{N}(\mathfrak{p}') = p$.

One can show that there are only finitely many prime ideals $\mathfrak{p}$ in K such that $\mathfrak{p} \notin I_\alpha$ and $\mathfrak{p} | (n + \alpha)\mathfrak{a}$ for some nonnegative integer n.

Lemma 1. *Let $N \geq 10^7$ and $M = [10^{-6} N]$ be positive integers. For each algebraic irrational number α of degree k, let q be the smallest positive integer such that $q\alpha$ is an algebraic integer. Define $c(\alpha) = 2kq$ and*

$$\begin{aligned} \mathfrak{T}_N := \mathfrak{T}_N(\alpha) := \{&N < n \leq N + M : \\ &\exists \mathfrak{p}_n \in I_\alpha (\mathfrak{p}_n | (n + \alpha)\mathfrak{a}, p_n = \mathcal{N}(\mathfrak{p}_n) > c(\alpha)(N + M))\}. \end{aligned} \tag{2.1}$$

Then there exists an integer $N_0 > 10^7$ depending on α such that

$$\sharp\mathfrak{T}_N \geq \left(1-\left(\frac{1}{2k}+\frac{\sqrt{3}}{4k^{3/2}}+\frac{3}{8k^2}\right)\right)M \tag{2.2}$$

for $N \geq N_0$. Each prime ideal $\mathfrak{p}_n$ in (2.1) satisfies

$$\mathfrak{p}_n \nmid \prod_{\substack{0\leq m\leq N+M\\ m\neq n}} (m+\alpha)\mathfrak{a} \tag{2.3}$$

and

$$(p_n) \nmid \prod_{0\leq m\leq N+M} (m+\alpha)\mathfrak{a}, \tag{2.4}$$

where (p_n) is the principal ideal of $K=\mathbb{Q}(\alpha)$ generated by p_n.

Remark. The first inequality of (2.15) is slightly better than (2.2). For $k=2$ at least 54 percent of integers in $(N, N+M]$ are contained in $\mathfrak{T}_N$. The proportion in (2.2) is an increasing sequence of k, whose value at $k=3$ is approximately 0.708. Hence, for any algebraic irrational α we see that

$$\sharp\mathfrak{T}_N(\alpha) > 0.54M$$

holds for $N \geq N_0$.

Proof. For an integer $n \geq 0$, we may write

$$(n+\alpha)\mathfrak{a} = \mathfrak{b}\prod_{\mathfrak{p}\in I_\alpha} \mathfrak{p}^{u_n(\mathfrak{p})}, \tag{2.5}$$

where the $u_n(\mathfrak{p})$ are nonnegative integers and $\mathfrak{b}$ is the product of all the prime factors of $(n+\alpha)\mathfrak{a}$ which are not in I_α. Define

$$\mathfrak{S}_N := \{N < n \leq N+M : p^{u_n(\mathfrak{p})} \leq M \text{ for all } \mathfrak{p}\in I_\alpha\}$$

for integers $N > 10^7$ and $M = [10^{-6}N]$. We first want to find an upper bound for $S_N := \sharp\mathfrak{S}_N$.

Define

$$\sigma(n) := \sum_{\substack{\mathfrak{p}^v|(n+\alpha)\mathfrak{a}\\ p^v\leq M}} \log p.$$

Since the norm of $\mathfrak{b}$ is bounded,

$$\sigma(n) = \sum_{\substack{\mathfrak{p}|(n+\alpha)\mathfrak{a}\\ \mathfrak{p}\in I_\alpha}} u_n(\mathfrak{p})\log p + O(1)$$

for $n \in \mathfrak{S}_N$. By taking the norm in (2.5) we have that

$$\sum_{\substack{\mathfrak{p} | (n+\alpha)\mathfrak{a} \\ \mathfrak{p} \in I_\alpha}} u_n(\mathfrak{p}) \log p = \log \mathcal{N}((n+\alpha)\mathfrak{a}) + O(1).$$

Since $\mathcal{N}((n+\alpha)\mathfrak{a}) \gg n^k$, we have that

$$\sigma(n) \geq k \log n + O(1)$$

for $n \in \mathfrak{S}_N$. Let $\rho := S_N/M \leq 1$, then

$$\sum_{n \in \mathfrak{S}_N} \sigma(n) \geq (k \log M + O(1)) S_N = (k\rho + o(1)) M \log M. \tag{2.6}$$

We next find an upper bound for the sum in (2.6). We split $\sigma(n)$ into three parts as

$$\sigma(n) = \sigma_1(n) + \sigma_2(n) + \sigma_3(n), \tag{2.7}$$

where $\sigma_1,\ \sigma_2,\ \sigma_3$ are the sums of $\log p$ with $\mathfrak{p}^v | (n+\alpha)\mathfrak{a}$ in the sets

$$\begin{aligned} &\sigma_1: \ v > 1,\ p^v \leq M, \\ &\sigma_2: \ v = 1,\ M^{1/2} \leq p \leq M, \\ &\sigma_3: \ v = 1,\ p < M^{1/2}. \end{aligned}$$

Following the proof of the lemma in [**2**], we find that

$$\sum_{n \in \mathfrak{S}_N} \sigma_1(n) = o(M \log M), \tag{2.8}$$

$$\sum_{n \in \mathfrak{S}_N} \sigma_2(n) \leq \left(\frac{1}{2} + o(1)\right) M \log M, \tag{2.9}$$

and

$$\sum_{n \in \mathfrak{S}_N} (\sigma_3(n))^2 \leq \left(\frac{3}{8} + o(1)\right) M \log^2 M. \tag{2.10}$$

By (2.6)–(2.9), we have

$$\sum_{n \in \mathfrak{S}_N} \sigma_3(n) \geq \left(k\rho - \frac{1}{2} + o(1)\right) M \log M. \tag{2.11}$$

If the right-hand side of (2.11) is negative, then

$$\rho \leq \frac{1}{2k} + o(1). \tag{2.12}$$

Otherwise, by (2.10), (2.11) and the Cauchy–Schwarz inequality

$$\left| \sum_{n \in \mathfrak{S}_N} \sigma_3(n) \right|^2 \leq S_N \sum_{n \in \mathfrak{S}_N} (\sigma_3(n))^2,$$

and we find that

$$\left(k\rho - \frac{1}{2} + o(1) \right)^2 M^2 \log^2 M \leq \left(\frac{3\rho}{8} + o(1) \right) M^2 \log^2 M.$$

From this we obtain

$$8k^2\rho^2 - (8k+3)\rho + 2 \leq o(1).$$

Thus,

$$S_N = \rho M \leq \left(\frac{8k+3+\sqrt{48k+9}}{16k^2} + o(1) \right) M. \tag{2.13}$$

The number of $n \in (N, N+M] \setminus \mathfrak{S}_N$ is

$$\geq M - \sharp \mathfrak{S}_N \geq \left(1 - \frac{8k+3+\sqrt{48k+9}}{16k^2} + o(1) \right) M$$

by (2.13). For such n, there is a prime ideal $\mathfrak{p}$ such that

$$\mathfrak{p}^v | (n+\alpha)\mathfrak{a}, \quad p^v > M \tag{2.14}$$

for some integer $v \geq 1$. We also see that

$$\mathfrak{p}^v \nmid (m+\alpha)\mathfrak{a}$$

for $m \neq n$ and $N < m \leq N+M$. For if $\mathfrak{p}^v | (m+\alpha)\mathfrak{a}$, then

$$(m-n)\mathfrak{a} \subset (m+\alpha)\mathfrak{a} + (n+\alpha)\mathfrak{a} \subset \mathfrak{p}^v.$$

Since $\mathfrak{p} \nmid \mathfrak{a}$, $\mathfrak{p}^v | m-n$ and so $p^v | m-n$. However, this contradicts $0 \neq |m-n| < M < p^v$. Since the number of $\mathfrak{p}$ with $p \leq c(\alpha)(N+M)$ is $o(M)$ by the prime ideal theorem, we have

$$\begin{aligned} \sharp \mathfrak{T}_N &\geq \left(1 - \frac{8k+3+\sqrt{48k+9}}{16k^2} + o(1) \right) M \\ &\geq \left(1 - \left(\frac{1}{2k} + \frac{\sqrt{3}}{4k^{3/2}} + \frac{3}{8k^2} \right) \right) M \end{aligned} \tag{2.15}$$

for sufficiently large N. This proves (2.2).

We next prove (2.3) for each $\mathfrak{p}_n$ in (2.1). Suppose that $\mathfrak{p}_n | (m+\alpha)\mathfrak{a}$ for some $0 \leq m \leq N+M$ and $m \neq n$. Similarly to the previous paragraph, we see that $p_n | m-n$, which contradicts $0 < |m-n| \leq N+M < p_n$.

Finally, we prove (2.4) for each $\mathfrak{p}_n$ in (2.1). Let $p = p_n$ and $n_1 = n$. Then it is enough to show that there are no positive integers $n_2, \dots, n_t \le N + M$ with $t \le k$ such that

$$(p) \mid \prod_{i=1}^{t} (n_i + \alpha)\mathfrak{a}. \tag{2.16}$$

Suppose that (2.16) holds. Let q be the smallest positive integer such that $\beta = q\alpha$ is an integer of K. Put $m_i = qn_i$ for $i \le t$, then (2.16) implies

$$\prod_{i=1}^{t} (m_i + \beta) \in (p). \tag{2.17}$$

One can expand the product

$$\prod_{i=1}^{t} (m_i + \beta) = \sum_{i=1}^{t+1} S_{t+1-i}(\vec{m})\beta^{i-1}, \tag{2.18}$$

where $S_j(\vec{m})$ $(0 \le j \le t)$ are the elementary symmetric polynomials in $m_1, \dots, m_t$ and $\vec{m} = (m_1, \dots, m_t)$. Note that $S_0(\vec{m}) = 1$. Let $\{\gamma_1, \dots, \gamma_k\}$ be a basis for the integer ring $\mathfrak{O}_K$ of K over $\mathbb{Z}$. Then there exist integers $a_{ij} \in \mathbb{Z}$ $(1 \le i, j \le k)$ such that

$$\beta^{i-1} = \sum_{j=1}^{k} a_{ij}\gamma_j. \tag{2.19}$$

By (2.17)–(2.19) we see that

$$\sum_{j=1}^{k} \left(\sum_{i=1}^{t+1} a_{ij} S_{t+1-i}(\vec{m}) \right) \gamma_j \in (p).$$

Since $\gamma_1, \dots, \gamma_k$ is a basis for $\mathfrak{O}_K$, it follows that

$$\sum_{i=1}^{t+1} a_{ij} S_{t+1-i}(\vec{m}) \equiv 0 \pmod{p} \tag{2.20}$$

for every $1 \le j \le k$.

If $t < k$, one can write this system with additional zeros (if necessary) as

$$A^T \begin{bmatrix} S_t(\vec{m}) \\ \vdots \\ S_0(\vec{m}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.21}$$

in $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, where A^T is the transpose of the matrix $A := (a_{ij})$. Since both of

$\{\gamma_1, \dots, \gamma_k\}$ and $\{\beta^0, \dots, \beta^{k-1}\}$ are linearly independent over $\mathbb{Q}$, the matrix A is invertible. Note that each a_{ij} depends only on α. Thus, the determinant of A is not divisible by a prime p if $p > |\det A|$. Hence, $A \pmod p$ is also invertible, and by (2.21)

$$S_i(\vec{m}) \equiv 0 \pmod p$$

for every $0 \le i \le t$ and sufficiently large $p > c(\alpha)(N + M)$. This contradicts the fact $S_0(\vec{m}) = 1$.

Next we consider the case $t = k$. Since k is the degree, we have

$$\beta^k = \sum_{i=0}^{k-1} c_i \beta^i \tag{2.22}$$

for some $c_i \in \mathbb{Z}$. By (2.17), (2.18) and (2.22), we find that

$$\prod_{i=1}^{k}(m_i + \beta) = \sum_{i=1}^{k+1} S_{k+1-i}(\vec{m})\beta^{i-1} = \sum_{i=1}^{k} \left(S_{k+1-i}(\vec{m}) + c_{i-1}\right)\beta^{i-1} \in (p).$$

By the argument in the case $t < k$, we deduce that

$$S_{k+1-i}(\vec{m}) + c_{i-1} \equiv 0 \pmod p$$

for every $1 \le i \le k$. In particular,

$$S_1(\vec{m}) + c_{k-1} = q(n_1 + \cdots + n_k) + c_{k-1} \equiv 0 \pmod p. \tag{2.23}$$

Since $N < n_1 \le N + M$ and $n_i \ge 0$ for all i, it is easy to see that

$$q(n_1 + \cdots + n_k) + c_{k-1} \ne 0$$

for sufficiently large N. Thus

$$0 < |q(n_1 + \cdots + n_k) + c_{k-1}| \le qk(N + M) + |c_{k-1}| < c(\alpha)(N + M) < p$$

for sufficiently large N. But this contradicts (2.23). This completes the proof of (2.4) and the lemma. □

2.2. Construction of χ.

We next investigate dependencies among the numbers n and $n + \alpha_j$. First recall that $\alpha_{r+1}, \dots, \alpha_{r+\ell}$ are independent transcendental parameters and claim that the values $\chi(n + \alpha_j)$ for $n \ge 0$ and $j = r + 1, \dots, r + \ell$ can be chosen to be any numbers in S^1, independent of the values $\chi(n)$ for $n \ge 1$ and $\chi(n + \alpha_j)$ for $n \ge 0$ and $j = 1, \dots, r$. Let A_j be a finite subset of $\mathbb{Z}_{\ge 0}$ for each j. Suppose that

$$\prod_{j=1}^{r+\ell} \prod_{n \in A_j} (n + \alpha_j)^{m_{n,j}} \cdot \prod_{n \in A_0} n^{m_{n,0}} = 1$$

for $m_{n,j} \in \mathbb{Z}$. Then we see that

$$\prod_{j=r+1}^{r+\ell} \prod_{n \in A_j} (n+\alpha_j)^{m_{n,j}} \in \mathbb{Q}(\alpha_1, \ldots, \alpha_r)$$

is algebraic over $\mathbb{Q}$ and a root of a polynomial with rational coefficients. Since $\alpha_{r+1}, \ldots, \alpha_{r+\ell}$ are algebraically independent over $\mathbb{Q}$, we must have

$$m_{n,j} = 0$$

for all $n \in A_j$ and $j = r+1, \ldots, r+\ell$. This proves the claim.

Next consider the algebraic parameters $\alpha_1, \ldots, \alpha_r$. For each $j \leq r$, we claim that the values $\chi(n+\alpha_j)$ for $n \geq 0$ are independent of the values $\chi(n+\alpha_i)$ for $n \geq 0$, $i \neq j$ and $i \leq r$, but may depend on the values $\chi(n)$ for $n \geq 1$. Suppose that

$$\prod_{j=1}^{r} \prod_{n \in A_j} (n+\alpha_j)^{m_{n,j}} \cdot \prod_{n \in A_0} n^{m_{n,0}} = 1$$

for $m_{n,j} \in \mathbb{Z}$. For each $j \leq r$, we have

$$\prod_{n \in A_j} (n+\alpha_j)^{m_{n,j}} = \prod_{i \neq j} \prod_{n \in A_i} (n+\alpha_i)^{-m_{n,i}} \cdot \prod_{n \in A_0} n^{-m_{n,0}} \in \mathbb{Q}$$

by (1.3). Hence

$$\prod_{n \in A_j} (n+\alpha_j)^{m_{n,j}} = q_j$$

for some $q_j \in \mathbb{Q}$. This proves the claim.

There is a positive integer N_0 such that Lemma 1 holds for all $\alpha_1, \ldots, \alpha_r$ and $N \geq N_0$. Let $\delta = 10^{-2}$. Then there exists a real number $\sigma_0 > 1$ such that

$$\sum_{n=0}^{N_0} \frac{1}{(n+\alpha_j)^{\sigma_0}} + |z_j| < \delta \sum_{n > N_0} \frac{1}{(n+\alpha_j)^{\sigma_0}} \tag{2.24}$$

for all $j = 1, \ldots, r$,

$$\zeta(\sigma_0, \alpha_j : \chi) = \sum_{n=0}^{\infty} \frac{\chi(n+\alpha_j)}{(n+\alpha_j)^{\sigma_0}} = z_j \tag{2.25}$$

for all $j = r+1, \ldots, r+\ell$ and some $\chi(n+\alpha_j) \in S^1$, and

$$\log \zeta(\sigma_0 : \chi) = \sum_{p} -\log\left(1 - \frac{\chi(p)}{p^{\sigma_0}}\right) = z_0 \tag{2.26}$$

for some $\chi(p) \in S^1$. Here, (2.24) is due to

$$\lim_{\sigma\to 1+}\sum_{n=0}^{\infty}\frac{1}{(n+\alpha_j)^{\sigma}}=\infty,$$

and (2.25) and (2.26) are explained in Chapter XI [**8**], using the theory of summation of convex curves. Thus, we only need to define $\chi(n+\alpha_j)$ for $n\geq 0$ and each $j=1,\ldots,r$.

Let j be a positive integer $\leq r$. We will construct a totally multiplicative function ψ from the set of fractional ideals in $\mathbb{Q}(\alpha_j)$ to S^1 and then let

$$\chi(n+\alpha_j)=\psi(n+\alpha_j)$$

for $n\geq 0$ by a restriction of ψ. We first determine the values of ψ on the set

$$P_j(N):=\{\mathfrak{p}:\mathfrak{p}|(n+\alpha_j)\mathfrak{a}_j \text{ for some } 0\leq n\leq N, \text{ or } \mathfrak{p}|\mathfrak{a}_j\}.$$

Make a list of the prime ideals $\mathfrak{p}\in P_j(N_0)$ and define $\psi(\mathfrak{p})$ inductively by the condition

$$\psi(p)=\psi(\mathfrak{p}_1)\cdots\psi(\mathfrak{p}_\nu)$$

for $(p)=\mathfrak{p}_1\cdots\mathfrak{p}_\nu$. Note that $\psi(p)=\chi(p)$ for a rational prime p is already determined in (2.26). Let

$$\psi(\mathfrak{p})=1$$

if there is a conjugate $\mathfrak{p}'$ of $\mathfrak{p}$ such that $\psi(\mathfrak{p}')$ is not yet defined, and otherwise let

$$\psi(\mathfrak{p})=\psi(p)\prod_{\mathfrak{p}'\neq\mathfrak{p}}\psi(\mathfrak{p}')^{-1}.$$

Hence, we have defined $\psi(\mathfrak{p})$ for all $\mathfrak{p}\in P_j(N_0)$ and $\chi(n+\alpha_j)$ for $n\leq N_0$ and $j=1,\ldots,r$.

Define $M_i=[10^{-6}N_i]$ and $N_{i+1}=N_i+M_i$ for $i\geq 0$. Let $N_i<n\leq N_{i+1}$ and assume that we have defined $\psi(\mathfrak{p})$ for all $\mathfrak{p}\in P_j(N_i)$. We want to determine $\psi(n+\alpha_j)$ by defining the values at prime ideal divisors of $(n+\alpha_j)$. If $\psi(\mathfrak{p})$ is already defined for all $\mathfrak{p}|(n+\alpha_j)\mathfrak{a}_j$, then $\psi(n+\alpha_j)$ is determined by multiplicativity. If there is a $\mathfrak{p}|(n+\alpha_j)\mathfrak{a}_j$ such that $\psi(\mathfrak{p})$ is not defined yet, but $n\notin\mathfrak{T}_{N_i}(\alpha_j)$, then we simply let $\psi(\mathfrak{p})=1$. If $n\in\mathfrak{T}_{N_i}(\alpha_j)$, then by Lemma 1 there is a prime ideal $\mathfrak{p}_n|(n+\alpha_j)\mathfrak{a}_j$ in (2.1) that satisfies (2.3) and (2.4). Thus, we may choose $\psi(\mathfrak{p}_n)$ to be any number in S^1. It follows that we may choose $\psi(n+\alpha_j)$ for $n\in\mathfrak{T}_{N_i}(\alpha_j)$ to be any number in S^1, too. By repeating this inductive process, we can determine $\psi(\mathfrak{p})$ for all $\mathfrak{p}$ and $\psi(n+\alpha_j)$ for all $n\geq 0$ and $j=1,\ldots,r$. It is important to notice that for $j=1,\ldots,r$, we can choose the value $\chi(n+\alpha_j)$ to be any number in S^1 if and only if $n\in\mathfrak{T}_{N_i}(\alpha_j)$ for some $i\geq 0$.

Now we define $\chi(n+\alpha_j)$ for $n\in\mathfrak{T}_{N_i}(\alpha_j)$, $1\leq j\leq r$ and $i\geq 0$ such that

$$\left|\sum_{n=0}^{N_i}\frac{\chi(n+\alpha_j)}{(n+\alpha_j)^{\sigma_0}}-z_j\right|<\delta\sum_{n=N_i+1}^{\infty}\frac{1}{(n+\alpha_j)^{\sigma_0}} \tag{2.27}$$

holds for all $i\geq 0$, which implies that

$$\zeta(\sigma_0, \alpha_j : \chi) = \sum_{n=0}^{\infty} \frac{\chi(n+\alpha_j)}{(n+\alpha_j)^{\sigma_0}} = z_j. \tag{2.28}$$

We prove (2.27) by induction on i. First, by (2.24) we see that (2.27) holds for $i = 0$. Suppose now that (2.27) holds for $i \geq 0$. We prove (2.27) for $i+1$ by making suitable choices of the values $\chi(n+\alpha_j) \in S^1$ for $n \in \mathfrak{T}_{N_i}(\alpha_j)$. For any $n,\ n' \in \mathfrak{T}_{N_i}(\alpha_j)$ with $n > n'$,

$$1 < \frac{(n+\alpha_j)^{\sigma_0}}{(n'+\alpha_j)^{\sigma_0}} < \left(\frac{N_i + M_i + \alpha_j}{N_i + \alpha_j}\right)^2 < 2.$$

Since $\sharp \mathfrak{T}_{N_i}(\alpha_j) > 5$, the sum

$$\sum_{n \in \mathfrak{T}_{N_i}(\alpha_j)} \frac{\chi(n+\alpha_j)}{(n+\alpha_j)^{\sigma_0}},$$

with $|\chi(n+\alpha_j)| = 1$, takes any value z in the disc

$$|z| \leq \sum_{n \in \mathfrak{T}_{N_i}(\alpha_j)} \frac{1}{(n+\alpha_j)^{\sigma_0}} =: S_{3,j}(N_i).$$

Let $\mathfrak{U}_{N_i}(\alpha_j)$ be the set of integers in $N_i < n \leq N_i + M_i$ with $n \notin \mathfrak{T}_{N_i}(\alpha_j)$, and let

$$\Lambda_j(N_i) = \sum_{n \leq N_i \text{ or } n \in \mathfrak{U}_{N_i}(\alpha_j)} \frac{\chi_j(n+\alpha_j)}{(n+\alpha_j)^{\sigma_0}} - z_j.$$

If $|\Lambda_j(N_i)| \leq S_{3,j}(N_i)$, then there exist $\chi(n+\alpha_j) \in S^1$ for all $n \in \mathfrak{T}_{N_i}(\alpha_j)$ such that

$$\sum_{n \in \mathfrak{T}_{N_i}(\alpha_j)} \frac{\chi(n+\alpha_j)}{(n+\alpha_j)^{\sigma_0}} = -\Lambda_j(N_i).$$

Thus,

$$\left| \sum_{n \leq M_i + N_i} \frac{\chi_j(n+\alpha_j)}{(n+\alpha_j)^{\sigma_0}} - z_j \right| = 0,$$

which obviously implies (2.27) for $i+1$.

If $|\Lambda_j(N_i)| > S_{3,j}(N_i)$, then there exist $\chi(n+\alpha_j) \in S^1$ for all $n \in \mathfrak{T}_{N_i}(\alpha_j)$ such that

$$\sum_{n \in \mathfrak{T}_{N_i}(\alpha_j)} \frac{\chi(n+\alpha_j)}{(n+\alpha_j)^{\sigma_0}} = -S_{3,j}(N_i) \frac{\Lambda_j(N_i)}{|\Lambda_j(N_i)|}.$$

Thus,

$$\left| \sum_{n \le N_i + M_i} \frac{\chi(n+\alpha_j)}{(n+\alpha_j)^{\sigma_0}} - z_j \right| = |\Lambda_j(N_i)| - S_{3,j}(N_i). \tag{2.29}$$

Define

$$S_{1,j}(N_i) := \left| \sum_{n \le N_i} \frac{\chi(n+\alpha_j)}{(n+\alpha_j)^{\sigma_0}} - z_j \right| \quad \text{and} \quad S_{2,j}(N_i) := \sum_{n \in \mathfrak{U}_{N_i}(\alpha_j)} \frac{1}{(n+\alpha_j)^{\sigma_0}}.$$

Then

$$|\Lambda_j(N_i)| - S_{3,j}(N_i) \le S_{1,j}(N_i) + S_{2,j}(N_i) - S_{3,j}(N_i). \tag{2.30}$$

By Lemma 1 we see that

$$\frac{S_{3,j}(N_i)}{S_{2,j}(N_i)} \ge \frac{(1/2+\delta)M_i(N_i+M_i+\alpha_j)^{-\sigma_0}}{(1/2-\delta)M_i(N_i+\alpha_j)^{-\sigma_0}} > \frac{1+\delta}{1-\delta}.$$

Thus,

$$S_{3,j}(N_i) - S_{2,j}(N_i) > \delta(S_{2,j}(N_i) + S_{3,j}(N_i)). \tag{2.31}$$

The inductive hypothesis (2.27) is

$$S_{1,j}(N_i) < \delta(S_{2,j}(N_i) + S_{3,j}(N_i) + S_{4,j}(N_i)), \tag{2.32}$$

where

$$S_{4,j}(N_i) = \sum_{n > N_{i+1}} \frac{1}{(n+\alpha_j)^{\sigma_0}}.$$

Thus we have

$$S_{1,j}(N_i) + S_{2,j}(N_i) - S_{3,j}(N_i) < \delta S_{4,j}(N_i). \tag{2.33}$$

Therefore, (2.27) for $i+1$ holds by (2.29), (2.30) and (2.33).

2.3. Completion of the proof.

Let η be a real number satisfying $0 < \eta < (\sigma_0 - 1)/2$. Given $\epsilon > 0$, there exists $N > 0$ such that

$$\max\left\{ \max_{0 \le j \le r+\ell} \sum_{n > N} \frac{1}{(n+\alpha_j)^{1+\eta}}, \sum_{p > N} -\log\left(1 - \frac{1}{p^{1+\eta}}\right) \right\} \le \frac{\epsilon}{3}.$$

By the construction of χ in Subsection 2.2, we can apply Kronecker's approximation theorem, to see that there exists a real number t such that

$$|e^{-it\log p} - \chi(p)| \le \frac{\epsilon}{A}$$

for $p \leq N$, and

$$|e^{-it\log(n+\alpha_j)} - \chi(n+\alpha_j)| < \frac{\epsilon}{A}$$

for $n \leq N$ and $1 \leq j \leq r + \ell$, where

$$A := \max\left\{3\sum_p \frac{1}{p^{1+\eta}}\left(1-\frac{1}{p^{1+\eta}}\right)^{-1},\ \max_{j\leq r}\left\{3\sum_n \frac{1}{(n+\alpha_j)^{1+\eta}}\right\}\right\}.$$

Therefore, we have

$$\begin{aligned}
&|\zeta(s+it,\alpha_j) - \zeta(s,\alpha_j:\chi)| \\
&\quad \leq \sum_{n>N} \frac{2}{(n+\alpha_j)^{1+\eta}} + \sum_{n\leq N} \frac{|\chi(n+\alpha_j) - (n+\alpha_j)^{-it}|}{(n+\alpha_j)^{1+\eta}} \\
&\quad \leq \frac{2\epsilon}{3} + \frac{\epsilon}{A}\sum_{n\leq N}\frac{1}{(n+\alpha_j)^{1+\eta}} \\
&\quad \leq \epsilon
\end{aligned}$$

for all $1 \leq j \leq r+\ell$ and $\Re s \geq 1+\eta$, and also

$$\begin{aligned}
&|\log\zeta(s+it) - \log\zeta(s:\chi)| \\
&\quad \leq 2\sum_{p>N} -\log\left(1-\frac{1}{p^{1+\eta}}\right) + \sum_{p\leq N}\sum_{m=1}^{\infty}\frac{1}{m}\frac{|\chi(p)^m - p^{-imt}|}{p^{m(1+\eta)}} \\
&\quad \leq \frac{2\epsilon}{3} + \frac{\epsilon}{A}\sum_{p\leq N}\sum_{m=1}^{\infty}\frac{1}{p^{m(1+\eta)}} \\
&\quad \leq \epsilon
\end{aligned}$$

for $\Re s \geq 1+\eta$. Put

$$\zeta_0(s) = \log\zeta(s), \qquad \zeta_0(s:\chi) = \log\zeta(s:\chi)$$

and

$$\zeta_j(s) = \zeta(s,\alpha_j), \qquad \zeta_j(s:\chi) = \zeta(s,\alpha_j:\chi)$$

for $j = 1,\ldots,r+\ell$, and

$$g_j(s) := \zeta_j(s:\chi) - z_j$$

for $j = 0,\ldots,r+\ell$. Then we have just proved that for any $\epsilon > 0$, there exists a real number t such that

$$|\zeta_j(s+it) - z_j - g_j(s)| \leq \epsilon \tag{2.34}$$

for all $j = 0,\ldots,r+\ell$ and $\Re s \geq 1+\eta$.

Since the functions $g_j(s)$ for $j \leq r+\ell$ have a common zero at $s=\sigma_0$ by (2.25), (2.26) and (2.28), there exists $0<\epsilon_0<\eta$ such that

$$g_j(s) \neq 0$$

for $0<|s-\sigma_0| \leq \epsilon_0$ for all $j \leq r+\ell$. Given $0<\epsilon_1<\epsilon_0$, let

$$M := \max_{0 \leq j \leq r+\ell} \max_{\Re s \geq 1+\eta} |\zeta_j'(s)| + 1$$

and

$$\epsilon_2 = \min_{0 \leq j \leq r+\ell} \min_{|s-\sigma_0|=\epsilon_1/M} |g_j(s)| > 0.$$

By (2.34) there exists a real number t such that

$$|\zeta_j(s+it) - z_j - g_j(s)| < \epsilon_2$$

for all $j \leq r+\ell$ and $\Re s \geq 1+\eta$. Thus, for each $j \leq r+\ell$ we have

$$|\zeta_j(s+it) - z_j - g_j(s)| < |g_j(s)|$$

on the circle $|s-\sigma_0| = \epsilon_1/M$. Therefore, by Rouché's theorem, there is a complex number s_j with $|s_j-\sigma_0| < \epsilon_1/M$ such that $\zeta_j(s_j+it) = z_j$. Moreover, we find that

$$|\zeta_j(s_1+it) - z_j| = \left| \int_{s_j}^{s_1} \zeta_j'(s+it) ds \right| < M|s_j - s_1| < 2\epsilon_1.$$

Thus, we have shown that given $\epsilon > 0$, there exists a complex number s_1+it with $|s_1-\sigma_0| < \epsilon/(2M)$ such that

$$|\zeta_j(s_1+it) - z_j| < \epsilon$$

for all $j \leq r+\ell$. This completes the proof of the theorem. □

3. Evidence of the denseness of $\zeta(s,\alpha)$ in the critical strip.

THEOREM 2. *Let α be an algebraic irrational number with $0<\alpha<1$ of degree $k>2$ and let $1/2<\sigma_0<1$ be fixed. Then for any complex number z, there exist a character χ on $\{n+\alpha\}_{n\geq 0}$ and an increasing sequence $\{N_i\}_{i\geq 0} \subset \mathbb{N}$ such that $N_i \to \infty$ as $i \to \infty$ and*

$$\lim_{i\to\infty} \sum_{n=0}^{N_i} \frac{\chi(n+\alpha)}{(n+\alpha)^{\sigma_0}} = z.$$

Notice that the limit in Theorem 2 is not a usual sum

$$\sum_{n=0}^{\infty} \frac{\chi(n+\alpha)}{(n+\alpha)^{\sigma_0}},$$

because from the proof in Subsection 3.1 one cannot see that the above sum is convergent. However, we can find a convergent rearrangement for algebraic irrational α of degree > 2 as follows.

Corollary 1. *Let α be an algebraic irrational number with $0 < \alpha < 1$ of degree $k > 2$ and let $1/2 < \sigma_0 < 1$ be fixed. Then the set of all convergent rearrangements of the sums $\sum_n \chi(n+\alpha)/(n+\alpha)^{\sigma_0}$ for all characters χ on the set $\{n+\alpha\}_{n\geq 0}$ is $\mathbb{C}$.*

We will prove Theorem 2 in Subsection 3.1 and Corollary 1 in Subsection 3.2.

3.1. Proof of Theorem 2.

Let N_0 be the integer in Lemma 1 and let $M_i = [10^{-6}N_i]$ and $N_{i+1} = N_i + M_i$ for all $i \geq 0$. The set $\mathfrak{T}_{N_i}$ is defined in (2.1) and we let $\mathfrak{V}_{N_i} = (N_i, N_{i+1}] \setminus \mathfrak{T}_{N_i}$. Given a complex number z, we want to find a character χ on $\{n+\alpha\}_{n\geq 0}$ such that

$$\lim_{i\to\infty} \sum_{n=0}^{N_i} \frac{\chi(n+\alpha)}{(n+\alpha)^{\sigma_0}} = z.$$

We instead define a character χ on the set of fractional ideals of $\mathbb{Q}(\alpha)$ and then restrict it to $\{n+\alpha\}_{n\geq 0}$. Since it is totally multiplicative, we only need to define it for all prime ideals.

Define $\chi(n+\alpha) = 1$ for all $n \leq N_0$. (We may choose $\chi(\mathfrak{p}) = 1$ for all $\mathfrak{p}|(n+\alpha)\mathfrak{a}$ or $\mathfrak{p}|\mathfrak{a}$.) We next inductively define $\chi(n+\alpha)$ for $N_i < n \leq N_{i+1}$ assuming that all $\chi(n+\alpha)$ for $n \leq N_i$ are defined. Since the fractional ideal $(n+\alpha)$ for each $n \in \mathfrak{T}_{N_i}$ has an independent prime ideal factor $\mathfrak{p}_n$ by Lemma 1, the values $\chi(n+\alpha)$ for $n \in \mathfrak{T}_{N_i}$ can be any complex numbers in $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Since χ is multiplicative, the values $\chi(n+\alpha)$ for $n \in \mathfrak{V}_{N_i}$ are determined by the induction hypothesis.

By (2.2) for $k > 2$, one can find a subset $\mathfrak{U}_{N_i}$ of $\mathfrak{T}_{N_i}$ such that $2\sharp\mathfrak{V}_{N_i} = \sharp\mathfrak{U}_{N_i}$. For each $n \in \mathfrak{V}_{N_i}$ and for any two elements $n', n'' \in \mathfrak{U}_{N_i}$, one can define two values $\chi(n'+\alpha)$ and $\chi(n''+\alpha)$ in S^1 such that

$$\frac{\chi(n+\alpha)}{(n+\alpha)^{\sigma_0}} + \frac{\chi(n'+\alpha)}{(n'+\alpha)^{\sigma_0}} + \frac{\chi(n''+\alpha)}{(n''+\alpha)^{\sigma_0}} = 0. \tag{3.1}$$

Thus, it is easy to find a character χ such that

$$\sum_{\substack{n\in\mathfrak{U}_{N_i}\\ n\in\mathfrak{V}_{N_i}}} \frac{\chi(n+\alpha)}{(n+\alpha)^{\sigma_0}} = 0.$$

Define

$$\Lambda_i = \sum_{n\leq N_i} \frac{\chi(n+\alpha)}{(n+\alpha)^{\sigma_0}} - z.$$

The sum

$$\sum_{n \in \mathfrak{T}_{N_i} \setminus \mathfrak{U}_{N_i}} \frac{\chi(n+\alpha)}{(n+\alpha)^{\sigma_0}}$$

can be any complex number in the set

$$\left\{ z \in \mathbb{C} : |z| \leq \sum_{n \in \mathfrak{T}_{N_i} \setminus \mathfrak{U}_{N_i}} \frac{1}{(n+\alpha)^{\sigma_0}} \right\}$$

by suitable choices of the values $\chi(n+\alpha) \in S^1$ for $n \in \mathfrak{T}_{N_i} \setminus \mathfrak{U}_{N_i}$. So we can define $\chi(n+\alpha) \in S^1$ so that

$$\sum_{n \in \mathfrak{T}_{N_i} \setminus \mathfrak{U}_{N_i}} \frac{\chi(n+\alpha)}{(n+\alpha)^{\sigma_0}} = -\Lambda_i$$

if

$$|\Lambda_i| \leq \sum_{n \in \mathfrak{T}_{N_i} \setminus \mathfrak{U}_{N_i}} \frac{1}{(n+\alpha)^{\sigma_0}},$$

and

$$\sum_{n \in \mathfrak{T}_{N_i} \setminus \mathfrak{U}_{N_i}} \frac{\chi(n+\alpha)}{(n+\alpha)^{\sigma_0}} = -\frac{\Lambda_i}{|\Lambda_i|} \sum_{n \in \mathfrak{T}_{N_i} \setminus \mathfrak{U}_{N_i}} \frac{1}{(n+\alpha)^{\sigma_0}}$$

if

$$|\Lambda_i| > \sum_{n \in \mathfrak{T}_{N_i} \setminus \mathfrak{U}_{N_i}} \frac{1}{(n+\alpha)^{\sigma_0}}.$$

In both cases we have

$$\begin{aligned} |\Lambda_{i+1}| &= \left| \Lambda_i + \sum_{n \in \mathfrak{T}_{N_i} \setminus \mathfrak{U}_{N_i}} \frac{\chi(n+\alpha)}{(n+\alpha)^{\sigma_0}} \right| \\ &\leq \left| \Lambda_i - \frac{\Lambda_i}{|\Lambda_i|} \sum_{n \in \mathfrak{T}_{N_i} \setminus \mathfrak{U}_{N_i}} \frac{1}{(n+\alpha)^{\sigma_0}} \right| \\ &= \left| |\Lambda_i| - \sum_{n \in \mathfrak{T}_{N_i} \setminus \mathfrak{U}_{N_i}} \frac{1}{(n+\alpha)^{\sigma_0}} \right| \\ &\leq |\Lambda_i|. \end{aligned}$$

Since the sum

$$\sum_{n \in \mathfrak{T}_{N_i} \setminus \mathfrak{U}_{N_i}} \frac{1}{(n+\alpha)^{\sigma_0}} \gg N_i^{1-\sigma_0} \to \infty$$

as $N_i \to \infty$, there is an integer J such that

$$|\Lambda_J| \le \sum_{n \in \mathfrak{T}_{N_J} \setminus \mathfrak{U}_{N_J}} \frac{1}{(n+\alpha)^{\sigma_0}}.$$

Then we see that

$$\sum_{n \in \mathfrak{T}_{N_J} \setminus \mathfrak{U}_{N_J}} \frac{\chi(n+\alpha)}{(n+\alpha)^{\sigma_0}} = -\Lambda_J$$

and

$$\Lambda_i = 0$$

for all $i > J$. Thus

$$\lim_{i\to\infty} \Lambda_i = \lim_{i\to\infty} \sum_{n \le N_i} \frac{\chi(n+\alpha)}{(n+\alpha)^{\sigma_0}} - z = 0.$$

This completes the proof. □

3.2. Proof of Corollary 1.

We follow the notation in the proof of Theorem 2. Given a complex number z, we have constructed a character χ in Theorem 2 such that

$$\lim_{i\to\infty} \sum_{n \le N_i} \frac{\chi(n+\alpha)}{(n+\alpha)^{\sigma_0}} = z.$$

We want to define a rearrangement $\{m_n\}_{n\ge 0}$ of $\mathbb{Z}_{\ge 0}$ and a character ψ such that

$$\sum_{n=0}^{\infty} \frac{\psi(m_n+\alpha)}{(m_n+\alpha)^{\sigma_0}} = z.$$

By the proof of Theorem 2, there exists a character ψ, an integer J and a rearrangement $\{m_n : n \ge 0\}$ of $\mathbb{Z}_{\ge 0}$ such that for $i \ge J$

(1) $m_n = n$ and $\psi(n+\alpha) = \chi(n+\alpha)$ for $n \le N_J$,

(2) m_n permutes the integers in $(N_i, N_{i+1}]$,

(3) $m < m'$ for every $m \in \mathfrak{U}_{N_i}$ and $m' \in \mathfrak{T}_{N_i} \setminus \mathfrak{U}_{N_i}$,

(4) for $\mathfrak{V}_{N_i} = \{m_{N_i+3\ell-2} : \ell = 1, \dots, L\}$ and $\mathfrak{U}_{N_i} = \{m_{N_i+3\ell-j} : \ell = 1, \dots, L$ and $j = 0, 1\}$, we have that

$$\frac{\psi(m_{N_i+3\ell-2}+\alpha)}{(m_{N_i+3\ell-2}+\alpha)^{\sigma_0}} + \frac{\psi(m_{N_i+3\ell-1}+\alpha)}{(m_{N_i+3\ell-1}+\alpha)^{\sigma_0}} + \frac{\psi(m_{N_i+3\ell}+\alpha)}{(m_{N_i+3\ell}+\alpha)^{\sigma_0}} = 0,$$

(5) $\psi(m_n+\alpha) = (-1)^n$ for $m_n \in \mathfrak{T}_{N_i} \setminus \mathfrak{U}_{N_i} = \{m_{N_i+3L+j} : j = 1, \dots, M_i - 3L\}$,

(6) $m_n < m_{n'}$ for every $m_n, m_{n'} \in \mathfrak{T}_{N_i} \setminus \mathfrak{U}_{N_i}$ with $n < n'$.

Then it is easy to show that the sum

$$\sum_{n=0}^{\infty} \frac{\chi(m_n + \alpha)}{(m_n + \alpha)^{\sigma_0}}$$

is convergent to z.

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Yoonbok LEE
Department of Mathematics
Research Institute of Basic Sciences
Incheon National University
119 Academy-ro, Yeonsu-gu
Incheon, Korea
E-mail: leeyb@inu.ac.kr, leeyb131@gmail.com

Hidehiko MISHOU
Department of Mathematics and Technology
School of System Design
Tokyo Denki University
5 Senju-Asahi-cho, Adachi-ku
Tokyo 120-8551, Japan
E-mail: h_mishou@mail.dendai.ac.jp