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Ohno-type identities for multiple harmonic sums

By Shin-ichiro SEKI and Shuji YAMAMOTO

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Abstract. We establish Ohno-type identities for multiple harmonic (q-)sums which generalize Hoffman's identity and Bradley's identity. Our result leads to a new proof of the Ohno-type relation for $\mathcal{A}$-finite multiple zeta values recently proved by Hirose, Imatomi, Murahara and Saito. As a further application, we give certain sum formulas for $\mathcal{A}_2$- or $\mathcal{A}_3$-finite multiple zeta values.

1. Introduction.

Let N be a positive integer. Euler [**5**] proved the following identity for the N-th harmonic number:

$$\sum_{m=1}^{N} \frac{(-1)^{m-1}}{m}\binom{N}{m} = \sum_{n=1}^{N}\frac{1}{n}. \tag{1}$$

It is known today that there are various generalizations of Euler's identity. We call a tuple of positive integers an index. For an index $\boldsymbol{k} = (k_1, \ldots, k_r)$, we write it in the form

$$\boldsymbol{k} = (\{1\}^{a_1-1}, b_1+1, \ldots, \{1\}^{a_{s-1}-1}, b_{s-1}+1, \{1\}^{a_s-1}, b_s),$$

where $a_1, \ldots, a_s, b_1, \ldots, b_s$ are positive integers and $\{1\}^a$ means $1, \ldots, 1$ repeated a times, and then we define its Hoffman dual $\boldsymbol{k}^\vee$ by

$$\boldsymbol{k}^\vee := (a_1, \{1\}^{b_1-1}, a_2+1, \{1\}^{b_2-1}, \ldots, a_s+1, \{1\}^{b_s-1}).$$

Let $\boldsymbol{k} = (k_1, \ldots, k_r)$ and $\boldsymbol{k}^\vee = (l_1, \ldots, l_s)$. After Roman [**12**] (the case $r = 1$) and Hernandez [**1**] (the case $s = 1$), Hoffman [**8**] proved

$$\sum_{1 \le m_1 \le \cdots \le m_r \le N} \frac{(-1)^{m_r-1}}{m_1^{k_1}\cdots m_r^{k_r}}\binom{N}{m_r} = \sum_{1\le n_1\le\cdots\le n_s\le N}\frac{1}{n_1^{l_1}\cdots n_s^{l_s}}. \tag{2}$$

There are also q-analogs of these identities. Let q be a real number satisfying $0 < q < 1$. For an integer m, we define the q-integer $[m]_q := (1-q^m)/(1-q)$. When $0 \le m \le N$, we define the q-factorial $[m]_q! := \prod_{a=1}^{m}[a]_q$ ($[0]_q! := 1$) and the q-binomial

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coefficient $\binom{N}{m}_q := [N]_q!/[m]_q![N-m]_q!$. Van Hamme [**19**] proved a q-analog of Euler's identity (1)

$$\sum_{m=1}^{N} \frac{(-1)^{m-1} q^{m(m+1)/2}}{[m]_q} \binom{N}{m}_q = \sum_{n=1}^{N} \frac{q^n}{[n]_q}.$$

After Dilcher [**4**] (the case $r=1$) and Prodinger [**11**] (the case $s=1$), Bradley [**3**] proved a q-analog of Hoffman's identity (2)

$$\sum_{1 \le m_1 \le \cdots \le m_r \le N} \frac{q^{(k_1-1)m_1+\cdots+(k_r-1)m_r}}{[m_1]_q^{k_1} \cdots [m_r]_q^{k_r}} \cdot (-1)^{m_r-1} q^{m_r(m_r+1)/2} \binom{N}{m_r}_q = \sum_{1 \le n_1 \le \cdots \le n_s \le N} \frac{q^{n_1+\cdots+n_s}}{[n_1]_q^{l_1} \cdots [n_s]_q^{l_s}}. \tag{3}$$

The equality (2) or (3) is a kind of duality for multiple harmonic (q-)sums. Since the duality relations for (q-)multiple zeta values are generalized to Ohno's relations ([**9**], [**2**]), it is natural to ask whether (and how) we can generalize (2) and (3) to Ohno-type identities. This question was considered by Oyama [**10**] and more recently by Hirose, Imatomi, Murahara and Saito [**7**]. More precisely, they treated identities of the $\mathcal{A}$-finite multiple zeta values, that is, congruences modulo prime numbers.

In this article, we prove Ohno-type identities which generalize (3) (Theorem 2.1) and (2) (Corollary 2.2). We stress that our formulas are true identities, not congruences. This allows us to give, besides a new proof of Hirose–Imatomi–Murahara–Saito's relation for $\mathcal{A}$-finite multiple zeta values, sum formulas for $\mathcal{A}_2$- or $\mathcal{A}_3$-finite multiple zeta values, which are congruences modulo square or cube of primes.

2. Main results.

2.1. Ohno-type identity.

For a tuple of non-negative integers $\boldsymbol{e} = (e_1, \ldots, e_r)$, we define its weight $\mathrm{wt}(\boldsymbol{e})$ and depth $\mathrm{dep}(\boldsymbol{e})$ to be $e_1 + \cdots + e_r$ and r, respectively. Let $J_{e,r}$ be the set of all tuples of non-negative integers $\boldsymbol{e}$ such that $\mathrm{wt}(\boldsymbol{e}) = e$, $\mathrm{dep}(\boldsymbol{e}) = r$, and set $J_{*,r} := \bigcup_{e=0}^{\infty} J_{e,r}$. For $\boldsymbol{e}_1, \boldsymbol{e}_2 \in J_{*,r}$, $\boldsymbol{e}_1 + \boldsymbol{e}_2$ denotes the entrywise sum. Similarly, let $I_{k,r}$ be the set of all indices $\boldsymbol{k}$ such that $\mathrm{wt}(\boldsymbol{k}) = k$, $\mathrm{dep}(\boldsymbol{k}) = r$, and set $I_{*,r} := \bigcup_{k=0}^{\infty} I_{k,r}$. By convention, $I_{*,0} = \{\emptyset\}$ is the set consisting only of the empty index.

For $\boldsymbol{k} = (k_1, \ldots, k_r) \in I_{*,r}$ and $\boldsymbol{e} = (e_1, \ldots, e_r) \in J_{*,r}$, put

$$b(\boldsymbol{k}; \boldsymbol{e}) := \prod_{i=1}^{r} \binom{k_i + e_i + \delta_{i1} + \delta_{ir} - 2}{e_i},$$

where δ_{ij} is Kronecker's delta. Here, we use the convention that

$$\binom{e-1}{e} = \begin{cases} 1 & (e = 0), \\ 0 & (e > 0). \end{cases}$$

For a positive integer N, $\boldsymbol{k}=(k_1,\dots,k_r)\in I_{*,r}$ and $\boldsymbol{e}=(e_1,\dots,e_r)\in J_{*,r}$, we define the multiple harmonic q-sums $H_N^{\star}(\boldsymbol{k};q)$ and $z_N^{\star}(\boldsymbol{k};\boldsymbol{e};q)$ by

$$H_N^{\star}(\boldsymbol{k};q):=\sum_{1\le m_1\le\cdots\le m_r\le N}\frac{q^{(k_1-1)m_1+\cdots+(k_r-1)m_r}}{[m_1]_q^{k_1}\cdots[m_r]_q^{k_r}}\cdot(-1)^{m_r-1}q^{m_r(m_r+1)/2}\binom{N}{m_r}_q,$$
$$z_N^{\star}(\boldsymbol{k};\boldsymbol{e};q):=\sum_{1\le m_1\le\cdots\le m_r\le N}\frac{q^{(e_1+1)m_1+\cdots+(e_r+1)m_r}}{[m_1]_q^{k_1+e_1}\cdots[m_r]_q^{k_r+e_r}}.$$

We set $z_N^{\star}(\boldsymbol{k};q):=z_N^{\star}(\boldsymbol{k};\{0\}^r;q)$ and $z_N^{\star}(\emptyset;q):=1$. The first main result is the following:

THEOREM 2.1. *Let N be a positive integer, e a non-negative integer and $\boldsymbol{k}\in I_{*,r}$ an index. Set $s:=\mathrm{dep}(\boldsymbol{k}^{\vee})$. Then we have*

$$\sum_{\boldsymbol{e}\in J_{e,r}}b(\boldsymbol{k};\boldsymbol{e})H_N^{\star}(\boldsymbol{k}+\boldsymbol{e};q)=\sum_{j=0}^{e}z_N^{\star}(\{1\}^{e-j};q)\sum_{\boldsymbol{e'}\in J_{j,s}}z_N^{\star}(\boldsymbol{k}^{\vee};\boldsymbol{e'};q). \tag{4}$$

The case $e=0$ gives Bradley's identity $H_N^{\star}(\boldsymbol{k};q)=z_N^{\star}(\boldsymbol{k}^{\vee};q)$. We will prove (4) by using a certain *connected sum* in Section 3, based on the same idea used in another paper of the authors [**17**]. This proof is new even if one specializes it to Hoffman's identity.

Let

$$H_N^{\star}(\boldsymbol{k}):=\lim_{q\to1}H_N^{\star}(\boldsymbol{k};q)=\sum_{1\le m_1\le\cdots\le m_r\le N}\frac{(-1)^{m_r-1}}{m_1^{k_1}\cdots m_r^{k_r}}\binom{N}{m_r},$$
$$\zeta_N^{\star}(\boldsymbol{k}):=\lim_{q\to1}z_N^{\star}(\boldsymbol{k};q)=\sum_{1\le m_1\le\cdots\le m_r\le N}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}. \tag{5}$$

By taking the limit $q\to1$ in (4), we obtain the following:

COROLLARY 2.2. *Let N be a positive integer, e a non-negative integer and $\boldsymbol{k}\in I_{*,r}$ an index. Set $s:=\mathrm{dep}(\boldsymbol{k}^{\vee})$. Then we have*

$$\sum_{\boldsymbol{e}\in J_{e,r}}b(\boldsymbol{k};\boldsymbol{e})H_N^{\star}(\boldsymbol{k}+\boldsymbol{e})=\sum_{j=0}^{e}\zeta_N^{\star}(\{1\}^{e-j})\sum_{\boldsymbol{e'}\in J_{j,s}}\zeta_N^{\star}(\boldsymbol{k}^{\vee}+\boldsymbol{e'}). \tag{6}$$

The case $e=0$ gives Hoffman's identity $H_N^{\star}(\boldsymbol{k})=\zeta_N^{\star}(\boldsymbol{k}^{\vee})$.

For an application of (6), we recall $\mathcal{A}$-finite multiple zeta values. First we define a $\mathbb{Q}$-algebra $\mathcal{A}$ by

$$\mathcal{A}:=\Bigg(\prod_{p\colon\text{prime}}\mathbb{Z}/p\mathbb{Z}\Bigg)\Bigg/\Bigg(\bigoplus_{p\colon\text{prime}}\mathbb{Z}/p\mathbb{Z}\Bigg).$$

For a positive integer N and an index $\boldsymbol{k}=(k_1,\dots,k_r)\in I_{*,r}$, we define the multiple harmonic sum $\zeta_N(\boldsymbol{k})$ by

$$\zeta_N(\boldsymbol{k}) := \sum_{1\le m_1<\cdots<m_r\le N} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$$

(compare with $\zeta_N^\star(\boldsymbol{k})$ given in (5)). We set $\zeta_N(\emptyset)=\zeta_N^\star(\emptyset)=1$ by convention. Then the $\mathcal{A}$-finite multiple zeta values $\zeta_{\mathcal{A}}(\boldsymbol{k})$ and $\zeta_{\mathcal{A}}^\star(\boldsymbol{k})$ are defined by

$$\zeta_{\mathcal{A}}(\boldsymbol{k}) := \big(\zeta_{p-1}(\boldsymbol{k}) \bmod p\big)_p, \quad \zeta_{\mathcal{A}}^\star(\boldsymbol{k}) := \big(\zeta_{p-1}^\star(\boldsymbol{k}) \bmod p\big)_p \in \mathcal{A}.$$

Since $(-1)^{m-1}\binom{p-1}{m} \equiv -1 \pmod p$ holds for any prime p greater than m, we have

$$\big(H_{p-1}^\star(\boldsymbol{k}) \bmod p\big)_p = -\zeta_{\mathcal{A}}^\star(\boldsymbol{k}).$$

Moreover, it is known that $\zeta_{\mathcal{A}}^\star(\{1\}^e)=0$ for $e>0$, while $\zeta_{\mathcal{A}}^\star(\emptyset)=1$. Hence we obtain the following relation among $\mathcal{A}$-finite multiple zeta values as a corollary of (6).

Corollary 2.3 (Hirose–Imatomi–Murahara–Saito [**7**]). *Let e be a non-negative integer and $\boldsymbol{k}\in I_{*,r}$ an index. Set $s:=\operatorname{dep}(\boldsymbol{k}^\vee)$. Then we have*

$$\sum_{\boldsymbol{e}\in J_{e,r}} b(\boldsymbol{k};\boldsymbol{e})\zeta_{\mathcal{A}}^\star(\boldsymbol{k}+\boldsymbol{e}) = -\sum_{e'\in J_{e,s}} \zeta_{\mathcal{A}}^\star(\boldsymbol{k}^\vee+\boldsymbol{e'}).$$

2.2. Sum formulas for finite multiple zeta values.

Before stating our second main result, let us recall the sum formulas for $\mathcal{A}$-finite multiple zeta values. First, it is easily seen that

$$\sum_{\boldsymbol{k}\in I_{k,r}} \zeta_{\mathcal{A}}(\boldsymbol{k}) = \sum_{\boldsymbol{k}\in I_{k,r}} \zeta_{\mathcal{A}}^\star(\boldsymbol{k}) = 0, \tag{7}$$

but this is not an analog of the sum formula for the multiple zeta values [**6**], since the admissibility condition $k_r\ge 2$ is ignored in (7). A more precise analog (and its generalization) is due to Saito–Wakabayashi [**14**]. For integers k,r and i satisfying $1\le i\le r<k$, we put $I_{k,r,i} := \{(k_1,\dots,k_r)\in I_{k,r} \mid k_i\ge 2\}$ and $B_{\boldsymbol{p}-k} := (B_{p-k} \bmod p)_p \in \mathcal{A}$, where B_n denotes the n-th Seki–Bernoulli number. Note that $B_{\boldsymbol{p}-k}=0$ if k is even.

Theorem 2.4 (Saito–Wakabayashi [**14**]). *Let k,r and i be integers satisfying $1\le i\le r<k$. Then, in the ring $\mathcal{A}$, we have equalities*

$$\sum_{\boldsymbol{k}\in I_{k,r,i}} \zeta_{\mathcal{A}}(\boldsymbol{k}) = (-1)^i\left\{\binom{k-1}{i-1} + (-1)^r\binom{k-1}{r-i}\right\}\frac{B_{\boldsymbol{p}-k}}{k},$$

$$\sum_{\boldsymbol{k}\in I_{k,r,i}} \zeta_{\mathcal{A}}^\star(\boldsymbol{k}) = (-1)^i\left\{\binom{k-1}{r-i} + (-1)^r\binom{k-1}{i-1}\right\}\frac{B_{\boldsymbol{p}-k}}{k}.$$

In particular, if k is even, we see that

$$\sum_{\boldsymbol{k}\in I_{k,r,i}} \zeta_{\mathcal{A}}(\boldsymbol{k}) = \sum_{\boldsymbol{k}\in I_{k,r,i}} \zeta_{\mathcal{A}}^\star(\boldsymbol{k}) = 0. \tag{8}$$

Our aim is to lift the identities (7) and (8) in $\mathcal{A}$, which represent systems of congruences modulo (almost all) primes p, to congruences modulo p^2 or p^3, by using the identity (6).

Let n be a positive integer. In accordance with [**13**], [**16**], [**21**], we define a $\mathbb{Q}$-algebra $\mathcal{A}_n$ by

$$\mathcal{A}_n := \left(\prod_{p\colon \text{prime}} \mathbb{Z}/p^n\mathbb{Z} \right) \Big/ \left(\bigoplus_{p\colon \text{prime}} \mathbb{Z}/p^n\mathbb{Z} \right)$$

and the $\mathcal{A}_n$-finite multiple zeta values $\zeta_{\mathcal{A}_n}(\boldsymbol{k})$ and $\zeta^\star_{\mathcal{A}_n}(\boldsymbol{k})$ by

$$\zeta_{\mathcal{A}_n}(\boldsymbol{k}) := (\zeta_{p-1}(\boldsymbol{k}) \bmod p^n)_p, \quad \zeta^\star_{\mathcal{A}_n}(\boldsymbol{k}) := (\zeta^\star_{p-1}(\boldsymbol{k}) \bmod p^n)_p \in \mathcal{A}_n.$$

We use the symbol $B_{\boldsymbol{p}-k}$ again to denote the element $(B_{p-k} \bmod p^n)_p$ of $\mathcal{A}_n$, and put $\boldsymbol{p} := (p \bmod p^n)_p \in \mathcal{A}_n$. Then our second main result is the following:

THEOREM 2.5 (= Proposition 4.6 + Theorem 5.2 + Theorem 4.7). *Let k, r be positive integers satisfying $r \le k$. Then, in the ring $\mathcal{A}_2$, we have*

$$\sum_{\boldsymbol{k}\in I_{k,r}} \zeta_{\mathcal{A}_2}(\boldsymbol{k}) = (-1)^{r-1}\binom{k}{r}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}, \quad \sum_{\boldsymbol{k}\in I_{k,r}} \zeta^\star_{\mathcal{A}_2}(\boldsymbol{k}) = \binom{k}{r}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}.$$

If k is odd, in the ring $\mathcal{A}_3$, we have

$$\sum_{\boldsymbol{k}\in I_{k,r}} \zeta_{\mathcal{A}_3}(\boldsymbol{k}) = (-1)^{r}\frac{k+1}{2}\binom{k}{r}\frac{B_{\boldsymbol{p}-k-2}}{k+2}\boldsymbol{p}^2, \quad \sum_{\boldsymbol{k}\in I_{k,r}} \zeta^\star_{\mathcal{A}_3}(\boldsymbol{k}) = -\frac{k+1}{2}\binom{k}{r}\frac{B_{\boldsymbol{p}-k-2}}{k+2}\boldsymbol{p}^2.$$

Furthermore, let i be an integer satisfying $1 \le i \le r$ and we assume that k is even and greater than r. Then the equalities

$$\sum_{\boldsymbol{k}\in I_{k,r,i}} \zeta_{\mathcal{A}_2}(\boldsymbol{k}) = (-1)^{r-1}\frac{a_{k,r,i}}{2}\cdot\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}, \quad \sum_{\boldsymbol{k}\in I_{k,r,i}} \zeta^\star_{\mathcal{A}_2}(\boldsymbol{k}) = \frac{b_{k,r,i}}{2}\cdot\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}$$

hold in $\mathcal{A}_2$. Here the coefficients $a_{k,r,i}$ and $b_{k,r,i}$ are given by

$$a_{k,r,i} := \binom{k-1}{r} + (-1)^{r-i}\left\{(k-r)\binom{k}{i-1} + \binom{k-1}{i-1} + (-1)^{r-1}\binom{k-1}{r-i}\right\},$$
$$b_{k,r,i} := \binom{k-1}{r} + (-1)^{i-1}\left\{(k-r)\binom{k}{r-i} + \binom{k-1}{r-i} + (-1)^{r-1}\binom{k-1}{i-1}\right\}.$$

We will prove this theorem in Section 4 and Section 5.

3. The proof of Theorem 2.1.

DEFINITION 3.1 (connected sum). Let N be a positive integer, q a real number satisfying $0 < q < 1$ and x an indeterminate. Let $r > 0$ and $s \ge 0$ be integers. For $\boldsymbol{k} = (k_1, \ldots, k_r) \in J_{*,r}$ satisfying $k_1, \ldots, k_{r-1} \ge 1$ and $\boldsymbol{l} = (l_1, \ldots, l_s) \in I_{*,s}$, we define a formal power series $Z^\star_N(\boldsymbol{k}; \boldsymbol{l}; q; x)$ in x by

$$Z_N^\star(\boldsymbol{k};\boldsymbol{l};q;x) := \sum_{1\le m_1\le\cdots\le m_r\le n_1\le\cdots\le n_s\le n_{s+1}=N} F_1(\boldsymbol{k};\boldsymbol{m};q;x)C(m_r,n_1,q,x)F_2(\boldsymbol{l};\boldsymbol{n};q;x),$$

where

$$F_1(\boldsymbol{k};\boldsymbol{m};q;x) := \frac{[m_1]_q}{[m_1]_q-q^{m_1}x}\prod_{i=1}^{r}\frac{q^{(k_i-1)m_i}}{[m_i]_q([m_i]_q-q^{m_i}x)^{k_i-1}}\cdot\frac{[m_r]_q}{[m_r]_q-q^{m_r}x},$$

$$C(m_r,n_1,q,x) := (-1)^{m_r-1}q^{m_r(m_r+1)/2}\frac{\prod_{h=1}^{n_1}([h]_q-q^hx)}{[m_r]_q![n_1-m_r]_q!},$$

$$F_2(\boldsymbol{l};\boldsymbol{n};q;x) := \prod_{j=1}^{s}\frac{q^{n_j}}{([n_j]_q-q^{n_j}x)[n_j]_q^{l_j-1}}$$

for $\boldsymbol{m}=(m_1,\dots,m_r)$ and $\boldsymbol{n}=(n_1,\dots,n_s)$.

Remark 3.2. The sum $Z_N^\star(\boldsymbol{k};\boldsymbol{l};q;x)$ consists of two parts

$$\sum_{1\le m_1\le\cdots\le m_r\le N} F_1(\boldsymbol{k};\boldsymbol{m};q;x) \quad\text{and}\quad \sum_{1\le n_1\le\cdots\le n_s\le N} F_2(\boldsymbol{l};\boldsymbol{n};q;x),$$

connected by the factor $C(m_r,n_1,q,x)$ (and the relation $m_r\le n_1$). We call it a connected sum with the connector $C(m_r,n_1,q,x)$. In [**17**], another type of connected sums is used to give a new proof of Ohno's relation for the multiple zeta values and Bradley's q-analog of it.

Theorem 3.3. *For $(k_1,\dots,k_r)\in J_{*,r}$ with $k_1,\dots,k_{r-1}\ge 1$ and $(l_1,\dots,l_s)\in I_{*,s}$, we have*

$$Z_N^\star(k_1,\dots,k_r+1;l_1,\dots,l_s;q;x) = Z_N^\star(k_1,\dots,k_r;1,l_1,\dots,l_s;q;x). \tag{9}$$

Moreover, if $s>0$, we also have

$$Z_N^\star(k_1,\dots,k_r+1,0;l_1,\dots,l_s;q;x) = Z_N^\star(k_1,\dots,k_r;1+l_1,\dots,l_s;q;x). \tag{10}$$

Proof. The equality (9) follows from the telescoping sum

$$\begin{aligned}
&\frac{q^m}{[m]_q-q^mx}\cdot C(m,n,q,x)\\
&\quad=\sum_{a=m+1}^{n}\left(\frac{q^m}{[m]_q-q^mx}\cdot C(m,a,q,x)-\frac{q^m}{[m]_q-q^mx}\cdot C(m,a-1,q,x)\right)\\
&\qquad\quad+\frac{q^m}{[m]_q-q^mx}\cdot C(m,m,q,x)\\
&\quad=\sum_{a=m}^{n}C(m,a,q,x)\cdot\frac{q^a}{[a]_q-q^ax}
\end{aligned}$$

applied to $m=m_r$, $n=n_2$ and $a=n_1$ in the definition of $Z_N^\star(k_1,\dots,k_r;1,l_1,\dots,l_s;q;x)$. Similarly, the equality (10) follows from the telescoping sum

$$\begin{aligned}
&\frac{q^m}{[m]_q}\sum_{a=m}^{n} q^{-a}C(a,n,q,x)\\
&\quad=\frac{q^m}{[m]_q}\sum_{a=m}^{n}\left(\frac{[a]_q}{q^a}\cdot C(a,n,q,x)\cdot\frac{1}{[n]_q}-\frac{[a+1]_q}{q^{a+1}}\cdot C(a+1,n,q,x)\cdot\frac{1}{[n]_q}\right)\\
&\quad=C(m,n,q,x)\cdot\frac{1}{[n]_q}
\end{aligned}$$

applied to $m=m_r$, $n=n_1$, $a=m_{r+1}$ in the definition of $Z_N^\star(k_1,\dots,k_r+1,0;l_1,\dots,l_s;q;x)$. □

Remark 3.4. Let r,s be as in Definition 3.1 and we omit q and x from notation. If we put $\widetilde{Z_N^\star}(k_1,\dots,k_r;l_1,\dots,l_s):=Z_N^\star(k_1,\dots,k_r-1;l_1,\dots,l_s)$ for $(k_1,\dots,k_r)\in I_{*,r}$ and $(l_1,\dots,l_s)\in I_{*,s}$, then we can rewrite the transport relations (9) and (10) into the following symmetrical form:

$$\begin{aligned}
\widetilde{Z_N^\star}(k_1,\dots,k_r+1;l_1,\dots,l_s)&=\widetilde{Z_N^\star}(k_1,\dots,k_r;1,l_1,\dots,l_s),\\
\widetilde{Z_N^\star}(k_1,\dots,k_r,1;l_1,\dots,l_s)&=\widetilde{Z_N^\star}(k_1,\dots,k_r;1+l_1,\dots,l_s)\qquad(s>0).
\end{aligned}$$

Corollary 3.5. *Let N be a positive integer and $\boldsymbol{k}=(k_1,\dots,k_r)$ an index. We define $P_N(\boldsymbol{k};q;x)$, $Q_N(\boldsymbol{k};q;x)$ and $R_N(q;x)$ by*

$$\begin{aligned}
P_N(\boldsymbol{k};q;x)&:=\sum_{1\le m_1\le\cdots\le m_r\le N}\frac{[m_1]_q}{[m_1]_q-q^{m_1}x}\prod_{i=1}^{r}\frac{q^{(k_i-1)m_i}}{[m_i]_q([m_i]_q-q^{m_i}x)^{k_i-1}}\cdot\frac{[m_r]_q}{[m_r]_q-q^{m_r}x}\\
&\qquad\cdot(-1)^{m_r-1}q^{m_r(m_r+1)/2}\binom{N}{m_r}_q,\\
Q_N(\boldsymbol{k};q;x)&:=\sum_{1\le m_1\le\cdots\le m_r\le N}\prod_{i=1}^{r}\frac{q^{m_i}}{([m_i]_q-q^{m_i}x)[m_i]_q^{k_i-1}},\\
R_N(q;x)&:=\prod_{h=1}^{N}\left(1-\frac{q^hx}{[h]_q}\right)^{-1}.
\end{aligned}$$

Then we have

$$P_N(\boldsymbol{k};q;x)=Q_N(\boldsymbol{k}^\vee;q;x)R_N(q;x). \tag{11}$$

Proof. By applying equalities in Theorem 3.3 $\mathrm{wt}(\boldsymbol{k})$ times, we see that

$$Z_N^\star(\boldsymbol{k};\emptyset;q;x)=\cdots=Z_N^\star(0;\boldsymbol{k}^\vee;q;x)$$

holds by the definition of the Hoffman dual. For example,

$$Z_N^\star(1,1,2;\emptyset)\overset{(9)}{=}Z_N^\star(1,1,1;1)\overset{(9)}{=}Z_N^\star(1,1,0;1,1)\overset{(10)}{=}Z_N^\star(1,0;2,1)\overset{(10)}{=}Z_N^\star(0;3,1)$$

(here we abbreviated $Z_N^\star(\boldsymbol{k};\boldsymbol{l};q;x)$ as $Z_N^\star(\boldsymbol{k};\boldsymbol{l})$). By definition, we have

$$\begin{aligned} Z_N^\star(\boldsymbol{k};\emptyset;q;x) &= \sum_{1\le m_1\le\cdots\le m_r\le N} F_1(\boldsymbol{k};\boldsymbol{m};q;x)C(m_r,N,q,x) \\ &= P_N(\boldsymbol{k};q;x)R_N(q;x)^{-1} \end{aligned}$$

and

$$\begin{aligned} Z_N^\star(0;\boldsymbol{k}^\vee;q;x) &= \sum_{1\le m\le n_1\le\cdots\le n_s\le N} \frac{q^{-m}[m]_q}{[m]_q-q^m x} C(m,n_1,q,x)F_2(\boldsymbol{k}^\vee;\boldsymbol{n};q;x) \\ &= Q_N(\boldsymbol{k}^\vee;q;x). \end{aligned}$$

In the last equality, we have used the partial fraction decomposition

$$\sum_{m=1}^{n_1} \frac{[m]_q}{[m]_q-q^m x}\cdot\frac{(-1)^{m-1}q^{m(m-1)/2}}{[m]_q![n_1-m]_q!} = \frac{1}{\prod_{h=1}^{n_1}([h]_q-q^h x)}.$$

The proof is complete. □

Proof of Theorem 2.1. By using the expansion formula

$$\frac{1}{([m]_q-q^m x)^k} = \sum_{e=0}^{\infty}\binom{k+e-1}{e}\frac{q^{em}x^e}{[m]_q^{k+e}}$$

for a positive integer m and a non-negative integer k, we see that

$$P_N(\boldsymbol{k};q;x) = \sum_{e=0}^{\infty}\sum_{\boldsymbol{e}\in J_{e,r}} b(\boldsymbol{k};\boldsymbol{e})H_N^\star(\boldsymbol{k}+\boldsymbol{e};q)x^e$$

and

$$Q_N(\boldsymbol{k}^\vee;q;x) = \sum_{e=0}^{\infty}\sum_{\boldsymbol{e}\in J_{e,s}} z_N^\star(\boldsymbol{k}^\vee;\boldsymbol{e};q)x^e.$$

Since $R_N(q;x)=\sum_{e=0}^{\infty} z_N^\star(\{1\}^e;q)x^e$, we obtain the identity (4) by comparing the coefficients of x^e in (11). □

4. Sum formulas for $\mathcal{A}_2$-finite multiple zeta values.

4.1. Auxiliary facts.

We prepare some known facts for finite multiple zeta values.

Proposition 4.1 ([**8**, Theorems 6.1, 6.2], [**20**, Theorems 3.1, 3.5]). *Let k_1, k_2 and k_3 be positive integers, and assume that $l := k_1+k_2+k_3$ is odd. Then*

$$\zeta_{\mathcal{A}}^\star(k_1,k_2) = (-1)^{k_2}\binom{k_1+k_2}{k_1}\frac{B_{\boldsymbol{p}-k_1-k_2}}{k_1+k_2}, \tag{12}$$

$$\zeta_{\mathcal{A}}^\star(k_1,k_2,k_3) = \frac{1}{2}\left\{(-1)^{k_3}\binom{l}{k_3}-(-1)^{k_1}\binom{l}{k_1}\right\}\frac{B_{\boldsymbol{p}-l}}{l}. \tag{13}$$

Proposition 4.2 ([**22**], [**20**, Theorem 3.2]). *Let k, r, k_1 and k_2 be positive integers, and assume that $l := k_1 + k_2$ is even. Then*

$$\zeta^{\star}_{\mathcal{A}_2}(\{k\}^r) = k\frac{B_{\boldsymbol{p}-rk-1}}{rk+1}\boldsymbol{p}, \tag{14}$$

$$\zeta^{\star}_{\mathcal{A}_2}(k_1, k_2) = \frac{1}{2}\left\{(-1)^{k_1}k_2\binom{l+1}{k_1} - (-1)^{k_2}k_1\binom{l+1}{k_2} + l\right\}\frac{B_{\boldsymbol{p}-l-1}}{l+1}\boldsymbol{p}. \tag{15}$$

Proposition 4.3 ([**15**, Corollary 3.16 (42)]). *Let n be a positive integer and $\boldsymbol{k} = (k_1, \ldots, k_r)$ an index. Then*

$$\sum_{j=0}^{r}(-1)^j\zeta_{\mathcal{A}_n}(k_j, \ldots, k_1)\,\zeta^{\star}_{\mathcal{A}_n}(k_{j+1}, \ldots, k_r) = 0. \tag{16}$$

4.2. Computations of sums for $\mathcal{A}_2$-finite multiple zeta values.

Definition 4.4. Let k, r and i be positive integers satisfying $i \le r \le k$. We define four sums $S_{k,r}$, $S^{\star}_{k,r}$, $S_{k,r,i}$ and $S^{\star}_{k,r,i}$ in $\mathcal{A}_2$ by

$$S_{k,r} := \sum_{\boldsymbol{k}\in I_{k,r}}\zeta_{\mathcal{A}_2}(\boldsymbol{k}), \qquad S^{\star}_{k,r} := \sum_{\boldsymbol{k}\in I_{k,r}}\zeta^{\star}_{\mathcal{A}_2}(\boldsymbol{k}),$$
$$S_{k,r,i} := \sum_{\boldsymbol{k}\in I_{k,r,i}}\zeta_{\mathcal{A}_2}(\boldsymbol{k}), \qquad S^{\star}_{k,r,i} := \sum_{\boldsymbol{k}\in I_{k,r,i}}\zeta^{\star}_{\mathcal{A}_2}(\boldsymbol{k}).$$

For an index $\boldsymbol{k} = (k_1, \ldots, k_r)$, we set $\boldsymbol{k}^+ := (k_1, \ldots, k_{r-1}, k_r + 1)$. We can calculate $S^{\star}_{k,r}$ and $S^{\star}_{k,r,i}$ by using the following identity.

Corollary 4.5. *Let e be a non-negative integer, $\boldsymbol{k} \in I_{*,r}$ an index and $s := \operatorname{dep}(\boldsymbol{k}^\vee)$. Then we have*

$$\sum_{j=0}^{e}\zeta^{\star}_{\mathcal{A}_2}(\{1\}^{e-j})\sum_{\boldsymbol{e}'\in J_{j,s}}\zeta^{\star}_{\mathcal{A}_2}(\boldsymbol{k}^\vee + \boldsymbol{e}')$$
$$= \sum_{\boldsymbol{e}\in J_{e,r}} b(\boldsymbol{k};\boldsymbol{e})\Big\{-\zeta^{\star}_{\mathcal{A}_2}(\boldsymbol{k}+\boldsymbol{e}) - \zeta^{\star}_{\mathcal{A}_2}(\boldsymbol{k}+\boldsymbol{e},1)\boldsymbol{p} + \zeta^{\star}_{\mathcal{A}_2}((\boldsymbol{k}+\boldsymbol{e})^+)\boldsymbol{p}\Big\}. \tag{17}$$

Proof. Since a congruence

$$(-1)^{m-1}\binom{p-1}{m} \equiv -1 - \sum_{m\le n\le p-1}\frac{p}{n} + \frac{p}{m} \pmod{p^2}$$

holds for any odd prime p and any positive integer m with $m < p$ (cf. [**16**, Lemma 4.1]), this corollary is a direct consequence of (6). □

Proposition 4.6. *For positive integers k and r such that $r \le k$, we have*

$$(-1)^{r-1}S_{k,r} = S^{\star}_{k,r} = \binom{k}{r}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}.$$

Proof. Let $\boldsymbol{k} = (r)$ and $e = k - r$ in (17). Then $\boldsymbol{k}^\vee = (\{1\}^r)$ and we have

$$\sum_{j=0}^{k-r} \zeta_{\mathcal{A}_2}^{\star}(\{1\}^{k-r-j}) S_{j+r,r}^{\star} = \binom{k}{r}\Big\{-\zeta_{\mathcal{A}_2}(k) - \zeta_{\mathcal{A}_2}^{\star}(k,1)\boldsymbol{p} + \zeta_{\mathcal{A}_2}(k+1)\boldsymbol{p}\Big\}. \tag{18}$$

For $0 \leq j < k - r$, $\zeta_{\mathcal{A}_2}^{\star}(\{1\}^{k-r-j}) S_{j+r,r}^{\star} = 0$ since both $\zeta_{\mathcal{A}_2}^{\star}(\{1\}^{k-r-j})$ and $S_{j+r,r}^{\star}$ are divisible by $\boldsymbol{p}$ by (14) and (7). Therefore, the left hand side of (18) is equal to $S_{k,r}^{\star}$. On the other hand, the right hand side of (18) is equal to

$$\binom{k}{r}\left\{-k\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p} + \binom{k+1}{k}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}\right\} = \binom{k}{r}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}$$

by (14) and (12). Hence we obtain the second equality of the proposition.

By taking $\sum_{\boldsymbol{k}\in I_{k,r}}$ of (16), we obtain

$$S_{k,r}^{\star} + \sum_{j=1}^{r-1}(-1)^j \sum_{l=j}^{k-r+j} S_{l,j} S_{k-l,r-j}^{\star} + (-1)^r S_{k,r} = 0.$$

We see that $S_{l,j} S_{k-l,r-j}^{\star} = 0$ for $1 \leq j \leq r-1$ and $j \leq l \leq k-r+j$, since both $S_{l,j}$ and $S_{k-l,r-j}^{\star}$ are divisible by $\boldsymbol{p}$ by (7). This gives $(-1)^{r-1} S_{k,r} = S_{k,r}^{\star}$. □

Next we compute $S_{k,r,i}^{\star}$ and $S_{k,r,i}$.

Theorem 4.7. *Let k, r and i be positive integers satisfying $i \leq r < k$, and assume that k is even. Then we have*

$$S_{k,r,i} = (-1)^{r-1}\frac{a_{k,r,i}}{2}\cdot\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}, \quad S_{k,r,i}^{\star} = \frac{b_{k,r,i}}{2}\cdot\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p},$$

where

$$a_{k,r,i} = \binom{k-1}{r} + (-1)^{r-i}\left\{(k-r)\binom{k}{i-1} + \binom{k-1}{i-1} + (-1)^{r-1}\binom{k-1}{r-i}\right\},$$

$$b_{k,r,i} = \binom{k-1}{r} + (-1)^{i-1}\left\{(k-r)\binom{k}{r-i} + \binom{k-1}{r-i} + (-1)^{r-1}\binom{k-1}{i-1}\right\}.$$

Proof. Let $\boldsymbol{k} = (i, r-i+1)$ and $e = k-r-1$ in (17). Then $\boldsymbol{k}^\vee = (\{1\}^{i-1}, 2, \{1\}^{r-i})$ and we have

$$\begin{aligned}
&\sum_{j=0}^{k-r-1} \zeta_{\mathcal{A}_2}^{\star}(\{1\}^{k-r-1-j}) S_{j+r+1,r,i}^{\star} \\
&= \sum_{e=0}^{k-r-1}\binom{i+e-1}{e}\binom{k-i-e-1}{k-r-1-e}\Big\{-\zeta_{\mathcal{A}_2}^{\star}(i+e, k-i-e) \\
&\qquad - \zeta_{\mathcal{A}_2}^{\star}(i+e, k-i-e, 1)\boldsymbol{p} + \zeta_{\mathcal{A}_2}^{\star}(i+e, k-i-e+1)\boldsymbol{p}\Big\}.
\end{aligned} \tag{19}$$

For $0 \le j < k-r-1$, we see that $\zeta^\star_{\mathcal{A}_2}(\{1\}^{k-r-1-j})S^\star_{j+r+1,r,i}$ is a rational multiple of $B_{\boldsymbol{p}-k+r+j}B_{\boldsymbol{p}-j-r-1}\boldsymbol{p}$ by (14) and Theorem 2.4. Since k is even, one of $B_{\boldsymbol{p}-k+r+j}$ or $B_{\boldsymbol{p}-j-r-1}$ is zero. Therefore, the left hand side of (19) is equal to $S^\star_{k,r,i}$.

On the other hand, we can calculate the right hand side of (19) as follows. By (15), (12) and (13), we have

$$\begin{aligned}
&-\zeta^\star_{\mathcal{A}_2}(i+e,k-i-e)-\zeta^\star_{\mathcal{A}_2}(i+e,k-i-e,1)\boldsymbol{p}+\zeta^\star_{\mathcal{A}_2}(i+e,k-i-e+1)\boldsymbol{p}\\
&\quad=\Bigg[-\frac{1}{2}\left\{(-1)^{i+e}(k-i-e)\binom{k+1}{i+e}-(-1)^{k-i-e}(i+e)\binom{k+1}{k-i-e}+k\right\}\\
&\qquad\quad-\frac{1}{2}\left\{-(k+1)-(-1)^{i+e}\binom{k+1}{i+e}\right\}+(-1)^{k-i-e+1}\binom{k+1}{i+e}\Bigg]\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}\\
&\quad=\frac{1}{2}\left[1-(-1)^{i+e}(k-i-e+1)\binom{k+1}{i+e}+(-1)^{i+e}(i+e)\binom{k+1}{k-i-e}\right]\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}\\
&\quad=\frac{1}{2}\left[1+(-1)^{i-1+e}\binom{k+1}{i+e+1}\right]\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}.
\end{aligned}$$

Therefore, the right hand side of (19) is equal to

$$\frac{1}{2}\sum_{e=0}^{k-r-1}\binom{i+e-1}{e}\binom{k-i-e-1}{k-r-1-e}\left[1+(-1)^{i-1+e}\binom{k+1}{i+e+1}\right]\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}. \tag{20}$$

By comparing the coefficient of x^{k-r-1} in $(1-x)^{-i}(1-x)^{-(r-i+1)} = (1-x)^{-(r+1)}$, we see that

$$\sum_{e=0}^{k-r-1}\binom{i+e-1}{e}\binom{k-i-e-1}{k-r-1-e}=\binom{k-1}{r},$$

and by using the partial fraction decomposition

$$F(x):=\sum_{e=0}^{k-r-1}\frac{(-1)^e}{e!(k-r-1-e)!}\cdot\frac{1}{x+e}=\frac{1}{x(x+1)\cdots(x+k-r-1)},$$

we see that

$$\begin{aligned}
&\sum_{e=0}^{k-r-1}\binom{i+e-1}{e}\binom{k-i-e-1}{k-r-1-e}\cdot(-1)^{i-1+e}\binom{k+1}{i+e+1}\\
&\quad=(-1)^{i-1}\frac{(k+1)!}{(i-1)!(r-i)!}\sum_{e=0}^{k-r-1}\frac{(-1)^e}{e!(k-r-1-e)!(i+e)(i+e+1)(k-i-e)}\\
&\quad=(-1)^{i-1}\frac{(k+1)!}{(i-1)!(r-i)!}\left\{\frac{1}{k}F(i)-\frac{1}{k+1}F(i+1)+\frac{(-1)^{r-1}}{k(k+1)}F(r-i+1)\right\}\\
&\quad=(-1)^{i-1}\left\{(k-r)\binom{k}{r-i}+\binom{k-1}{r-i}+(-1)^{r-1}\binom{k-1}{i-1}\right\}.
\end{aligned}$$

Thus we have proved the desired formula for $S^\star_{k,r,i}$.

Let us take the sum $\sum_{\boldsymbol{k}\in I_{k,r,r+1-i}}$ of (16). Then we obtain

$$S^\star_{k,r,r+1-i}+\sum_{j=1}^{r-i}(-1)^j\sum_{l=j}^{k-r+j-1}S_{l,j}S^\star_{k-l,r-j,r+1-i-j}$$
$$+\sum_{j=r-i+1}^{r-1}(-1)^j\sum_{l=j+1}^{k-r+j}S_{l,j,j+i-r}S^\star_{k-l,r-j}+(-1)^rS_{k,r,i}=0. \tag{21}$$

We know that $S_{l,j}S^\star_{k-l,r-j,r+1-i-j}$ is a rational multiple of $B_{\boldsymbol{p}-l-1}B_{\boldsymbol{p}-k+l}\boldsymbol{p}$ for $1\le j\le r-i$ and we also know that $S_{l,j,j+i-r}S^\star_{k-l,r-j}$ is a rational multiple of $B_{\boldsymbol{p}-l}B_{\boldsymbol{p}-k+l-1}\boldsymbol{p}$ for $r-i+1\le j\le r-1$ by Theorem 2.4 and Proposition 4.6. Since k is even, these are zero for every l. Therefore, we have

$$S_{k,r,i}=(-1)^{r-1}\,\frac{b_{k,r,r+1-i}}{2}\cdot\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}=(-1)^{r-1}\,\frac{a_{k,r,i}}{2}\cdot\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}. \qquad \square$$

5. Sum formulas for $\mathcal{A}_3$-finite multiple zeta values.

For positive integers k and r such that $r\le k$, we set

$$T_{k,r}:=\sum_{\boldsymbol{k}\in I_{k,r}}\zeta_{\mathcal{A}_3}(\boldsymbol{k}),\quad T^\star_{k,r}:=\sum_{\boldsymbol{k}\in I_{k,r}}\zeta^\star_{\mathcal{A}_3}(\boldsymbol{k}).$$

We can calculate $T^\star_{k,r}$ by using the following identity.

COROLLARY 5.1. *Let e be a non-negative integer, $\boldsymbol{k}\in I_{*,r}$ an index and $s:=\operatorname{dep}(\boldsymbol{k}^\vee)$. Then we have*

$$\sum_{j=0}^{e}\zeta^\star_{\mathcal{A}_3}(\{1\}^{e-j})\sum_{\boldsymbol{e}'\in J_{j,s}}\zeta^\star_{\mathcal{A}_3}(\boldsymbol{k}^\vee+\boldsymbol{e}')$$
$$=\sum_{\boldsymbol{e}\in J_{e,r}}b(\boldsymbol{k};\boldsymbol{e})\Big\{-\zeta^\star_{\mathcal{A}_3}(\boldsymbol{k}+\boldsymbol{e})-\zeta^\star_{\mathcal{A}_3}(\boldsymbol{k}+\boldsymbol{e},1)\boldsymbol{p}+\zeta^\star_{\mathcal{A}_3}((\boldsymbol{k}+\boldsymbol{e})^+)\boldsymbol{p}$$
$$-\zeta^\star_{\mathcal{A}_3}(\boldsymbol{k}+\boldsymbol{e},1,1)\boldsymbol{p}^2+\zeta^\star_{\mathcal{A}_3}((\boldsymbol{k}+\boldsymbol{e})^+,1)\boldsymbol{p}^2\Big\}. \tag{22}$$

PROOF. Since a congruence

$$(-1)^{m-1}\binom{p-1}{m}$$
$$\equiv -1-\left(\sum_{m\le n\le p-1}\frac{1}{n}-\frac{1}{m}\right)p-\left(\sum_{m\le n_1\le n_2\le p-1}\frac{1}{n_1n_2}-\frac{1}{m}\sum_{m\le n\le p-1}\frac{1}{n}\right)p^2 \pmod{p^3}$$

holds for any odd prime p and any positive integer m with $m<p$ (cf. [**16**, Lemma 4.1]), this corollary is a direct consequence of (6). $\square$

From now on, we assume that k is odd. We recall a formula

$$\zeta_{\mathcal{A}_3}(k) = -\frac{k(k+1)}{2} \cdot \frac{B_{\boldsymbol{p}-k-2}}{k+2}\boldsymbol{p}^2 \tag{23}$$

proved by Sun [**18**, Theorem 5.1]. Here, $\boldsymbol{p}^2 = (p^2 \bmod p^3)_p \in \mathcal{A}_3$.

THEOREM 5.2. *Let k and r be positive integers satisfying $r \leq k$, and assume that k is odd. Then we have*

$$(-1)^{r-1}T_{k,r} = T^\star_{k,r} = -\frac{k+1}{2}\binom{k}{r}\frac{B_{\boldsymbol{p}-k-2}}{k+2}\boldsymbol{p}^2.$$

PROOF. Let $\boldsymbol{k} = (r)$ and $e = k - r$ in (22). Then $\boldsymbol{k}^\vee = (\{1\}^r)$ and we have

$$\begin{aligned}&\sum_{j=0}^{k-r}\zeta^\star_{\mathcal{A}_3}(\{1\}^{k-r-j})T^\star_{j+r,r}\\ &= \binom{k}{r}\left\{-\zeta_{\mathcal{A}_3}(k) - \zeta^\star_{\mathcal{A}_3}(k,1)\boldsymbol{p} + \zeta^\star_{\mathcal{A}_3}(k+1)\boldsymbol{p} - \zeta^\star_{\mathcal{A}_3}(k,1,1)\boldsymbol{p}^2 + \zeta^\star_{\mathcal{A}_3}(k+1,1)\boldsymbol{p}^2\right\}.\end{aligned} \tag{24}$$

Let us fix $0 \leq j < k - r$. By (14) and Proposition 4.6, $\zeta^\star_{\mathcal{A}_3}(\{1\}^{k-r-j})$ and $T^\star_{j+r,r}$ are divisible by $\boldsymbol{p}$. Furthermore, if $j+r$ is even (resp. odd), then $\zeta^\star_{\mathcal{A}_3}(\{1\}^{k-r-j})$ (resp. $T^\star_{j+r,r}$) is divisible by $\boldsymbol{p}^2$. Therefore, $\zeta^\star_{\mathcal{A}_3}(\{1\}^{k-r-j})T^\star_{j+r,r} = 0$ and we see that the left hand side of (24) is equal to $T^\star_{k,r}$. On the other hand, by using Proposition 4.1, Proposition 4.2 and (23), we see that the right hand side of (24) is equal to

$$\begin{aligned}&\binom{k}{r}\left[\frac{k(k+1)}{2} - \frac{1}{2}\left\{-\binom{k+2}{k} + k^2 + 3k + 1\right\} + (k+1)\right.\\ &\qquad \left. - \frac{1}{2}\left\{-(k+2) + \binom{k+2}{k}\right\} - (k+2)\right]\frac{B_{\boldsymbol{p}-k-2}}{k+2}\boldsymbol{p}^2\\ &= -\frac{k+1}{2}\binom{k}{r}\frac{B_{\boldsymbol{p}-k-2}}{k+2}\boldsymbol{p}^2.\end{aligned}$$

Hence we obtain the second equality of the theorem. By taking $\sum_{\boldsymbol{k}\in I_{k,r}}$ of (16), we obtain

$$T^\star_{k,r} + \sum_{j=1}^{r-1}(-1)^j\sum_{l=j}^{k-r+j}T_{l,j}T^\star_{k-l,r-j} + (-1)^rT_{k,r} = 0.$$

Let us fix $1 \leq j \leq r-1$ and $j \leq l \leq k-r+j$. By Proposition 4.6, $T_{l,j}$ and $T^\star_{k-l,r-j}$ are divisible by $\boldsymbol{p}$. Furthermore, if l is odd (resp. even), then $T_{l,j}$ (resp. $T^\star_{k-l,r-j}$) is divisible by $\boldsymbol{p}^2$. Therefore, we see that $T_{l,j}T^\star_{k-l,r-j} = 0$ and this gives $(-1)^{r-1}T_{k,r} = T^\star_{k,r}$. □

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Shin-ichiro SEKI
Mathematical Institute
Tohoku University
6-3 Aoba, Aramaki, Aoba-Ku
Sendai 980-8578, Japan
E-mail: shinichiro.seki.b3@tohoku.ac.jp

Shuji YAMAMOTO
Department of Mathematics
Faculty of Science and Technology
Keio University
3-14-1 Hiyoshi, Kohoku-ku
Yokohama 223-8522, Japan
E-mail: yamashu@math.keio.ac.jp