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Apéry–Fermi pencil of $K3$-surfaces and 2-isogenies

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Abstract. Given a generic $K3$-surface Y_k of the Apéry–Fermi pencil, we use the Kneser–Nishiyama technique to determine all its non isomorphic elliptic fibrations. These computations lead to determine those fibrations with 2-torsion sections T. We classify the fibrations such that the translation by T gives a Shioda–Inose structure. The other fibrations correspond to a $K3$-surface identified by its transcendental lattice. The same problem is solved for a singular member Y_2 of the family showing the differences with the generic case. In conclusion we put our results in the context of relations between 2-isogenies and isometries on the singular surfaces of the family.

1. Introduction.

The Apéry–Fermi pencil $\mathcal{F}$ is realized with the affine equations

$$X+\frac{1}{X}+Y+\frac{1}{Y}+Z+\frac{1}{Z}=k, \quad k\in\mathbb{C},$$

and taking $k=s+1/s$, is seen as the Fermi threefold $\mathcal{Z}$ with compactification denoted $\bar{\mathcal{Z}}$ [**27**].

The projection $\pi_s:\bar{\mathcal{Z}}\to\mathbb{P}^1(s)$ is called the Fermi fibration. In their paper [**27**], Peters and Stienstra proved that for $s\notin\{0,\infty,\pm1,3\pm2\sqrt{2},-3\pm2\sqrt{2}\}$ the fibers of the Fermi fibration are $K3$-surfaces with the Néron–Severi lattice of the generic fiber isometric to $M_6=E_8[-1]\oplus E_8[-1]\oplus U\oplus\langle-2\times6\rangle$ and transcendental lattice isometric to $T=U\oplus\langle2\times6\rangle$ (U denotes the hyperbolic lattice and E_8 the unique positive definite even unimodular lattice of rank 8). Hence this family appears as a family of M_6-polarized $K3$-surfaces Y_k with period $t\in\mathcal{H}$. And we deduce from a result of Dolgachev [**12**] the following property. Let $E_t=\mathbb{C}/(\mathbb{Z}+t\mathbb{Z})$ and $E'_t=\mathbb{C}/(\mathbb{Z}+(-1/6t)\mathbb{Z})$ be the corresponding pair of isogenous elliptic curves. Then there exists a canonical involution τ on Y_k such that $Y_k/(\tau)$ is birationally isomorphic to the Kummer surface $E_t\times E'_t/(\pm1)$.

This result is linked to the Shioda–Inose structure of $K3$-surfaces with Picard number 19 and 20 described first by Shioda and Inose [**33**] and extended by Morrison [**21**](Corollary 6.4).

As observed by Elkies [**14**], the base of the pencil of $K3$-surfaces can be identified with the elliptic modular curve $X_0(6)/\langle w_2,w_3\rangle$ where w_2 and w_3 denote the Atkin–Lehner involutions [**1**]. Indeed it can be derived from Peters and Stienstra [**27**].

Shioda considers the problem whether every Shioda–Inose structure can be extended to a sandwich, that is, given a $K3$-surface S, if there exists a unique Kummer surface

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$K = Km(C_1 \times C_2)$ with two rational maps of degree 2, $S \to K$ and $K \to S$ where C_1 and C_2 are elliptic curves. In [**31**] Shioda proved a "Kummer sandwich theorem", for an elliptic $K3$-surface S (with a section) with two II^*-fibres.

In van Geemen–Sarti [**16**], Comparin–Garbagnati [**10**], Koike [**18**] and Schütt [**28**] (3.5, 4.4, 5.4), sandwich Shioda–Inose structures are constructed via elliptic fibrations with 2-torsion sections.

Recently Bertin and Lecacheux [**5**] found all the elliptic fibrations of a singular member Y_2 of $\mathcal{F}$ (i.e. of Picard number 20) and observed that many of its elliptic fibrations are endowed with 2-torsion sections. Considering the minimal resolution of the quotient of Y_2 by the symplectic involution defined by the 2-torsion section, a question arises: are the corresponding involutions all Morrison–Nikulin? (see Section 4, Definition 4.1). Observing also that the Shioda's Kummer sandwiching between a $K3$-surface S and its Kummer surface K is in fact a 2-isogeny between two elliptic fibrations of S and K, we extended the above question to the generic member Y_k of the family $\mathcal{F}$ and obtained the following results.

Theorem 1.1. *Suppose Y_k is a generic $K3$-surface of the family with Picard number* 19.

Let $\pi : Y_k \to \mathbb{P}^1$ be an elliptic fibration with a torsion section of order 2 *which defines an involution i of Y_k (van Geemen–Sarti involution) then the minimal resolution of the quotient Y_k/i is either the Kummer surface K_k associated to Y_k given by its Shioda–Inose structure or a surface S_k with transcendental lattice $T(S_k) = \langle -2 \rangle \oplus \langle 2 \rangle \oplus \langle 6 \rangle$ and Néron–Severi lattice $NS(S_k) = U \oplus E_8[-1] \oplus E_7[-1] \oplus \langle -2 \rangle \oplus \langle -6 \rangle$, which is not a Kummer surface. Thus, π leads to an elliptic fibration either of K_k or of S_k. Moreover there exist some genus* 1 *fibrations $\theta : K_k \to \mathbb{P}^1$ without section such that their Jacobian variety satisfies $J_\theta(K_k) = S_k$.*

More precisely, among the elliptic fibrations of Y_k (up to automorphisms) 12 *of them have a two-torsion section. And only* 7 *of them possess a Morrison–Nikulin involution i such that $Y_k/i = K_k$.*

Remark 1.1. The fact that S_k is not a Kummer surface follows from a result of Morrison [**21**].

The $K3$-surface S_k is the Hessian $K3$-surface of a general cubic surface with 3 nodes studied by Dardanelli and van Geemen [**13**].

Theorem 1.2. *In the Apéry–Fermi pencil, the $K3$-surface Y_2 is singular, meaning that its Picard number is* 20. *Moreover Y_2 has many more* 2*-torsion sections than the generic $K3$-surface Y_k; hence among its* 20 *van Geemen–Sarti involutions,* 13 *of them are Morrison–Nikulin involutions,* 5 *are symplectic automorphisms of order* 2 *(self-involutions) and the two remaining ones exchange two elliptic fibrations of Y_2.*

The specializations to Y_2 of the 7 *Morrison–Nikulin involutions of a generic member Y_k are verified among the* 13 *Morrison–Nikulin involutions of Y_2. The specializations of the* 5 *remaining involutions between Y_k and the $K3$-surface S_k are among the* 7 *van Geemen–Sarti involutions of Y_2 which are not Morrison–Nikulin.*

Remark 1.2. The fact that the specializations to Y_2 of the 7 Morrison–Nikulin involutions of Y_k are Morrison–Nikulin involutions of Y_2 can be deduced from a general result of Schütt [**28**].

This theorem provides an example of a Kummer surface K_2 defined by the product of two isogenous elliptic curves (actually the same elliptic curve of j-invariant equal to 8000), having many fibrations of genus one whose Jacobian surface is not a Kummer surface. A similar result but concerning a Kummer surface defined by two non-isogenous elliptic curves has been exhibited by Keum [**17**].

Throughout the paper we use the following result [**36**]. If E denotes an elliptic fibration with a 2-torsion point $(0,0)$:

$$E : y^2 = x^3 + Ax^2 + Bx,$$

the quotient curve $E/\langle(0,0)\rangle$ has a Weierstrass equation of the form

$$E/\langle(0,0)\rangle : y^2 = x^3 - 2Ax^2 + (A^2 - 4B)x.$$

The paper is organized as follows.

In Section 2 we recall the Kneser–Nishiyama method and use it to find all the 27 elliptic fibrations of a generic $K3$-surface of the family $\mathcal{F}$. In Section 3, using Elkies's method of “2-neighbors” [**15**], we exhibit an elliptic parameter giving a Weierstrass equation of the elliptic fibration. The results are summarized in Table 2. Thus we obtain all the Weierstrass equations of the 12 elliptic fibrations with 2-torsion sections. Their 2-isogenous elliptic fibrations are computed in Section 5 with their Mordell–Weil groups and discriminants. Section 4 recalls generalities about Nikulin involutions and Shioda–Inose structure. Section 5 is devoted to the proof of Theorem 1.1 while Section 6 is concerned with the proof of Theorem 1.2.

It is not easy to obtain a theorem similar to Theorem 1.2 for other singular $K3$-surfaces of the family, in particular to get all their fibrations. Nevertheless, in the last Section 7, as a corollary of a result of Boissière, Sarti and Veniani [**7**], we shall explain why the existence of symplectic automorphisms of order two cannot be expected in all the singular $K3$-surfaces of the family. Precisely we prove this existence only on the singular $K3$-surfaces Y_2 and Y_{10}.

Computations were performed using partly the computer algebra system PARI [**26**] and mostly the computer algebra system MAPLE and the Maple Library “Elliptic Surface Calculator” written by Kuwata [**20**].

2. Elliptic fibrations of the family.

We refer to [**5**], [**29**] for definitions concerning lattices, primitive embeddings, orthogonal complement of a sublattice into a lattice. We recall only what is essential for understanding this section and Section 5.2.

2.1. Discriminant forms.

Let L be a non-degenerate lattice. The *dual lattice* L^* of L is defined by

$$L^* := \mathrm{Hom}(L, \mathbb{Z}) = \{x \in L \otimes \mathbb{Q} /\ b(x,y) \in \mathbb{Z} \text{ for all } y \in L\}$$

and the *discriminant group* G_L by

$$G_L := L^*/L.$$

This group is finite if and only if L is non-degenerate. In the latter case, its order is equal to the absolute value of the lattice determinant $\mid \det(G(e)) \mid$ for any basis e of L. A lattice L is *unimodular* if G_L is trivial.

Let G_L be the discriminant group of a non-degenerate lattice L. The bilinear form on L extends naturally to a $\mathbb{Q}$-valued symmetric bilinear form on L^* and induces a symmetric bilinear form

$$b_L : G_L \times G_L \to \mathbb{Q}/\mathbb{Z}.$$

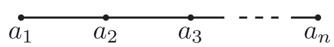
If L is even, then b_L is the symmetric bilinear form associated to the quadratic form defined by

$$\begin{array}{c} q_L : G_L \ \rightarrow \ \mathbb{Q}/2\mathbb{Z} \\ q_L(x+L) \mapsto x^2 + 2\mathbb{Z}. \end{array}$$

The latter means that $q_L(na) = n^2 q_L(a)$ for all $n \in \mathbb{Z}$, $a \in G_L$ and $b_L(a,a') = (1/2)(q_L(a+a') - q_L(a) - q_L(a'))$, for all $a, a' \in G_L$, where $1/2 : \mathbb{Q}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is the natural isomorphism. The pair $\boldsymbol{(G_L, b_L)}$ (resp. $\boldsymbol{(G_L, q_L)}$) is called the *discriminant bilinear* (resp. *quadratic*) *form* of L.

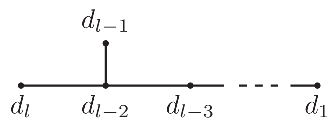
The *lattices* $A_n = \langle a_1, a_2, \ldots, a_n \rangle$ $(n \geq 1)$, $D_l = \langle d_1, d_2, \ldots, d_l \rangle$ $(l \geq 4)$, $E_p = \langle e_1, e_2, \ldots, e_p \rangle$ $(p = 6, 7, 8)$ defined by the following *Dynkin diagrams* are called the *root lattices*. All the vertices a_j, d_k, e_l are roots and two vertices a_j and a'_j are joined by a line if and only if $b(a_j, a'_j) = 1$. We use Bourbaki's definitions [**8**]. The discriminant groups of these root lattices are given below.

$\mathbf{A_n}$, $\mathbf{G_{A_n}}$.



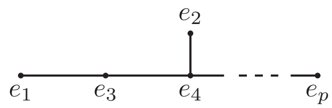
Set
$[1]_{A_n} = (1/n+1)\sum_{j=1}^{n}(n-j+1)a_j$
then $A_n^* = \langle A_n, [1]_n \rangle$ and
$G_{A_n} = A_n^*/A_n \simeq \mathbb{Z}/(n+1)\mathbb{Z}$.
$q_{A_n}([1]_{A_n}) = -n/(n+1)$.

$\mathbf{D_l}$, $\mathbf{G_{D_l}}$.



Set
$[1]_{D_l} = (1/2)\big(\sum_{i=1}^{l-2} i d_i + (1/2)(l-2)d_{l-1} + (1/2)l d_l\big)$
$[2]_{D_l} = \sum_{i=1}^{l-2} d_i + (1/2)(d_{l-1} + d_l)$
$[3]_{D_l} = (1/2)\big(\sum_{i=1}^{l-2} i d_i + (1/2)l d_{l-1} + (1/2)(l-2)d_l\big)$
then $D_l^* = \langle D_l, [1]_{D_l}, [3]_{D_l} \rangle$,
$G_{D_l} = D_l^*/D_l = \langle [1]_{D_l} \rangle \simeq \mathbb{Z}/4\mathbb{Z}$ if l is odd,
$G_{D_l} = D_l^*/D_l = \langle [1]_{D_l}, [2]_{D_l} \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if l is even.
$q_{D_l}([1]_{D_l}) = -(l/4)$, $q_{D_l}([2]_{D_l}) = -1$, $b_{D_l}([1],[2]) = -(1/2)$.

$\mathbf{E_p}$, $\mathbf{G_{E_p}}$. $p = 6, 7, 8$.



Set

$[1]_{E_6} = \eta_6 = -(1/3)(2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6)$,

$[1]_{E_7} = \eta_7 = -(1/2)(2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6 + 3e_7)$,

then $E_6^* = \langle E_6, \eta_6 \rangle$, $E_7^* = \langle E_7, \eta_7 \rangle$, $E_8^* = E_8$.

$G_{E_6} = E_6^*/E_6 \simeq \mathbb{Z}/3\mathbb{Z}$, $G_{E_7} = E_7^*/E_7 \simeq \mathbb{Z}/2\mathbb{Z}$,

$q_{E_6(\eta_6)} = -(4/3)$, $q_{E_7(\eta_7)} = -(3/2)$.

Let L be a Niemeier lattice (i.e. an unimodular lattice of rank 24). Denote L_{root} its root lattice. We often write $L = Ni(L_{\text{root}})$. Elements of L are defined by the glue code composed of glue vectors. Take for example $L = Ni(A_{11}D_7E_6)$. Its glue code is generated by the glue vector $[1, 1, 1]$ where the first 1 means $[1]_{A_{11}}$, the second 1 means $[1]_{D_7}$ and the third 1 means $[1]_{E_6}$. In the glue code $\langle [1, (0, 1, 2)] \rangle$, the notation $(0, 1, 2)$ means any circular permutation of $(0, 1, 2)$. Niemeier lattices, their root lattices and glue codes used in the paper are given in Table 1 (glue codes are taken from Conway and Sloane [**11**]).

Table 1. Some Niemeier lattices and their glue codes [**11**].

L_{root}	L/L_{root}	glue vectors
E_8^3	(0)	0
$D_{16}E_8$	$\mathbb{Z}/2\mathbb{Z}$	$\langle [1, 0] \rangle$
$D_{10}E_7^2$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\langle [1, 1, 0], [3, 0, 1] \rangle$
$A_{17}E_7$	$\mathbb{Z}/6\mathbb{Z}$	$\langle [3, 1] \rangle$
D_{24}	$\mathbb{Z}/2\mathbb{Z}$	$\langle [1] \rangle$
D_{12}^2	$(\mathbb{Z}/2\mathbb{Z})^2$	$\langle [1, 2], [2, 1] \rangle$
D_8^3	$(\mathbb{Z}/2\mathbb{Z})^3$	$\langle [1, 2, 2], [1, 1, 1], [2, 2, 1] \rangle$
$A_{15}D_9$	$\mathbb{Z}/8\mathbb{Z}$	$\langle [2, 1] \rangle$
E_6^4	$(\mathbb{Z}/3\mathbb{Z})^2$	$\langle [1, (0, 1, 2)] \rangle$
$A_{11}D_7E_6$	$\mathbb{Z}/12\mathbb{Z}$	$\langle [1, 1, 1] \rangle$
D_6^4	$(\mathbb{Z}/2\mathbb{Z})^4$	$\langle$even permutations of $[0, 1, 2, 3]\rangle$
$A_9^2D_6$	$\mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\langle [2, 4, 0], [5, 0, 1], [0, 5, 3] \rangle$
$A_7^2D_5^2$	$\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	$\langle [1, 1, 1, 2], [1, 7, 2, 1] \rangle$

2.2. Kneser–Nishiyama technique.

We use the Kneser–Nishiyama method to determine all the elliptic fibrations of Y_k. For further details we refer to [**3**], [**5**], [**25**], [**29**]. In [**3**], [**5**], [**25**] only singular $K3$ (i.e. of Picard number 20) are considered. In this paper we follow [**29**] we briefly recall.

Given an elliptic $K3$-surface S, recall that $H^2(S, \mathbb{Z})$ with respect to the cup-product has the structure of an even lattice of rank 22 and the frame $W(S)$ is the orthogonal complement in the Néron Severi lattice $NS(S)$ of the lattice generated by the zero section and the general fiber. Nishiyama aims at embedding the frames $W(S)$ of all elliptic fibrations into Niemeier lattices that are negative definite lattices of rank 24. For this purpose, Nishiyama determines an even negative definite lattice M such that

$$q_M = -q_{NS(S)}, \quad \text{rank}(M) + \rho(S) = 26,$$

$\rho(S)$ being the Picard number of S.

By Nikulin ([**23**], Corollary 1.6.2), $M \oplus W(S)$ has a Niemeier lattice as an overlattice for each frame $W(S)$ of an elliptic fibration on S. Thus one has to determine the (inequivalent) primitive embeddings of M into Niemeier lattices L. To achieve this, it is essential to consider the root lattices involved. In each case, the orthogonal complement of M into L gives the corresponding frame $W(S)$.

Let us describe how to determine M in the case of the Apéry–Fermi pencil.

Let $T(Y_k)$ be the transcendental lattice of Y_k, that is the orthogonal complement of $NS(Y_k)$ in $H^2(Y_k, \mathbb{Z})$. The lattice $T(Y_k)$ is an even lattice of rank $r = 22 - \rho(Y_k) = 3$ and signature $(2, 1)$. Let $t := r - 2 = 1$. By Nikulin's theorem ([**23**], Theorem 1.12.4), $T(Y_k)[-1]$ admits a primitive embedding into the following indefinite unimodular lattice:

$$T(Y_k)[-1] \hookrightarrow U \oplus E_8[-1],$$

where U denotes the hyperbolic lattice and E_8 the unique positive definite even unimodular lattice of rank 8. Define M as the orthogonal complement of a primitive embedding of $T(Y_k)[-1]$ in $U \oplus E_8[-1]$. Since

$$T(Y_k)[-1] = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -12 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

it suffices to get a primitive embedding of (-12) into $E_8[-1]$. From Nishiyama [**25**] (p. 335) we find the following primitive embedding:

$$v = \langle 9e_2 + 6e_1 + 12e_3 + 18e_4 + 15e_5 + 12e_6 + 8e_7 + 4e_8 \rangle \hookrightarrow E_8[-1],$$

giving $(v)^{\perp}_{E_8[-1]} = A_2 \oplus D_5$. Now the primitive embedding of $T(Y_k)[-1]$ in $U \oplus E_8[-1]$ is defined by $U \oplus v$; hence $M = (U \oplus v)^{\perp}_{U \oplus E_8[-1]} = A_2 \oplus D_5$. By construction, this lattice M is negative definite of rank equal to $8 - 1 = 7 = r + 4 = 26 - \rho(Y_k)$ and its discriminant form satisfies by Nikulin ([**23**], Proposition 1.6.1),

$$q_M = -q_{T(Y_k)[-1]} = q_{T(Y_k)} = -q_{NS(Y_k)}.$$

Hence M takes exactly the shape required for Nishiyama's technique.

All the elliptic fibrations come from all the primitive embeddings of $M = A_2 \oplus D_5$ into all the Niemeier lattices L. Since M is a root lattice, a primitive embedding of M into L is in fact a primitive embedding into L_{root}. Whenever the primitive embedding is given by a primitive embedding of A_2 and D_5 in two different factors of L_{root}, or for the primitive embedding of M into E_8, we use Nishiyama's results [**25**] (4.1 and p. 335). Otherwise we have to determine the primitive embeddings of M into D_l for $l = 8, 9, 10, 12, 16, 24$. This is done in the following lemma.

LEMMA 2.1. *We obtain the following primitive embeddings.*

1. $A_2 \oplus D_5 = \langle d_8, d_6, d_7, d_5, d_4, d_1, d_2\rangle \hookrightarrow D_8$

   $\langle d_8, d_6, d_7, d_5, d_4, d_1, d_2\rangle^{\perp}_{D_8} = \langle 2d_1+4d_2+6d_3+6d_4+6d_5+6d_6+3d_7+3d_8\rangle = (-12)$

2. $A_2 \oplus D_5 = \langle d_9, d_7, d_8, d_6, d_5, d_3, d_2\rangle \hookrightarrow D_9$

   $\langle d_9, d_7, d_8, d_6, d_5, d_3, d_2\rangle^{\perp}_{D_9} = \langle d_9+d_8+2d_7+2d_6+2d_5+2d_4+d_3-d_1, d_3+2d_2+3d_1\rangle$

   with Gram matrix $\begin{pmatrix} -4 & 6 \\ 6 & -12 \end{pmatrix}$ *of determinant* 12.

3. $A_2 \oplus D_5 = \langle d_n, d_{n-2}, d_{n-1}, d_{n-3}, d_{n-4}, d_{n-7}, d_{n-6}\rangle \hookrightarrow D_n, n \geq 10$

   $\langle d_n, d_{n-2}, d_{n-1}, d_{n-3}, d_{n-4}, d_{n-7}, d_{n-6}\rangle^{\perp}_{D_n} = \langle a, d_{n-6}+2d_{n-7}+3d_{n-8}, d_{n-9}, ..., d_1\rangle$

   with $a = d_n + d_{n-1} + 2(d_{n-2} + \cdots + d_2) + d_1$.

$$\left((A_2 \oplus D_5)^{\perp}_{D_n}\right)_{root} = D_{n-8}.$$

We have also the relation $2.[2]_{D_n} = a + d_1$, *a being the above root.*

THEOREM 2.1. *There are* 27 *elliptic fibrations on the generic* $K3$*-surface of the Apéry–Fermi pencil* (*i.e. with Picard number* 19). *They are obtained from all the non isomorphic primitive embeddings of* $A_2 \oplus D_5$ *into the various Niemeier lattices. Among them,* 4 *fibrations have rank* 0, *precisely with the type of singular fibers and torsion*:

$$\begin{array}{cl} A_{11}2A_2 2A_1 & 6-torsion \\ E_6 D_{11} & 0-torsion \\ E_7 A_5 D_5 & 2-torsion \\ E_8 E_6 A_3 & 0-torsion. \end{array}$$

The list together with the rank and torsion is given in Table 2.

REMARK 2.1. Fibrations of rank 0 are also already computed in [**30**].

PROOF. The torsion groups can be computed as explained in [**5**] or [**3**]. Let us recall briefly the method.

Denote ϕ a primitive embedding of $M = A_2 \oplus D_5$ into a Niemeier lattice L. Define $W = (\phi(M))^{\perp}_{L}$ and $N = (\phi(M))^{\perp}_{L_{\text{root}}}$. We observe that $W_{\text{root}} = N_{\text{root}}$. Thus computing N then N_{root} gives the type of singular fibers. Recall also that the torsion part of the Mordell–Weil group is

$$\overline{W_{\text{root}}}/W_{\text{root}} (\subset W/N)$$

and can be computed in the following way [**3**]: let $l + L_{\text{root}}$ be a non trivial element of L/L_{root}. If there exist $k \neq 0$ and $u \in L_{\text{root}}$ such that $k(l+u) \in N_{\text{root}}$, then $l + u \in W$ and the class of l is a torsion element.

We use also several facts.

Table 2. The elliptic fibrations of the Apéry–Fermi family.

L_{root}	L/L_{root}			type of Fibers	Rk	Tors.
E_8^3	(0)					
	#1	$A_2 \subset E_8$	$D_5 \subset E_8$	$E_6A_3E_8$	0	(0)
	#2	$A_2 \oplus D_5 \subset E_8$		E_8E_8	1	(0)
$D_{16}E_8$	$\mathbb{Z}/2\mathbb{Z}$					
	#3	$A_2 \subset E_8$	$D_5 \subset D_{16}$	E_6D_{11}	0	(0)
	#4	$A_2 \oplus D_5 \subset E_8$		D_{16}	1	$\mathbb{Z}/2\mathbb{Z}$
	#5	$D_5 \subset E_8$	$A_2 \subset D_{16}$	A_3D_{13}	1	(0)
	#6	$A_2 \oplus D_5 \subset D_{16}$		E_8D_8	1	(0)
$D_{10}E_7^2$	$(\mathbb{Z}/2\mathbb{Z})^2$					
	#7	$A_2 \subset E_7$	$D_5 \subset D_{10}$	$E_7A_5D_5$	0	$\mathbb{Z}/2\mathbb{Z}$
	#8	$A_2 \subset E_7$	$D_5 \subset E_7$	$A_5A_1D_{10}$	1	$\mathbb{Z}/2\mathbb{Z}$
	#9	$A_2 \oplus D_5 \subset D_{10}$		$E_7E_7A_1A_1$	1	$\mathbb{Z}/2\mathbb{Z}$
	#10	$D_5 \subset E_7$	$A_2 \subset D_{10}$	$A_1D_7E_7$	2	(0)
$A_{17}E_7$	$\mathbb{Z}/6\mathbb{Z}$					
	#11	$D_5 \subset E_7$	$A_2 \subset A_{17}$	A_1A_{14}	2	(0)
D_{24}	$\mathbb{Z}/2\mathbb{Z}$					
	#12	$A_2 \oplus D_5 \subset D_{24}$		D_{16}	1	(0)
D_{12}^2	$(\mathbb{Z}/2\mathbb{Z})^2$					
	#13	$A_2 \subset D_{12}$	$D_5 \subset D_{12}$	D_9D_7	1	(0)
	#14	$A_2 \oplus D_5 \subset D_{12}$		D_4D_{12}	1	$\mathbb{Z}/2\mathbb{Z}$
D_8^3	$(\mathbb{Z}/2\mathbb{Z})^3$					
	#15	$A_2 \subset D_8$	$D_5 \subset D_8$	$D_5A_3D_8$	1	$\mathbb{Z}/2\mathbb{Z}$
	#16	$A_2 \oplus D_5 \subset D_8$		D_8D_8	1	$\mathbb{Z}/2\mathbb{Z}$
$A_{15}D_9$	$\mathbb{Z}/8\mathbb{Z}$					
	#17	$A_2 \oplus D_5 \subset D_9$		A_{15}	2	$\mathbb{Z}/2\mathbb{Z}$
	#18	$D_5 \subset D_9$	$A_2 \subset A_{15}$	D_4A_{12}	1	(0)
E_6^4	$(\mathbb{Z}/3\mathbb{Z})^2$					
	#19	$A_2 \subset E_6$	$D_5 \subset E_6$	$A_2A_2E_6E_6$	1	$\mathbb{Z}/3\mathbb{Z}$
$A_{11}D_7E_6$	$\mathbb{Z}/12\mathbb{Z}$					
	#20	$A_2 \subset E_6$	$D_5 \subset D_7$	$A_2A_2A_1A_1A_{11}$	0	$\mathbb{Z}/6\mathbb{Z}$
	#21	$A_2 \subset A_{11}$	$D_5 \subset D_7$	$A_8A_1A_1E_6$	1	(0)
	#22	$A_2 \subset A_{11}$	$D_5 \subset E_6$	A_8D_7	2	(0)
	#23	$D_5 \subset E_6$	$A_2 \subset D_7$	$A_{11}D_4$	2	$\mathbb{Z}/2\mathbb{Z}$
D_6^4	$(\mathbb{Z}/2\mathbb{Z})^4$					
	#24	$A_2 \subset D_6$	$D_5 \subset D_6$	$A_3D_6D_6$	2	$\mathbb{Z}/2\mathbb{Z}$
$A_9^2D_6$	$\mathbb{Z}/2 \times \mathbb{Z}/10$					
	#25	$D_5 \subset D_6$	$A_2 \subset A_9$	A_6A_9	2	(0)
$A_7^2D_5^2$	$\mathbb{Z}/4 \times \mathbb{Z}/8$					
	#26	$D_5 \subset D_5$	$A_2 \subset D_5$	$A_1A_1A_7A_7$	1	$\mathbb{Z}/4\mathbb{Z}$
	#27	$D_5 \subset D_5$	$A_2 \subset A_7$	$D_5A_4A_7$	1	(0)

1. If the rank of the Mordell–Weil group is 0, then the torsion group is equal to W/N. Hence fibrations $\#1(A_3E_6E_8)$, $\#3(D_{11}E_6)$, $\#7(D_5A_5E_7)$, $\#20(A_{11}2A_12A_2)$ have respective torsion groups (0), (0), $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$.

2. If there is a singular fiber of type E_8, then the torsion group is (0). Hence the fibrations #1, #2 and #6 have no torsion.

3. Using Lemma 2.2 below and the shape of glue vectors we prove that fibrations #11, #18, #21, #22, #25, #27 have no torsion.

4. Using Lemma 2.3 below and the shape of glue vectors we can determine the torsion for elliptic fibrations #5, #10, #13, #15, #23.

LEMMA 2.2. *Suppose A_2 primitively embedded in A_n, $A_2 = \langle a_1, a_2\rangle \hookrightarrow A_n$. Then for all $k \neq 0$, $k[1]_{A_n} \notin ((A_2)^{\perp}_{A_n})_{root}$.*

PROOF. It follows from the fact that $[1]_{A_n}$ is not orthogonal to a_1. □

LEMMA 2.3. *Suppose A_2 primitively embedded in D_l, $A_2 = \langle d_l, d_{l-2}\rangle \hookrightarrow D_l$. Then $2.[2]_{D_l} \in ((A_2)^{\perp}_{D_l})_{root}$ but there is no k satisfying $k.[i]_{D_l} \in ((A_2)^{\perp}_{D_l})_{root}, i = 1, 3$.*

PROOF. It follows from Nishiyama [**25**] $(A_2)^{\perp}_{D_l} = \langle y, x_4, d_{l-4}, \dots, d_1\rangle$, $l \geq 5$, with $y = d_l + 2d_{l-1} + 2d_{l-2} + d_{l-3}$, $x_4 = d_l + d_{l-1} + 2(d_{l-2} + d_{l-3} + \cdots + d_2) + d_1$ and Gram matrix

$$L^4_{l-3} = \left(\begin{array}{c|ccccc} -4 & -1 & 1 & 0 & \dots & 0 \\ \hline -1 & & & & & \\ 1 & & & & & \\ 0 & & & D_{l-3} & & \\ \vdots & & & & & \\ 0 & & & & & \end{array}\right).$$

Moreover $((A_2)^{\perp}_{D_l})_{\mathrm{root}} = \langle x_4, d_{l-4}, \dots, d_1\rangle$. From there we compute easily the relation $2.[2]_{D_l} = x_4 + d_{l-4} + 2(d_{l-5} + \cdots + d_1)$. The last assertion follows from the fact that $[i]_{D_l}$ is not orthogonal to A_2. □

We now give some examples showing the method in detail.

2.2.1. Fibration #17.

It comes from a primitive embedding of $A_2 \oplus D_5$ into D_9 giving a primitive embedding of $A_2 \oplus D_5$ into $Ni(A_{15}D_9)$ with glue code $\langle [2, 1]\rangle$. Since by Lemma 2.1(2) $N_{\mathrm{root}} = A_{15}$, among the elements $k.[2, 1]$, only $4.[2, 1] = [8, 4.1 \in D_9]$ satisfies $2.[8, 0 + u] \in N_{\mathrm{root}} = A_{15}$ with $u = 4.1$. Hence the torsion group is $\mathbb{Z}/2\mathbb{Z}$.

2.2.2. Fibration #19.

It comes from a primitive embedding of $A_2 = \langle e_1, e_3\rangle$ into $E_6^{(1)}$ and $D_5 = \langle e_2, e_3, e_4, e_5, e_6\rangle$ into $E_6^{(2)}$ giving a primitive embedding of $A_2 \oplus D_5$ into $Ni(E_6^4)$. In that case $Ni(E_6^4)/E_6^4 \simeq (\mathbb{Z}/3\mathbb{Z})^2$ and the glue code is $\langle [1, (0, 1, 2)]\rangle$. Moreover $(D_5)^{\perp}_{E_6} = 3e_2 + 4e_1 + 5e_3 + 6e_4 + 4e_5 + 2e_6 = a$, $(A_2)^{\perp}_{E_6} = \langle e_2, y\rangle \oplus \langle e_5, e_6\rangle$ with $y = 2e_2 + e_1 + 2e_3 + 3e_4 + 2e_5 + e_6$. From the relation

$$[1]_{E_6} = -\frac{1}{3}(2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6)$$

we get

$$-3.[1]_{E_6} = a - 2e_1 - e_3 + e_5 + 2e_6 \in E_6$$

$$-3.[1]_{E_6} = 2y - e_2 + e_5 + 2e_6 \in (A_2)^{\perp}_{E_6}$$

and deduce that only $[1,0,1,2]$, $[2,0,2,1]$, $[0,0,0,0]$ contribute to the torsion thus the torsion group is $\mathbb{Z}/3\mathbb{Z}$.

2.2.3. Fibration #10.

The embeddings of $A_2 = \langle d_{10}, d_8\rangle$ into D_{10} and $D_5 = \langle e_2, e_3, e_4, e_5, e_6\rangle$ into $E_7^{(1)}$ lead to a primitive embedding of $A_2 \oplus D_5$ into $Ni(D_{10}E_7^2)$ satisfying $Ni(D_{10}E_7^2)/(D_{10}E_7^2) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ with glue code $\langle [1,1,0],[3,0,1]\rangle$. We deduce from Lemma 2.3 that no glue vector can contribute to the torsion which is therefore (0).

2.2.4. Fibration #18.

The embeddings of $A_2 = \langle a_1, a_2\rangle$ into A_{15} and $D_5 = \langle d_9, d_7, d_8, d_6, d_5\rangle$ into D_9 lead to a primitive embedding of $A_2 \oplus D_5$ into $Ni(A_{15}D_9)$ satisfying $Ni(A_{15}D_9)/(A_{15}D_9) \simeq (\mathbb{Z}/8\mathbb{Z})$ with glue code $\langle [2,1]\rangle$. We deduce from Lemma 2.2 that no glue vector can contribute to the torsion which is therefore (0).

2.2.5. Fibration #8.

The primitive embeddings of $A_2 = \langle e_1, e_3\rangle$ into $E_7^{(1)}$ and $D_5 = \langle e_2, e_3, e_4, e_5, e_6\rangle$ into $E_7^{(2)}$ lead to a primitive embedding of $A_2 \oplus D_5$ into $Ni(D_{10}E_7^2)$ satisfying $Ni(D_{10}E_7^2)/(D_{10}E_7^2) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ with glue code $\langle [1,1,0],[3,0,1]\rangle$. From Nishiyama [**25**] we get $(A_2)^{\perp}_{E_7^{(1)}} = \langle e_2, y, e_7, e_6, e_5\rangle \simeq A_5$ with $y = 2e_2 + e_1 + 2e_3 + 3e_4 + 2e_5 + e_6$ and $(D_5)^{\perp}_{E_7} = \langle (-4), e_2 + e_3 + 2(e_4 + e_5 + e_6 + e_7) = (-2)\rangle$. Hence $N = D_{10} \oplus A_5 \oplus (-4) \oplus A_1$ and $W_{\text{root}} = N_{\text{root}} = D_{10} \oplus A_5 \oplus A_1$. Now

$$-2\eta_7 = -2.[1]_{E_7} = 2y - e_2 + e_5 + 2e_6 + 3e_7 \in ((A_2)^{\perp}_{E_7})_{\text{root}}$$

and for all $k \neq 0$, $k.[1]_{E_7} \notin (D_5)^{\perp}_{E_7}$. Hence only the generator $[1,1,0]$ can contribute to the torsion group which is therefore $\mathbb{Z}/2\mathbb{Z}$.

2.2.6. Fibration #24.

The primitive embeddings $A_2 = \langle d_6, d_4\rangle$ into $D_6^{(1)}$ and $D_5 = \langle d_6, d_5, d_4, d_3, d_2\rangle$ into $D_6^{(2)}$ give a primitive embedding of $A_2 \oplus D_5$ into $L = Ni(D_6^4)$ with $L/L_{\text{root}} \simeq (\mathbb{Z}/2\mathbb{Z})^4$ and glue code $\langle$ even permutations of $[0,1,2,3]\rangle$. From Nishiyama [**25**] we get $(A_2)^{\perp}_{D_6} = \langle y = 2d_5 + d_6 + 2d_4 + d_3, x_4 = d_5 + d_6 + 2(d_4 + d_3) + d_2, d_2, d_1\rangle$, $((A_2)^{\perp}_{D_6})_{\text{root}} = \langle x_4, d_2, d_1\rangle \simeq A_3$ and $(D_5)^{\perp}_{D_6} = \langle x_6'\rangle = \langle d_5 + d_6 + 2(d_4 + d_3 + d_2 + d_1) = (-4)\rangle$. We deduce $N_{\text{root}} = A_3 \oplus D_6 \oplus D_6$. From the relations $2.[2]_{D_6} = x_4 + d_2 + 2d_1$ and $2.[3]_{D_6} = y + x_4 + d_2 + d_1$ we deduce that the glue vectors having 1, 2, 3 or 0 in the first position may belong to W. From the relation $2.[2]_{D_6} = x_6'$ we deduce that only glue vectors with 2 or 0 in the second position may belong to W. Finally only the glue vectors $[0,2,3,1], [1,0,3,2], [1,2,0,3], [2,0,1,3], [2,2,2,2], [3,0,2,1], [3,2,1,0], [0,0,0,0]$ belong to W. Since y and x_6' are not roots, only glue vectors with 0 or 2 in the first position and 0 in the second position may contribute to torsion that is $[2,0,1,3], [0,0,0,0]$. Hence the torsion group is $\mathbb{Z}/2\mathbb{Z}$.

2.2.7. Fibration #26.

The primitive embeddings of $A_2 = \langle d_5, d_3\rangle$ into $D_5^{(1)}$ and D_5 into $D_5^{(2)}$ give a primitive embedding into $L = Ni(A_7^2 D_5^2)$ with $L/L_{\text{root}} \simeq \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and glue code $\langle[1,1,1,2],[1,7,2,1]\rangle$. From Nishiyama we get $(A_2)^{\perp}_{D_5} = \langle y, x_4, d_1\rangle$ with $y = 2d_4+d_5+2d_3+d_2$, $x_4 = d_5+d_4+2d_3+2d_2+d_1$ and Gram matrix $M_2^4 = \begin{pmatrix} -4 & -1 & 1 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}$ of determinant 12. We also deduce $N_{\text{root}} = E_7^2 A_1^2$, $2.[2]_{D_5} = x_4 + d_1 \in ((A_2)^{\perp}_{D_5})_{\text{root}}$. Moreover neither $k.[1]_{D_5}$ nor $k.[3]_{D_5}$ belongs to $(A_2)^{\perp}_{D_5}$. Thus only glue vectors with 2 or 0 in the third position can belong to W and eventually contribute to torsion, that is $[2,2,2,0]$, $[4,4,0,0]$, $[6,6,2,0]$, $[2,6,0,2]$, $[6,2,0,2]$, $[4,0,2,2]$, $[0,4,2,2]$, $[0,0,0,0]$. Since there is no $u_4 \in D_5$ satisfying $2.(2+u_4) = 0$ or $4.(2+u_4) = 0$, glue vectors with the last component equal to 2 cannot satisfy $k(l+u) \in N_{\text{root}}$ with $l \in L$ and $u \in L_{\text{root}} = A_7^2 D_5^2$. Hence only the glue vectors generated by $\langle[2,2,2,0]\rangle$ contribute to torsion and the torsion group is therefore $\mathbb{Z}/4\mathbb{Z}$. $\square$

3. Weierstrass Equations for all the elliptic fibrations of Y_k.

The method can be found in [**5**], [**15**]. We follow also the same kind of computations used for Y_2 given in [**5**]. We give only explicit computations for 4 examples, #19, #2, #9, and #16. For #2 and #9 it was not obvious to find a rational point on the quartic curve. All the results are given in Table 3. For the 2 or 3-neighbor method [**15**] we give in the third column the starting fibration and in the forth the elliptic parameter. The terms in the elliptic parameter refer to the starting fibration.

3.1. Fibration #19.

We take $u = XY/Z$ as a parameter of an elliptic fibration and with the birational transformation

$$x = -u(1+uZ)(u+Y), \quad y = u^2((u+Y)(uY-1)Z + Y(Y+2u+k) - 1)$$

we obtain a Weierstrass equation

$$y^2 + ukyx + u^2\left(u^2+uk+1\right)y = x^3,$$

where the point $(x=0, y=0)$ is a 3-torsion point and the point $(-u^2, -u^2)$ is of infinite order. The singular fibers are of type $IV^*(0,\infty)$, $I_3(u^2+uk+1)$ and $I_1(27u^2 - k(k^2-27)u + 27)$. Moreover if $k = s + 1/s$ the two singular fibers of type I_3 are above $u = -s$ and $-1/s$.

3.2. Fibration #2.

Using the 3-neighbor method from fibration #19 we construct a new fibration with a fiber of type II^* and the parameter $m = ys/(u+s)^2$. Then we obtain a cubic C_m in w, u, with $x = w(u+s)$

$$C_m : (s+u)m^2 + u\left(s^2w + u^2s + w + u\right)m - w^3s^2 = 0.$$

From some component of the fiber of type I_3 at $u=-s$ we obtain the rational point on $C_m : \omega_m = (u_1 = (ms-1)/(s-m), w_1 = m(s^2-1)/s(s-m))$ which is not a flex point. The first stage is to obtain a quartic equation $Qua : y^2 = ax^4+bx^3+cx^2+dx+e^2$. First we observe that ω_m is on the line $w = u+1/s$, so we replace w by K with $w = u+1/s+K$ and $u = u_1+T$. The transformation $K = WT$ gives an equation of degree two in T, with constant term $fW+g$ where f and g belong to $\mathbb{Q}(s,m)$. With the change variable $Wf+g = x$ we have an equation $M(x)T^2+N(x)T+x = 0$. The discriminant of the quadratic equation in T is $N(x)^2-4xM(x)$, a polynomial of degree 4 in x and constant term a square. Easily we obtain the form Qua.

From the quartic form, setting $y = e+dx/2e+x^2X'$, $x = (8e^3X'-4ce^2+d^2)/Y'$ we get

$$Y'^2+4e\,(dX'-be)\,Y+4e^2\left(8e^3X'-4ce^2+d^2\right)\left(X'^2-a\right)=0.$$

Finally the following Weierstrass equation follows from standard transformations where we replace m by t

$$\begin{aligned}&Y^2-X^3+\frac{1}{3}t^4(s^2+1)(s^6+219s^4-21s^2+1)X\\&\quad-\frac{2t^5}{27}\left(-864s^5t^2+(s^4+14s^2+1)(s^8-548s^6+198s^4-44s^3+1)t-864s^5\right)=0,\end{aligned}$$

with a section Φ of height 12 corresponding to $(8e^3X'-4ce^2+d^2)=0$ and $Y'=0$. The coordinates of Φ, too long, are omitted but we can follow the previous computation to obtain it.

Writing the above form as

$$y^2 = x^3-3\alpha x+\left(t+\frac{1}{t}\right)-2\beta$$

we recover the values of the j invariants of the two elliptic curves for the Shioda–Inose structure (see Paragraph 4.5.1 and Corollary 4.1 below).

3.3. Fibration #9.

Let $g = XY/Z^2$. Eliminating X and writing $Y = ZU$ we obtain an equation of bidegree 2 in U and Z. If $k = s+1/s$ there is a rational point $U=-1$, $Z=-s/g$ on the previous curve. By standard transformations we get a Weierstrass equation

$$y^2 = x^3+\frac{1}{4}g^2\left(s^4+14s^2+1\right)x^2+s^2g^3\left(g+s^2\right)\left(gs^2+1\right)x$$

and a rational point

$$\begin{aligned}x&=\frac{s^2(g-1)^2\left(g+s^2\right)\left(s^2g+1\right)}{\left(s^2-1\right)^2},\\ y&=\frac{1}{2}\frac{s^2\left(g^2-1\right)\left(g+s^2\right)\left(s^2g+1\right)\left(2g^2s^2+g\left(s^4-6s^2+1\right)+2s^2\right)}{\left(s^2-1\right)^3}.\end{aligned}$$

The singular fibers are of type $2III^*(\infty, 0)$, $2I_2(-s^2, -1/s^2)$, $4I_1$.

3.4. Fibration #16.

Using the fibration #9 we consider the parameter $t = x/g(g + s^2)$ and obtain a Weierstrass equation

$$Y^2 = X^3 + \left(4t\left(t^2 + s^2\right) + t^2\left(s^4 + 14s^2 + 1\right)\right) X^2 + 16s^6t^4X.$$

The singular fibers are of type $I_4^*(\infty, 0)$, $4I_1$.

4. Nikulin involutions and Shioda–Inose structure.

4.1. Background.

Let S be a $K3$-surface.

The lattice $H^2(S, \mathbb{Z})$ admits a Hodge decomposition of weight two

$$H^2(S, \mathbb{C}) \simeq H^{2,0} \oplus H^{1,1} \oplus H^{0,2}.$$

Similarly, the transcendental lattice $T(S)$ has a Hodge decomposition of weight two

$$T(S) \otimes \mathbb{C} \simeq T^{2,0} \oplus T^{1,1} \oplus T^{0,2}.$$

An isomorphism between two lattices that preserves their bilinear form and their Hodge decomposition is called a *Hodge isometry.*

An automorphism of a $K3$-surface S is called *symplectic* if it acts on $H^{2,0}(S)$ trivially. Such automorphisms were studied by Nikulin in [**24**] who proved that a symplectic involution i (*Nikulin involution*) has eight fixed points and that the minimal resolution $Y \to S/i$ of the eight nodes is again a $K3$-surface.

We have then the rational quotient map $p : S \to Y$ of degree 2. The transcendental lattices $T(S)$ and $T(Y)$ are related by the chain of inclusions

$$2T(Y) \subseteq p^*T(S) = T(S)(2) \subseteq T(Y),$$

which preserves the quadratic forms and the Hodge structures.

In this paper, $K3$-surfaces are given as elliptic surfaces. If we have a 2-torsion section τ, we consider the symplectic involution i (*van Geemen–Sarti involution*) given by the fiberwise translation by τ. In this situation, the rational quotient map $S \to Y$ is just an isogeny of degree 2 between elliptic curves over $\mathbb{C}(t)$, and we have a rational map $Y \to S$ of degree 2 as the dual isogeny.

4.2. Fibrations of some Kummer surfaces.

Let E_l be an elliptic curve with invariant j, defined by a Weierstrass equation in the Legendre form

$$E_l : y^2 = x(x - 1)(x - l).$$

Then l satisfies the equation $j = 256((1 - l + l^2)^3/l^2(l - 1)^2)$. For a fixed j the six values of l are given by l or $1/l, 1 - l, (l - 1)/l, -1/(l - 1), (l - 1)/l$.

Table 3. Weierstrass equations of the elliptic fibrations of Y_k.

	Weierstrass Equation	From	Param.
#1	$y^2 + tkyx + t^2k(t+1)y = x^3 - t^4(t+1)^3$ $II^*(\infty), IV^*(0), I_4(-1), 2I_1$ $r = 0$		$\frac{Y(X+Z)^2(Z+Y)}{XZ^3}$
#2	$y^2 = x^3 - \frac{1}{3}t^4(s^2+1)(s^6+219s^4-21s^2+1)x$ $+\frac{2}{27}t^5(-864s^5t^2+(s^4+14s^2+1)(s^8-548s^6+198s^4-44s^2+1)t-864s^5)$ $2II^*(\infty,0), 4I_1$ $x_P = \Phi$	#19	$\frac{ys}{(s+t)^2}$
#3	$y^2 = x^3 + \frac{1}{4}t(4t^2s^4 + (s^4-10s^2+1)t + 12)x^2$ $-t^2(2ts^2-3)x + t^3$ $I_7^*(\infty), IV^*(0), 3I_1$ $r = 0$	#7	$\frac{x}{s^4t^2}$
#4	$y^2 = x^3 +$ $\left(\frac{1}{2}t^3 - \frac{1}{24}\left(s^2+1\right)\left(s^6+219s^4-21s^2+1\right)t\right.$ $\left.+\frac{1}{216}\left(s^8-548s^6+198s^4-44s^2+1\right)\left(s^4+14s^2+1\right)\right)x^2 + 16s^{10}x$ $I_{12}^*(\infty), 6I_1$ x_P	#2	$\frac{x}{2t^2}$
#5	$y^2 - k(t+1)yx + ky = x^3 + (t^3-3)x^2 + 3x - 1$ $I_9^*(\infty), I_4(0), 5I_1$ $x_P = 0$	#1	$\frac{x}{t^2}$
#6	$y^2 = x^3 + \left(\frac{1}{4}t^2\left(s^4+14s^2+1\right) + t^3s^2\right)x^2$ $+t^4s^2(s^4+1)x + t^5s^6$ $I_4^*(\infty), II^*(0), 4I_1$ x_P	#9	$\frac{x}{(t+s^2)(ts^2+1)}$
#7	$y^2 = x^3 + \frac{1}{4}t(t(s^4-10s^2+1)+8s^4)x^2 - t^2s^2(t-s^2)^3x$ $III^*(\infty), I_1^*(0), I_6(s^2), 2I_1$ $r = 0$	#15	$\frac{x}{t}$
#8	$y^2 - k(t-1)yx = x(x-1)(x-t^3)$ $I_6^*(\infty), I_6(0), I_2(1), 4I_1$ $x_P = 1$		$\frac{(X+Z)(Y+Z)}{XZ}$
#9	$y^2 = x^3 + \frac{1}{4}t^2(s^4+14s^2+1)x^2 + t^3s^2(t+s^2)(ts^2+1)x$ $2III^*(\infty,0), 2I_2\left(-s^2, -\frac{1}{s^2}\right), 2I_1$ $x_P = \frac{s^2(t-1)^2(t+s^2)(ts^2+1)}{(s^2-1)^2}$		$\frac{XY}{Z^2}$
#10	$y^2 + t(s^2+1)(x+t^2s^2)y = (x-t^3s^2)(x^2+t^3s^4)$ $I_3^*(\infty), III^*(0), I_2(-1), 4I_1$ $x_{P_1} = t^3s^2, \quad x_{P_2} = 0$		$\frac{XY}{(Y+Z)Z}$
#11	$y^2 + t(st-1-s^2)yx - s^3y = x^2(x+s(t(s^2-1)-s(s^2+1)))$ $I_{15}(\infty), I_2(s), 7I_1$ $x_{P_1} = st, \quad x_{P_2} = -s^3t + s^2(s^2+1)$	#8	$\frac{y+sx}{xt}$
#12	$y^2 = x^3 + t\left(t^2s^2 + \frac{1}{4}(s^4+14s^2+1) + (s^4+1)\right)x^2$ $-\left(2t^2s^4+\frac{1}{2}s^2(s^4+14s^2+1)t+s^2(s^4+1)\right)x+ts^6+\frac{1}{4}s^4(s^4+14s^2+1)$ $I_{12}^*(\infty), 6I_1$ x_P	#14	$\frac{x}{t^2} + \frac{s^2}{t}$
#13	$y^2 = x^3 + \frac{1}{4}t(4t^2 + (s^4-10s^2+1)t + 4s^4)x^2$ $+2t^3s^4(t-s^4)x + t^5s^8$ $I_5^*(\infty), I_3^*(0), 4I_1$ $x_P = -ts^4$	#15	$\frac{x}{t^2}$

Table 4. Weierstrass equations of the elliptic fibrations of Y_k.

No	Weierstrass Equation	From	Param.
#14	$y^2 = x^3 + \left(t^3(s^4+1) + \frac{1}{4}t^2(s^4+14s^2+1) + ts^2\right)x^2 + s^4t^6x$ $I_0^*(\infty), I_8^*(0), 4I_1\left(-\frac{1}{4}, -4\frac{s^2}{(s^2-1)^2}, ..\right)$ $x_P = \frac{s^4(2t+1)^2}{(s^2-1)^2}$	#9	$\frac{x}{t(t+s^2)(ts^2+1)}$
#15	$(y-tx)(y-s^2tx) = x(x-ts^2)(x-ts^2(t+1)^2)$ $I_4^*(\infty), I_1^*(0), I_4(-1), 3I_1\left(\frac{1}{4}\left(\frac{s^2-1}{s}\right)^2, ..\right)$ $x_P = s^2t$		$\frac{(XY+1)Z}{X}$
#16	$y^2 = x^3 + t(4(t^2+s^2) + t(s^4+14s^2+1))x^2 + 16s^6t^4x$ $I_4^*(\infty,0), 4I_1$ $x_P = \frac{-4ts^6(t+1)^2}{(t+s^2)^2}$	#9	$\frac{x}{t(t+s^2)}$
#17	$y^2 - \frac{1}{2}(s^4+14s^2+1-s^2t^2)yx = x(x-4s^2)(x-4s^6)$ $I_{16}(\infty), 8I_1\left(\pm\frac{s^2\pm 4s-1}{s}, ..\right)$ $x_{P_1} = 4s^2; \quad x_{P_2} = \frac{4s^4(ts+s^2-1)^2}{(ts+1-s^2)^2}$	#16	$\frac{y}{ts^2x}$
#18	$y^2 + (-t^2+(s^2-1)t-2s^2)yx + s^4t^2y = x^2(x-s^4)$ $I_{13}(\infty), I_0^*(0), 5I_1$ $x_P = 0$	#15	$\frac{y-tx}{t(x-ts^2)}$
#19	$y^2 + ktyx + t^2(t^2+tk+1)y = x^3$ $2IV^*(\infty,0), 2I_3\left(-s, -\frac{1}{s}\right), 2I_1$ $x_P = -t^2$		$\frac{XY}{Z}$
#20	$y^2 - yx(t^2-kt+1) = x(x-1)(x+t^2-tk)$ $I_{12}(\infty), 2I_3\left(s, \frac{1}{s}\right), 2I_2(0,k), 2I_1$ $r = 0$		$X+Y+Z$
#21	$y^2 = x^3 + \frac{1}{4}t^2(t^2+2(s^2-1)t+(s^4-10s^2+1))x^2$ $+\frac{1}{2}t^3s^4(t-(s^2-1))x + \frac{1}{4}s^8t^4$ $I_9(\infty), IV^*(0), 2I_2(1,-s^2), 3I_1$ $x_P = s^2t^2$	#15	$\frac{y-s^2x-ts^4(t^2-1)}{x-ts^2(t+1)^2}$
#22	$y^2 + (t(1-s^2)+s^2)yx + t^3s^2y = x(x-s^2t)(x+t^2s^2(1-t))$ $I_3^*(\infty), I_9(0), 6I_1$ $x_{P_1} = 1, \quad x_{P_2} = s^2t$		$\frac{Z(XYZ+s)}{1+YZ}$
#23	$y^2 + (2t^2-tk+1)yx = x(x-t^2)(x-t^4)$ $I_0^*(\infty), I_{12}(0), 6I_1\left(\frac{1}{k\pm 2}, ..\right)$ $x_{P_1} = t^2, \quad x_{p_2} = \frac{(tk-1)^2}{k^2-4}$		$\frac{1}{X+Y}$
#24	$y^2 + (s^2+1)tyx = x(x-t^2s^2)(x-s^2t(t+1)^2)$ $2I_2^*(\infty,0), I_4(-1), 4I_1$ $x_{P_1} = t+1; \quad x_{P_2} = t^2s^2$		$\frac{Z}{Y}$
#25	$y^2 + (s+t)(ts+1)yx + t^2s^2(t(s^2-1)+s)y =$ $x(x-st)(x-t^2s(t-s))$ $I_7(\infty), I_{10}(0), 7I_1$ $x_{P_1} = ts; \quad x_{P_2} = -t^2s^2$		$\frac{Y-s}{XY+sZ}$
#26	$y^2 + (ts-1)(t-s)xy = x(x-t^2s^2)^2$ $2I_8(\infty,0), I_2\left(s, \frac{1}{s}\right), 4I_1$ $x_P = ts$		Z
#27	$y^2 - (t(s^2-1)+s^2)yx + t^3s^2(t+1)y =$ $x^2(x+t^2s^2(t+1))$ $I_1^*(\infty), I_8(0), I_5(-1)4I_1$ $x_P = 0$		$\frac{Z-s}{X+Y}$

Consider the Kummer surface K given by $E_{l_1} \times E_{l_2}/\pm 1$ and choose as equation for K

$$x_1(x_1-1)(x_1-l_1)t^2 = x_2(x_2-1)(x_2-l_2).$$

Following [**19**] we can construct different elliptic fibrations. In the general case we can consider the three elliptic fibrations F_i of K defined by the elliptic parameters m_i, with corresponding types of singular fibers

$$F_6 : m_6 = \frac{x_1}{x_2} \qquad 2I_2^*, 4I_2$$

$$F_8 : m_8 = \frac{(x_2-l_2)(x_1-x_2)}{l_2(l_2-1)x_1(x_1-1)} \qquad III^*, I_2^*, 3I_2, I_1$$

$$F_5 : m_5 = \frac{(x_1-x_2)(l_2(x_1-l_1)+(l_1-1)x_2)}{(l_2x_1-x_2)(x_1-l_1+(l_1-1)x_2)} \qquad I_6^*, 6I_2.$$

In the special case when $E_1 = E_2$ and $j_1 = j_2 = 8000$ we obtain the following fibrations

$$F_6 : l_1 = l_2 = 3+2\sqrt{2} \qquad m_6 = \frac{x_1}{x_2} \qquad 2I_2^*, I_4, 2I_2$$

$$F_8 : l_1 = 3+2\sqrt{2}, l_2 = \frac{1}{l_1} \qquad m_8 = \frac{(x_2-l_2)(x_1-x_2)}{l_2(l_2-1)x_1(x_1-1)} \qquad III^*, I_2^*, I_4, I_2, I_1$$

$$G_8 : l_1 = 3+2\sqrt{2}, l_2 = l_1 \qquad m_8 \qquad III^*, I_3^*, 3I_2$$

$$F_5 : l_1 = l_2 = 3+2\sqrt{2} \qquad m_5 = \frac{(x_1-x_2)(l_2(x_1-l_1)+(l_1-1)x_2)}{(l_2x_1-x_2)(x_1-l_1+(l_1-1)x_2)} \qquad I_6^*, I_4, 4I_2.$$

4.3. Nikulin involutions and Kummer surfaces.

Proposition 4.1. *Consider a family $S_{a,b}$ of $K3$-surfaces with an elliptic fibration, a two-torsion section defining an involution i and two singular fibers of type I_4^*,*

$$S_{a,b} : Y^2 = X^3 + \left(t + \frac{1}{t} + a\right) X^2 + b^2 X.$$

Then the $K3$-surface $S_{a,b}/i$ is the Kummer surface $(E_1 \times E_2)/(\pm Id)$ where the j_i invariants of the elliptic curves E_i, $i = 1,2$ are given by the formulae

$$j_1 j_2 = 4096 \frac{\left(a^2 - 3 + 12b^2\right)^3}{b^2}$$

$$(j_2 - 1728)(j_1 - 1728) = \frac{1024a^2\left(2a^2 - 9 - 72b^2\right)^2}{b^2}.$$

Proof. Recall that if E_i, $i = 1,2$ are two elliptic curves in the Legendre form

$$E_i : y^2 = x(x-1)(x-l_i),$$

the Kummer surface K

$$K : (E_1 \times E_2)/(\pm Id)$$

is defined by the following equation

$$x_1(x_1-1)(x_1-l_1)t^2 = x_2(x_2-1)(x_2-l_2).$$

The Kummer surface K admits an elliptic fibration with parameter $u = m_6 = x_1/x_2$ and Weierstrass equation H_u

$$H_u : Y^2 = X(X - u(u-1)(ul_2 - l_1))(X - u(u-l_1)(l_2u - 1)).$$

The 2-isogenous curve $S_{a,b}/\langle(0,0)\rangle$ has the following Weierstrass equation

$$Y^2 = X\left(X - t\left(t^2 + (a-2b)t + 1\right)\right)\left(X - t\left(t^2 + (a+2b)t + 1\right)\right)$$

with two singular fibers of type I_2^* above 0 and ∞.

We easily prove that $S_{a,b}/\langle(0,0)\rangle$ and H_u are isomorphic on the field $\mathbb{Q}(\sqrt{w_2})$ where

$$l_1 = w_1'w_2 = \frac{w_2}{w_1}, \quad l_2 = \frac{1}{w_1'w_2'} = w_1w_2 \text{ and } \ t = w_1u,$$

w_1, w_1' and w_2, w_2' being respectively the roots of polynomials $t^2 + (a-2b)t + 1$ and $t^2 + (a+2b)t + 1$.

Recall that the modular invariant j_i of the elliptic curve E_i is linked to l_i by the relation

$$j_i = 256\frac{\left(1 - l_i + l_i^2\right)^3}{l_i^2(1-l_i)^2}.$$

By elimination of w_1 and w_2, it follows the relations between j_1 and j_2

$$j_1j_2 = 4096\frac{\left(a^2 - 3 + 12b^2\right)^3}{b^2}$$
$$(j_2 - 1728)(j_1 - 1728) = \frac{1024a^2\left(2a^2 - 9 - 72b^2\right)^2}{b^2}. \qquad \square$$

In the Fermi family, the $K3$-surface Y_k has the fibration #16 with two singular fibers I_4^*, a 2-torsion point and Weierstrass equation

$$y^2 = x^3 + x^2t\left(4\left(t^2 + s^2\right) + t\left(s^4 + 14s^2 + 1\right)\right) + 16t^4s^6x.$$

Taking

$$y = y't^3\left(2\sqrt{s}\right)^3, \quad x = x't^2\left(2\sqrt{s}\right)^2 \text{ and } t = t's,$$

we obtain the following Weierstrass equation

$$y'^2 = x'^3 + \left(t' + \frac{1}{t'} + \frac{1}{4}\frac{s^4 + 14s^2 + 1}{s}\right) + s^4x'.$$

By the previous proposition with $a = (1/4)(s^4 + 14s^2 + 1)/s$, $b = s^2$, we derive the corollary below.

COROLLARY 4.1. *The surface obtained with the* 2*-isogeny of kernel* $\langle(0,0)\rangle$ *from fibration* #16*, is the Kummer surface associated to the product of two elliptic curves of* j*-invariants* j_1, j_2 *satisfying*

$$j_1 j_2 = \frac{\left(s^2+1\right)^3\left(s^6+219s^4-21s^2+1\right)^3}{s^{10}}$$

$$\left(j_1-12^3\right)\left(j_2-12^3\right) = \frac{\left(s^4+14s^2+1\right)^2\left(s^8-548s^6+198s^4-44s^2+1\right)^2}{s^{10}}.$$

REMARK 4.1. If $s=1$ we find $j_1=j_2=8000$.

REMARK 4.2. If $b=1$ we obtain the family of surfaces studied by Narumiya and Shiga, [**22**]. Morover if $a=9/4$ (resp. 4) we find the two modular surfaces associated to the modular groups $\Gamma_1(7)$ (resp. $\Gamma_1(8)$). In these two cases we get $j_1=j_2=-3375$ (resp. $j_1=j_2=8000$).

REMARK 4.3. With the same method we can consider a family of $K3$-surfaces with Weierstrass equations

$$E_v : Y^2+XY-\left(v+\frac{1}{v}-k\right)Y = X^3-\left(v+\frac{1}{v}-k\right)X^2,$$

singular fibers of type $2I_1^*, 2I_4, 2I_1$ and the point $P_v=(0,0)$ of order 4. The elliptic curve $E_v'=E_v/\langle 2P_v\rangle$ has singular fibers of type $2I_2^*, 4I_2$. An analog computation gives $E_v' \equiv (E_1\times E_2)/(\pm Id)$ and

$$j_1 j_2 = \left(256k^2-16k-767\right)^3$$

$$\left(j_1-12^3\right)\left(j_2-12^3\right) = (32k-1)^2\left(128k^2-8k-577\right)^2.$$

4.4. Shioda–Inose structure.

DEFINITION 4.1. A $K3$-surface S has a Shioda–Inose structure if there is a symplectic involution i on S with rational quotient map $S \overset{p}{\to} Y$ such that Y is a Kummer surface and p^* induces a Hodge isometry $T(S)(2)\simeq T(Y)$.

Such an involution i is called a Morrison–Nikulin involution.

An equivalent criterion is that S admits a (Nikulin) involution interchanging two orthogonal copies of $E_8[-1]$ in $NS(S)$.

Or even more abstractly: $2E_8[-1]\hookrightarrow NS(S)$ [**21**] (Theorem 6.3).

Applying this criterion to fibrations #17 and #8 and the van Geemen–Sarti involution we get the following result.

PROPOSITION 4.2. *The translation by the two-torsion point of the fibrations* #17 *and* #8 *endowes* Y_k *with a Shioda–Inose structure.*

PROOF. Fibration #17 has a fiber of type I_{16} at $t=\infty$. The idea [**16**] is to use the components $\Theta_{-2}, \Theta_{-1}, \Theta_0, \Theta_1, \Theta_2, \Theta_3, \Theta_4$ of I_{16} and the zero section to generate a lattice of type $E_8[-1]$. The two-torsion section intersects Θ_8 and the translation by the two-torsion point on the fiber I_{16} transforms Θ_n in Θ_{n+8}. The translation maps the lattice $E_8[-1]$ on an another disjoint $E_8[-1]$ lattice and defines a Shioda–Inose structure.

For fibration #8, the fiber above $t = 0$ is of type I_6 and the section of order 2 specializes to the singular point $(0, 0)$. Then after a blow up, it will not meet the 0-component. If we denote $\Theta_{0,i}$, $0 \leq i \leq 5$, the six components, then the zero section meets $\Theta_{0,0}$ and the 2-torsion section meets $\Theta_{0,3}$. The translation by the 2-torsion section induces the permutation $\Theta_{0,i} \to \Theta_{0,i+3}$.

The fiber above $t = \infty$ is of type I_6^*. The simple components are denoted $\Theta_{\infty,0}, \Theta_{\infty,1}$ and $\Theta_{\infty,2}, \Theta_{\infty,3}$; the double components are denoted C_i with $0 \leq i \leq 6$ and $\Theta_{\infty,0}.C_0 = \Theta_{\infty,1}.C_0 = 1$; $\Theta_{\infty,2}.C_6 = \Theta_{\infty,3}.C_6 = 1$. Then the 2-torsion section intersects $\Theta_{\infty,2}$ or $\Theta_{\infty,3}$ and the translation by the 2-torsion section induces the transposition $C_i \longleftrightarrow C_{6-i}$.

The class of the components C_0, C_1, C_2, $\Theta_{\infty,0}, \Theta_{\infty,1}$, the zero section, $\Theta_{0,0}$ and $\Theta_{0,1}$ define a copy of $E_8[-1]$. The Nikulin involution defined by the two-torsion section maps this $E_8[-1]$ to another copy of $E_8[-1]$ orthogonal to the first one; so the Nikulin involution is a Morrison–Nikulin involution. □

4.5. Base change and van Geemen–Sarti involutions.

If a $K3$-surface S has an elliptic fibration with two fibers of type II^*, this fibration can be realized by a Weierstrass equation of type

$$E_h : y^2 = x^3 - 3\alpha x + \left(h + \frac{1}{h} - 2\beta\right).$$

Moreover Shioda [**31**] deduces the "Kummer sandwiching", $\mathrm{K} \to S \to \mathrm{K}$, identifying the Kummer $\mathrm{K} = E_1 \times E_2/\pm 1$ with the help of the j-invariants of the two elliptic curves E_1, E_2 and giving the following elliptic fibration of K

$$y^2 = x^3 - 3\alpha x + \left(t^2 + \frac{1}{t^2} - 2\beta\right).$$

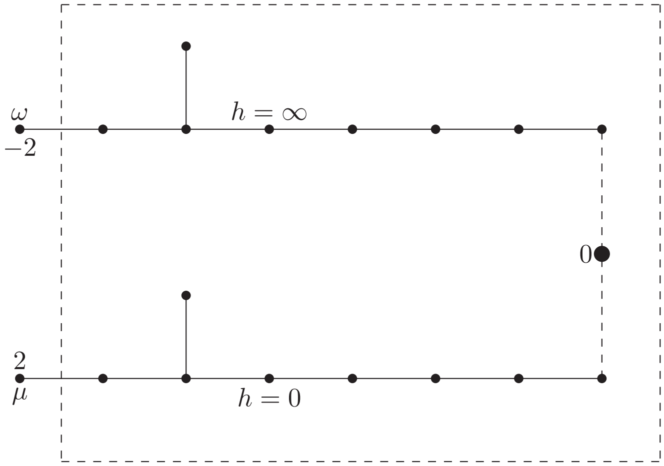
This can be viewed as a base change of the fibration E_h of S.

4.5.1. Alternate elliptic fibration.

We shall now use an alternate elliptic fibration of S ([**29**], Example 13.6) to show that this construction is indeed a 2-isogeny between two elliptic fibrations of S and K. In the next picture we consider a divisor D of type I_{12}^* composed of the zero section 0 and the components of the II^* fibers enclosed in dashed lines. The far double components of the II^* fibres can be chosen as sections of the new fibration. Take ω as the zero section and μ for the other one. More precisely with the new parameter $u := x$ (x from E_h) and the variables $Y = yh$ and $X = h$, we obtain the Weierstrass equation

$$Y^2 = X^3 + \left(u^3 - 3\alpha u - 2\beta\right) X^2 + X.$$

The section μ defined by $X = h = 0$ and $Y = 0$ is a two-torsion section.



In this equation, if we substitute $X(= h)$ by t^2, we obtain an equation in W, t with $Y = Wt^2$, which is the equation for the 2-isogenous elliptic curve. Indeed the birational transformation

$$y = 4\,Y + 4\,U^3 + 2\,UA, \qquad x = 2\,\frac{Y + U^3}{U}$$

with inverse

$$U = \frac{1}{2}\,\frac{y}{x + A}, \qquad Y = \frac{1}{8}\,\frac{\left(-y^2 + 2\,x^3 + 4\,x^2A + 2\,xA^2\right)y}{(x + A)^3}$$

transforms the curve $Y^2 = U^6 + AU^4 + BU^2$ in the Weierstrass form

$$y^2 = (x + A)\left(x^2 - 4B\right).$$

This is an equation for the 2-isogenous curve of the curve $Y^2 = X^3 + AX^2 + BX$ [**36**]. On the curve $Y^2 = U^6 + AU^4 + BU^2$, the involution $U \mapsto -U$ means adding the two-torsion point $(x = -A, y = 0)$.

Using this above process with $A = (u^3 - 3\alpha u - 2\beta)$, the 2-isogenous curve E_u has a Weierstrass equation

$$E_u : Y^2 = \left(X + \left(u^3 - 3\alpha u - 2\beta\right)\right)\left(X^2 - 4\right)$$

with singular fibers of type $I_6^*, 6I_2$.

The coefficients α and β can be computed using the j-invariants

$$\alpha^3 = J_1J_2;\;\; \beta^2 = (1 - J_1)(1 - J_2);\;\; j_i = 1728J_i.$$

If the elliptic curve is put in the Legendre form $y'^2 = x'(x' - 1)(x' - l)$ then $j = 256((1 - l + l^2)^3/l^2(l - 1)^2)$, so

$$\alpha^3 = \frac{16}{729}\,\frac{\left(1 - l_1 + l_1^2\right)^3\left(1 - l_2 + l_2^2\right)^3}{l_1^2(l_1 - 1)^2 l_2^2(l_2 - 1)^2}$$

$$\beta = \frac{1}{27}\frac{(2l_1-1)(l_1-2)(2l_2-1)(l_2-2)(l_1+1)(l_2+1)}{l_1l_2(l_1-1)(l_2-1)}.$$

On the Kummer surface $E_1 \times E_2/\pm 1$ of equation

$$X_1(X_1-1)(X_1-l_1)Z^2 = X_2(X_2-1)(X_2-l_2)$$

we consider an elliptic fibration (case $\mathcal{J}_5$ of [**19**]) with the parameter

$$z = \frac{(l_2X_1 - X_2)(X_1 - l_1 + X_2(l_1-1))}{X_2(X_1-1)}$$

(in fact $z = -(l_1(l_2-1))/(m_5-1)$ cf. 4.2) and obtain the Weierstrass equation

$$\begin{aligned} Y^2 = {} & (X - 2l_1l_2(l_1-1)(l_2-1))(X + 2l_1l_2(l_1-1)(l_2-1)) \\ & \big(X + 4z^3 + 4(-2l_1l_2 + l_1 + l_2 + 1)z^2 \\ & + 4(l_1l_2-1)(l_1l_2 - l_1 - l_2)z + 2l_1l_2(l_1-1)(l_2-1)\big). \end{aligned}$$

Substituting $z = w - (1/3)(-2l_1l_2 + l_1 + l_2 + 1)$ it follows

$$\begin{aligned} Y^2 = {} & (X - 2l_1l_2(l_1-1)(l_2-1))(X + 2l_1l_2(l_1-1)(l_2-1)) \\ & \left(X + 4w^3 - \frac{4}{3}\left(l_2^2 - l_2 + 1\right)\left(l_1^2 - l_1 + 1\right)w \right. \\ & \left. + \frac{2}{27}(l_2-2)(2l_2-1)(l_1-2)(2l_1-1)(l_2+1)(l_1+1)\right). \end{aligned}$$

Up to an automorphism of this Weierstrass form we recover the equation of E_u.

The previous results can be used to show the following proposition

Proposition 4.3. *The translation by the two-torsion point of the elliptic fibration #4 gives to Y_k a Shioda–Inose structure.*

5. Proof of Theorem 1.1.

Notation 5.1. If we consider an elliptic fibration ϕ of a $K3$-surface S with a two-torsion point T, we will write $\phi(T)$ for the elliptic fibration of S/i if i denotes the involution given by the fiberwise translation by T.

From the Shioda–Tate formula (cf. e.g. [**32**], Corollary 1.7]) we have the relation

$$12 = \frac{|\Delta| \prod m_v^{(1)}}{|\mathrm{Tor}|^2}$$

where Δ is the determinant of the height-matrix of a set of generators of the Mordell–Weil group, $m_v^{(1)}$ the number of simple components of a singular fiber and $|\mathrm{Tor}|$ the order of the torsion group of the Mordell–Weil group. From a set of infinite sections this formula allows us to determine generators of the Mordell–Weil group except for fibration #4. Using the 2-isogeny we determine also the Mordell–Weil group of $\#n(T)$. The discriminant is either 12×2 or 12×8.

Proposition 5.1. *The translation by the two-torsion point of the fibration #16 gives to Y_k a Shioda–Inose structure.*

Proof. From the Proposition 4.1, the translation by the two-torsion point of #16 gives to the quotient a Kummer structure. The fibration #16 is of rank one, its Mordell–Weil group is generated by the point P of x-coordinate x_P in Table 4, and the two-torsion point. By computation we can see that the Mordell–Weil group of the 2-isogenous curve on $\mathbb{C}(t)$ is generated by the image of P and torsion sections. So we can compute the discriminant of the Néron–Severi group which is 12×8. The second condition, $T(Y_k)(2) \simeq T(K_k)$, is then verified. □

Remark 5.1. The $K3$-surface of Picard number 20 given by the elliptic fibration

$$Y^2 = X^3 - \left(t + \frac{1}{t} - \frac{3}{2}\right) X^2 + \frac{1}{16} X$$

or

$$y^2 = x^3 - \frac{1}{2}\, t \left(2\, t^2 + 2 - 3\, t\right) x^2 + \frac{1}{16}\, t^4 x$$

has rank 1. The Mordell–Weil group is generated by $(0,0)$ and $P = (x = 1/4, y = (t-1)^2/8)$. The determinant of the Néron–Severi group is equal to 12. By computation we find that the image of P by the 2-isogeny is equal to $2Q$ with $Q = (t(t-1)(t^2-t+1), -t^3(t-1)(t^2-t+1))$ of height $3/4$. The determinant of the Néron–Severi group of the 2-isogenous curve is then 12 not 12×2^2. So the involution induced by the two-torsion point is not a Nikulin–Morrison involution. Moreover the 2-isogenous elliptic curve is a fibration of the Kummer surface $E \times E/\pm 1$ where $j(E) = 0$.

For fibrations $\#n(T)$ with discriminant of the transcendental lattice 12×8 we prove the Shioda–Inose structure in the following way: from corollary 4.1 this is true for $\#16(T)$, from Proposition 4.3 this is true for $\#4(T)$ and from Proposition 4.2 this is true for $\#17(T)$, $\#8(T)$. The other fibrations $\#n(T)$ can be obtained by 2- or 3-neighbor method from $\#16(T)$, $\#8(T)$ or $\#17(T)$. The results are given in the Table 5. In the second column are written the Weierstrass equations for the $\#n$ elliptic fibration and its 2-isogenous fibration, singular fibers and the x-coordinates of generators of the Mordell lattice of $\#n(T)$. In the third column we give the starting fibration for the 2- or 3-neighbor method and in the last column the parameter used from the starting fibration.

5.1. The $K3$-surface S_k.

For the remaining fibrations, (discriminant 12×2), using also the 2- or 3-neighbor method, they are proved to lie on the same surface S_k. Except for the case #7 the results are collected in the Table 6 with the same format. The case #7 needs an intermediate fibration explained in the next paragraph.

Starting with fibration $\#7(T)$ and using the parameter $m_7 = y/xt(t-s^2)$ it follows the Weierstrass equation

Table 5. Fibrations with discriminant 12×8 (Fibrations of the Kummer K_k).

No	Weierstrass Equation	From	Param.
#4	see Prop 9		
#8	$y^2 - k(t-1)yx = x(x-1)\left(x-t^3\right)$ $I_6^*(\infty), I_6(0), I_2(1), 4I_1$ $y^2 = x^3 + \frac{1}{2}\left(4t^3 - t^2k^2 + 2tk^2 + 4 - k^2\right)x^2$ $+\frac{1}{16}(t-1)^2\left(4t^2 + t\left(4-k^2\right) + (k-2)^2\right)$ $\left(4t^2 + t\left(4-k^2\right) + (k+2)^2\right)x$ $I_3^*(\infty), I_3(0), I_4(1), 4I_2$ $x_Q = -\frac{1}{4}(t-1)\left(4t^2 + t\left(4-k^2\right) + (k-2)^2\right)$		
#16	$y^2 = x^3 + t\left(4\left(t^2+s^2\right) + t\left(s^4+14s^2+1\right)\right)x^2 + 16s^6t^4x$ $2I_4^*(\infty,0), 4I_1$ $y^2 = x\left(x - t\left(4\left(t^2+s^2\right) + t\left(s^4+14s^2+1+8s^3\right)\right)\right)$ $\left(x - t\left(4\left(t^2+s^2\right) + t\left(s^4+14s^2+1-8s^3\right)\right)\right)$ $2I_2^*(\infty,0), 4I_2$ $x_Q = \frac{t^2\left(\left(t^2+s^2\right)\left(3+s^2\right)+t\left(-s^4+8s^2+1\right)\right)^2}{(t+1)^2\left(t+s^2\right)^2}$		
#17	$y^2 - \frac{1}{2}\left(s^4+14s^2+1-s^2t^2\right)yx = x\left(x-4s^2\right)\left(x-4s^6\right)$ $I_{16}(\infty), 8I_1\left(\pm\frac{s^2\pm 4s-1}{s}, ..\right)$ $y^2 = x\left(x - \left(t^2s^2 - \left(s^4+14s^2+1\right) \pm 8s\left(s^2+1\right)\right)\right)$ $\left(x - \left(ts+s^2\pm 4s-1\right)\left(ts-s^2\pm 4s+1\right)\right)$ $I_8(\infty), 8I_2\left(\pm\frac{s^2\pm 4s-1}{s}, ..\right)$ $x_{Q_1} = \frac{1}{16}\left(ts+s^2-4s-1\right)\left(ts-s^2+4s+1\right)$ $\left(t^2s^2 - \left(s^4+14s^2+1\right) + 8s\left(s^2+1\right)\right)$ $x_{Q_2} = \frac{1}{16}\frac{(s-1)^2}{(s+1)^2}\left(ts+s^2+4s-1\right)$ $\left(ts-s^2+4s+1\right)\left(t^2s^2 - \left(s^4+14s^2+1\right) - 8s\left(s^2+1\right)\right)$		
#23	$y^2 + \left(2t^2 - tk + 1\right)yx = x\left(x-t^2\right)\left(x-t^4\right)$ $I_0^*(\infty), I_{12}(0), 6I_1\left(\frac{1}{k\pm 2}, ..\right)$ $y^2 = x\left(x + \frac{1}{4}(t(k-2)-1)\left(4t^2-(k+2)t+1\right)\right)$ $\left(x + \frac{1}{4}(t(k+2)-1)\left(4t^2-(k-2)t+1\right)\right)$ $I_0^*(\infty), I_6(0), 6I_2\left(\frac{1}{k\pm 2}, ..\right)$ $x_{Q_1} = \frac{-1}{4}\left(4t^2-(k-2)t+1\right)(t(k-2)-1);$ $x_{Q_2} = \frac{-1}{4}\frac{k-2}{k+2}(t(k+2)-1)\left(4t^2-(k-2)t+1\right)$	#8 #26 #24	$\frac{y-y_{2Q}+\frac{k}{2}(x-x_{2Q})}{t(x-x_{2Q})}$ $\frac{2y-(t-s)(ts-1)x}{t\left(x-ts(ts-1)^2\right)}$ $\frac{(y-y_{2Q_2})+\frac{s^2+1}{2}(x-x_{2Q_2})}{(t+1)(x-x_{2Q_2})}$
#24	$y^2 + \left(s^2+1\right)tyx = x\left(x-t^2s^2\right)\left(x-s^2t(t+1)^2\right)$ $2I_2^*(\infty,0), I_4(-1), 4I_1$ $y^2 = x^3 + \frac{1}{2}t\left(4t^2s^2 - t\left(s^4-10s^2+1\right) + 4s^2\right)x^2$ $+\frac{1}{16}t^2\left(4t^2s^2 + \left(8s^2-(s-1)^4\right)t + 4s^2\right)$ $\left(4t^2s^2 + \left(8s^2-(s+1)^4\right)t + 4s^2\right)x$ $2I_1^*(\infty,0), 5I_2(-1,)$ $x_{Q_1} = \frac{1}{4}(2t^2s^2 + t\left(s^2-1\right) - 2)^2;$ $x_{Q_2} = -\frac{1}{4}t\left(4t^2s^2 + \left(8s^2-(s+1)^4\right)t + 4s^2\right)$		
#26	$y^2 + (ts-1)(t-s)xy = x\left(x-t^2s^2\right)^2$ $2I_8(\infty,0), I_2\left(s,\frac{1}{s}\right), 4I_1$ $y^2 = x\left(x+4t^2s^2\right)\left(x+\frac{1}{4}(t-s)^2(st-1)^2\right)$ $4I_4\left(\infty,0,s,\frac{1}{s}\right), 4I_2$ $x_{Q_1} = ts(ts-1)^2$		

Table 6. Fibrations with discriminant 12×2 (Fibrations of S_k).

No	Weierstrass Equation	From	Param.
#7	$y^2 = x^3 + \frac{1}{4}t\left(t\left(s^4-10s^2+1\right)+8s^4\right)x^2 - t^2s^2\left(t-s^2\right)^3 x$ $III^*(\infty), I_1^*(0), I_6\left(s^2\right), 2I_1$ $y^2 = x^3 - \frac{1}{2}t\left(t\left(s^4-10s^2+1\right)+8s^4\right)x^2$ $+\frac{1}{16}t^3\left(64t^2s^2+\left(s^8-20s^6-90s^4-20s^2+1\right)t +16s^4\left(s^2+1\right)^2\right)x$ $III^*(\infty), I_2^*(0), I_3\left(s^2\right), 2I_2$		
#9	$y^2 = x^3 + \frac{1}{4}\left(s^4+14s^2+1\right)t^2x^2 + t^3s^2\left(s^2+t\right)\left(ts^2+1\right)x$ $2III^*(\infty,0), 2I_2\left(-s^2,-\frac{1}{s^2}\right), 2I_1$ $y^2 = x^3 - \frac{1}{2}\left(s^4+14s^2+1\right)t^2x^2$ $-\frac{1}{16}t^3\left(64t^2s^4+t\left(-s^8+36s^6-198s^4+36s^2-1\right)+64s^4\right)x$ $2III^*(\infty,0), 2I_1\left(-s^2,-\frac{1}{s^2}\right), 2I_2$ $x_Q = \frac{1}{4}\frac{(t+1)^2\left(2t^2s^2+t\left(s^4-6s^2+1\right)+2s^2\right)^2}{\left(s^2-1\right)^2(t-1)^2}$	#20	$\frac{y}{(ts-1)^4}$
#14	$y^2 = x^3 + \left(t^3\left(s^4+1\right)+\frac{1}{4}t^2\left(s^4+14s^2+1\right)+ts^2\right)x^2 + t^6s^4x$ $I_0^*(\infty), I_8^*(0), 4I_1\left(\frac{-1}{4}, \frac{-4s^2}{\left(s^2-1\right)^2}, ..\right)$ $y^2 = x\left(x-\frac{1}{4}\left(s^2+1\right)^2t^3-\frac{1}{4}\left(s^4+14s^2+1\right)t^2-ts^2\right)$ $\left(x-\frac{1}{4}\left(s^2-1\right)^2t^3-\frac{1}{4}\left(s^4+14s^2+1\right)t^2-ts^2\right)$ $I_0^*(\infty), I_4^*(0), 4I_2\left(\frac{-1}{4}, \frac{-4s^2}{\left(s^2-1\right)^2}, ..\right)$ $x_Q = \frac{1}{4}t^2\left(s^2-1\right)^2(4t+1)$	#15	$\frac{t^2s^2}{x+t^3s^2-\frac{1}{4}t^2\left(s^2-1\right)^2}$
#15	$(y-tx)\left(y-s^2tx\right) = x\left(x-ts^2\right)\left(x-ts^2(t+1)^2\right)$ $I_4^*(\infty), I_1^*(0), I_4(-1), 3I_1\left(\frac{1}{4}\left(\frac{s^2-1}{s}\right)^2, ..\right)$ $y^2 = x\left(x+t^3s^2-\frac{1}{4}t^2(s^2-1)^2\right)$ $\left(x+t^3s^2-\frac{1}{4}(s^2-4s-1)(s^2+4s-1)t^2+4ts^2\right)$ $2I_2^*(\infty,0), 4I_2\left(-1, \frac{1}{4}\left(\frac{s^2-1}{s}\right)^2, ..\right)$ $x_Q = \frac{1}{4}t^2(s^2-1)^2$	#20	$\frac{y}{x(t-s)^2}$
#20	$y^2-(t^2s-\left(s^2+1\right)t+3s)yx - s^2(t-s)(ts-1)y = x^3$ $I_{12}(\infty), 2I_3\left(s, \frac{1}{s}\right), 2I_2\left(0, \frac{s^2+1}{s}\right), 2I_1$ $y^2 + \left(t^2s-(s^2+1)t-3s\right)yx - s(t-s)^2(ts-1)^2y = x^3$ $3I_6\left(\infty, s, \frac{1}{s}\right), 2I_2, 2I_1\left(0, \frac{s^2+1}{s}\right)$		

$$Y^2 + 2\left(m_7^2s^2-2\right)YX - 16m_7^4s^4Y = \left(X-8m_7^2s^2\right)$$
$$\left(X+8m_7^2s^2\right)\left(X+m_7^2\left(s^4-6s^2+1\right)-4\right)$$

with singular fibers $I_8(\infty), IV^*(0), 8I_1$.

Then the parameter $m_{15} = Y/(X+8m_7^2s^2)$ leads to the fibration #15(T).

For the last part of Theorem 1.1 we give properties of S_k. First we prove that S_k is the Jacobian variety of some genus 1 fibrations of K_k.

Starting with the fibration #26(T) and Weierstrass equation

$$y^2 = x\left(x+4t^2s^2\right)\left(x+\frac{1}{4}(t-s)^2(ts-1)^2\right),$$

the new parameter $m := y/t(x + 1/4(t - s)^2(ts - 1)^2)$ defines an elliptic fibration of #26(T) with Weierstrass equation

$$E_m : Y^2 - m\left(s^2+1\right)YX = X\left(X - s^2m^2\right)\left(X + \frac{1}{4}(2m-s)^2(2m+s)^2\right)$$

and singular fibers are of type $4I_4(0, \pm(1/2)s, \infty), 8I_1$.

Then setting as new parameter $n = X/m^2$, it follows a genus one curve in m and Y. Its equation, of degree 2 in Y, can be transformed into

$$w^2 = -16n\left(-n+s^2\right)m^4 + n\left(s^4(8+n) - 10ns^2 + n(1+4n)\right)m^2 - ns^4\left(-n+s^2\right).$$

Let us recall the formula giving the Jacobian of a genus one curve defined by the equation $y^2 = ax^4 + bx^3 + cx^2 + dx + e$. If $c_4 = 2^4(12ae - 3bd + c^2)$ and $c_6 = 2^5(72ace - 27ad^2 - 27b^2e + 9bcd - 2c^3)$, then the equation of the Jacobian curve is

$$\bar{y}^2 = \bar{x}^3 - 27c_4\bar{x} - 54c_6.$$

In our case we obtain

$$y^2 = x\left(x + n^3s^2 - \frac{1}{4}n^2\left(s^2-1\right)^2\right)$$
$$\left(x + n^3s^2 - \frac{1}{4}\left(s^2-4s-1\right)\left(s^2+4s-1\right)n^2 + 4ns^2\right),$$

which is precisely the fibration #15(T).

Remark 5.2. Using the new parameter $p_1 = Y/m^2(X + 1/4(2m - s)^2(2m + s)^2)$ another result can be derived from E_m leading to

$$E_{p_1} : Y^2 - 2s(2p_1-1)(2p_1+1)YX =$$
$$X\left(X + 64s^2p_1^2\right)(X + (2sp_1+1)(2sp_1-1)(s+2p_1)(s-2p_1)),$$

with singular fibers $2I_0^*, 4I_2, 4I_1$. From E_{p_1} and the new parameter $k = X/p_1^2$ we obtain a genus one fibration whose Jacobian is #14(T).

Starting from the fibration #26(T) (previous equation), the parameter $q = x/t^2$ leads to a genus one fibration whose Jacobian is a fibration of S_k leading to #15(T).

5.2. Transcendental and Néron–Severi lattices of the surface S_k.

We shall indentify the $K3$-surface S_k by its transcendental and Néron–Severi lattices. As a corollary, using again the Kneser–Nishiyama technique this allows to recover the yet known elliptic fibrations of S_k but also all of them. We shall see that all the fibrations can be obtained from the primitive embeddings of $M = A_1 \oplus A_1 \oplus A_5$ into the various Niemeier lattices. Since M is composed of 3 root lattices of type A_n, the $K3$-surface S_k will possess probably more twice elliptic fibrations than Y_k.

Lemma 5.1. *The $K3$-surface S_k has the transcendental lattice*

$$T(S_k) = \langle -2\rangle \oplus \langle 2\rangle \oplus \langle 6\rangle$$

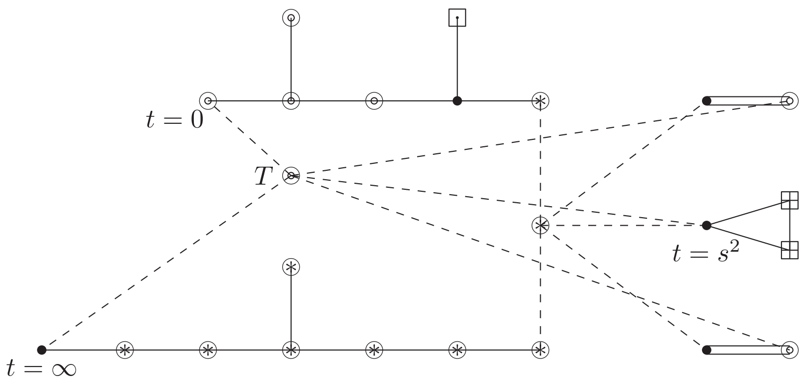
and Néron–Severi lattice

$$NS(S_k) = U \oplus E_8[-1] \oplus E_7[-1] \oplus \langle -2\rangle \oplus \langle -6\rangle.$$

PROOF. We consider the elliptic fibration #7(T) with Weierstrass equation given in Table 6 and draw the graph of the singular fibers, the zero and two-torsion sections of the elliptic fibration

$$Y^2 = X^3 + \left(\left(-\frac{1}{2}\,s^4 + 5\,s^2 - \frac{1}{2}\right)t^2 - 4\,s^4 t\right)X^2$$
$$\left(4\,s^2t^5 + \left(\frac{1}{16}\,s^8 - \frac{45}{8}\,s^4 - \frac{5}{4}\,s^2 - \frac{5}{4}\,s^6 + \frac{1}{16}\right)t^4 + s^4\left(s^2+1\right)^2 t^3\right)X$$

with singular fibers $III^*(\infty), I_2^*(0), I_3(s^2), 2I_2(t_1,t_2)$.



With the parameter $m = X/t$ we obtain another fibration with singular fibers $II^*(\infty)$ (shown by ⊛), $I_2^*(0)$ (shown by ⊚), $I_3((1/4)s^2(s^2-1)^2)$ (part of it shown by ⊞), $I_2(4s^4)$ (part of it shown by ⊡), $I_1(\sigma_0)$, where $\sigma_0 = -(s^2-6s+1)(s^2+6s+1)(s^2+1)^4/1728s^2$.

This new fibration Σ_k has no torsion, rank 0, Weierstrass equation

$$y^2 = x^3 + 2m\left(\left(-s^4+10s^2-1\right)m + 2s^4\left(s^2+1\right)^2\right)x^2$$
$$+\left(m-4s^4\right)m^3\left(s^8-20s^6-90s^4-20s^2+1\right)x + 256m^5s^2\left(m-4s^4\right)^2$$

and Néron–Severi group

$$NS(\Sigma_k) = U \oplus E_8 \oplus D_6 \oplus A_2 \oplus A_1.$$

By Morrison ([**21**], Corollary 2.10 ii), the Néron–Severi group of an algebraic $K3$-surface X with $12 \leq \rho(X) \leq 20$ is uniquely determined by its signature and discriminant form. Thus we compute $q_{NS(S_k)}$ with the help of the fibration Σ_k. From

$$\frac{D_6^*}{D_6} = \langle [1]_{D_6}, [3]_{D_6} \rangle \text{ and } q_{D_6}([1]_{D_6}) = q_{D_6}([3]_{D_6}) = -\frac{3}{2},$$

we deduce the discriminant form, since $b_{D_6}([1]_{D_6}, [3]_{D_6}) = 0$,

$$\begin{aligned}(G_{NS(S_k)}, q_{NS(S_k)})\\ &= \mathbb{Z}/2\mathbb{Z}\left(-\frac{3}{2}\right) \oplus \mathbb{Z}/2\mathbb{Z}\left(-\frac{3}{2}\right) \oplus \mathbb{Z}/3\mathbb{Z}\left(-\frac{2}{3}\right) \oplus \mathbb{Z}/2\mathbb{Z}\left(-\frac{1}{2}\right) \text{mod}.2\mathbb{Z}\\ &= \mathbb{Z}/2\mathbb{Z}\left(\frac{1}{2}\right) \oplus \mathbb{Z}/6\mathbb{Z}\left(-\frac{1}{6}\right) \oplus \mathbb{Z}/2\mathbb{Z}\left(-\frac{1}{2}\right).\end{aligned}$$

From Morrison ([**21**], Theorem 2.8 and Corollary 2.10) there is a unique primitive embedding of $NS(S_k)$ into the $K3$-lattice $\Lambda = E_8[-1]^2 \oplus U^3$, whose orthogonal complement is by definition the transcendental lattice $T(S_k)$. Now from Nikulin([**23**], Proposition 1.6.1), it follows

$$G_{NS(S_k)} \simeq (G_{NS(S_k)})^{\perp} = G_{T(S_k)}, \quad q_{T(S_k)} = -q_{NS(S_k)}.$$

In other words the discriminant form of the transcendental lattice is

$$(G_{T(S_k)}, q_{T(S_k)}) = \mathbb{Z}/2\mathbb{Z}\left(-\frac{1}{2}\right) \oplus \mathbb{Z}/6\mathbb{Z}\left(\frac{1}{6}\right) \oplus \mathbb{Z}/2\mathbb{Z}\left(\frac{1}{2}\right).$$

From this last relation we prove that $T(S_k) = \langle -2 \rangle \oplus \langle 6 \rangle \oplus \langle 2 \rangle$. Denoting T' the lattice $T' = \langle -2 \rangle \oplus \langle 6 \rangle \oplus \langle 2 \rangle$, we observe that T' and $T(S_k)$ have the same signature and discriminant form. Since $|\det(T')| = 24$ is small, there is only one equivalence class of forms in a genus, meaning that such a transcendental lattice is, up to isomorphism, uniquely determined by its signature and discriminant form ([**11**], p. 395).

Now computing a primitive embedding of $T(S_k)$ into Λ, since by Morrison ([**21**], Corollary 2.10 i) this embedding is unique, its orthogonal complement provides $NS(S_k)$. Take the primitive embedding $\langle -2 \rangle = \langle e_2 \rangle \hookrightarrow E_8$, $\langle 2 \rangle = \langle u_1 + u_2 \rangle \hookrightarrow U$, $\langle 6 \rangle = \langle u_1 + 3u_2 \rangle \hookrightarrow U$, (u_1, u_2) denoting a basis of U. Hence we deduce

$$NS(S_k) = U \oplus E_8[-1] \oplus E_7[-1] \oplus \langle -2 \rangle \oplus \langle -6 \rangle. \qquad \square$$

Corollary 5.1. *All the elliptic fibrations of S_k are obtained from the primitive embeddings of $A_1 \oplus A_1 \oplus A_5$ in the various Niemeier lattices.*

Proof. In that purpose, embed $T(S_k)[-1]$ into $U \oplus E_8[-1]$ in the following way: $(-2) \oplus (-6)$ primitively embedded in $E_8[-1]$ as in Nishiyama ([**25**], p. 334) and $\langle 2 \rangle = \langle u_1 + u_2 \rangle \hookrightarrow U$. We obtain $M = (T(S_k)[-1])^{\perp}_{U \oplus E_8[-1]} = A_1 \oplus A_1 \oplus A_5$. Now all the elliptic fibrations of S_k are obtained from the primitive embeddings of M into the various Niemeier lattices, as explained in Section 2. $\square$

Using their Weierstrass equations and a 2-neighbor method [**15**], it was proved in the previous subsection that all the fibrations #7(T), #9(T), #14(T), #15(T), #20(T) are on the same $K3$-surface. Using the Kneser–Nishiyama method we can identify each of these elliptic fibrations with a primitive embedding into a certain Niemeier lattice.

This identification will be performed comparing singular fibers and Mordell–Weil lattices.

5.2.1.

Take the primitive embedding into $Ni(D_{10}E_7^2)$, given by $A_5=\langle e_2,e_4,e_5,e_6,e_7\rangle\hookrightarrow E_7$ and $A_1^2=\langle d_{10},d_7\rangle\hookrightarrow D_{10}$.

Since $(A_5)^{\perp}_{E_7}=A_2$ and $(A_1^2)^{\perp}_{D_{10}}=A_1\oplus A_1\oplus D_6$, it follows $N=N_{\text{root}}=2A_1A_2D_6E_7$, $\det N=24\times 4$, thus the rank is 0 and the torsion group $\mathbb{Z}/2\mathbb{Z}$. Hence this fibration can be identified with the elliptic fibration $\#7(T)$.

5.2.2.

The primitive embedding is into $Ni(D_{10}E_7^2)$, given by

$$A_5\oplus A_1^2=\langle d_{10},d_8,d_7,d_6,d_5,d_{10}+d_9+2(d_8+d_7+d_6+d_5+d_4)+d_3,d_3\rangle\hookrightarrow D_{10}.$$

We get

$$\left(A_5\oplus A_1^2\right)^{\perp}_{D_{10}}=(-6)\oplus\langle x\rangle\oplus\langle d_1\rangle=(-6)\oplus A_1\oplus A_1$$

with

$$x=d_9+d_{10}+2(d_8+d_7+d_6+d_5+d_4+d_3+d_2)+d_1$$

and

$$(-6)=3d_9+2d_{10}+4d_8+3d_7+2d_6+d_5.$$

Thus $N_{\text{root}}=A_1A_1E_7^2$ and the rank of the fibration is 1. Since $2[2]_{D_{10}}=x+d_1$ and there is no other relation with $[1]_{D_{10}}$ or $[3]_{D_{10}}$, among the glue vectors $\langle[1,1,0]\rangle,\langle[3,0,1]\rangle$ generating $Ni(D_{10}E_7^2)$, only $\langle[2,1,1]\rangle$ contributes to torsion.

Hence the torsion group is $\mathbb{Z}/2\mathbb{Z}$. Moreover the 2-torsion section is

$$2F+0+[[2],[1],[1]]$$

with height $4-(1/2+1/2+3/2+3/2)=0$. The infinite section is

$$3F+0+[(-6),0,0]$$

with height 6. Hence this fibration can be identified with the fibration $\#9(T)$.

5.2.3.

Take the primitive embedding into $Ni(D_8^3)$, given by $A_5=\langle d_8,d_6,d_5,d_4,d_3\rangle\hookrightarrow D_8^{(1)}$ and $A_1^2=\langle d_8,d_1\rangle\hookrightarrow D_8^{(2)}$. We compute $(A_5)^{\perp}_{D_8}=(-6)\oplus\langle x_1=(-2)\rangle\oplus\langle d_1\rangle$ with $x_1=d_7+d_8+2(d_6+d_5+d_4+d_3+d_2)+d_1$

$$\begin{aligned}(A_1^2)^{\perp}_{D_8}&=\langle d_7\rangle\oplus\langle x_1=d_7+d_8+2(d_6+d_5+d_4+d_3+d_2)+d_1\rangle\\&\quad\oplus\langle d_5,d_4,x_3=d_7+d_8+2d_6+d_5,d_3\rangle=A_1\oplus A_1\oplus D_4.\end{aligned}$$

We deduce $N_{\text{root}} = 4A_1D_4D_8$ (hence the fibration has rank 1) and the relations

$$2[2]_{D_8} = x_1 + d_1 \tag{1}$$
$$2([2]_{D_8} - (d_1 + d_2)) = x_3 + 2d_3 + 2d_4 + d_5 \tag{2}$$
$$2[3]_{D_8} = x_1 + 2x_3 + d_3 + 2d_4 + d_5 + d_7 \tag{3}$$
$$2([1]_{D_8} - (d_6 + d_8)) = x_1 + x_3 + d_3 + 2d_4 + 2d_5 + d_7 \tag{4}$$
$$2([1]_{D_8} - (d_6 + d_7 + d_8)) = 2x_3 + 3d_5 + 4d_4 + 3d_3 + 2d_2 + d_1 - d_7. \tag{5}$$

Thus, among the glue vectors $\langle[1,2,2],[1,1,1],[2,2,1]\rangle$ generating the Niemeier lattice, only vectors $\langle[0,3,3],[2,1,2]\rangle$ contribute to torsion and the torsion group is $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.

From relations (1) to (5) we deduce the various contributions and heights of the following sections (see Table 7).

Table 7. Contributions and heights of the sections of 5.2.3.

$0+2F+$	Cont. A_1	Cont. A_1	Cont. A_1	Cont. A_1	Cont. D_4	Cont. D_6	ht.
$[0,[3],[3]]$	0	0	1/2	1/2	1	2	0
$[[2],[2]-(d_1+d_2),[1]]$	1/2	1/2	0	0	1	2	0
$[[2],[1]-(d_6+d_8),[2]]$	1/2	1/2	1/2	1/2	1	1	0
$[[2]-(d_1+d_2),[1]-(d_6+d_7+d_8),[2]]$	0	0	1/2	0	1	1	3/2

Hence this fibration can be identified with the fibration #14(T).

5.2.4.
The primitive embedding is into $Ni(D_8^3)$ and given by

$$A_5 = \langle d_8, d_6, d_5, d_4, d_3\rangle \hookrightarrow D_8^{(1)} \quad A_1 = \langle d_8\rangle \hookrightarrow D_8^{(2)} \quad A_1 = \langle d_8\rangle \hookrightarrow D_8^{(3)}.$$

As previously $(A_5)^{\perp}_{D_8} = \langle(-6)\rangle \oplus \langle x_1\rangle \oplus \langle d_1\rangle$; we get also $\langle d_8\rangle^{\perp}_{D_8} = \langle d_7\rangle \oplus \langle x_4 = d_7 + d_8 + 2d_6 + d_5, d_5, d_4, d_3, d_2, d_1\rangle = A_1 \oplus D_6$. Hence $N_{\text{root}} = 4A_12D_6$, and the rank is 1. Moreover it follows the relations

$$2[2]_{D_8} = x_1 + d_1 \tag{6}$$
$$2[2]_{D_8} = x_3 + d_5 + 2d_4 + 2d_3 + 2d_2 + 2d_1 \tag{7}$$
$$2([1]_{D_8} - (d_5 + d_6 + d_7 + d_8)) = 2x_3 + d_5 + 4d_4 + 3d_3 + 2d_2 + d_1 - d_7 \tag{8}$$
$$2[3]_{D_8} = 3x_3 + d_7 + 2d_5 + 4d_4 + 3d_3 + 2d_2 + d_1 \in A_1 \oplus D_6. \tag{9}$$

We deduce that among the glue vectors generating $Ni(D_8^3)$, only $\langle[0,3,3],[2,1,2]\rangle$ contribute to torsion. So the torsion group is $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. From relations (6) to (9) we deduce the various contributions and heights of the following sections (see Table 8).

Table 8. Contributions and heights of the sections of 5.2.4.

$0+2F+$	Cont. A_1	Cont. A_1	Cont. A_1	Cont. D_6	Cont. A_1	Cont. D_6	ht.
[0,[3],[3]]	0	0	1/2	1+1/2	1/2	1+1/2	0
[[2],[1]$-(d_5+d_6+d_7+d_8)$,[2]]	1/2	1/2	1/2	1+1/2	0	1	0
[[2],[2],[1]$-(d_5+d_6+d_7+d_8)$]	1/2	1/2	0	1	1/2	1+1/2	0
[[3],0,[3]]	0	1/2	0	0	1/2	1+1/2	3/2

Hence this fibration can be identified with the fibration #15(T).

5.2.5.
Take the primitive embedding onto $Ni(A_5^4D_4)$ given by $A_5 \hookrightarrow A_5$, $A_1 \oplus A_1 = \langle d_4, d_1\rangle \hookrightarrow D_4$. Since $\langle d_4, d_1\rangle^{\perp}_{D_4} = A_1^2$, we get $N = N_{\text{root}} = 3A_5 2A_1$; thus the rank of the fibration is 0 and since $\det(N) = 24 \times 6^2$, the torsion group is $\mathbb{Z}/6\mathbb{Z}$.
This fibration can be identified with the fibration #20(T).

Remark 5.3. From fibration #20(T) the surface S_k appears to be a double cover of the rational elliptic modular surface associated to the modular group $\Gamma_0(6)$ given in Beauville's paper [**2**]

$$(x+y)(y+z)(z+x)(t-s)(ts-1) = 8sxyz.$$

6. Proof of Theorem 1.2.

We recall first on Table 9 the results obtained by Bertin and Lecacheux in [**5**].

Comparing to the fibrations of the family, you remark more elliptic fibrations with 2-torsion sections on Y_2. Some of them are specializations for $s=1$ of the generic ones. They are denoted for example #17($18-m$) which means the following: it is the fibration ($18-m$) in the last Table of [**5**] (m denotes the elliptic parameter of the fibration numbered 18) and the specialization for $k=2$ of the fibration #17 of the generic case. Those generic elliptic fibrations with Morrison–Nikulin involutions possess specializations to Y_2 with Morrison–Nikulin involutions, by a Schütt's Lemma [**28**], namely #4($16-o$), #8($9-r$), #16($14-t$), #17($18-m$), #23($2-k$), #24($5-d$) a), #26($1-s$). Others (#5($17-q$), ($24-\psi$), #10($10-e$), #15($6-p$) c), #24($5-d$) b), c)) are specific to K_2 and cannot be deduced from elliptic fibrations of the generic Kummer. To identify them, we have to use the distinguished property of Y_2, that is, Y_2 is a singular $K3$-surface with Picard number 20.

Hence Y_2 inherits of a Shioda–Inose structure, that is the quotient of Y_2 by an involution is isomorphic to a Kummer surface K_2 realized from the product of CM elliptic curves [**33**], [**34**] provided in the following way.

Since the transcendental lattice of Y_2 is $T(Y_2) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$, we get $b^2-4ac = -8$, $\tau_1 = (-b+\sqrt{b^2-4ac})/2a$, $\tau_2 = (b+\sqrt{b^2-4ac})/2$, hence $\tau_1 = \tau_2 = i\sqrt{2}$.

We deduce $K_2 = E \times E/\pm 1$ with $E = \mathbb{C}/(\mathbb{Z}+\tau_1\mathbb{Z})$ and $j(E) = j(i\sqrt{2}) = 8000$. The fact that the two CM elliptic curves are equal and satisfy $j(E) = 8000$ can be obtained also by specialization from the Shioda–Inose structure of the family (see Remark 4.1).

Table 9. The elliptic fibrations of Y_2.

L_{root}	L/L_{root}			Fibers	R	Tor.
E_8^3	(0)					
	#1$(11-f)$	$A_1 \subset E_8$	$D_5 \subset E_8$	$E_7A_3E_8$	0	(0)
	#2$(13-h)$	$A_1 \oplus D_5 \subset E_8$		$A_1E_8E_8$	1	(0)
E_8D_{16}	$\mathbb{Z}/2\mathbb{Z}$					
	#3$(30-\phi)$	$A_1 \subset E_8$	$D_5 \subset D_{16}$	E_7D_{11}	0	(0)
	#4$(16-o)$	$A_1 \oplus D_5 \subset E_8$		A_1D_{16}	1	$\mathbb{Z}/2\mathbb{Z}$
	#5$(17-q)$	$D_5 \subset E_8$	$A_1 \subset D_{16}$	$A_3A_1D_{14}$	0	$\mathbb{Z}/2\mathbb{Z}$
	#6$(25-\delta)$	$A_1 \oplus D_5 \subset D_{16}$		$E_8A_1D_9$	0	(0)
$E_7^2D_{10}$	$(\mathbb{Z}/2\mathbb{Z})^2$					
	#7$(29-\beta)$	$A_1 \subset E_7$	$D_5 \subset D_{10}$	$E_7D_6D_5$	0	$\mathbb{Z}/2\mathbb{Z}$
	#8$(9-r)$	$A_1 \subset E_7$	$D_5 \subset E_7$	$D_6A_1D_{10}$	1	$\mathbb{Z}/2\mathbb{Z}$
	$(24-\psi)$	$A_1 \oplus D_5 \subset E_7$		E_7D_{10}	1	$\mathbb{Z}/2\mathbb{Z}$
	#9$(12-g)$	$A_1 \oplus D_5 \subset D_{10}$		$E_7E_7A_1A_3$	0	$\mathbb{Z}/2\mathbb{Z}$
	#10$(10-e)$	$D_5 \subset E_7$	$A_1 \subset D_{10}$	$A_1A_1D_8E_7$	1	$\mathbb{Z}/2\mathbb{Z}$
E_7A_{17}	$\mathbb{Z}/6\mathbb{Z}$					
	$(21-c)$	$A_1 \oplus D_5 \subset E_7$		A_{17}	1	$\mathbb{Z}/3\mathbb{Z}$
	#11$(19-n)$	$D_5 \subset E_7$	$A_1 \subset A_{17}$	A_1A_{15}	2	(0)
D_{24}	$\mathbb{Z}/2\mathbb{Z}$					
	#12$(23-i)$	$A_1 \oplus D_5 \subset D_{24}$		A_1D_{17}	0	(0)
D_{12}^2	$(\mathbb{Z}/2\mathbb{Z})^2$					
	#13$(26-\pi)$	$A_1 \subset D_{12}$	$D_5 \subset D_{12}$	$A_1D_{10}D_7$	0	$\mathbb{Z}/2\mathbb{Z}$
	#14$(22-u)$	$A_1 \oplus D_5 \subset D_{12}$		$A_1D_5D_{12}$	0	$\mathbb{Z}/2\mathbb{Z}$
D_8^3	$(\mathbb{Z}/2\mathbb{Z})^3$					
	#15$(6-p)$	$A_1 \subset D_8$	$D_5 \subset D_8$	$A_1D_6A_3D_8$	0	$(\mathbb{Z}/2)^2$
	#16$(14-t)$	$A_1 \oplus D_5 \subset D_8$		$A_1D_8D_8$	1	$\mathbb{Z}/2\mathbb{Z}$
D_9A_{15}	$\mathbb{Z}/8\mathbb{Z}$					
	#17$(18-m)$	$A_1 \oplus D_5 \subset D_9$		$A_1A_1A_1A_{15}$	0	$\mathbb{Z}/4\mathbb{Z}$
	#18$(28-\alpha)$	$D_5 \subset D_9$	$A_1 \subset A_{15}$	D_4A_{13}	1	(0)
E_6^4	$(\mathbb{Z}/3\mathbb{Z})^2$					
	#19$(8-b)$	$A_1 \subset E_6$	$D_5 \subset E_6$	$A_5E_6E_6$	1	$\mathbb{Z}/3\mathbb{Z}$
$A_{11}E_6D_7$	$\mathbb{Z}/12\mathbb{Z}$					
	#20$(7-w)$	$A_1 \subset E_6$	$D_5 \subset D_7$	$A_5A_1A_1A_{11}$	0	$\mathbb{Z}/6\mathbb{Z}$
	#21$(27-\mu)$	$A_1 \subset A_{11}$	$D_5 \subset D_7$	$A_9A_1A_1E_6$	1	(0)
	$(20-j)$	$A_1 \oplus D_5 \subset D_7$		$A_{11}E_6A_1$	0	$\mathbb{Z}/3\mathbb{Z}$
	#22$(15-l)$	$A_1 \subset A_{11}$	$D_5 \subset E_6$	A_9D_7	2	(0)
	#23$(2-k)$	$D_5 \subset E_6$	$A_1 \subset D_7$	$A_{11}A_1D_5$	1	$\mathbb{Z}/4\mathbb{Z}$
D_6^4	$(\mathbb{Z}/2\mathbb{Z})^4$					
	#24$(5-d)$	$A_1 \subset D_6$	$D_5 \subset D_6$	$A_1D_4D_6D_6$	1	$(\mathbb{Z}/2)^2$
$D_6A_9^2$	$\mathbb{Z}/2 \times \mathbb{Z}/10$					
	#25$(3-v)$	$D_5 \subset D_6$	$A_1 \subset A_9$	A_7A_9	2	(0)
$D_5^2A_7^2$	$\mathbb{Z}/4 \times \mathbb{Z}/8$					
	#26$(1-s)$	$D_5 \subset D_5$	$A_1 \subset D_5$	$A_1A_3A_7A_7$	0	$\mathbb{Z}/8\mathbb{Z}$
	#27$(4-a)$	$D_5 \subset D_5$	$A_1 \subset A_7$	$D_5A_5A_7$	1	(0)

The elliptic curve E can be also put in the Legendre form:

$$E \quad y^2 = x(x-1)(x-l),$$

l satisfying the equation $j = 8000 = 256(1-l+l^2)^3/l^2(l-1)^2$. Thus $l = 3 \pm 2\sqrt{2}$ or $l = -2 \pm 2\sqrt{2}$ or $l = (1 \pm \sqrt{2})/2$.

Proposition 6.1. *The elliptic fibrations* $(24-\psi)(T)$ *and* $\#10(10-e)(T)$ *are elliptic fibrations of* K_2.

Proof. It follows from the 4.2 fibration F_8 that the fibration $\#10(10-e)(T)$ with Weierstrass equation

$$Y^2 = X^3 - 2U^2(U-1)X^2 + U^3(U+1)^2(U-4)X,$$

singular fibers $III^*(0)$, $I_2^*(\infty)$, $I_4(-1)$, $I_2(4)$, $I_1(-1/2)$, and $\mathbb{Z}/2\mathbb{Z}$-torsion is an elliptic fibration of K_2. Similarly from the 4.2 fibration G_8, we deduce that the elliptic fibration $(24-\psi)(T)$ with Weierstrass equation

$$Y^2 = X^3 + 2\left(t+5t^2\right)X^2 + t^2(4t+1)\left(t^2+6t+1\right)X,$$

singular fibers $III^*(\infty)$, $I_3^*(0)$, $3I_2(-1/4, t^2+6t+1)$ and $\mathbb{Z}/2\mathbb{Z}$-torsion is an elliptic fibration of K_2. □

To achieve the proof of Theorem 1.2 we need also the following lemma.

Lemma 6.1. *The Kummer surface* K_2 *has exactly* 4 *extremal elliptic fibrations given by Shimada–Zhang* [**35**] *with the type of their singular fibers and their torsion group*

(1) $E_7\ A_7\ A_3\ A_1$, $\mathbb{Z}/2\mathbb{Z}$,

(2) $D_9\ A_7\ A_1\ A_1$, $\mathbb{Z}/2\mathbb{Z}$,

(3) $D_6\ D_5\ A_7$, $\mathbb{Z}/2\mathbb{Z}$,

(4) $A_7\ A_3\ A_3\ A_3\ A_1\ A_1$, $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$.

From Lemma 6.1 (2) we obtain the fibration $\#5(17-q)(T)$ and from Lemma 6.1 (3) the fibration $\#15(6-p)(T)$.

We notice also that fibrations $\#17(18-m)(T)$ and $\#26(1-s)(T)$ obtained by specialization are also fibration (4) of Lemma 6.1 and fibration $\#23(2-k)(T)$, by a 2-neighbor process of parameter $m = X/(k^2(k^2+4))$, gives fibration (3) of Lemma 6.1.

Finally, by a 2-neighbor process of parameter $m = X/(d^2(d+1))$, fibrations $\#24(5-d)(T)$ b) and c) give fibration $\#16(14-t)(T)$, hence are elliptic fibrations of K_2.

Corollary 6.1. *As a byproduct of the proof we get Weierstrass equations for extremal fibrations of Lemma* 6.1 (2), (3), (4).

All these previous results are listed in Table 10. Other Van-Geemen–Sarti involutions are in the Table 11, where we can see easily on the equations the self-involutions and isogenies which exchange two elliptic fibrations of Y_2.

7. 2-isogenies and isometries.

Theorem 1.2, where the 2-isogenous $K3$-surfaces of Y_2 are either its Kummer surface K_2 or Y_2 itself, cannot be generalized *in extenso* to all the other singular $K3$-surfaces

Table 10. Morrison–Nikulin involutions of Y_2 (fibrations of K_2).

No	Weierstrass Equation	From or to
$\#4(16-o)$	$y^2 = x^3 + (o^3 - 5o^2 + 2)x^2 + x$ $I_{12}^*(\infty), I_2(0), 4I_1(1, 5, o^2 - 4o - 4)$ $Y^2 = X(X - o^3 + 5o^2)(X - o^3 + 5o^2 - 4)$ $I_6^*(\infty), I_4(0), 4I_2(1, 5, o^2 - 4o - 4)$	Spec.$\#4(T)$
$\#5(17-q)$	$y^2 = x^3 + (q^3 + q^2 + 2q - 2)x^2 + (1 - 2q)x$ $I_{10}^*(\infty), I_4(0), I_2(\frac{1}{2}), 2I_1(q^2 + 2q + 5)$ $Y^2 = X^3 - 2(q^3 + q^2 + 2q - 2)X^2 + Xq^4(q^2 + 2q + 5)$ $I_5^*(\infty), I_8(0), 2I_2(q^2 + 2q + 5), I_1(\frac{1}{2})$	Lemma 6.1(2)
$\#8(9-r)$	$y^2 = x^3 - r(r^2 - r + 2)x^2 + r^3x$ $I_6^*(\infty), I_2^*(0), I_2(1), 2I_1(\pm 2i)$ $Y^2 = X^3 + 2r(r^2 - r + 2)X^2 + (r - 1)^2r^2(r^2 + 4)X$ $I_3^*(\infty), I_1^*(0), I_4(1), 2I_2(\pm 2i)$	Spec.$\#8(T)$
$(24-\psi)$	$y^2 = x^3 - (\psi + 5\psi^2)x^2 - \psi^5x$ $I_6^*(\infty), III^*(0), 3I_1(-\frac{1}{4}, \psi^2 + 6\psi + 1)$ $Y^2 = X^3 + 2(5\psi^2 + \psi)X^2 + X\psi^2(4\psi + 1)(\psi^2 + 6\psi + 1)$ $III^*(\infty), I_3^*(0), 3I_2(-(\frac{1}{4}), \psi^2 + 6\psi + 1)$	Prop. 6.1
$\#10(10-e)$	$y^2 = x(x^2 - e^2(e - 1)x + e^3(2e + 1))$ $I_4^*(\infty), III^*(0), 2I_2(-1, -\frac{1}{2}), I_1(4)$ $Y^2 = X^3 + 2e^2(e - 1)X^2 + e^3(e - 4)(e + 1)^2X$ $I_2^*(\infty), III^*(0), I_4(-1), I_2(4), I_1(-\frac{1}{2})$	Prop. 6.1
$\#15(6-p)$ $T = (p, 0)$	$y^2 = x(x - p)(x - p(p + 1)^2)$ $I_4^*(\infty), I_2^*(0), I_4(-1), I_2(-2)$ $Y^2 = X^3 - p(p^2 + 2p - 1)X^2 - p^3(p + 2)X$ $I_2^*(\infty), I_1^*(0), I_8(-1), I_1(-2)$	Lemma 6.1(3)
$\#16(14-t)$	$y^2 = x^3 + t(t^2 + 4t + 1)x^2 + t^4x$ $2I_4^*(\infty, 0), I_2(-1), 2I_1(t^2 + 6t + 1 = 0)$ $Y^2 = X^3 - 2t(t^2 + 4t + 1)X^2 + t^2(t + 1)^2(t^2 + 6t + 1)X$ $2I_2^*(\infty, 0), I_4(-1), 2I_2(t^2 + 6t + 1)$	Spec.$\#16(T)$
$\#17(18-m)$	$y^2 = x(x^2 + x(\frac{1}{4}(m^2 - 4)^2 - 2) + 1)$ $I_{16}(\infty), 3I_2(0, \pm 2), 2I_1(\pm 2\sqrt{2})$ $Y^2 = X(X - \frac{1}{4}m^4 + 2m^2)(X - \frac{1}{4}m^4 + 2m^2 - 4)$ $I_8(\infty), 3I_4(0, \pm 2), 2I_2(\pm 2\sqrt{2})$	Lemma 6.1(4) Spec.$\#17(T)$
$\#23(2-k)$	$y^2 = x^3 + x^2(\frac{1}{4}k^4 - k^3 + k^2 - 2k) + k^2x$ $I_{12}(\infty), I_1^*(0), I_2(2), 3I_1(4, \pm 2i)$ $Y^2 = X^3 - (\frac{1}{2}k^4 - 2k^3 - 4k)X^2 + \frac{k^3(k-4)(k^2+4)(k-2)^2}{16}X$ $I_6(\infty), I_2^*(0), I_4(2), 3I_2(4, \pm 2i)$	$\frac{X}{k^2(k^2+4)}$ to Lemma 6.1(3)
$\#24(5-d)$ $a)\, T = (0, 0)$ $b)\, T = (d + d^2, 0)$ $c)\, T = (d^2 + d^3, 0)$	$y^2 = x(x - (d + d^2))(x - (d^3 + d^2))$ $2I_2^*(\infty, 0), I_0^*(-1), I_2(1)$ $a) : Y^2 = X^3 + 2d(d + 1)^2X^2 + d^2(d^2 - 1)^2X$ $2I_1^*(\infty, 0), I_0^*(-1), I_4(1)$ $b) : Y^2 = X^3 + 2d(d + 1)(d - 2)X^2 + d^4(d + 1)^2X$ $I_1^*(\infty), I_4^*(0), I_0^*(-1), I_1(1)$ $c) : Y^2 = X^3 - 2d(d + 1)(2d - 1)X^2 + Xd^2(d + 1)^2$	$a)$Spec.$\#24(T)$ $b) m = \frac{X}{d^2(d+1)}$ to $\#16(14-t)(T)$ $c)$ similar to $b)$
$\#26(1-s)$	$y^2 = x^3 + x^2(\frac{1}{4}(s - 1)^4 - 2s^2) + s^4x$ $2I_8(\infty, 0), I_4(1), I_2(-1), 2I_1(3 \pm 2\sqrt{2})$ $Y^2 = X(X - \frac{1}{4}(s - 1)^4 + 4s^2)(X - \frac{1}{4}(s - 1)^4)$ $I_8(1), 3I_4(0, -1, \infty), 2I_2(3 \pm 2\sqrt{2})$	Spec.$\#26(T)$ Lemma 6.1(4)

Table 11. Self and exchanging involutions of Y_2.

No	Weierstrass Equation
$\#7(29-\beta)$	$y^2 = x^3 + 2\beta^2(\beta-1)x^2 + \beta^3(\beta-1)^2x$ $I_2^*(\infty), III^*(0), I_1^*(-1)$ $Y^2 = X^3 - 4\beta^2(\beta-1)X^2 + 4\beta^3(\beta-1)^3X$ $I_1^*(\infty), III^*(0), I_2^*(1)$
$\#9(12-g)$	$y^2 = x^3 + 4g^2x^2 + g^3(g+1)^2x$ $2III^*(\infty,0), I_4(-1), I_2(1)$ $Y^2 = X^3 - 8g^2X^2 - 4g^3(g-1)^2X$ $2III^*(\infty,0), I_4(1), I_2(-1)$
$\#13(26-\pi)$	$y^2 = x^3 + x^2\pi(\pi^2-2\pi-2) + \pi^2(2\pi+1)x$ $I_6^*(\infty), I_3^*(0), I_2(-1/2), I_1(4)$ $Y^2 = X^3 - 2X^2\pi(\pi^2-2\pi-2) + \pi^5(\pi-4)X$ $I_6^*(0), I_3^*(\infty), I_2(4), I_1(-1/2)$
$\#14(22-u)$	$y^2 = x^3 + u(u^2+4u+2)x^2 + u^2x$ $I_8^*(\infty), I_1^*(0), I_2(-2), I_1(-4)$ $Y^2 = \left(X - u(u-2)^2\right)X(X-4u)$ $I_4^*(\infty), I_2^*(0), I_4(-2), I_2(-4)$
$\#15(6-p)$ $a)\, T=(0,0)$ $b)\, T=(p(p+1)^2,0)$	$y^2 = x(x-p)(x-p(p+1)^2)$ $I_4^*(\infty), I_2^*(0), I_4(-1), I_2(-2)$ $a): Y^2 = X(X+4p+p^3+4p^2)(X+p^3)$ $I_4^*(0), I_2^*(\infty), I_4(-2), I_2(-1)$ $b): Y^2 = X^3 - 2p(2p^2+4p+1)X^2 + p^2X$ $I_8^*(\infty), I_1^*(0), I_2(-1), I_1(-2)$
$\#20(7-w)$	$y^2 = x^3 - (2-w^2-\frac{1}{4}w^4)x^2 - (w^2-1)x$ $I_{12}(\infty), I_6(0), 2I_2(\pm1), 2I_1(\pm 2i\sqrt{2})$ $Y^2 = X^3 + 2(2-w^2-\frac{1}{4}w^4)X^2 + \frac{1}{16}w^6(w^2+8)X$ $I_{12}(0), I_6(\infty), 2I_2(\pm 2i\sqrt{2}), 2I_1(\pm1)$

of the Apéry–Fermi family. The first reason is the difficulty of exhibiting all the elliptic fibrations of singular $K3$-surfaces Y_k. For example, considering the $K3$-surface Y_{10}, these elliptic fibrations are given by all the primitive embeddings into Niemeier lattices of the lattice $A_1 \oplus A_2 \oplus N$ where N is not a root lattice. The fact that A_1 (resp. A_2) embeds primitively into all the A_n, (resp. A_n, $n \geq 2$) leads to considerably many more elliptic fibrations on Y_{10} than on Y_2. The fact that N is not a root lattice requires new accurate techniques we shall explain in a forthcoming paper. Another reason is the relation with a Theorem of Boissière, Sarti and Veniani [**7**], telling when p-isogenies (p prime) between complex projective $K3$-surfaces X and Y define isometries between their rational transcendental lattices $T(X)_{\mathbb{Q}}$ and $T(Y)_{\mathbb{Q}}$ (these lattices are isometric if there exists $M \in \mathrm{Gl}(n,\mathbb{Q})$ satisfying $T(X)_{\mathbb{Q}} = M^tT(Y)_{\mathbb{Q}}M$). Let us recall the part of their Theorem related to 2-isogenies.

THEOREM 7.1 ([**7**]). *Let $\gamma: X \to Y$ be a 2-isogeny between complex projective $K3$-surfaces X and Y. Then $rk(T(Y)_{\mathbb{Q}}) = rk(T(X)_{\mathbb{Q}}) =: r$.*

1. *If r is odd, there is no isometry between $T(Y)_{\mathbb{Q}}$ and $T(X)_{\mathbb{Q}}$.*

2. *If r is even, there exists an isometry between $T(Y)_{\mathbb{Q}}$ and $T(X)_{\mathbb{Q}}$ if and only if $T(Y)_{\mathbb{Q}}$ is isometric to $T(Y)_{\mathbb{Q}}(2)$. This property is equivalent to the following: for every*

prime number q congruent to 3 or 5 modulo 8, the q-adic valuation $\nu_q(\det T(Y))$ is even.

As a corollary we deduce the following result.

THEOREM 7.2. *Among the singular $K3$-surfaces of the Apéry–Fermi family defined for k rational integer, only Y_2 and Y_{10} possess symplectic automorphisms of order 2 that are "self 2-isogenies".*

PROOF. The singular $K3$-surfaces of the Apéry–Fermi family defined for k rational integer are

$$Y_0, \quad Y_2, \quad Y_3, \quad Y_6, \quad Y_{10}, \quad Y_{18}, \quad Y_{102}, \quad Y_{198}.$$

This list has been computed numerically by Boyd [**9**]. Using the notation [**35**], that is writing the transcendental lattice $T(Y) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ as $T(Y) = [a\ b\ c]$ we get:

$$T(Y_0) = [4 \quad 2 \quad 4] \qquad T(Y_2) = [2 \quad 0 \quad 4] \qquad T(Y_6) = [2 \quad 0 \quad 12].$$

They are obtained by specialization of fibration #20 for $k = 0$, 2 and 6. For $k = 0$ the elliptic fibration has rank 0 and singular fibers of type I_{12}, I_4, $2I_3$, $2I_1$. For $k = 2$, the transcendental lattice is already known. For $k = 6$, the elliptic fibration has rank 0 and type of singular fibers I_{12}, $2I_3$, $3I_2$. Now using Shimada–Zhang table [**35**], we derive the previous announced transcendental lattices.

The transcendental lattices $T(Y_3)$ and $T(Y_{18})$ were computed in the paper [**4**]. With the method used there, we can compute the transcendental lattices of Y_{10}, Y_{102} and Y_{198}. We obtain:

$$T(Y_3) = [2 \quad 1 \quad 8] \qquad T(Y_{10}) = [6 \quad 0 \quad 12] \qquad T(Y_{18}) = [10 \quad 0 \quad 12]$$
$$T(Y_{102}) = [12 \quad 0 \quad 26] \qquad T(Y_{198}) = [12 \quad 0 \quad 34].$$

Applying Bessière, Sarti and Veniani's Theorem, we conclude that only Y_2 and Y_{10} may have self isogenies. By Theorem 1.2, Y_2 has self isogenies. We shall prove that Y_{10} satisfies the same property.

Consider the following elliptic fibration of rank 0 of Y_{10} (one of the elliptic fibrations of Y_{10} obtained in a forthcoming paper):

$$y^2 = x^3 + x^2\left(9(t+5)(t+3) + (t+9)^2\right) - xt^3(t+5)^2$$

with singular fibers $III^*(\infty)$, $I_6(0)$, $I_4(-5)$, $I_3(-9)$, $I_2(-4)$ and 2-torsion. Its 2-isogenous curve has a Weierstrass equation

$$Y^2 = X^3 + X^2\left(-20u^2 - 180u - 432\right) + 4X(u+4)^2(u+9)^3$$

with singular fibers $III^*(\infty)$, $I_6(-9)$, $I_4(-4)$, $I_3(0)$, $I_2(-5)$, rank 0 and 2-torsion. Hence this 2-isogeny defines an automorphism of order 2 of Y_{10} given by $t + u = 9$, $x = -X/2$, $y = iY/2\sqrt{2}$. □

Moreover we observe that

$$T(Y_2) = [2 \quad 0 \quad 4], \qquad T(Y_2)_{\mathbb{Q}} = [2 \quad 0 \quad 1],$$
$$T(K_2) = [4 \quad 0 \quad 8], \qquad T(K_2)_{\mathbb{Q}} = [2 \quad 0 \quad 1],$$

Similarly

$$T(Y_{10})_{\mathbb{Q}} = [6 \quad 0 \quad 3], \qquad T(K_{10})_{\mathbb{Q}} = [3 \quad 0 \quad 6].$$

Hence we suspect some relations between the rational transcendental lattices of K_i and of S_i for singular Y_i. We give some examples of such relations in the following proposition.

Proposition 7.1. *Even if the 2-isogenies from Y_0, Y_6 are not isometries, the following rational transcendental lattices satisfy the relations*

1. $T(K_0)_{\mathbb{Q}} = T(S_0)_{\mathbb{Q}}$,
2. $T(K_6)_{\mathbb{Q}} = T(S_6)_{\mathbb{Q}}$.

Moreover the $K3$-surfaces S_3 and K_3 are the same surface.

Proof. 1. For $k = 0$ we get two elliptic fibrations of rank 0, namely #20 and #8. The fibration #8(T) gives a rank 0 elliptic fibration of K_0 with Weierstrass equation

$$y^2 = x^3 + 2x^2\left(t^3+1\right) + x(t-1)^2\left(t^2+t+1\right)^2,$$

type of singular fibers D_7, $3A_3$, A_2, 4-torsion and $T(K_0) = [8\ 4\ 8]$, using Shimada–Zhang's list [**35**]. On the other end the fibration #20(T) gives a rank 0 elliptic fibration of S_0

$$y^2 = x\left(x - \frac{1}{4}(t-3i)(t+i)^3\right)\left(x - \frac{1}{4}(t+3i)(t-i)^3\right) \qquad (i^2=-1)$$

with type of singular fibers $3A_5$ $(\infty, \pm i)$, $3A_1$ $(0, \pm 3i)$, torsion group $\mathbb{Z}/2 \times \mathbb{Z}/6$, 3-torsion points being $((1/4)(t^2+1)^2, \pm(1/2)(t^2+1)^2)$. Hence, by Shimada–Zhang's list [**35**], $T(S_0) = [2\ 0\ 6]$. Now we can easily deduce the relation

$$\begin{pmatrix} 1/2 & 0 \\ 1/2 & -1 \end{pmatrix}\begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}.$$

2. For $k = 6$ the elliptic fibration #20 has rank 0 and #20(T) gives a rank 0 elliptic fibration of S_6:

$$y^2 = x^3 + x^2\left(-\frac{t^4}{2} + 6t^3 - 21t^2 + 18t + \frac{3}{2}\right) + x\frac{(t-3)^2}{16}\left(t^2-6t+1\right)^3,$$

with singular fibers $2I_6(t^2-6t+1)$, $I_6(\infty)$, $I_4(3)$, $2I_1(0,6)$, and torsion group $\mathbb{Z}/6\mathbb{Z}$. Using Shimada–Zhang's list [**35**], we find $T(S_6) = [4\ 0\ 6]$. Since

$$T(K_6) = \begin{pmatrix} 4 & 0 \\ 0 & 24 \end{pmatrix} \underset{\mathbb{Q}}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

and

$$T(S_6) = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \underset{\mathbb{Q}}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

we get straightforward

$$T(K_6)_{\mathbb{Q}} = T(S_6)_{\mathbb{Q}}.$$

Finally we prove that S_3 and K_3 are the same surface. Consider the elliptic fibration #20 of Y_3 with Weierstrass equation

$$y^2 = x^3 + \frac{1}{4}\left(t^4 - 6t^3 + 15t^2 - 18t - 3\right)x^2 - t(t-3)x,$$

singular fibers $I_{12}(\infty)$, $2I_3(t^2-3t+1)$, $2I_2(0,3)$, $2I_1(t^2-3t+9)$, rank 1 and 6-torsion. The infinite section $P_3 = (t, -(1/2)t(t^2-3t+3))$, of height $5/4$ generates the free part of the Mordell–Weil group, since $\det(T(Y_3)) = 15$ by the previous theorem and by the Shioda–Tate formula

$$\det(T(Y_3)) = \frac{5}{4}\frac{12 \times 3^2 \times 2^2}{6^2} = 15.$$

Its 2-isogenous curve has Weierstrass equation

$$y^2 = x^3 + \left(-\frac{1}{2}t^4 + 3t^3 - \frac{15}{2}t^2 + 9t + \frac{3}{2}\right)x^2 + \frac{1}{16}\left(t^2 - 3t + 9\right)\left(t^2 - 3t + 1\right)x,$$

singular fibers $3I_6(\infty, t^2-3t+1)$, $2I_2(t^2-3t+9)$, $2I_1(3,0)$, rank 1 and 6-torsion. The section Q_3 image by the 2-isogeny of the infinite section P_3 is an infinite section of height $5/2$. Since neither Q_3 nor $Q_3 + (0,0)$ are 2-divisible, the section Q_3 generates the free part of the Mordell–Weil group. Hence by the Shioda–Tate formula, it follows

$$\det(T(S_3)) = \frac{5}{2}\frac{6^3 \times 2^2}{6^2} = 60 = \det(T(K_3)).$$

We can show that K_3 and S_3 are the same surface. To prove this property we show that a genus one fibration is indeed an elliptic fibration. We start with the fibration of K_3 obtained from #26(T) and parameter $m = y/t(x + (1/4)(t-s)^2(ts-1)^2)$. If $k = 3$ and $s = s_3 := (3+\sqrt{5})/2$ we get E_m. Then changing $X = s_3^2 x$ and $Y = s_3^3 y$ it follows

$$y^2 - 3myx = x\left(x - m^2\right)\left(x - \frac{1}{8}\left(\left(112 - 48\sqrt{5}\right)m^4 - 16m^2 + 7 + 3\sqrt{5}\right)\right).$$

The next fibration is obtained with the parameter $n = x/m^2$. Now if $w = y/m^2$ it gives the following quartic in w and m

$$w^2 - 3mnw + 2\left(3\sqrt{5}-7\right)n(n-1)m^4 - n(n-1)(n-2)m^2 - \frac{1}{9}\left(3\sqrt{5}+7\right)n(n-1).$$

Notice the point $(w = -(1/4)(7+3\sqrt{5})n(n+1)(n-1), m = (1/4)(2+\sqrt{5})(2n-1+\sqrt{5}))$ on this quartic, so it is an elliptic fibration of K_3 which is #15(T). □

Remark 7.1. The Kummer surface K_0 is nothing else than the Schur quartic [**7**] (Section 6.3) with equation

$$x^4 - xy^3 = z^4 - zt^3.$$

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