J. Math. Soc. Japan
Vol. 72, No. 1 (2020) pp. 213–249
doi: 10.2969/jmsj/80238023

Generalization of Schläfli formula to the volume of a spherically faced simplex

By Kazuhiko Aomoto and Yoshinori Machida

(Received Apr. 2, 2018)
(Revised Sep. 24, 2018)

Abstract. The purpose of this paper is to present a variational formula of Schläfli type for the volume of a spherically faced simplex in the Euclidean space. It is described in terms of Cayley–Menger determinants and their differentials involved with hypersphere arrangements. We derive it as a limit of fundamental identities for hypergeometric integrals associated with hypersphere arrangements obtained by the authors in the preceding article.

1. Introduction.

In Milnor's expository article in [**16**], the following formula is stated:

For a geodesic polyhedron P^n in the n-dimensional, spherical, Euclidean or Lobachevsky space of constant curvature K, the differential of the volume $V_n(P^n)$ of P^n is expressed as

$$KdV_n(P^n) = \frac{1}{n-1}\sum_F V_{n-2}(F)\,d\theta_F,$$

where the right hand side is to be summed over all $(n-2)$-dimensional faces, θ_F is the dihedral angle between the $(n-1)$-dimensional faces which meet at F, and $V_k(\cdot)$ stands for k-dimensional volume.

This classical formula originates in Schläfli's work since the mid-19th century (see [**23**], [**24**]). It is an interesting problem to extend its differential equality to more general (not necessarily geodesic) figures in the space of constant curvature. Hypergeometric integrals are intimately related to this problem. The first author has shown (see [**3**]) that the volume formula for a pseudo-simplex with spherical faces in the $(n+1)$-dimensional fundamental unit hypersphere can be deduced by limit procedure from a differential equality satisfied by hypergeometric integrals associated with the corresponding arrangement of n-dimensional hyperspheres. The classical Schläfli formula is its special case.

In this article, we give a new variational formula for the volume of a pseudo-simplex with spherical faces in the Euclidean space. See Theorems 1 and 2. To derive it, we apply the variational formula obtained in [**7**] which is involved in hypergeometric integrals associated with hypersphere arrangements. This procedure can be done by regularization of

2010 *Mathematics Subject Classification.* Primary 33C70; Secondary 14F40, 14H70.
Key Words and Phrases. hypergeometric integral, hypersphere arrangement, twisted rational de Rham cohomology, Cayley–Menger determinant, contiguity relation, Gauss–Manin connection, Schläfli formula.

integrals (the method of generalized functions), i.e., by taking the zero limit of exponents for hypergeometric integrals (see [**10**]). A hypersphere arrangement in the n-dimensional Euclidean space can be realized by the stereographic projection as the restriction to the fundamental unit hypersphere of a hyperplane arrangement in the $(n+1)$-dimensional Euclidean space. The theory of hypergeometric integrals associated with hypersphere arrangements has been developed in this framework in terms of twisted rational de Rham cohomology (see [**5**], [**7**]). It is described in terms of Cayley–Menger determinants.

In Theorem 20, we make a correction to some errors in the variation volume formula in [**3**] which is an extension of the Schläfli formula (see (52) in this article) of a geodesic simplex in the unit hypersphere (refer to [**2**], [**3**], [**12**], [**13**], [**14**], [**21**], [**23**], [**24**], [**27**]; also refer to [**9**], [**21**], [**22**] related to the Bellows conjecture).

Acknowledgements. The authors would like to appreciate a careful reading and several valuable suggestions by the referee. According to these suggestions, Theorems 1 and 2 have been made more precise and clearer.

They also would like to appreciate a useful suggestion due to Masahiko Ito for drawing the figures.

2. Main theorems.

Let $\mathcal{A} = \{S_1, \ldots, S_{n+1}\}$ be an arrangement of $(n-1)$-dimensional hyperspheres in $\mathbf{R}^n$, where S_j has center O_j and radius r_j. Let $\mathcal{N}$ be the set of all non-empty subsets of $N := \{1, \ldots, n+1\}$. Each ordered sequence $J = \{j_1, \ldots, j_p\}$ with $j_\nu \in N$ $(1 \le \nu \le p)$ defines a subset of N. By abuse of terminology, we may also say J belongs to $\mathcal{N}$ and write $J \in \mathcal{N}$ without any confusion. For each $J = \{j_1, \ldots, j_p\} \in \mathcal{N}$, we define the Cayley–Menger determinants by

$$B(0\,J) = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & \rho^2_{j_1 j_1} & \cdots & \rho^2_{j_1 j_p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \rho^2_{j_p j_1} & \cdots & \rho^2_{j_p j_p} \end{vmatrix}, \quad B(0 \star J) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & r^2_{j_1} & \cdots & r^2_{j_p} \\ 1 & r_{j_1^2} & \rho^2_{j_1 j_1} & \cdots & \rho^2_{j_1 j_p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r^2_{j_p} & \rho^2_{j_p j_1} & \cdots & \rho^2_{j_p j_p} \end{vmatrix},$$

where ρ_{jk} is the distance between O_j and O_k. Their values are independent of the ordering of $j_1, \ldots, j_p$, and hence depend only on the unordered set J.

Suppose $\mathcal{A}$ satisfies the condition

$$(\mathcal{H}1) \quad (-1)^{|J|} B(0\,J) > 0, \quad (-1)^{|J|+1} B(0 \star J) > 0 \quad (J \in \mathcal{N}),$$

where $|J|$ is the cardinality of J. Refer to Section 5, Examples, Figure 1. Under $(\mathcal{H}1)$, the following facts are known.

1. The complement to the hypersphere arrangement $\mathcal{A}$,

$$X := \mathbf{R}^n \Big\backslash \bigcup_{j \in N} S_j$$

has $2^{n+1} - 1$ bounded components and a unique unbounded one. (This property is a consequence from the following fact : $\mathcal{A}$ is the image of the standard stereographic projection from the intersection of a central $(n+1)$-dimensional hyperplane arrangement $\hat{\mathcal{A}} = \bigcup_{j \in N} \hat{H}_j$ with the fundamental unit hypersphere $\hat{S}_0$ in $\mathbf{R}^{n+1}$ such that the center of $\hat{\mathcal{A}}$, i.e., the common point $\bigcap_{j \in N} \hat{H}_j$ lies in the inside of $\hat{S}_0$. See the Introduction in [**7**] for more details.)

2. Let $\mathcal{N}_0$ be the set of all non-empty proper subsets of N. For each $J \in \mathcal{N}_0$, the intersection

$$S_J := \bigcap_{j \in J} S_j$$

is an $(n - |J|)$-dimensional sphere, where a 0-dimensional sphere is a set of two points. S_N denotes the empty set by convention.

3. For each $J \in \mathcal{N}$, denote by D_J^- and D_J^+ the intersections $\bigcap_{j \in J} D_j^-$ and $\bigcap_{j \in J} D_j^+$, where D_j^- denotes the closure $\{f_j \leq 0\}$ of the inside of S_j which is a closed ball in $\mathbf{R}^n$ and D_j^+ the closure $\{f_j \geq 0\}$ of the exterior part of S_j respectively.

Let D be the closure of a bounded component of X. Then there exists a $J \in \mathcal{N}$ such that D can be represented as

$$D = D_J^- \cap D_{J^c}^+,$$

where J^c denotes the complement of J in N. Remark that in the case $J = N$, $D = D_N^-$ because of $J^c = \emptyset$.

For each $K \in \mathcal{N}_0$, the intersection

$$DS_K := S_K \cap D$$

is a non-empty connected subset of the boundary ∂D, called an $(n - |K|)$-dimensional face of D which is homeomorphic to an $(n - |K|)$-dimensional cell. If $|K| = n$, then DS_K consists of a single point, called a vertex of D.

4. Up to isometry, we can take a Euclidean coordinate system $(x_1, \ldots, x_n)$ on $\mathbf{R}^n$ such that

(a) O_{n+1} is at the origin $(x_1, \ldots, x_n) = (0, \ldots, 0)$,

(b) the x_{n-j+1}-coordinate of O_j is negative for every $j = 1, \ldots, n$,

(c) there exist constants $c_1, \ldots, c_{j-1}$ such that $S_{n-j+2,\ldots,n+1}$ is the intersection of S_{n+1} with $(n - j + 1)$-dimensional plane $x_1 = c_1, \ldots, x_{j-1} = c_{j-1}$ for every $j = 2, \ldots, n$ (see Section 2 for more details).

We consider an infinitesimal deformation of the arrangement $\mathcal{A}$. To describe them, we put

$$\theta_j = -\frac{1}{2}\, d \log r_j^2,\ J = \{j\}; \quad \theta_{jk} = \frac{1}{2}\, d \log \rho_{jk}^2,\ J = \{j, k\}.$$

To define θ_J for $|J| \geq 3$, we introduce yet another Cayley–Menger determinant: for an ordered subset $(j_1, \ldots, j_p) \subset N$,

$$B\begin{pmatrix} 0 & \star & j_2 & \cdots & j_p \\ 0 & j_1 & j_2 & \cdots & j_p \end{pmatrix} = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & r_{j_1}^2 & r_{j_2}^2 & \cdots & r_{j_p}^2 \\ 1 & \rho_{j_2 j_1}^2 & \rho_{j_2 j_2}^2 & \cdots & \rho_{j_2 j_p}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_{j_p j_1}^2 & \rho_{j_p j_2}^2 & \cdots & \rho_{j_p j_p}^2 \end{vmatrix}.$$

It does not depend on the ordering of $j_2, \ldots, j_p$.

For each $J = \{j_1, \ldots, j_p\} \in \mathcal{N}$ with $p \geq 3$, let

$$\theta_J = \frac{(-1)^p}{2} \sum_{(k_1, \ldots, k_p)} \prod_{q=3}^{p} \frac{B\begin{pmatrix} 0 & \star & k_{q-1} & \ldots & k_1 \\ 0 & k_q & k_{q-1} & \ldots & k_1 \end{pmatrix}}{B(0\, k_q\, \ldots\, k_1)} \cdot d \log \rho_{k_1 k_2}^2,$$

where the ordered p-tuple $(k_1, \ldots, k_p)$ runs over all permutations of $j_1, \ldots, j_p$ such that $k_1 < k_2$.

We denote by v_J the $(n - |J|)$-dimensional spherical volume of the face DS_J. The volume of a vertex is 1 by convention.

An exterior differential $d_{\mathcal{A}}$ denotes the total differential corresponding to the infinitesimal deformation of the arrangement $\mathcal{A}$.

Then we have the following.

Theorem 1. *Under $(\mathcal{H}1)$, the n-dimensional Euclidean volume $v(D)$ for $D = D_J^- \cap D_{J^c}^+$ is given by*

$$\text{(i)} \quad n!\, v(D) = -\sum_{K \in \mathcal{N}_0} (n - |K|)! (-1)^{|K \cap J|} \sqrt{\frac{(-1)^{|K|+1} B(0 \star K)}{2^{|K|}}}\, v_K - (-1)^{|J|} \sqrt{\frac{(-1)^{n+1} B(0\, N)}{2^n}}, \tag{1}$$

while its variational version is given by

$$\text{(ii)} \quad (n-1)!\, d_{\mathcal{A}} v(D) = -\sum_{K \in \mathcal{N}_0} (n - |K|)!\, (-1)^{|K \cap J^c|} \sqrt{\frac{(-1)^{|K|+1} B(0 \star K)}{2^{|K|}}}\, v_K\, \theta_K - (-1)^{|J^c|} \sqrt{\frac{(-1)^{n+1} B(0\, N)}{2^n}}\, \theta_N. \tag{2}$$

In particular, in the case where $J = N$, i.e., $D = D_N^-$, the above formulae (i) and (ii) reduce to the following:

$$\text{(i)} \quad n!\, v(D) = -\sum_{K \in \mathcal{N}_0} (n - |K|)! (-1)^{|K|} \sqrt{\frac{(-1)^{|K|+1} B(0 \star K)}{2^{|K|}}}\, v_K$$

$$+ (-1)^n \sqrt{\frac{(-1)^{n+1} B(0\,N)}{2^n}}, \tag{3}$$

$$\text{(ii)} \quad (n-1)!\, d_{\mathcal{A}} v(D) = - \sum_{K \in \mathcal{N}_0} (n - |K|)! \sqrt{\frac{(-1)^{|K|+1} B(0 \star K)}{2^{|K|}}}\, v_K\, \theta_K$$
$$- \sqrt{\frac{(-1)^{n+1} B(0\,N)}{2^n}}\, \theta_N. \tag{4}$$

In place of $(\mathcal{H}1)$, we can also consider the following condition

$(\mathcal{H}2)$ (i) $(-1)^{|J|} B(0\,J) > 0, \quad (-1)^{|J|+1} B(0 \star J) > 0 \quad (J \in \mathcal{N}_0)$,
(ii) $(-1)^{n+1} B(0\,N) > 0, \quad (-1)^{n+2} B(0 \star N) < 0 \quad (J = N)$,
(iii) f_j is positive on $\bigcap_{k \in \partial_j N} S_k \quad (j \in \mathcal{N})$.

Here $\partial_j N$ denotes the deletion of the element j from N.

In this case, D_N^- is empty. On the other hand, D_N^+ has two connected components $D_N'^+$ and $D_N''^+$, $D_N'^+$ is bounded, while $D_N''^+$ is unbounded. The closure of every other connected component of $\mathbf{R}^n - \bigcup_{j \in N} S_j$ can be expressed as $D = D_J^- \cap D_{J^c}^+$ $(J \in \mathcal{N}_0)$ as above. Refer to Section 5, Examples, Figure 2.

Under this circumstance, the following is valid.

THEOREM 2. *Under $(\mathcal{H}2)$, the volume $v(D)$ for $D = D_N'^+$ is given by*

$$\text{(i)} \quad n!\, v(D) = - \sum_{K \in \mathcal{N}_0} (n - |K|)! \sqrt{\frac{(-1)^{|K|+1} B(0 \star K)}{2^{|K|}}}\, v_K + \sqrt{\frac{(-1)^{n+1} B(0\,N)}{2^n}}, \tag{5}$$

while its variational version is given by

$$\text{(ii)} \quad (n-1)!\, d_{\mathcal{A}} v(D) = - \sum_{K \in \mathcal{N}_0} (n - |K|)!\, (-1)^{|K|} \sqrt{\frac{(-1)^{|K|+1} B(0 \star K)}{2^{|K|}}}\, v_K\, \theta_K$$
$$+ (-1)^{n+1} \sqrt{\frac{(-1)^{n+1} B(0\,N)}{2^n}}\, \theta_N. \tag{6}$$

3. Preliminaries.

1. Let $n+1$ real quadratic polynomials f_j of n variables $x = (x_1, \dots, x_n)$ in $\mathbf{R}^n$:

$$f_j(x) = Q(x) + \sum_{\nu=1}^{n} 2\alpha_{j\nu} x_\nu + \alpha_{j0} \quad (1 \le j \le n+1)$$

be given, where $Q(x)$ denotes the quadratic form

$$Q(x) = \sum_{\nu=1}^{n} x_\nu^2,$$

and $\alpha_{j\nu} \in \mathbf{R}$, $\alpha_{j0} \in \mathbf{R}$.

Let S_j be the $(n-1)$-dimensional hypersphere defined by $f_j(x)=0$. Denote by O_j the center of S_j, by r_j the radius of S_j and by ρ_{jk} the distance of O_j and O_k.

Then

$$r_j^2 = -\alpha_{j0} + \sum_{\nu=1}^{n} \alpha_{j\nu}^2,$$
$$\rho_{jk}^2 = \sum_{\nu=1}^{n} (\alpha_{j\nu} - \alpha_{k\nu})^2.$$

2. Let $\mathcal{A} = \bigcup_{j=1}^{n+1} S_j$ be the arrangement of hyperspheres consisting of $n+1$ hyperspheres S_j.

We define Cayley–Menger determinants associated with $\mathcal{A}$.

Definition 3. Denote by J and K the two ordered sequences of p indices $J = \{j_1,\dots,j_p\}$, $K=\{k_1,\dots,k_p\} \in \mathcal{N}$. Cayley–Menger determinants associated with $\mathcal{A}$ are given by the following system of determinants (see [**8**], [**11**], [**15**], [**25**]):

$$B\begin{pmatrix} 0 & j_1 & \dots & j_p \\ 0 & k_1 & \dots & k_p \end{pmatrix} := \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & \rho_{j_1k_1}^2 & \rho_{j_1k_2}^2 & \cdots & \rho_{j_1k_p}^2 \\ 1 & \rho_{j_2k_1}^2 & \rho_{j_2k_2}^2 & \cdots & \rho_{j_2k_p}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_{j_pk_1}^2 & \rho_{j_pk_2}^2 & \cdots & \rho_{j_pk_p}^2 \end{vmatrix},$$

$$B\begin{pmatrix} \star & j_1 & \dots & j_p \\ 0 & k_1 & \dots & k_p \end{pmatrix} := \begin{vmatrix} 1 & r_{k_1}^2 & r_{k_2}^2 & \dots & r_{k_p}^2 \\ 1 & \rho_{j_1k_1}^2 & \rho_{j_1k_2}^2 & \cdots & \rho_{j_1k_p}^2 \\ 1 & \rho_{j_2k_1}^2 & \rho_{j_2k_2}^2 & \cdots & \rho_{j_2k_p}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_{j_pk_1}^2 & \rho_{j_pk_2}^2 & \cdots & \rho_{j_pk_p}^2 \end{vmatrix},$$

$$B\begin{pmatrix} \star & j_1 & \dots & j_p \\ \star & k_1 & \dots & k_p \end{pmatrix} := \begin{vmatrix} 0 & r_{k_1}^2 & r_{k_2}^2 & \dots & r_{k_p}^2 \\ r_{j_1}^2 & \rho_{j_1k_1}^2 & \rho_{j_1k_2}^2 & \cdots & \rho_{j_1k_p}^2 \\ r_{j_2}^2 & \rho_{j_2k_1}^2 & \rho_{j_2k_2}^2 & \cdots & \rho_{j_2k_p}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{j_p}^2 & \rho_{j_pk_1}^2 & \rho_{j_pk_2}^2 & \cdots & \rho_{j_pk_p}^2 \end{vmatrix},$$

$$B\begin{pmatrix} 0 & \star & j_2 & \dots & j_p \\ 0 & k_1 & k_2 & \dots & k_p \end{pmatrix} := \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & r_{k_1}^2 & r_{k_2}^2 & \dots & r_{k_p}^2 \\ 1 & \rho_{j_2k_1}^2 & \rho_{j_2k_2}^2 & \cdots & \rho_{j_2k_p}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_{j_pk_1}^2 & \rho_{j_pk_2}^2 & \cdots & \rho_{j_pk_p}^2 \end{vmatrix},$$

$$B\begin{pmatrix} 0 & \star & j_1 & \dots & j_p \\ 0 & \star & k_1 & \dots & k_p \end{pmatrix} := \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & r_{k_1}^2 & r_{k_2}^2 & \dots & r_{k_p}^2 \\ 1 & r_{j_1}^2 & \rho_{j_1k_1}^2 & \rho_{j_1k_2}^2 & \dots & \rho_{j_1k_p}^2 \\ 1 & r_{j_2}^2 & \rho_{j_2k_1}^2 & \rho_{j_2k_2}^2 & \dots & \rho_{j_2k_p}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{j_p}^2 & \rho_{j_pk_1}^2 & \rho_{j_pk_2}^2 & \dots & \rho_{j_pk_p}^2 \end{vmatrix}.$$

These determinants will be abbreviated to $B\binom{0\ J}{0\ K}$, $B\binom{\star\ J}{0\ K}$, $B\binom{\star\ J}{\star\ K}$, $B\binom{0\ \star\ \partial_{j_1}J}{0\ k_1\ \partial_{k_1}K}$, $B\binom{0\ \star\ J}{0\ \star\ K}$ respectively. Here $\partial_{j_1}J$ denotes the deletion of the element j_1 from the sequence $J = \{j_1, \dots, j_p\}$. When $J = K$, then $B\binom{0\ J}{0\ J}$, $B\binom{\star\ J}{\star\ J}$, $B\binom{0\ \star\ J}{0\ \star\ J}$ are simply written by $B(0\,J)$, $B(\star\ J)$, $B(0 \star J)$ respectively.

For example, $B(0\,j) = -1$, $B(0\,j\,k) = 2\rho_{jk}^2$, $B(0 \star j) = 2r_j^2$ and

$$\begin{aligned} B(0 \star j\,k) &= r_j^4 + r_k^4 + \rho_{jk}^4 - 2r_j^2\,r_k^2 - 2r_j^2\,\rho_{jk}^2 - 2r_k^2\,\rho_{jk}^2, \\ B(0\,j\,k\,l) &= \rho_{jk}^4 + \rho_{jl}^4 + \rho_{kl}^4 - 2\rho_{jk}^2\,\rho_{jl}^2 - 2\rho_{jk}^2\rho_{kl}^2 - 2\rho_{jl}^2\,\rho_{kl}^2. \end{aligned}$$

3. Assume the condition $(\mathcal{H}1)$. Put $D := D_J^- \cap D_{J^c}^+$ for $J \in \mathcal{N}$:

$$D:\ f_j \le 0 \quad (j \in J), \qquad f_j \ge 0 \quad (j \in J^c).$$

It is a non-empty spherically faced n-simplex, which will be called a pseudo n-simplex in the sequel. The boundary of D consists of the $(n-p)$-dimensional faces $DS_K = D \cap S_K$, $1 \le p \le n$, where, for any $K \in \mathcal{N}$ such that $|K| = p$, the intersection $S_K = \bigcap_{j\in K} S_j$ defines an $(n-p)$-dimensional sphere.

D is a conformal image by the stereographic projection of a spherical n-cell in $\hat{S}_0$ surrounded by $n+1$ pieces of intersections with the real hyperplanes $\hat{H}_j$ $(j \in N)$ (see [**4**] Section 4 and the Introduction in [**7**] for details).

The orientation of $\mathbf{R}^n$ and D is determined such that the standard n-form ϖ is positive:

$$\varpi = dx_1 \wedge \dots \wedge dx_n > 0.$$

In particular, for $j \in N$, $\bigcap_{k\in\partial_j N} S_k$ consists of two points denoted by $\{P_j, P_j'\}$:

$$f_k = 0 \quad (k \in \partial_j N) \quad \text{at } P_j \text{ and } P_j'.$$

In the special case $J = N$ so that $D = D_N^-$, there exists the unique point P_j in ∂D_N^- such that

$$\{P_j\} = \bigcap_{k\in\partial_j N} S_k \cap \partial D_N^- \quad (\partial D_N^- \text{ denotes the boundary of } D_N^-).$$

The other point P_j' is outside D_N^-. It holds that

$$[f_j]_{P_j} < 0, \quad [f_j]_{P'_j} > 0.$$

4. By a change of coordinates via parallel displacement, we can take Euclidean coordinates $(x_1, \ldots, x_n)$ such that O_{n+1} coincides with the origin, i.e., $\alpha_{n+1,\nu} = 0$ $(1 \le \nu \le n)$. Then

$$\det^2\big((\alpha_{j\nu})_{1\le j,\nu\le n}\big) = (-1)^{n+1}\frac{B(0\,N)}{2^n} > 0,$$

the matrix $(\alpha_{j\nu})_{1\le j,\nu\le n} \in GL(n, \mathbf{R})$. Due to the Iwasawa decomposition for $GL(n, \mathbf{R})$, by orthogonal transformation we can find new Euclidean coordinates $(x_1, \ldots, x_n)$ such that the polynomials f_j have the following expressions:

$$f_j(x) = Q(x) + \sum_{\nu=1}^{n+1-j} 2\alpha_{j\nu}x_\nu + \alpha_{j0} \quad (1 \le j \le n), \tag{7}$$

$$f_{n+1}(x) = Q(x) + \alpha_{n+1\,0}, \tag{8}$$

here $\alpha_{j\,n+1-j} > 0\,(1 \le j \le n)$ and that O_j satisfies the condition:

x_{n+1-j}-coordinate of O_j is negative for every j $(1 \le j \le n)$.

We have the equalities

$$\prod_{j=p}^{n} \alpha_{j\,n+1-j} = \sqrt{\frac{(-1)^{n-p}\,B(0\,p\ldots n\,n+1)}{2^{n-p+1}}} \quad (1 \le p \le n). \tag{9}$$

Lemma 4. *For $J \in \mathcal{N}_0$ $(|J| \le n-1)$, S_J is a sphere of dimension $n - |J|$. Its radius r_J equals*

$$r_J = \sqrt{-\frac{1}{2}\,\frac{B(0 \star J)}{B(0\,J)}}.$$

By the use of coordinates x_j, $S_{n-|J|+2\ldots n+1}$ represent the spheres of dimension $n - |J|$ satisfying the following equalities:

- $S_{n+1} : Q(x) = r_{n+1}^2$,
- $S_{n-|J|+2\ldots n+1} : x_1 = c_1, \ldots, x_{|J|-1} = c_{|J|-1},\ \displaystyle\sum_{j=|J|}^{n} x_j^2 = r_{n-|J|+2\ldots n+1}^2$,

  with $c_1, \ldots, c_{|J|-1}$ being constants,
- $S_{2\ldots n+1}$: two points P_1, P'_1 such that $P_1 \in D_1^-, P'_1 \in D_1^+$.

In the same way, for any $j \in N$, $S_{\partial_j N}$ consists of two points P_j, P'_j such that $P_j \in D_j^-, P'_j \in D_j^+$.

5. Let $\Delta[O_1, \ldots, O_{n+1}]$ denote the n-dimensional linear simplex with the vertices $O_1, \ldots, O_{n+1}$. Then the following holds true.

Lemma 5. *Every $P_j\,(1 \le j \le n+1)$ lies in the same side of the simplex $\Delta[O_1,\dots,O_{n+1}]$ relative to the hyperplane including the $(n-1)$-dimensional face $\Delta[O_1,\dots,O_{j-1},O_{j+1},\dots,O_{n+1}]$.*

Proof. Since the statement is invariant under isometry, we have only to prove that the x_n-coordinate of P_1 is negative: $x_n(P_1) < 0$ with respect to the above coordinates. Note that P_1, P_1' is symmetric with respect to the reflection $x_n \to -x_n$. Since that $x_n(O_1) < 0$, $f_1(P_1) < 0$, $f_1(P_1') > 0$, we have the inequality between the distance: $\mathrm{dis}(P_1, O_1) < \mathrm{dis}(P_1', O_1)$. This means P_1 lies in the lower side of the hyperplane $x_n = 0$ including the face $\Delta[O_2,\dots,O_{n+1}]$. □

Denote by $\Delta[P_1,P_2,\dots,P_{n+1}]$ and $\widetilde{\Delta}[P_1,P_2,\dots,P_{n+1}]$ be the linear n-simplex with faces supported by linear subspaces and the pseudo n-simplex with spherical faces both with vertices P_j respectively. The latter coincides with D_N^- as a set. This pseudo n-simplex denoted by $\widetilde{\Delta}[P_1,\dots,P_{n+1}]$ is uniquely determined by the sequence $P_1,\dots,P_{n+1}$.

Indeed, $\widetilde{\Delta}[P_1,\dots,P_{n+1}]$ is the image by the stereographic projection of a spherical n-simplex $\hat{\Delta}$ in the fundamental unit hypersphere $\hat{S}_0 \subset \mathbf{R}^{n+1}$. $\hat{\Delta}$ is defined as follows. Let $\bigcup_{j\in N} \hat{H}_j$ be a real central arrangement of hyperplanes in $\mathbf{R}^{n+1}$ whose center is the intersection $\bigcap_{j\in N} \hat{H}_j$, $\hat{H}_j$ being defined by a linear function on $\mathbf{R}^{n+1}$:

$$\hat{H}_j : \hat{f}_j(\xi) = 0 \quad (\xi \in \mathbf{R}^{n+1}).$$

The assumption $(\mathcal{H}1)$ means that the center is in the inside of $\hat{S}_0$ (see [**4**] Lemma 3.1 and [**7**] Lemma 3). Denote by $\hat{H}_j^{\pm}$ the closed half space in $\mathbf{R}^{n+1}$ divided by $\hat{H}_j$ such that the function $\hat{f}_j$ is non-negative or non-positive on $\hat{H}_j^{\pm}$. Then $\hat{\Delta}$ is the intersection of $\hat{S}_0$ with the cone $\bigcap_{j\in N} \hat{H}_j^-$ whose summit $\bigcap_{j\in\mathcal{N}} \hat{H}_j$ lies in the inside of $\hat{S}_0$ (refer to the Introduction in [**7**]). $\hat{\Delta}$ is non-empty. Hence $\widetilde{\Delta}[P_1,\dots,P_{n+1}]$ is non-empty. Its orientation depends on the ordering of $P_1,\dots,P_{n+1}$.

By definition, the following properties are valid.

Lemma 6. (i)

$$(-1)^{n(n+1)/2+\nu-1}\, df_1 \wedge \cdots \widehat{df_\nu} \cdots \wedge df_{n+1} \quad (1 \le \nu \le n+1)$$

is positive or negative at P_ν or P_ν'.

(ii) *The pseudo n-simplex $\widetilde{\Delta}[P_1,P_2,\dots,P_{n+1}]$ has the sign of orientation $(-1)^{n(n-1)/2}$ such that*

$$\widetilde{\Delta}[P_1,P_2,\dots,P_{n+1}] = (-1)^{n(n-1)/2}\, D_N^-.$$

Proof. Indeed, we can show that

$$df_2 \wedge \cdots \wedge df_{n+1} = 2^n(-1)^{(n-1)(n-2)/2} \prod_{j=2}^{n} \alpha_{j\,n+1-j}\, x_n\, \varpi. \tag{10}$$

Seeing that $x_n < 0$ at P_1 and that $x_n > 0$ at P'_1, (i) holds true for $\nu = 1$. The same holds true for every ν by symmetry relative to isometry. Thus (i) is proved. The property (ii) follows from the following fact. The orientation of $\tilde{\Delta}[P_1, \dots, P_{n+1}]$ can then be identified with the orientation of $\hat{\Delta}$ which is opposite in sign to the ordered arrangement of signed halfspaces $\langle \hat{H}^-_{n+1}, \dots, \hat{H}^-_1 \rangle$ in $\mathbf{R}^{n+1}$. □

6. Denote by f_J the product $\prod_{j \in J} f_j$. The residues of the forms ϖ / f_J along S_J can be computed explicitly as follows.

Proposition 7. *For $J = \{j_1, \dots, j_p\} \in \mathcal{N}_0$, we have*

$$\begin{aligned} \mathrm{Res}_{S_J}\left[\frac{\varpi}{f_J}\right] &= \left[\frac{\varpi}{df_{j_1} \wedge \cdots \wedge df_{j_p}}\right]_{S_J} \\ &= \frac{(-1)^{(p-1)(p-2)/2}}{\sqrt{(-1)^{p-1} 2^p B(0 \star J)}} \varpi_J, \quad (1 \leq p \leq n) \end{aligned} \tag{11}$$

where ϖ_J denote the standard spherical volume elements on S_J respectively such that

- $$\varpi_{n+1} = \frac{\sum_{\nu=1}^{n} (-1)^{\nu-1} x_\nu dx_1 \wedge \cdots \widehat{dx_\nu} \cdots \wedge dx_n}{r_{n+1}}, \tag{12}$$
- $$\varpi_{n-p+2 \dots n+1} = \frac{\sum_{\nu=1}^{n-p+1} (-1)^{\nu-1} x_{p+\nu-1} dx_p \wedge \cdots \widehat{dx_{p+\nu-1}} \cdots \wedge dx_n}{r_{n-p+2 \dots n+1}} \quad (1 \leq p \leq n-1), \tag{13}$$
- $$\varpi_{2 \dots n+1} = \mp 1 \quad \text{at } P_1 \text{ or } P'_1, \tag{14}$$

and that

$$r_{n-p+2 \dots n+1} = \sqrt{-\frac{1}{2} \frac{B(0 \star n-p+2 \dots n+1)}{B(0\, n-p+2 \dots n+1)}}, \tag{15}$$

ϖ_J are obtained respectively from $\varpi_{n-p+2 \dots n+1}$ by permutations of elements in the set of indices N.

Proof. Because of symmetry, it is sufficient to prove (11) in the case where $J = \{n-p+2, \dots, n+1\}$.

First prove (11) in the case of (12) and (13). Since

$$\begin{aligned} & df_{n-p+2} \wedge \cdots \wedge df_{n+1} \\ &\quad = 2^p (-1)^{(p-1)(p-2)/2} \prod_{j=n-p+2}^{n} \alpha_{j\, n-j+1} \sum_{j=p}^{n} x_j dx_1 \wedge \cdots \wedge dx_{p-1} \wedge dx_j, \end{aligned} \tag{16}$$

(9) (13) and (15) imply

$$df_{n-p+2} \wedge \cdots \wedge df_{n+1} \wedge \varpi_{n-p+2 \dots n+1}$$

$$= 2^p(-1)^{(p-1)(p-2)/2} \prod_{j=n-p+2}^{n} \alpha_{j\,n-j+1} \frac{\left(\sum_{j=p}^{n} x_j^2\right)}{r_{n-p+2\,\dots\,n+1}}\, \varpi$$

$$= 2^p(-1)^{(p-1)(p-2)/2} \sqrt{\frac{(-1)^{p-1}B(0\star\; n-p+2\;\dots\; n+1)}{2^p}}\,\varpi \quad (1 \le p \le n-1). \tag{17}$$

Hence, along S_J it follows that

$$\left[\frac{\varpi}{df_{n-p+2}\wedge\cdots\wedge df_{n+1}}\right]_{S_J} = \frac{(-1)^{(p-1)(p-2)/2}}{\sqrt{(-1)^{p+1}\,2^p\,B(0\star\; n-p+2\;\dots\; n+1)}}\,\varpi_{n-p+2\,\dots\,n+1}.$$

On the other hand, when $p = n$, in view of (9), (10) and $x_n < 0$ at P_1 and $x_n > 0$ at P_1' respectively, we have the identity

$$x_n = \begin{cases} -r_{2\,\dots\,n+1} < 0 & \text{at} \quad P_1, \\ r_{2\,\dots\,n+1} > 0 & \text{at} \quad P_1'. \end{cases}$$

Hence, at P_1 and P_1',

$$\left[\frac{\varpi}{df_2\wedge\cdots\wedge df_{n+1}}\right] = \mp\frac{(-1)^{(n-1)(n-2)/2}}{\sqrt{(-1)^{n+1}\,2^n\,B(0\star 2\;\dots\; n+1)}} \tag{18}$$

respectively. □

Notation 1. For $J \in \mathcal{N}$, denote by F_J the rational n-form and by $W_0(J)\varpi$ a linear combination of F_K $(K \subset J)$ as follows:

$$F_J = \frac{\varpi}{f_J},$$

$$W_0(J)\varpi = -\sum_{\nu\in J} B\begin{pmatrix} 0 & \star & \partial_\nu J \\ 0 & \nu & \partial_\nu J\end{pmatrix} F_{\partial_\nu J} + B(0\star J)\,F_J.$$

Remark that F_J is also a linear combination of $W_0(K)\varpi\,(K\subset J,\ |K|\ge 1)$.

The following Lemma can be proved by a direct calculation (see [**7**] Lemma 12).

Lemma 8.

$$\sum_{\nu=1}^{n+1}(-1)^{\nu-1}\frac{df_1}{f_1}\wedge\cdots\widehat{\frac{df_\nu}{f_\nu}}\cdots\wedge\frac{df_{n+1}}{f_{n+1}} = \frac{2^n\,(-1)^{n(n-1)/2+1}}{\sqrt{(-1)^{n+1}\,2^n\,B(0\,N)}}\,W_0(N)\varpi. \tag{19}$$

The following proposition gives the values of f_j at the points P_j and P_j'.

Proposition 9. *The values of $1/f_j$ at P_j and P_j' are negative and positive respectively. They are evaluated as*

$$\left[\frac{1}{f_j}\right]_{P_j} = \frac{(-1)^{n+1}\sqrt{B(0\star\partial_j N)\,B(0\,N)} + B\begin{pmatrix} 0 & \star & \partial_j N \\ 0 & j & \partial_j N\end{pmatrix}}{B(0\star N)} < 0, \tag{20}$$

$$\left[\frac{1}{f_j}\right]_{P'_j} = \frac{(-1)^n\sqrt{B(0\star\partial_j N)\,B(0\,N)} + B\begin{pmatrix} 0 & \star & \partial_j N \\ 0 & j & \partial_j N\end{pmatrix}}{B(0\star N)} > 0. \tag{21}$$

Due to the product formula for resultant,

$$\left[\frac{1}{f_j}\right]_{P_j}\left[\frac{1}{f_j}\right]_{P'_j} = -\frac{B(0\,\partial_j N)}{B(0\star N)} < 0.$$

Proof. For simplicity, we may assume that $j = 1$. First notice that $f_1 \neq 0$ at P_1. By taking the residues of both sides of (19) at P_1 (see [**26**]), we have from (18)

$$\begin{aligned}
1 &= \mathrm{Res}_{P_1}\frac{df_2}{f_2}\wedge\cdots\wedge\frac{df_{n+1}}{f_{n+1}} \\
&= \frac{2^n(-1)^{n(n-1)/2+1}}{\sqrt{(-1)^{n+1}\,2^n\,B(0\,N)}}\left\{-B\begin{pmatrix} 0 & \star & \partial_1 N \\ 0 & 1 & \partial_1 N\end{pmatrix} + B(0\star N)\left[\frac{1}{f_1}\right]_{P_1}\right\}\mathrm{Res}_{P_1}\left[\frac{\varpi}{f_2\,\ldots\,f_{n+1}}\right] \\
&= \frac{(-1)^{n+1}}{\sqrt{B(0\star\partial_1 N)\;B(0\,N)}}\left\{-B\begin{pmatrix} 0 & \star & \partial_1 N \\ 0 & 1 & \partial_1 N\end{pmatrix} + B(0\star N)\left[\frac{1}{f_1}\right]_{P_1}\right\}.
\end{aligned} \tag{22}$$

We can solve the equation (22) with respect to $[1/f_1]_{P_1}$ and gets the formula (20). (21) can be deduced in a similar way. $\square$

4. Proof of main theorems.

Main theorems are a consequence from some identities proved in [**7**] concerning hypergeometric integrals defined on the n-dimensional complex affine space $\mathbf{C}^n$. The proofs are given by "regularization procedure of integrals using generalized functions" (refer to [**10**]).

Fix $J = \{j_1,\ldots,j_p\} \in \mathcal{N}$ such that $|J| = p \geq 1$. Denote $J^c = N\backslash J = \{j_{p+1},\ldots,j_{n+1}\}$. The bounded domain $D = D_J^- \cap D_{J^c}^+$ contains the vertices $P_{j_1},\ldots,P_{j_p}$ and $P'_{j_{p+1}},\ldots,P'_{j_{n+1}}$ so that D supports the pseudo n-simplex $\widetilde{\Delta}[P_{j_1},\ldots,P_{j_p}, P'_{j_{p+1}},\ldots, P'_{j_{n+1}}]$ with the vertices $P_{j_1},\ldots,P_{j_p}$, $P'_{j_{p+1}},\ldots,P'_{j_{n+1}}$. Its orientation is determined as follows.

$$\widetilde{\Delta}[P_{j_1},\ldots,P_{j_p},P'_{j_{p+1}},\ldots,P'_{j_{n+1}}] = -(-1)^{n(n+1)/2+|J|}\,\mathrm{sgn}(J\,J^c)\,D_J^-\cap D_{J^c}^+,$$

where $\mathrm{sgn}(J\,J^c)$ denotes the sign of the permutation $\{J\,J^c\} = \{j_1,\ldots,j_{n+1}\}$ of the sequence $N = \{1,2,\ldots,n+1\}$.

For example, in the case $n = 2$ (see Section 5, Example, Figure 1),

$$\begin{aligned}
&\widetilde{\Delta}[P_1,P'_2,P'_3] = -D_1^-\cap D_{23}^+, && \widetilde{\Delta}[P_1,P_2,P'_3] = D_{12}^-\cap D_3^+, \\
&\widetilde{\Delta}[P_2,P'_1,P'_3] = D_2^-\cap D_{13}^+, && \widetilde{\Delta}[P_1,P_3,P'_2] = -D_{13}^-\cap D_2^+, \\
&\widetilde{\Delta}[P_3,P'_1,P'_2] = -D_3^-\cap D_{12}^+, && \widetilde{\Delta}[P_2,P_3,P'_1] = D_{23}^-\cap D_1^+.
\end{aligned}$$

In D, in the neighborhood of DS_K, it follows that $f_k \leq 0$ for $k \in K\cap J$ and $f_k \geq 0$

for $k \in K \cap J^c$ from the definition.

Suppose that the system of exponents $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ are given such that all $\lambda_j > 0$.

Let $\Phi(x)$ be the multiplicative meromorphic function

$$\Phi(x) = \prod_{j \in N} f_j^{\lambda_j}.$$

For each $J \in \mathcal{N}$ $(1 \leq |J|)$, consider the integral of the absolute value $|\Phi(x)|$ over the domain $D = D_J^- \cap D_{J^c}^+$:

$$\mathcal{J}_\lambda(\varphi) = \int_D |\Phi(x)| \varphi \varpi,$$

where we take the branch of $\Phi(x)$ such that $\Phi(x) > 0$ for $x \in D$. There exists a twisted n-cycle $\mathfrak{z}$ such that

$$\mathcal{J}_\lambda(\varphi) = \int_{\mathfrak{z}} \Phi(x)\, \varphi\, \varpi.$$

Then the following proposition holds true (refer to [**7**]).

Proposition 10. *For each $D = D_J^- \cap D_{J^c}^+$, the following identity holds true*:

$$(2\lambda_\infty + n)\, \mathcal{J}_\lambda(1) = \sum_{K \in \mathcal{N}, K \neq \emptyset} (-1)^{|K|} \frac{\prod_{j \in K} \lambda_j}{\prod_{\nu=1}^{|K|-1} (\lambda_\infty + n - \nu)} \int_D |\Phi(x)| W_0(K)\, \varpi,$$

where the sum ranges over the family of all unordered non-empty sets $K \in \mathcal{N}$ and $\lambda_\infty = \sum_{j=1}^{n+1} \lambda_j$.

On the other hand, the variation of $\mathcal{J}_\lambda(1)$ is defined by

$$d_{\mathcal{A}} \mathcal{J}_\lambda(1) = \sum_{j=1}^{n+1} \sum_{\nu=0}^{n} d\alpha_{j\nu}\, \frac{\partial}{\partial_{\alpha_{j\nu}}} \mathcal{J}_\lambda(1).$$

We want to give an explicit variation formula for $\mathcal{J}_\lambda(1)$ with respect to the parameters r_j^2, ρ_{kl}^2. To do that, it is necessary to introduce the system of special one forms θ_J appearing in Section 2. We reproduce them here.

Definition 11. We define the following:

$$\theta_j = -\frac{1}{2}\, d \log r_j^2,$$

$$\theta_{jk} = \frac{1}{2}\, d \log\, \rho_{jk}^2,$$

$$\theta_J = (-1)^p \sum_{j,k \in J, j<k} \frac{1}{2}\, d \log B(0\, j\, k) \cdot$$

$$\sum_{\mu_1,\dots,\mu_{p-2}} \prod_{\nu=1}^{p-2} \frac{B\begin{pmatrix} 0 & \star & \mu_{\nu-1} & \dots & \mu_1 & j & k \\ 0 & \mu_\nu & \mu_{\nu-1} & \dots & \mu_1 & j & k \end{pmatrix}}{B(0\,\mu_\nu\,\mu_{\nu-1}\,\dots\,\mu_1\,j\,k)}, \quad (2 \le p \le n+1,\ |J| = p)$$

where $\mu_1, \dots, \mu_{p-2}$ ranges over the family of all ordered sequences consisting of $p-2$ different elements of $\partial_j \partial_k J$.

Then we have the following (refer to [**7**]).

Proposition 12. *For each $D = D_J^- \cap D_{J^c}^+$, we have*

$$d_{\mathcal{A}} \mathcal{J}_\lambda(1) = \sum_{K \in \mathcal{N}, K \neq \emptyset} \frac{\prod_{j \in K} \lambda_j}{\prod_{\nu=1}^{|K|-1} (\lambda_\infty + n - \nu)} \theta_K \int_D |\Phi(x)| W_0(K) \varpi.$$

Remark. Let ε_j $(j \in \mathcal{N})$ be the standard basis of the lattice $\mathbf{Z}^{n+1}$ in $\mathbf{C}^{n+1}$, the λ-space. For each $J \in \mathcal{N}$, the integral

$$\mathcal{J}_\lambda \left(\frac{1}{\prod_{j \in J} f_j} \right) = \mathcal{J}_{\lambda - \sum_{j \in J} \varepsilon_j}(1)$$

analytically depends on the parameters r_j^2, ρ_{jk}^2 and λ. Differential or difference linear relations among $\mathcal{J}_{\lambda+\eta}(1)$ $(\lambda \in \mathbf{C}^{n+1}, \eta \in \mathbf{Z}^{n+1})$ are called generally "contiguity relation" among hypergeometric functions (see [**5**], [**7**]). The identities stated in Propositions 10 and 12 are particular cases of them.

Let $v(D)$ be the volume of the domain $D = D_J^- \cap D_{J^c}^+$:

$$v(D) = \int_D \varpi.$$

Further, let $v_K = v_K(D)$ $(K \in \mathcal{N}_0)$ be the volume of the $(n - |K|)$-dimensional face DS_K of D:

$$v_K = \int_{DS_K} |\varpi_K|.$$

Theorem 1 is an immediate consequence of Propositions 10 and 12 tending $\lambda_j \to 0$ for all positive λ_j.

Proof of Theorem 1. Since both identities (1) and (2) in Theorem 1 can be proved in the same way, we only give a proof for the latter identity (2).

Proposition 12 shows

$$\begin{aligned} d_{\mathcal{A}} \mathcal{J}_\lambda(1) = &\sum_{K \in \mathcal{N}_0} \frac{\prod_{j \in K} \lambda_j}{\prod_{\nu=1}^{|K|-1} (\lambda_\infty + n - \nu)} \int_D |\Phi(x)| W_0(K) \varpi\, \theta_K \\ &+ \frac{\prod_{j \in N} \lambda_j}{\prod_{\nu=1}^{n} (\lambda_\infty + n - \nu)} \int_D |\Phi(x)|\, W_0(N) \varpi\, \theta_N. \end{aligned} \tag{23}$$

Let us take the limit for $\lambda_j = \tau$, $\tau \to 0\,(1 \le j \le n+1)$ on both sides of (23).
In the left hand side of (23), we have

$$\lim_{\tau\to 0} d_{\mathcal{A}}\mathcal{J}_\lambda(1) = d_{\mathcal{A}}v(D).$$

In the right hand side of (23), remark that

$$\lim_{\tau\downarrow 0}\prod_{k\in K}\lambda_k\int_D|\Phi(x)|\varphi(x)F_L = 0,$$

provided $L \subsetneqq K$.

In the sum in the right hand side, first, fix $j \in J$ and take $K = \{j\}$. Seeing that $f_j \le 0$ in D, due to Proposition 7, the following equalities hold by the method of generalized functions (see [**10**] Chapter III, 2):

$$\begin{aligned}\lim_{\tau\to 0}\lambda_j\int_D|\Phi(x)|\,\varphi(x)\,W_0(j)\,\varpi &= \lim_{\tau\to 0}\lambda_j\int_D|\Phi(x)|\,\varphi(x)\,B(0\star j)\,\frac{\varpi}{f_j}\\ &= -B(0\star j)\,\lim_{\tau\to 0}\lambda_j\int_D|\Phi(x)|\,\varphi(x)\frac{\varpi}{|f_j|}\\ &= -\sqrt{\frac{B(0\star j)}{2}}\int_{S_j\cap\partial D}[\varphi]_{S_j}\,|\varpi_j|.\end{aligned}\tag{24}$$

Next, fix $j \in J^c$ and take $K = \{j\}$. Seeing that $f_j \ge 0$ in D, we have similarly

$$\lim_{\tau\to 0}\lambda_j\int_D|\Phi(x)|\,\varphi(x)\,W_0(j)\,\varpi = \sqrt{\frac{B(0\star j)}{2}}\int_{S_j\cap\partial D}[\varphi]_{S_j}\,|\varpi_j|.\tag{25}$$

In the case where $K = \{j,k\} \in J$ or $K = \{j,k\} \in J^c$, $f_j \le 0$, $f_k \le 0$ or $f_j \ge 0$, $f_k \ge 0$ in D. Hence,

$$\begin{aligned}\lim_{\tau\to 0}\lambda_j\lambda_k\int_D|\Phi(x)|\,\varphi(x)\,W_0(j\,k)\,\varpi &= B(0\star j\,k)\,\lim_{\tau\to 0}\lambda_j\lambda_k\int_D|\Phi(x)|\,\varphi(x)\,\frac{\varpi}{f_jf_k}\\ &= B(0\star j\,k)\,\lim_{\tau\to 0}\lambda_j\lambda_k\int_D|\Phi(x)|\,\varphi(x)\,\frac{\varpi}{|f_jf_k|}\\ &= B(0\star j\,k)\int_D|\Phi(x)|\,\varphi(x)\left|\frac{\varpi}{df_j\wedge df_k}\right|\\ &= -\sqrt{\frac{-B(0\star j\,k)}{4}}\int_{S_j\cap S_k\cap\partial D}[\varphi]_{S_{jk}}\,|\varpi_{jk}|.\end{aligned}\tag{26}$$

On the contrary, in the case where $j \in J, k \in J^c$ or $j \in J^c, k \in J$, we have

$$\lim_{\tau\to 0}\lambda_j\lambda_k\int_D|\Phi(x)|\,\varphi(x)\,W_0(j\,k)\,\varpi = \sqrt{\frac{-B(0\star j\,k)}{4}}\int_{S_j\cap S_k\cap\partial D}[\varphi]_{S_{jk}}\,|\varpi_{jk}|.\tag{27}$$

In general, we have for $K = \{k_1,\ldots,k_p\} \in \mathcal{N}_0$, $p \ge 3$,

$$\begin{aligned}\lim_{\tau\to 0}\prod_{j\in K}\lambda_j\int_D|\Phi(x)|\,\varphi(x)\,W_0(K)\varpi &= B(0\star K)\lim_{\tau\to 0}\prod_{j\in K}\lambda_j\int_D|\Phi(x)|\,\varphi(x)\,F_K\\ &=(-1)^{|K\cap J|}\,B(0\star K)\lim_{\tau\to 0}\prod_{j\in K}\lambda_j\int_D|\Phi(x)|\,\varphi(x)\,|F_K|\\ &=(-1)^{|K\cap J|}\,B(0\star K)\int_{S_K\cap\partial D}[\varphi]_{S_K}\left|\frac{\varpi}{df_{k_1}\wedge\cdots\wedge df_{k_p}}\right|_{S_K}\\ &=-(-1)^{|K\cap J^c|}\sqrt{\frac{(-1)^{|K|+1}\,B(0\star K)}{2^{|K|}}}\int_{S_K\cap\partial D}[\varphi]_{S_K}|\varpi_K|.\end{aligned}\tag{28}$$

As for the last term of the rigt hand side of (23), when $\tau\to 0$, the limit value is divided into the ones at $P_j\,(j\in J)$ or $P'_j\,(j\in J^c)$.

We may assume that D is divided into $(n+1)$ domains $D^*_j\,(j\in N)$ such that D^*_j include some neighborhoods of $P_j\,(j\in J)$ and $P'_j\,(j\in J^c)$ in D respectively:

$$D=\bigcup_{j\in N}D^*_j.$$

Then it follows that

$$\int_D|\Phi(x)|\varpi=\sum_{j\in N}\int_{D^*_j}|\Phi(x)|\,\varpi.$$

First take and fix $j\in J$. Consider the integral over D^*_j. Since $f_k<0$ $(k\in J)$ and $f_k>0$ $(k\in J^c)$ in the inside of D^*_j, and $[f_k]_{P_j}=0\,(k\in N, k\neq j)$ and $[f_j]_{P_j}<0$,

$$\begin{aligned}&\lim_{\tau\to 0}\frac{\prod_{k\in\mathcal{N}}\lambda_k}{\prod_{\nu=0}^{n-1}(\lambda_\infty+\nu)}\int_{D^*_j}|\Phi(x)|\;W_0(N)\,|\varpi|\\ &=\lim_{\tau\downarrow 0}\frac{(-1)^n}{(n+1)\,\prod_{\nu=1}^{n-1}((n+1)\varepsilon+\nu)}\\ &\times\left(-B\begin{pmatrix}0&\star&\partial_jN\\0&j&\partial_jN\end{pmatrix}+B(0\star N)\left[\frac{1}{f_j}\right]_{P_j}\right)\left|\left[\frac{\varpi}{df_1\wedge\cdots\wedge df_{j-1}\wedge df_{j+1}\wedge\cdots\wedge df_{n+1}}\right]_{P_j}\right|\\ &=-\frac{1}{(n+1)\,(n-1)!}\sqrt{\frac{(-1)^{n+1}\,B(0\,N)}{2^n}},\end{aligned}$$

in view of (20) and Proposition 7.

Next take and fix $j\in J^c$. The limit of the integral over each D^*_j still has the same value:

$$\begin{aligned}&\lim_{\tau\to 0}\frac{\prod_{k\in\mathcal{N}}\lambda_k}{\prod_{\nu=0}^{n-1}(\lambda_\infty+\nu)}\int_{D^*_j}|\Phi(x)|\;W_0(N)\,|\varpi|\\ &=\lim_{\tau\downarrow 0}\frac{(-1)^{n-1}}{(n-1)\,\prod_{\nu=1}^{n-1}((n+1)\varepsilon+\nu)}\end{aligned}$$

$$\times\left(-B\begin{pmatrix}0 & \star & \partial_j N\\ 0 & j & \partial_j N\end{pmatrix}+B(0\star N)\left[\frac{1}{f_j}\right]_{P_j'}\right)\left|\left[\frac{\varpi}{df_1\wedge\cdots\wedge df_{j-1}\wedge df_{j+1}\wedge\cdots\wedge df_{n+1}}\right]_{P_j'}\right|$$
$$=-\frac{1}{(n+1)\,(n-1)!}\sqrt{\frac{(-1)^{n+1}\,B(0\,N)}{2^n}},$$

in view of (21) and Proposition 7. Namely, for all $j\in N$,

$$\lim_{\tau\downarrow 0}\frac{\prod_{k=1}^{n+1}\lambda_k}{\prod_{\nu=0}^{n-1}(\lambda_\infty+\nu)}\int_{D_j^*}|\Phi(x)|\;W_0(N)\,\varpi$$
$$=-\frac{1}{(n+1)\,(n-1)!}\sqrt{\frac{(-1)^{n+1}\,B(0\,N)}{2^n}}. \tag{29}$$

(24) to (29) imply (2).

(1) can be proved in the same way from Proposition 10. In this way, Theorem 1 has been completely proved. □

We now assume the condition $(\mathcal{H}2)$ and go to the proof of Theorem 2.

Sketch of Proof of Theorem 2. There are $2^{n+1}-1$ non-empty bounded chambers as in the case of Theorem 1. Moreover D_N^- is empty and both P_j and P_j' are in the outside of S_j $(j\in N)$. This property follows from the following fact: The cone $\bigcap_{j\in N}\hat{H}_j^-$ together with its summit $\bigcap_{j\in N}\hat{H}_j$ $(j\in N)$ in $\mathbf{R}^{n+1}$ is in the outside of $\hat{S}_0$. D_N^- is the image by the stereographic projection of the set which is by definition the intersection of $\hat{S}_0$ with the cone $\bigcap_{j\in N}\hat{H}_j^-$. However this set is empty from the assumption $(\mathcal{H}2)$.

When $J\in\mathcal{N}_0$,

$$D_J^-\cap D_{J^c}^+:\quad f_j\le 0\quad (j\in J),\quad f_j\ge 0\quad (j\in J^c)$$

are non-empty connected domains.

When J is empty, $D_J^-\cap D_{J^c}^+=D_N^+$ consists of two connected components $D_N'^+$ and $D_N''^+$. $D_N'^+$ is bounded and $D_N''^+$ is unbounded, and $D_N'^+\cap D_N''^+=\emptyset$. $D_N'^+$ is the support of the pseudo n-simplex $\widetilde{\Delta}[P_1,\dots,P_{n+1}]$ such that $P_j\in S_{\partial_j N}$. The pseudo n-simplex $\widetilde{\Delta}[P_1,\dots,P_{n+1}]$ is the image by the stereographic projection of the spherical simplex $\hat{\Delta}$ in $\hat{S}_0$, which is determined as follows. The intersection of $\hat{S}_0$ with the cone $\bigcap_{j\in N}\hat{H}_j^+$ in $\mathbf{R}^{n+1}$ has two connected components in $\hat{S}_0$. We take as $\hat{\Delta}$ the connected component disjoint with the source point of the stereographic projection $(0,\dots,0,-1)$ in $\hat{S}_0$.

The orientation of $\widetilde{\Delta}[P_1,\dots,P_{n+1}]$ is the same as $D_N'^+$:

$$D_N^+=D_N'^+\cup D_N''^+:f_j\ge 0\quad (1\le j\le n+1),$$
$$\widetilde{\Delta}[P_1,\dots,P_{n+1}]=D_N'^+.$$

$S_{\partial_j N}$ consists of two points $\{P_j,\,P_j'\}$ (see Section 5, Example, Figure 2 for 2-dimensional case). Then

$$\left[\frac{1}{f_j}\right]_{P_j} > 0, \quad \left[\frac{1}{f_j}\right]_{P'_j} > 0$$

and

$$\left[\frac{1}{f_j}\right]_{P_j} \left[\frac{1}{f_j}\right]_{P'_j} = -\frac{B(0\,\partial_j N)}{B(0 \star N)} > 0,$$

because $(-1)^{n+1} B(0 \star N) < 0$. Under this circumstance, the proof of Theorem 2 can be done almost in the same way as Theorem 1. We omit and leave a reader to prove it. $\square$

Theorem 2 also follows from the identities in Theorem 1 by an analytic continuation (i.e., by a Picard–Lefschetz transformation of twisted cycles around the locus: $B(0\star J) = 0$) of $v(D_N^-)$ moving the parameters r_j^2, ρ_{jk}^2 such that

$$\begin{aligned}
&B(0 \star N) \longrightarrow -B(0 \star N),\\
&v(D_N^-) \longrightarrow (-1)^n\, v(D_N'^+),\\
&v(D_J^-) \longrightarrow (-1)^{|J|-1}\, v(D_J^-).
\end{aligned}$$

Remark. An elementary proof of Theorem 2 (i) (and therefore of Theorem 1 (i)) will be given in the Appendix.

5. Examples.

In the following, we give two simple examples of main theorems.

Example 1. In the n-dimensional Euclidean space, consider two hyperspheres S_1, S_2 with the centers O_1, O_2 and with radii r_1, r_2 such that the distance between O_1 and O_2 is equal to ρ_{12}.

Assume $S_1 \cap S_2$ is a non-empty $(n-2)$-dimensional sphere. $S_1 \cap S_2$ is contained in the hyperplane L which intersects the segment $\overline{O_1O_2}$ at a point M.

The radius h of $S_1 \cap S_2$, the distance O_1M and O_2M are expressed as

$$\begin{aligned}
h &= \frac{\sqrt{-B(0 \star 1\;2)}}{2\rho_{12}} = r_1 \sin\frac{1}{2}\psi_{12} = r_2 \sin\frac{1}{2}\psi_{21},\\
O_1M &= r_1 \cos\frac{1}{2}\psi_{12} = \frac{B\begin{pmatrix}0 & 2 & 1\\ 0 & \star & 1\end{pmatrix}}{2\rho_{12}},\\
O_2M &= r_2 \cos\frac{1}{2}\psi_{21} = \frac{B\begin{pmatrix}0 & 1 & 2\\ 0 & \star & 2\end{pmatrix}}{2\rho_{12}}
\end{aligned}$$

such that

$$\rho_{12} = r_1 \cos\frac{1}{2}\psi_{12} + r_2 \cos\frac{1}{2}\psi_{21},$$

where ψ_{12}, ψ_{21} satisfy $0 < \psi_{12} < \pi$, $0 < \psi_{21} < \pi$.

Denote by D_{12}^- the common domain (lens domain) surrounded by S_1, S_2. $S_1 \cap S_2$ is an $(n-2)$-dimensional sphere.

The volume $v(D_{12}^-)$ of D_{12}^- can be evaluated by an elementary calculus as follows:

$$v(D_{12}^-) = v_1 + v_2, \tag{30}$$

where

$$v_1 = \frac{1}{n-1} C_{n-2} r_1^n \int_{\cos(\psi_{12}/2)}^1 (1-\tau^2)^{(n-1)/2}\, d\tau,$$
$$v_2 = \frac{1}{n-1} C_{n-2} r_2^n \int_{\cos(\psi_{21}/2)}^1 (1-\tau^2)^{(n-1)/2}\, d\tau,$$

v_1, v_2 denote the volumes of the domains surrounded by S_1, L and S_2, L respectively, and C_{n-2} denotes the volume of the $(n-2)$-dimensional unit hypersphere:

$$C_{n-2} = \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)}.$$

The lower dimensional volumes of $S_1 \cap S_2$, $S_1 \cap \partial D_{12}^-$, $S_2 \cap \partial D_{12}^-$ equal respectively

$$v(S_1 \cap S_2) = C_{n-2}\, h^{n-2},$$
$$v(S_1 \cap \partial D_{12}^-) = \frac{\partial v}{\partial r_1} = C_{n-2}\, r_1^{n-1} \int_0^{\psi_{12}/2} \sin^{n-2} t\, dt, \tag{31}$$
$$v(S_2 \cap \partial D_{12}^-) = \frac{\partial v}{\partial r_2} = C_{n-2}\, r_2^{n-1} \int_0^{\psi_{21}/2} \sin^{n-2} t\, dt. \tag{32}$$

The integral in the right hand side can be expressed in the following expansion:

$$\int_0^{\psi_{jk}/2} \sin^{n-2} t\, dt = -\sum_{0\le 2\nu \le n-3} \cos\frac{\psi_{jk}}{2} \frac{(n-3)\cdots(n-2\nu+1)}{(n-2)\cdots(n-2\nu)} \left(\sin\frac{\psi_{jk}}{2}\right)^{n-3-2\nu}$$
$$+ \begin{cases} C'_{n-2}\left(1-\cos\dfrac{\psi_{jk}}{2}\right) \\ C'_{n-2}\dfrac{\psi_{jk}}{2} \end{cases} \quad (\{j,k\} = \{1,2\}), \tag{33}$$

where $2C'_{n-2}$ or $2\pi\, C'_{n-2}$ equals $\sqrt{\pi}\,\Gamma((n-1)/2)/\Gamma(n/2)$ according as n is odd or even.

Remark. In the case where n is odd, $\dim S_j \cap \partial D_{12}^-\, (j = 1, 2)$ is even, (31)–(33) are related with the generalized Gauss–Bonnet formula. Indeed, the second formula due to Allendoerfer–Weil (see [**1**]) can be applied to $v(S_1 \cap \partial D_{12}^-)$ or $v(S_2 \cap \partial D_{12}^-)$. The above formulae coincide with it.

The derivation of $v(D_{12}^-)$ with respect to r_1, r_2, ρ_{12} in (30) leads to the following formula

$$dv(D_{12}^-) = v(S_1 \cap \partial D_{12}^-)\, dr_1 + v(S_2 \cap \partial D_{12}^-)\, dr_2 - \frac{1}{n-1}\, v(S_1 \cap S_2) \sqrt{\frac{-B(0 \star 1\ 2)}{4}}\, \theta_{12} \tag{34}$$

with $\theta_{12} = (1/2) \log \rho_{12}^2$.

(34) is nothing else than a special case in the n dimensional space derived from (4) for $\tau \to 0$, after putting to be $\lambda_1 = \lambda_2 = \tau$ and $\lambda_j = 0$ $(3 \le j \le n+1)$.

In particular, in the case where $n = 2, 3$, the volumes $v(D_{12}^-)$ are simply written as

- $v(D_{12}^-) = \dfrac{1}{2}\, r_1^2\, (\psi_{12} - \sin \psi_{12}) + \dfrac{1}{2}\, r_2^2\, (\psi_{21} - \sin \psi_{21}) \quad (n = 2),$ (35)
- $v(D_{12}^-) = \pi r_1^3 \left(\dfrac{2}{3} - \cos \dfrac{1}{2}\psi_{12} + \dfrac{1}{3}\, \cos^3 \dfrac{1}{2}\, \psi_{12} \right) + \pi r_2^3 \left(\dfrac{2}{3} - \cos \dfrac{1}{2}\psi_{21} + \dfrac{1}{3}\, \cos^3 \dfrac{1}{2}\psi_{21} \right) \quad (n = 3),$ (36)

where ψ_{12}, ψ_{21} denote the angles at O_1, O_2 respectively subtended by the diameter of $S_1 \cap S_2$. Remark that

$$e^{i\psi_{12}/2} = \frac{B\begin{pmatrix} 0 & \star & 1 \\ 0 & 2 & 1 \end{pmatrix} + i\sqrt{-B(0 \star 1\ 2)}}{2\rho_{12}\, r_1},$$

$$e^{i\psi_{21}/2} = \frac{B\begin{pmatrix} 0 & \star & 2 \\ 0 & 1 & 2 \end{pmatrix} + i\sqrt{-B(0 \star 1\ 2)}}{2\rho_{12}\, r_2}.$$

In the case where $n = 2, 3$, the formula (34) becomes

- $dv(D_{12}^-) = r_1 \psi_{12}\, dr_1 + r_2 \psi_{21}\, dr_2 - \sqrt{-B(0 \star 1\ 2)}\, \dfrac{d\rho_{12}}{\rho_{12}},$ (37)
- $dv(D_{12}^-) = \dfrac{\pi r_1}{\rho_{12}} \{ r_2^2 - (r_1 - \rho_{12})^2 \}\, dr_1 + \dfrac{\pi r_2}{\rho_{12}} \{ r_1^2 - (r_2 - \rho_{12})^2 \}\, dr_2 - \dfrac{\pi}{4\rho_{12}^2}\, B(0 \star 1\ 2)\, d\rho_{12},$ (38)

in view of the identity

$$\frac{1}{2}\, d\psi_{jk} = \frac{1}{\sqrt{-B(0 \star j\ k)}} \left\{ -B\begin{pmatrix} 0 & \star & j \\ 0 & \star & k \end{pmatrix} \frac{dr_j}{r_j} + 2r_k\, dr_k - B\begin{pmatrix} 0 & j & k \\ 0 & \star & k \end{pmatrix} \frac{d\rho_{jk}}{\rho_{jk}} \right\} \tag{39}$$

for $j, k = 1, 2$ or $2, 1$ respectively.

Example 2. Assume that $n = 2$.

Then D_{123}^- is the pseudo-triangle $\tilde{\Delta}[P_1 P_2 P_3]$ with vertices $P_1 = (\xi_1, \xi_2)$, $P_2(\eta_1, \eta_2)$, $P_3(\zeta_1, \zeta_2)$ (see Figure 1), where

$$\xi_1 = -\frac{B\begin{pmatrix} 0 & 2 & 3 \\ 0 & \star & 3 \end{pmatrix}}{\sqrt{2B(0\,2\,3)}}, \quad \xi_2 = -\sqrt{\frac{-B(0\star 2\,3)}{2B(0\,2\,3)}},$$

$$\eta_1 = \frac{1}{B(0\,1\,3)\sqrt{2B(0\,2\,3)}} \left\{ -B\begin{pmatrix} 0 & 1 & 3 \\ 0 & \star & 3 \end{pmatrix} B\begin{pmatrix} 0 & 1 & 3 \\ 0 & 2 & 3 \end{pmatrix} - \sqrt{B(0\star 1\,3)\,B(0\,1\,2\,3)} \right\},$$

$$\eta_2 = \frac{1}{B(0\,1\,3)\sqrt{2B(0\,2\,3)}} \left\{ -B\begin{pmatrix} 0 & 1 & 3 \\ 0 & \star & 3 \end{pmatrix} \sqrt{-B(0\,1\,2\,3)} - B\begin{pmatrix} 0 & 1 & 3 \\ 0 & 2 & 3 \end{pmatrix} \sqrt{-B(0\star 1\,3)} \right\},$$

$$\zeta_1 = \frac{1}{B(0\,1\,2)\sqrt{2B(0\,2\,3)}} \left\{ B\begin{pmatrix} 0 & 1 & 2 \\ 0 & \star & 2 \end{pmatrix} B\begin{pmatrix} 0 & 1 & 2 \\ 0 & 3 & 2 \end{pmatrix} - B(0\,1\,2)\,B(0\,2\,3) \right.$$
$$\left. + \sqrt{B(0\,1\,2\,3)\,B(0\star 1\,2)} \right\},$$

$$\zeta_2 = \frac{1}{B(0\,1\,2)\sqrt{2B(0\,2\,3)}} \left\{ -B\begin{pmatrix} 0 & 1 & 2 \\ 0 & \star & 2 \end{pmatrix} \sqrt{-B(0\,1\,2\,3)} + B\begin{pmatrix} 0 & 1 & 2 \\ 0 & 3 & 2 \end{pmatrix} \sqrt{-B(0\star 1\,2)} \right\}.$$

Note that $\xi_2 < 0$, $\eta_1 < 0$.

The area of $\Delta(O_1O_3O_2)$ is expressed by

$$|\Delta(O_1O_3O_2)| = \frac{1}{2}\,|\delta|, \tag{40}$$

where δ denotes

$$\delta = \begin{vmatrix} 1 & \xi_1 & \xi_2 \\ 1 & \eta_1 & \eta_2 \\ 1 & \zeta_1 & \zeta_2 \end{vmatrix} = -\frac{1}{2}\sqrt{-B(0\,1\,2\,3)} < 0.$$

Denote by φ_j the angle of the triangle $\Delta(O_1O_3O_2)$ at the vertex O_j. Then

$$e^{i\varphi_j} = \frac{B\begin{pmatrix} 0 & k & j \\ 0 & l & j \end{pmatrix} + i\sqrt{-B(0\,1\,2\,3)}}{2\rho_{jk}\rho_{jl}} \quad (j, k, l \text{ different indices}). \tag{41}$$

Denote by P_1', P_2', P_3' the intersection points of $S_2 \cap S_3$, $S_3 \cap S_1$, $S_1 \cap S_2$ which are different from P_1, P_2, P_3 respectively. Also denote by ψ_{jk} the angle at O_j subtended by the arc $\widehat{P_kP_k'} \cap S_j$.

Then Theorem 1 (i) shows

LEMMA 13.

$$v(D) = \tilde{\Delta}(P_1P_3P_2)$$
$$= |\Delta(O_1O_3O_2)| - \sum_{j=1}^{3} |\Delta(O_1O_3O_2) \cap D_j| + \sum_{1\le j<k\le 3} |\Delta(O_1O_3O_2) \cap D_j^- \cap D_k^-|, \tag{42}$$

where owing to (30)

$$\begin{aligned}
|\Delta(O_1O_3O_2)\cap D_j^-| &= \frac{1}{2}\,r_j^2\varphi_j,\\
|\Delta(O_1O_3O_2)\cap D_j^-\cap D_k^-| &= \frac{1}{2}\,|D_j^-\cap D_k^-|\\
&= \frac{1}{4}\,r_j^2\,(\psi_{jk}-\sin\psi_{jk})+\frac{1}{4}\,r_k^2\,(\psi_{kj}-\sin\psi_{kj}).
\end{aligned}$$

Denote by ψ_1, ψ_2, ψ_3 the angles at O_j subtended by the sides $\widehat{P_2P_3}, \widehat{P_3P_1}, \widehat{P_1P_2}$ of the pseudo triangle $\tilde{\Delta}(P_1,P_3,P_2)$ respectively such that the arc length s_j of $\widehat{P_kP_l}$ is equal to

$$s_j = r_j\,\psi_j.$$

ψ_j are also related with ψ_{jk}, φ_j as follows:

$$\psi_j = \frac{1}{2}\,\psi_{jk}+\frac{1}{2}\,\psi_{jl}-\varphi_j.$$

On the other hand, since $\varphi_1+\varphi_2+\varphi_3 = 2\pi$,

$$\begin{aligned}
\psi_1+\psi_2+\psi_3 &= \frac{1}{2}\sum_{j\neq k}\psi_{jk}\\
&= 2\pi-\angle P_1P_3P_2-\angle P_2P_1P_3-\angle P_3P_2P_1.
\end{aligned}$$

This identity is a special case of the second Allendoerfer–Weil formula in the Euclidean plane (see [**1**] Theorem II). Furthermore, from (42),

$$2v(D) = r_1\,v_1+r_2\,v_2+r_3\,v_3-\frac{1}{2}\sum_{1\le j<k\le 3}\sqrt{-B(0\star j\,k)}\,v_{jk}+\frac{1}{2}\sqrt{-B(0\,1\,2\,3)},$$

with $v_j = r_j\psi_j$. This identity coincides with (3) in the two dimensional case.

Taking into consideration the identities (30), (31), (33) and the following equalities (j,k,l are different indices of $1,2,3$)

$$\begin{aligned}
dB(0\,1\,2\,3) &= -2\sum_{j<k} d\rho_{jk}^2\,B\begin{pmatrix}0 & j & l\\ 0 & k & l\end{pmatrix},\\
d\varphi_j &= \frac{1}{\sqrt{-B(0\,1\,2\,3)}}\left\{-B\begin{pmatrix}0 & j & k\\ 0 & l & k\end{pmatrix}\frac{d\rho_{jk}}{\rho_{jk}}-B\begin{pmatrix}0 & j & l\\ 0 & k & l\end{pmatrix}\frac{d\rho_{jl}}{\rho_{jl}}+2\rho_{kl}d\rho_{kl}\right\},
\end{aligned}$$

we get the formula

$$\begin{aligned}
dv(D) = \sum_{j=1}^{3} r_j\,\psi_j\,dr_j &- \frac{1}{2}\sum_{j<k}\sqrt{-B(0\star j\,k)}\,\frac{d\rho_{jk}}{\rho_{jk}}\\
&- \frac{1}{2\sqrt{-B(0\,1\,2\,3)}}\left\{\sum_{j<k}^{3} B\begin{pmatrix}0 & \star & j & k\\ 0 & l & j & k\end{pmatrix}\frac{d\rho_{jk}}{\rho_{jk}}\right\},
\end{aligned}$$

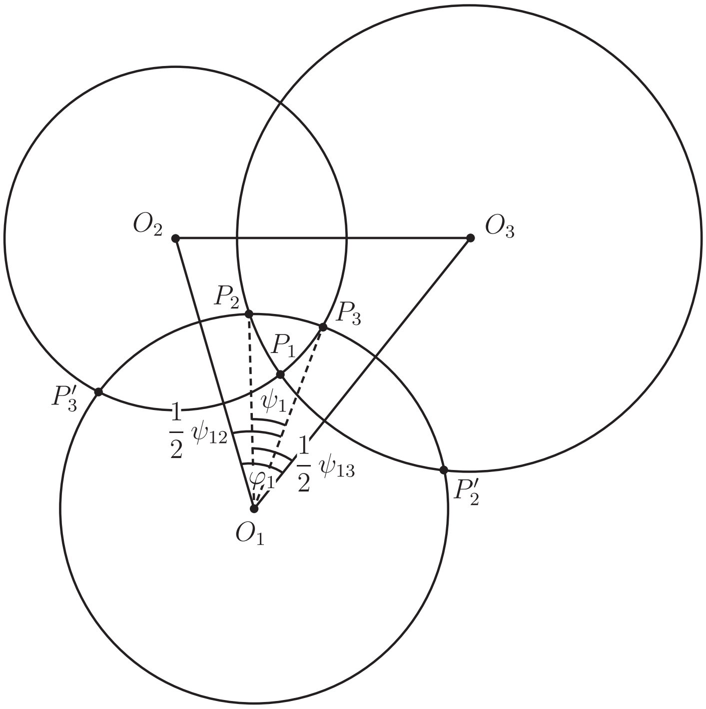

Figure 1. $B(0 \star 1\,2\,3) > 0$.

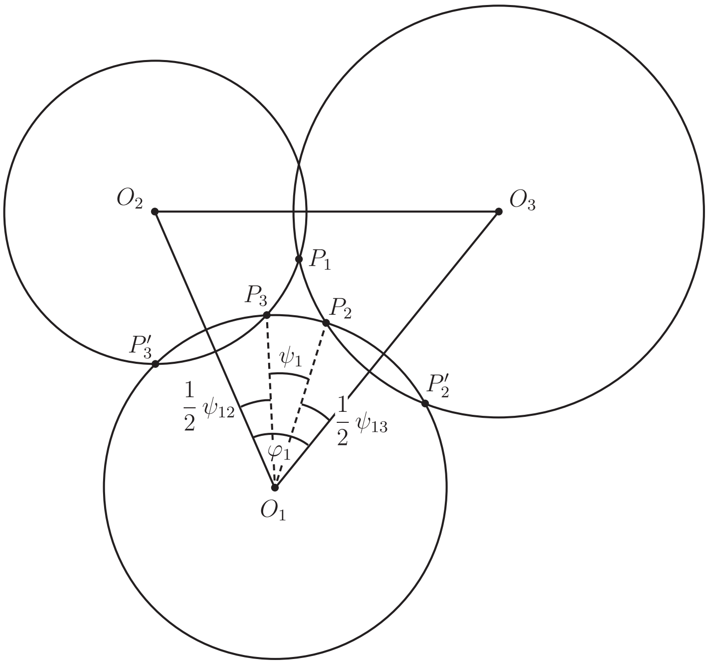

Figure 2. $B(0 \star 1\,2\,3) < 0$.

which is nothing else than (4) for $n = 2$ in view of the identity

$$\theta_{123} = -\frac{1}{B(0\,1\,2\,3)} \left\{ B\begin{pmatrix} 0 & \star & 1 & 2 \\ 0 & 3 & 1 & 2 \end{pmatrix} \frac{d\rho_{12}}{\rho_{12}} + B\begin{pmatrix} 0 & \star & 1 & 3 \\ 0 & 2 & 1 & 3 \end{pmatrix} \frac{d\rho_{13}}{\rho_{13}} + B\begin{pmatrix} 0 & \star & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \frac{d\rho_{23}}{\rho_{23}} \right\}.$$

6. Restriction to the unit hypersphere.

We assume further

$$f_{n+1}(x) = Q(x) - 1,$$

i.e., S_{n+1} is the unit hypersphere with center O_{n+1} at the origin.

We may assume the linear functions

$$f'_j(x) := f_j(x) - Q(x) + 1 = \sum_{\nu=1}^{n} u_{j\nu} x_\nu + u_{j0} \quad (1 \le j \le n)$$

are normalized such that the configuration matrix $A' = (a'_{jk})$ $(0 \le j, k \le n)$ of order $n+1$ consisting of

$$a'_{j0} = a'_{0j} = u_{j0},$$
$$a'_{jk} = \sum_{\nu=1}^{n} u_{j\nu} u_{k\nu} - u_{j0} u_{k0},$$

satisfies $a'_{00} = -1$, $a'_{jj} = 1$ $(1 \le j \le n)$. We put further

$$f'_{n+1} = 1 - Q(x).$$

For the set of indices $J = \{j_1, \dots, j_p\}$, $K = \{k_1, \dots, k_p\} \subset \{0, 1, \dots, n, n+1\}$, we denote by $A'\binom{J}{K}$ the subdeterminant with the $j_1, \dots, j_p$th rows and the $k_1, \dots, k_p$th columns. In particular, we abbreviate $A'\binom{J}{J}$ by $A'(J)$.

The family of the hyperplanes $H_j : f'_j(x) = 0$ define the arrangement of hyperplanes $\mathcal{A}' = \bigcup_{j=1}^{n} H_j$ which correspond to $\mathcal{A} = \bigcup_{j=1}^{n} S_j$, $S_j : f_j(x) = 0$, one-to-one.

The components of the matrix A' are described by the Cayley–Menger determinants as follows:

$$a'_{j0} = \frac{B\begin{pmatrix} 0 & j & n+1 \\ 0 & \star & n+1 \end{pmatrix}}{\sqrt{-B(0 \star j\, n+1)}}, \tag{43}$$

$$a'_{jk} = \frac{-B\begin{pmatrix} 0 & \star & j & n+1 \\ 0 & \star & k & n+1 \end{pmatrix}}{\sqrt{B(0 \star j\, n+1)\, B(0 \star k\, n+1)}}. \tag{44}$$

H_j has the same intersection with S_{n+1} as the intersection $S_j \cap S_{n+1}$.

From now on, we shall assume the condition $(\mathcal{H}1)$.

$(\mathcal{H}1)$ can be rephrased in terms of the minors of A' as follows:

$$(\mathcal{H}1) \quad A'(0\,J) < 0 \ (J \subset \partial_{n+1}N), \quad A'(J) > 0 \ (1 \leq |J|, J \subset \partial_{n+1}N).$$

Remark that it always holds: $-A'(0\,J) > A'(J) > 0$.

Since S_{n+1} is the unit hypersphere, we have the identity

$$B(0 \star n+1) = 2, \quad B(0 \star j\, n+1) = -1,$$

so that

$$a'_{jk} = -B(0 \star j\, k\, n+1) = -\cos\langle j, k\rangle,$$

where $\langle j, k\rangle$ denotes the angle subtended by S_j, S_k in S_{n+1}.

Let $D = D^-_{12\,\dots\,n+1}$ be the (non-empty) real n-dimensional domain defined by

$$D^-_{12\,\dots\,n+1} = \bigcap_{j=1}^{n+1} D^-_j, \quad D^-_j : f'_j \leq 0 \ (\subset \mathbf{R}^n) \quad (1 \leq j \leq n+1).$$

Then, for any $J \subset \partial_{n+1}N$ such that $|J| = p$, $1 \leq p \leq n-1$, the intersection $S_{J\,n+1} = S_{n+1} \cap \bigcap_{j \in J} S_j$ defines an $(n-p-1)$-dimensional sphere. In particular, $\bigcap_{k \in \partial_j \partial_{n+1} N} S_k$ consists of two points.

The orientation of $\mathbf{R}^n$ and D is determined such that the standard n-form ϖ is positive:

$$\varpi = dx_1 \wedge \cdots \wedge dx_n > 0.$$

We can define the standard volume form on S_{n+1} as

$$\varpi_{n+1} := \sum_{\nu=1}^{n} (-1)^\nu \, x_\nu dx_1 \wedge \cdots \widehat{dx_\nu} \cdots \wedge dx_n = 2\left[\frac{\varpi}{df'_{n+1}}\right]_{S_{n+1}}.$$

Let $\Phi'(x)$ be the multiplicative function

$$\Phi'(x) = \prod_{j \in \partial_{n+1}N} f'_j(x)^{\lambda_j} \quad (\lambda_j \in \mathbf{R}_{\geq 0}).$$

We take the value of the many valued function $\Phi'(x)$ such that $\Phi'(x) > 0$ at the infinity in $\mathbf{R}^n$.

Denote the twisted rational de Rham $(n-1)$-cohomology by $H^{n-1}_{\nabla}(X, \Omega^\bullet(*S))$ and its dual by $H_{n-1}(X, \mathcal{L}^*)$, where $\mathcal{L}^*$ denotes the dual local system on the complexification X of the space $S_{n+1} - \bigcup_{j \in \mathcal{A}} S_j$ associated with Φ'. The covariant differentiation ∇ is given by

$$\nabla\psi = d\psi + d\log\Phi' \wedge \psi.$$

The corresponding integral can be expressed as the pairing

$$H_{\nabla}^{n-1}(X,\Omega^{\bullet}(*S))\times H_{n-1}(X,\mathcal{L}^{*})\ni(\varphi,\mathfrak{z})\longrightarrow\mathcal{J}'_{\lambda}(\varphi)=\int_{\mathfrak{z}}\Phi'(x)\varphi(x)\,\varpi_{n+1}$$

for $\varphi\varpi\in\Omega^{n-1}(*S)$ and a twisted $(n-1)$-cycle $\mathfrak{z}$.

The following has been proved in [**3**].

Proposition 14. *$H_{\nabla}^{n-1}(X,\Omega^{\bullet}(*S))$ is of dimension 2^n and has a basis*

$$F'_J=\frac{\varpi_{n+1}}{f'_J},$$

where f'_J means the product $\prod_{j\in J}f'_j$ and J ranges over the family of all unordered subsets of indices such that $J\subset\partial_{n+1}N$ including the empty set $\emptyset$.

From now on, we choose a twisted cycle $(n-1)$-cycle $\mathfrak{z}$ such that

$$\int_{\mathfrak{z}}\Phi'(x)\,\varphi\,\varpi_{n+1}=\int_{D^{-}_{12\,\dots\,n+1}}|\Phi'(x)|\,\varphi\,\varpi_{n+1}\quad(\varphi\,\varpi_{n+1}\in\Omega^{n-1}(*S)).$$

$F'_{\emptyset}$ means ϖ_{n+1}, and we define

$$\mathcal{J}'_{\lambda}(\varphi)=\int_{\mathfrak{z}}\Phi'(x)\,\varphi\,\varpi_{n+1}.$$

The derivation of the integral $\mathcal{J}'_{\lambda}(\varphi)$ with respect to the parameters a'_{jk},a'_{j0} can be expressed as

$$\begin{aligned}d_{A'}\mathcal{J}'_{\lambda}(\varphi)&=\sum_{j=1}^{n}da'_{j0}\frac{\partial}{\partial a'_{j0}}\mathcal{J}'_{\lambda}(\varphi)+\sum_{1\le j,k\le n}da'_{jk}\frac{\partial}{\partial a'_{jk}}\mathcal{J}'_{\lambda}(\varphi)\\&=\int_{\mathfrak{z}}\Phi'(x)\,\nabla_{A'}(\varphi\,\varpi_{n+1}),\end{aligned}\tag{45}$$

where

$$\nabla_{A'}(\varphi\,\varpi_{n+1})=d_{A'}(\varphi\,\varpi_{n+1})+d_{A'}\log\Phi'(x)\wedge\varphi\,\varpi_{n+1}.$$

In addition to the above basis, it is convenient to introduce the following basis which we call "of second kind":

Definition 15. We define the following:

$$F'_{*,J}:=F'_J+\sum_{\nu\in J}\frac{A'\begin{pmatrix}0&\partial_\nu J\\ \nu&\partial_\nu J\end{pmatrix}}{A'(J)}F'_{\partial_\nu J}.$$

In particular, $F'_{*,\emptyset}=F'_{\emptyset}=\varpi_{n+1}$.

The differential one-forms defined below will play an essential role in the sequel.

DEFINITION 16. We define the following:

$$\theta'_j := da'_{j0},$$

$$\theta'_{jk} := da'_{jk} - \frac{A'\begin{pmatrix} 0 & k \\ j & k \end{pmatrix}}{A'(0\,k)}\, da'_{k0} - \frac{A'\begin{pmatrix} 0 & j \\ k & j \end{pmatrix}}{A'(0\,j)}\, da'_{j0}.$$

General θ'_J for $|J| \geq 3$ are defined by induction:

$$\theta'_J := -\sum_{\nu \in J} \frac{A'\begin{pmatrix} 0 & \partial_\nu J \\ \nu & \partial_\nu J \end{pmatrix}}{A'(0\,\partial_\nu J)}\, \theta'_{\partial_\nu J} \quad (3 \leq |J| \leq n).$$

Denote $\lambda'_\infty = \sum_{j=1}^n \lambda_j$ and $J = \{j_1, \ldots, j_p\}$, $|J| = p$.
The following fact has been proved in [**3**].

PROPOSITION 17. *The following variation formula holds*:

$$\nabla_{A'}(F'_\emptyset) \sim \sum_{p=1}^n \sum_{1 \leq j_1 < \cdots < j_p \leq n} \frac{\lambda_{j_1} \cdots \lambda_{j_p}}{\prod_{q=1}^{p-1}(-\lambda_\infty - n + q + 1)} (-1)^p\, \theta'_J \frac{A'(J)}{A'(0\,J)}\, F'_{*,J}. \tag{46}$$

(*The formula* (4.12) *in* [**3**] *has an error. In the right hand side, the sign* $(-1)^p$ *should be added as above to the original formula.*)

For example,

- $$\nabla_{A'}(F'_\emptyset) \sim -\lambda_1 \frac{1}{A'(0\,1)}\, \theta'_1\, (F'_1 + a'_{10}\, F'_\emptyset) - \lambda_2 \frac{1}{A'(0\,2)}\, \theta'_2\, (F'_2 + a'_{20}\, F'_\emptyset)$$
$$- \frac{\lambda_1 \lambda_2}{\lambda_\infty} \frac{A'(1\,2)}{A'(0\,1\,2)}\, \theta'_{123}\, F'_{*,12} \quad (n = 2),$$
- $$\nabla_{A'}(F'_\emptyset) \sim -\sum_{j=1}^3 \lambda_j da'_{j0} F'_{*,j} - \sum_{1 \leq j < k \leq 3} \frac{\lambda_j\, \lambda_k}{\lambda_\infty + 1} \frac{A'(j\,k)}{A'(0\,j\,k)}\, \theta'_{jk}\, F'_{*,jk}$$
$$- \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_\infty(\lambda_\infty + 1)}\, \theta'_{123} \frac{A'(1\,2\,3)}{A'(0\,1\,2\,3)}\, F'_{*,123} \quad (n = 3).$$

7. Analogue of Schläfli formula.

The variational formula for the volume of a spherically faced simplex in the unit hypersphere was presented in [**3**]. In addition to the formulae stated in Theorems 1 and 2, Theorem 20 in this section makes a completely integrable system.

However, some formulae stated there have a few errors. In this section, we present a correct version as in Theorem 20.

Let P_j $(1 \leq j \leq n)$ be the points in $\mathbf{R}^n$ such that

$$\{P_j\} = \bigcap_{k \in \partial_j N} H_k \cap S_{n+1}.$$

We can take the Euclidean coordinates $x_1, \ldots, x_n$ such that the polynomials f_j have the following expressions:

$$f'_j(x) = \sum_{\nu=1}^{n+1-j} u_{j\,\nu} x_\nu + u_{j\,0} \quad (1 \leq j \leq n). \tag{47}$$

We assume for simplicity that $u_{j\,n+1-j} = 2\alpha_{j\,n+1-j} > 0$ $(1 \leq j \leq n)$ and that P_j satisfies Lemma 5.

We have the equalities

$$\prod_{j=p+1}^{n} u_{j\,n-j+1} = \sqrt{-A'(0\,p+1\ldots n)} \quad (1 \leq p \leq n). \tag{48}$$

The affine subspace $\bigcap_{j=n-p+1}^{n} H_j$ contains the $(n-p-1)$-dimensional sphere $S_{n-p+1\,\ldots\,n\,n+1} = \bigcap_{j=n-p+1}^{n} S_j \cap S_{n+1}$ with radius

$$r_{n-p+1\,\ldots\,n\,n+1} = \sqrt{-\frac{A'(n-p+1\ldots n)}{A'(0\,n-p+1\ldots n)}}.$$

Denote by $\widetilde{\Delta}[P_1, P_2, \ldots, P_n]$ be the pseudo $(n-1)$-simplex in S_{n+1} with spherical faces with vertices P_j such that their sign of orientation is $(-1)^{n(n-1)/2}$. The support of $\widetilde{\Delta}[P_1, P_2, \ldots, P_n]$ coincides with $D = D^-_{12\,\ldots\,n+1}$.

By definition, the following properties are valid.

Lemma 18. (i)

$$df'_n \wedge \cdots \wedge df'_1 > 0$$

on D.

(ii) *The pseudo $(n-1)$-simplex $\widetilde{\Delta}[P_1, P_2, \ldots, P_n]$ has the sign $(-1)^{n(n-1)/2}$ of orientation such that*

$$\widetilde{\Delta}[P_1, P_2, \ldots, P_n] = (-1)^{n(n-1)/2}\, S_{n+1} \cap D.$$

Proof. Indeed, we can show that

$$df'_n \wedge \cdots \wedge df'_1 = \prod_{j=1}^{n} u_{j\,n-j+1}\, \varpi > 0. \tag{49}$$

(ii) follows from Lemma 5. □

Let $v_\emptyset$ be the volume of the pseudo $(n-1)$-simplex $\widetilde{\Delta}[P_1, P_2, \ldots, P_n]$ defined by

$$v_\emptyset = \int_{\widetilde{\Delta}[P_1,P_2,\ldots,P_n]} \varpi_{n+1} > 0,$$

where the orientation of $\widetilde{\Delta}[P_1,P_2,\ldots,P_n]$ is chosen such that ϖ_{n+1} should be positive on it.

We are interested in the variation formula for $v_\emptyset$, which can be expressed in terms of the lower dimensional volumes of the faces of $\widetilde{\Delta}[P_1,P_2,\ldots,P_n]$.

Every face of the pseudo simplex is included in some $S_{J\,n+1}$. $S_{J\,n+1}$ is defined as an $(n-p-1)$-dimensional sphere with radius

$$r_{J\,n+1} = \sqrt{-\frac{A'(J)}{A'(0\,J)}}.$$

We can consider the $(n-p-1)$-dimensional volume v_J ($|J| = p$) relative to the corresponding standard volume form $\varpi'_{J\,n+1}$ on the $(n-p-1)$-dimensional sphere:

$$v_J = \int_{\widetilde{\Delta}[P_1,P_2,\ldots,P_n]\cap S_J} |\varpi'_{J\,n+1}|,$$

where

$$|\varpi'_{J\,n+1}| = r^{n-p-1}_{J\,n+1}\,|\varpi_{J\,n+1}| > 0.$$

The orientation of $\widetilde{\Delta}[P_1,P_2,\ldots,P_n]\cap S_J$ is chosen such that $\varpi'_{J\,n+1}$ should be positive: $\varpi'_{J\,n+1} = |\varpi'_{J\,n+1}|$, $|\varpi'_{J\,n+1}|$ being the absolute value of $\varpi'_{J\,n+1}$.

When $J = \{n-p+1\ldots n\}$, we can give an explicit expression for $\varpi'_{n-p+1\,\ldots\,n+1}$ as follows:

$$f'_j(x) = 0 \quad (n-p+1 \le j \le n+1),$$
$$\sum_{j=p+1}^{n} x_j^2 = r^2_{n-p+1\,\ldots\,n+1},$$

where

$$r_{n-p+1\,\ldots\,n+1} = \sqrt{-\frac{A'(n-p+1\ldots n)}{A'(0\,n-p+1\ldots n)}}.$$

The standard volume form on $S_{n-p+1\,\ldots\,n+1}$ is given by

$$\begin{aligned}\varpi'_{n-p+1\,\ldots\,n+1} &= \sum_{\nu=p+1}^{n} (-1)^\nu \,\frac{x_\nu\, dx_{p+1}\wedge\cdots\widehat{dx_\nu}\cdots\wedge dx_n}{r_{n-p+1\,\ldots\,n+1}}\\ &= r^{n-p-1}_{n-p+1\,\ldots\,n+1}\,\varpi_{n-p+1\,\ldots\,n+1},\end{aligned} \tag{50}$$

where

$$\varpi_{n-p+1\,\dots\,n+1} = \sum_{\nu=p+1}^{n} (-1)^{\nu}\, \xi_\nu\, d\xi_{p+1} \wedge \cdots \widehat{d\xi_\nu} \cdots \wedge d\xi_n,$$

through the transformation

$$x_\nu = r_{n-p+1\,\dots\,n+1}\, \xi_\nu \quad (p+1 \le \nu \le n),$$

such that $\sum_{\nu=p+1}^{n} \xi_\nu^2 = 1$.

The following Lemma follows by definition of the residue formula.

Lemma 19. *For $J = \{j_1, \dots, j_p\}\, (1 \le j_1 < \cdots < j_p \le n)$,*

$$\left[\frac{\varpi_{n+1}}{df'_{j_p} \wedge \cdots \wedge df'_{j_1}}\right]_{S_{j_1\,\dots\,j_p}} = \frac{1}{\sqrt{A'(J)}}\, \varpi'_{J\,n+1}.$$

In particular,

$$\left[\frac{\varpi_{n+1}}{df'_n \wedge \cdots \widehat{df'_j} \cdots \wedge df'_1}\right]_{P_j} = \frac{(-1)^{n-j}}{\sqrt{A'(\partial_j \partial_{n+1} N)}} \quad (1 \le j \le n),$$

since $[f'_j]_{P_j}$ at the point P_j of $S_{\partial_j\,\partial_{n+1} N} \cap D$ is negative.

Proof. To prove Lemma 19, we may assume that $j_1 = n-p+1, \dots, j_p = n$ and f'_j are represented by the reduced form (47). A direct calculation and (48) show the following identity

$$\begin{aligned} d(1-Q(x)) \wedge df'_n \wedge \cdots \wedge df'_{n-p+1} \wedge \sum_{\nu=p+1}^{n} (-1)^{\nu} x_\nu dx_{p+1} \wedge \cdots \widehat{dx_\nu} \cdots \wedge dx_n \\ = 2 \prod_{q=1}^{p} u_{n-q+1\;q} \left(\sum_{\nu=p+1}^{n} x_\nu^2\right) \varpi \\ = 2\sqrt{-A'(0\; n-p+1 \dots n)}\; r^2_{n-p+1\,\dots\,n+1}\, \varpi. \end{aligned}$$

Hence,

$$\begin{aligned} \left[df'_n \wedge \cdots \wedge df'_{n-p+1} \wedge \sum_{\nu=p+1}^{n} (-1)^{\nu} x_\nu dx_{p+1} \wedge \cdots \widehat{dx_\nu} \cdots \wedge dx_n\right]_{S_{n+1}} \\ = \sqrt{-A'(0\; n-p+1 \dots n)}\; r^2_{n-p+1\,\dots\,n+1} \varpi_{n+1}. \end{aligned}$$

Namely,

$$\varpi'_{n-p+1\,\dots\,n+1} = \frac{\sum_{\nu=p+1}^{n} (-1)^{\nu} x_\nu dx_{p+1} \wedge \cdots \widehat{dx_\nu} \cdots \wedge dx_n}{r_{n-p+1\,\dots\,n+1}}$$

$$= \sqrt{A'(n-p+1\dots n)} \left[\frac{\varpi_{n+1}}{df'_n \wedge \cdots \wedge df'_{n-p+1}} \right]_{S_{n-p+1\,\dots\,n+1}}.$$

General volume forms $\varpi'_{J\,n+1}$ can be explicitly written by the use of suitable coordinates transformed by isometry. □

The next theorem has been essentially stated in [**3**, Theorem 8], but has some errors in the formulae (5.6) therein. Here we state a correct version, which follows from Proposition 17.

THEOREM 20. *For $v_\emptyset = v(D^-_{12\,\dots\,n})$, we have*

$$\begin{aligned} d_{A'} v_\emptyset = & -\sum_{p=1}^{n-1} \sum_{|J|=p} (-1)^p \frac{(n-p-1)!}{(n-2)!}\, \theta'_J \frac{\sqrt{A'(J)}}{A'(0\,J)}\, v_J \\ & + (-1)^n \frac{1}{(n-2)!} \frac{1}{\sqrt{-A'(0\,1\dots n)}}\, \theta'_{12\,\dots\,n}, \end{aligned} \tag{51}$$

where J ranges over the collection of unordered subsets of $\{1,2,\dots,n\}$ and $|J| = p$.

In particular, if all $a'_{j0} = 0$, then

$$\begin{aligned} &\theta'_j = 0, \quad \theta'_{jk} = da'_{jk}, \\ &\theta'_J = 0 \quad \text{for } |J| \geq 3. \end{aligned}$$

Therefore, in the case of $n \geq 2$, (51) reduces to the well-known identity due to Schläfli:

$$d_{\mathcal{A}'}\, v_\emptyset = -\sum_{j<k} \frac{1}{n-2} \frac{1}{\sqrt{A'(j\,k)}}\, v_{jk}\, da'_{jk}. \tag{52}$$

For elementary proofs, refer to [**13**] and [**16**].

To prove this theorem, we need the following lemma equivalent to Proposition 9.

LEMMA 21. *We have the identity*

$$\begin{aligned} \left[\frac{1}{f'_j}\right]_{P_j} &= \left[\frac{1}{f_j}\right]_{P_j} \\ &= \frac{\sqrt{-A'(\partial_j\partial_{n+1}N)\,A'(0\,\partial_{n+1}N)} + A'\begin{pmatrix} 0 & \partial_j\partial_{n+1}N \\ j & \partial_j\partial_{n+1}N \end{pmatrix}}{-A'(\partial_{n+1}N)} < 0, \end{aligned}$$

so that

$$\left[\frac{1}{f'_j}\right]_{P_j} + \frac{A'\begin{pmatrix} 0 & \partial_j\partial_{n+1}N \\ j & \partial_j\partial_{n+1}N \end{pmatrix}}{A'(\partial_{n+1}N)} = -\frac{\sqrt{-A'(\partial_j\partial_{n+1}N)\,A'(0\,\partial_{n+1}N)}}{A'(\partial_{n+1}N)}.$$

Proof of Theorem 20. Take λ_j such that all $\lambda_j = \tau > 0$ in the formula (46). Then (45) shows that

$$\begin{aligned} d_{A'} V_\emptyset &= \lim_{\tau\downarrow 0} d_{A'} \int_{\widetilde{\Delta}[P_1,\ldots,P_n]} |\Phi'(x)| \, \varpi_{n+1} \\ &= \lim_{\tau\downarrow 0} \int_{\mathfrak{z}} \Phi'(x) \, \nabla_{A'}(\varpi_{n+1}) \\ &= \lim_{\tau\downarrow 0} \int_{\widetilde{\Delta}[P_1,\ldots,P_n]} |\Phi'(x)| \, \nabla_{A'}(\varpi_{n+1}). \end{aligned}$$

In view of the formula (4.11) and the proof of Theorem 7 in [**2**], we have only to check the following fact:

$$\begin{aligned} &\lim_{\tau\downarrow 0} \frac{\prod_{j=1}^n \lambda_j}{\prod_{q=1}^{n-1}(-\lambda_\infty - n + q + 1)} \frac{A'(\partial_{n+1}N)}{A'(0\,\partial_{n+1}N)} \Bigg\{ \mathcal{J}_\lambda \left(\frac{1}{f_{\partial_{n+1}N}} \right) \\ &\quad + \sum_{j\in\partial_{n+1}N} \frac{A'\begin{pmatrix} 0 & \partial_j\partial_{n+1}N \\ j & \partial_j\partial_{n+1}N \end{pmatrix}}{A'(\partial_j\partial_{n+1}N)} \mathcal{J}_\lambda \left(\frac{1}{f_{\partial_j\partial_{n+1}N}} \right) \Bigg\} = \frac{(-1)^n}{(n-2)!\sqrt{-A'(0\,1\ldots n)}}. \end{aligned} \tag{53}$$

By the residue theorem, the left hand side reduces to n pieces of point measures at P_j and equals

$$\begin{aligned} &\lim_{\tau\downarrow 0} \frac{\tau^n}{\prod_{q=1}^{n-1}(-n\tau - n + q + 1)} \mathcal{J}(F'_{*,\partial_{n+1}N}) \frac{A'(\partial_{n+1}N)}{A'(0\,\partial_{n+1}N)} \\ &= -\lim_{\tau\downarrow 0} \frac{\tau^{n-1}}{n\prod_{q=1}^{n-2}(-n\tau - n + q + 1)} \mathcal{J}(F'_{*,\partial_{n+1}N}) \frac{A'(\partial_{n+1}N)}{A'(0\,\partial_{n+1}N)} \\ &= \sum_{j=1}^n \frac{(-1)^{n-1}}{n(n-2)!} \left\{ \left[\frac{1}{f'_j}\right]_{P_j} + \frac{A'\begin{pmatrix} 0 & \partial_j\partial_{n+1}N \\ j & \partial_j\partial_{n+1}N \end{pmatrix}}{A'(\partial_{n+1}N)} \right\} \frac{A'(\partial_{n+1}N)}{A'(0\,\partial_{n+1}N)} \left| \left[\frac{\varpi_{n+1}}{df'_n \wedge \cdots \widehat{df'_j} \cdots \wedge df'_1} \right]_{P_j} \right|. \end{aligned}$$

On the other hand, we have

$$\left[\frac{\varpi_{n+1}}{df'_n \wedge \cdots \widehat{df'_j} \cdots \wedge df'_1} \right]_{P_j} = \frac{(-1)^{n+1-j}}{\sqrt{A'(\partial_j\partial_{n+1}N)}}.$$

Each term in the summand of the right hand side does not depend on j and is equal to

$$\begin{aligned} &\frac{(-1)^{n-1}}{n(n-2)!} \left\{ \frac{\sqrt{-A'(\partial_j\partial_{n+1}N)\,A'(0\,\partial_{n+1}N)}}{A'(\partial_{n+1}N)} \right\} \frac{1}{\sqrt{A'(\partial_j\partial_{n+1}N)}} \\ &\quad = \frac{(-1)^{n-1}}{n\,(n-2)!} \frac{\sqrt{-A'(0\,\partial_{n+1}N)}}{A'(\partial_{n+1}N)}. \end{aligned}$$

Hence, the left hand side of (53) becomes

$$\lim_{\tau\downarrow 0} \frac{\tau^n}{\prod_{q=1}^{n-1}(-n\tau - n + q + 1)} \frac{A'(\partial_{n+1}N)}{A'(0\,\partial_{n+1}N)} \mathcal{J}(F'_{*,\partial_{n+1}N}) = \frac{(-1)^n}{(n-2)!} \frac{1}{\sqrt{-A'(0\,\partial_{n+1}N)}}.$$

In this way, we have proved Theorem 20. □

Remark. In three dimensional case, i.e., for $n = 3$, D^-_{123} is a pseudo triangle $\tilde{\Delta} P_1 P_2 P_3$ with circular arc sides. Theorem 20 shows the identity

$$\begin{aligned} d_{A'} v_\emptyset = \sum_{j=1}^{3} \theta'_j \frac{1}{A'(0\,j)} v_j - \sum_{j<k} \theta'_{jk} \frac{\sqrt{A'(j\,k)}}{A'(0\,j\,k)} \\ - \frac{1}{\sqrt{-A'(0\,1\,2\,3)}} \theta'_{123}. \end{aligned} \tag{54}$$

On the other hand, Gauss–Bonnet theorem shows the identity

$$v_\emptyset = 2\pi - \sum_{j=1}^{3} a'_{j0} v_j - \sum_{j<k} (\pi - \langle jk \rangle), \tag{55}$$

where $\langle jk \rangle$ denotes the angle of the triangle at P_l ($\{j, k, l\}$: a permutation of $\{1, 2, 3\}$) such that

$$a'_{jk} = -\cos\langle jk \rangle,$$

and a'_{j0} is the geodesic curvature of the arc $\partial D^-_{123} \cap S_j$.

We can see by a direct calculation that the differential of (55) coincides with (54). Gauss–Bonnet theorem was extended into a higher dimensional polyhedral domain by Allendoerfer–Weil (see the second formula in [**1**]). However, in the case of a spherically faced simplex, the formula (51) does not seem to generally coincide with the differential of the identity due to Allendoerfer–Weil.

Appendix. Elementary proof of Theorem 2 (i).

Denote by $P_j\,(1 \le j \le n+1)$ the vertex points of the n-simplex D'^+_N such that $P_j \in \partial D^+_N \cap \bigcap_{k \in \partial_j N} S_k$. For the ordered set $J = \{j_1, \ldots, j_p\} \in \mathcal{N}$ such that $j_1 > j_2 > \cdots > j_p$ ($|J| = p$) and $J^c = \{j^*_1 > \cdots > j^*_{n-p+1}\}$, $\widetilde{\Delta}[O_J,\, P_{J^c}]$ means the n-cell

$$\widetilde{\Delta}[O_J,\, P_{J^c}] = \widetilde{\Delta}\big[O_{j_1}, \ldots, O_{j_p}, P_{j^*_1}, \ldots, P_{j^*_{n-p+1}}\big]$$

with the vertices $O_{j_1}, \ldots, O_{j_p}$ and $P_{j^*_1}, \ldots, P_{j^*_{n-p+1}}$. Notice that $\widetilde{\Delta}[P_{j^*_1}, \ldots, P_{j^*_{n-p+1}}] = S_J \cap D^+_N$ is a pseudo $(p-1)$-simplex with the faces $S_k \cap S_J \cap D^+_N\,(k \in J^c)$ in the $(n-p)$-dimensional sphere $S_J = \bigcap_{j \in J} S_j$. As a set this cell consists of all segments joining any point of $(p-1)$-simplex $\Delta[O_{j_1}, \ldots, O_{j_p}]$ and the pseudo $(n-p)$-simplex $\widetilde{\Delta}[P_{j^*_1}, \ldots, P_{j^*_{n-p+1}}]$.

We have the cell decomposition of $\Delta[O_{n+1}, \ldots, O_2, O_1]$:

$$\Delta[O_{n+1},\dots,O_2,O_1] = -\sum_{p=0}^{n}\sum_{|J|=p}\widetilde{\Delta}[O_J,P_{J^c}]\,\varepsilon_J,$$

where ε_J denotes $(-1)^{\sum_{j\in J^c} j}\cdot(-1)^{(n-p)(n-p+1)/2}$.

For example, in the case $n=2$ (see Figure 2), this partition is simply represented as

$$\begin{aligned}\Delta[O_3,O_2,O_1] &= \widetilde{\Delta}[P_3,P_2,P_1]+\widetilde{\Delta}[O_1,P_2,P_3]+\widetilde{\Delta}[O_2,P_3,P_1]+\widetilde{\Delta}[O_3,P_1,P_2]\\ &\quad+\widetilde{\Delta}[O_3,O_2,P_1]+\widetilde{\Delta}[O_2,O_1,P_3]+\widetilde{\Delta}[O_1,O_3,P_2].\end{aligned}$$

Hence we have the identity for their volumes

$$v(\Delta[O_{n+1},\dots,O_1]) = \sum_{J\in\mathcal{N},\,|J|\le n} v(\widetilde{\Delta}[O_J,P_{J^c}]),$$

or equivalently,

$$v(\widetilde{\Delta}[P_N]) = v(\Delta[O_{n+1},\dots,O_1]) - \sum_{J\in\mathcal{N},\,1\le|J|\le n} v(\widetilde{\Delta}[O_J,P_{J^c}]).$$

The identity stated in Theorem 2 (i) is a direct consequence of the following Lemma.

Lemma 22.

$$v(\widetilde{\Delta}[O_J,P_{J^c}]) = \frac{(n-p)!}{n!}\sqrt{\frac{(-1)^{p+1}B(0\star J)}{2^p}}\,v_J.$$

Proof. Without losing generality, we may assume that f_j have the reduced form (7), (8) and $J=\{n+1,n,\dots,n-p+2\}$.

$O_j\,(n-p+2\le j\le n+1)$ can be expressed as

$$O_j = (-\alpha_{j1},\,\dots,\,-\alpha_{j\,n-j+1},0,\dots,0)\quad(\alpha_{j\,n-j+1}>0).$$

The pseudo $(n-p)$-simplex $\widetilde{\Delta}[P_{n-p+1},\dots,P_1]$ with support $D_N^+\cap S_J$ is defined by the equations for $\xi=(\xi_1,\dots,\xi_n)$:

$$f_j(\xi)=0\,(n-p+2\le j\le n+1),\quad f_k(\xi)\ge 0\,(1\le k\le n-p+1). \tag{56}$$

The coordinates $\xi_j\quad(1\le j\le p-1)$ are uniquely determined by (56) and denoted by γ_j.

ξ ranges over the $(n-p)$-dimensional sphere

$$\xi=(\gamma_1,\dots,\gamma_{p-1},\xi_p,\dots,\xi_n)$$

under the condition

$$f_k(\xi)\ge 0\quad(1\le k\le n-p+1),$$

$$\sum_{j=p}^{n+1} \xi_j^2 = r^2_{n-p+2\,\ldots\,n+1},$$

where $r_{n+1\,\ldots\,n-p+2}$ denotes the radius of the hypersphere S_J:

$$r_{n-p+2\,\ldots\,n+1} = \sqrt{\frac{(-1)^p\, B(0\, n-p+2\ \ldots\ n+1)}{2^{p-1}}}.$$

The n-pseudo simplex $\widetilde{\Delta}[O_{n+1}, \ldots, O_{n-p+2}, P_{n-p+1}, \ldots, P_1]$ consist of the union of the p-simplex $\Delta[O_{n+1}, \ldots, O_{n-p+2}, \xi]$ with $\xi \in \tilde{\Delta}[P_{n-p+1}, \ldots, P_1]$:

$$\widetilde{\Delta}[O_{n+1}, \ldots, O_{n-p+2}, P_{n-p+1}, \ldots, P_1] = \bigcup_{\xi \in \tilde{\Delta}[P_{n-p+1},\ldots,P_1]} \Delta[O_{n+1}, \ldots, O_{n-p+2}, \xi].$$

Namely, every point of $\widetilde{\Delta}[O_{n+1}, \ldots, O_{n-p+2}, P_{n-p+1}, \ldots, P_1]$ is parametrized by the expression:

$$\begin{aligned} x_j &= -\sum_{k=1}^{j} y_k\, \alpha_{n-k+1,j} + y_0\, \gamma_j \quad (1 \le j \le p-1),\\ x_j &= y_0\, \xi_j \quad (p \le j \le n) \end{aligned}$$

such that $y = (y_0, \ldots, y_{p-1})$ ranges over the p-convex set

$$\delta_p : y_j \ge 0\, (0 \le j \le p-1), \quad \sum_{j=0}^{p-1} y_j \le 1.$$

Hence, the volume of $v(\tilde{\Delta}[O_{n+1}, \ldots, O_{n-p+2}, P_{n-p+1}, \ldots, P_1])$ is the mixed volume of the $(p-1)$-simplex $\Delta[O_{n+1}, \ldots, O_{n-p+2}]$ and the pseudo $(n-p)$-simplex $\widetilde{\Delta}[P_{n-p+1}, \ldots, P_1]$. In view of (9), (13) and (15),

$$\begin{aligned} &v(\widetilde{\Delta}[O_{n+1}, \ldots, O_{n-p+2}, P_{n-p+1}, \ldots, P_1])\\ &\quad = \int_{\widetilde{\Delta}[O_{n+1},\ldots,O_{n-p+2},P_{n-p+1},\ldots,P_1]} |dx_1 \wedge \cdots \wedge dx_{p-1} \wedge dx_p \wedge \cdots \wedge dx_n|\\ &\quad = \prod_{j=1}^{p-1} \alpha_{n-j+1,j} \int_{\delta_p} y_0^{n-p} dy_1 \wedge \cdots \wedge dy_{p-1} \wedge dy_0,\\ &\int_{\widetilde{\Delta}[P_{n-p+1},\ldots,P_1]} \left| \sum_{\nu=p}^{n} (-1)^{\nu-p}\, \xi_\nu\, d\xi_p \wedge \cdots < d\xi_\nu > \cdots \wedge d\xi_n \right|\\ &\quad = \frac{(n-p)!}{n!} \sqrt{\frac{(-1)^{p+1}\, B(0 \star n-p+2 \ldots n+1)}{2^p}}\, v_{n-p+2\,\ldots\,n+1}, \end{aligned}$$

since

$$\int_{\delta_p} y_0^{n-p} dy_1 \wedge \cdots \wedge dy_{p-1} \wedge dy_0 = \frac{(n-p)!}{n!}.$$

In this way, Lemma 22 has been proved in the case where $J = \{n+1, \dots, n-p+2\}$. Therefore it also holds true for general J because of symmetry. □

Theorem 1 (i) can also be proved in a similar way.

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Kazuhiko Aomoto
Nagoya University
Nagoya-shi 464-8602, Japan
E-mail: kazuhiko@aba.ne.jp

Yoshinori Machida
Shizuoka University
Shizuoka-shi 422-8529, Japan
E-mail: yomachi212@gmail.com