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Isotropic quadrangular algebras

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Abstract. Quadrangular algebras arise in the theory of Tits quadrangles. They are anisotropic if and only if the corresponding Tits quadrangle is, in fact, a Moufang quadrangle. Anisotropic quadrangular algebras were classified in the course of classifying Moufang polygons. In this paper we extend the classification of anisotropic quadrangular algebras to a classification of isotropic quadrangular algebras satisfying a natural non-degeneracy condition.

1. Introduction.

The notion of a quadrangular algebra arose in the classification of Moufang quadrangles in [**12**], where quadrangular algebras played a role analogous to the role played by quadratic Jordan division algebras of degree 3 in the classification of Moufang hexagons. A formal definition and a purely algebraic classification of quadrangular algebras were given subsequently in [**13**].

The definition of a quadrangular algebra in [**13**] requires that a certain quadratic form q be anisotropic and that a second quadratic map π be anisotropic in the sense given in D2 of [**13**, Definition 1.17]. These two conditions are both satisfied by the quadrangular algebras that arise in the study of Moufang quadrangles. At the time, we saw no geometric interpretation of the notion of a quadrangular algebra without these two conditions and so we simply included them both in the definition.

In [**7**], we introduced the notion of a Tits polygon. This notion generalizes the notion of a Moufang polygon. In [**8**], we show that the root group data of a Tits quadrangle coming from an exceptional group has a natural parametrization by an algebraic structure satisfying all the properties of a quadrangular algebra except for the two anisotropic conditions (and these two anisotropic conditions do hold if and only if the Tits quadrangle is, in fact, a Moufang quadrangle). In light of this observation, we want to correct the definition of a quadrangular algebra by omitting these two conditions. We give the new definition in Definition 2.1 below.

We will say that a quadrangular algebra (in this new sense) is *anisotropic* if both of the omitted conditions do, in fact, hold and we will say that a quadrangular algebra is *isotropic* if either of these conditions fails to hold. The quadrangular algebras classified in [**13**] are thus the anisotropic quadrangular algebras. Our main goal in this paper is give

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the classification of isotropic quadrangular algebras. More precisely, we give the classification (in Theorem 5.10) of proper quadrangular algebras (as defined in Definition 5.4) such that the map h in Definition 2.1 is non-degenerate (as defined in Observation 5.8).

In an appendix, we indicate the connection between quadrangular algebras, buildings, Tits indices and the exceptional groups. In particular, we observe in the appendix that there is a natural correspondence (which can be described in terms of root group data and Tits indices) between the quadrangular algebras that appear in Theorem 5.10(i)–(ii) (up to isotopy) and the Tits quadrangles that arise from the exceptional groups (up to isomorphism). This correspondence, which is summarized in Table 1, and a characterization of this class of Tits quadrangles are the subject of [**8**].

2. Quadrangular algebras.

Here is our new definition of a quadrangular algebra:

DEFINITION 2.1. A *quadrangular algebra* is an ordered set

$$(K, L, q, f, \varepsilon, X, \cdot, h, \theta),$$

where K is a field, L is a vector space over K, q is a non-degenerate quadratic form on L (see Notation 2.2), f is the bilinear form associated with q, ε is an element of L such that $q(\varepsilon) = 1$, X is a non-trivial vector space over K, $(a, v) \mapsto a \cdot v$ is a map from $X \times L$ to X (which is denoted below, and, in general, simply by juxtaposition), h is a map from $X \times X$ to L and θ a map from $X \times L$ to L satisfying the following axioms:

(A1) The map $\cdot$ is bilinear (over K).

(A2) $a \cdot \varepsilon = a$ for all $a \in X$.

(A3) $(av)v^\sigma = q(v)a$ for all $a \in X$ and all $v \in L$, where

$$v^\sigma = f(v, \varepsilon)\varepsilon - v. \tag{2.1}$$

(B1) h is bilinear (over K).

(B2) $h(a, bv) = h(b, av) + f(h(a, b), \varepsilon)v$ for all $a, b \in X$ and all $v \in L$.

(B3) $f(h(av, b), \varepsilon) = f(h(a, b), v)$ for all $a, b \in X$ and all $v \in L$.

(C1) For each $a \in X$, the map $v \mapsto \theta(a, v)$ is linear (over K).

(C2) $\theta(ta, v) = t^2\theta(a, v)$ for all $t \in K$, all $a \in X$ and all $v \in L$.

(C3) There exists a function g from $X \times X$ to K such that

$$\theta(a + b, v) = \theta(a, v) + \theta(b, v) + h(a, bv) - g(a, b)v$$

for all $a, b \in X$ and all $v \in L$.

(C4) There exists a function ϕ from $X \times L$ to K such that

$$\theta(av, w) = q(v)\theta(a, w^\sigma)^\sigma - f(w, v^\sigma)\theta(a, v)^\sigma \\ + f(\theta(a, v), w^\sigma)v^\sigma + \phi(a, v)w$$

for all $a \in X$ and $v, w \in L$.

(D1) Let $\pi(a) = \theta(a, \varepsilon)$ for all $a \in X$. Then

$$a\theta(a, v) = (a\pi(a))v$$

for all $a \in X$ and all $v \in L$.

Notation 2.2. By the assumption in Definition 2.1 that q is *non-degenerate*, we mean that the restriction of q to the radical of f is anisotropic (as in [**11**, 8.2.3]). If $\mathrm{char}(K) \neq 2$ (in which case $q(v) = f(v, v)/2$ for all $v \in L$), it follows from this assumption that, in fact, f is non-degenerate.

Definition 2.3. Let $\Xi = (K, L, q, f, \varepsilon, X, \cdot, h, \theta)$ be a quadrangular algebra and let π be the map from X to L that appears in D1 of Definition 2.1. We will say that π is *anisotropic* if $\pi(a) \in \langle\varepsilon\rangle$ implies that $a = 0$. We will say that Ξ is *anisotropic* if both q and π are anisotropic and we will say that Ξ is *isotropic* if q or π fails to be anisotropic.

Observation 2.4. In [**13**, (1.3)], v^{-1} is defined to be $v^\sigma/q(v)$ for all $v \in L$ such that $q(v) \neq 0$. Thus A3 in Definition 2.1 and A3 in [**13**, Definition 1.17] coincide when q is anisotropic. In Definition 2.1 we have eliminated D2 of [**13**, Definition 1.17]. All the remaining axioms of Definition 2.1 and [**13**, Definition 1.17] are identical (except that we write all the scalars on the left in Definition 2.1). Thus the quadrangular algebras as defined in [**13**, Definition 1.17] are precisely the anisotropic quadrangular algebras as defined in Definition 2.3.

Remark 2.5. We have made two small changes in the notation: In Definition 2.1, we denote the basepoint of (K, L, q) by ε rather than 1, and we include the bilinear form f in the list of spaces and maps comprising the quadrangular algebra.

Remark 2.6. From now on, we will refer to the axioms $\mathrm{A1}, \mathrm{A2}, \ldots, \mathrm{D1}$ in Definition 2.1 without explicitly referencing Definition 2.1.

Observation 2.7. We mention that the algebraic parts of [**12**, Chapters 21–28] inspired a different set of axioms in [**2**]. The algebraic structures studied in [**2**] serve as parameter algebras for arbitrary Moufang quadrangles, not just the exceptional ones.

Observation 2.8. Let $C(q, \varepsilon)$ denote the Clifford algebra with basepoint as defined in [**5**] (or [**12**, Definition 12.47] or [**13**, Definition 2.21]). (By [**12**, (12.51)], $C(q, \varepsilon)$ is canonically isomorphic to the even Clifford algebra $C_0(q)$.) By A1–A3, the map $\cdot$ from $X \times L$ to X extends uniquely to a map from $X \times C(q, \varepsilon)$ to X making X into a right $C(q, \varepsilon)$-module, If we replace X by a non-zero submodule X_0 for $C(q, \varepsilon)$, then all the conditions in Definition 2.1 continue to hold. We apply this observation in Notation 4.14

and in the proofs of Propositions 7.3 and 11.15. In general, however, we cannot be certain that the restrictions of θ to $X_0 \times L_0$ and of h to $X_0 \times X_0$ are not identically zero, so the resulting quadrangular algebra might not be very interesting. We give an example of this phenomenon in Observation 4.15 below.

3. Composition algebras.

In this section we assemble some elementary observations about composition algebras that will be needed in the next section.

NOTATION 3.1. Let (C, K) be a composition algebra. Thus one of the following holds:

(i) C/K is a field extension, $\mathrm{char}(K) = 2$ and $C^2 \subset K$.

(ii) $C = K$ and $\mathrm{char}(K) \neq 2$.

(iii) C/K is a quadratic étale extension, i.e. C/K is either a separable field extension or $C = K \oplus K$.

(iv) C is quaternion and $K = Z(C)$.

(v) C is octonion and $K = Z(C)$.

Let n_C denote the norm of (C, K), t_C its trace and σ_C its standard involution. Thus $n_C(e) = e^\sigma e \in K$ and $t_C(e) = e + e^\sigma \in K$ for all $e \in C$, where $\sigma = \sigma_C$. In cases (i) and (ii), $\sigma_C = 1$ and in case (iii), $(s,t)^\sigma = (t,s)$ for all $(s,t) \in C$ if $C = K \oplus K$. The norm n_C is a quadratic form over K. In cases (i) and (ii), n_C is anisotropic; in case (iii), n_C is anisotropic if C is a field and hyperbolic if $C = K \oplus K$. Also in the remaining two cases, n_C is either hyperbolic or anisotropic. The composition algebra (C, K) is called *division* if n_C is anisotropic and *split* if either $\dim_K C = 1$ or (C, K) is in one of the cases (iii), (iv) or (v) and n_C is hyperbolic. We refer to $\dim_K C$ as the *dimension* of (C, K). If (C, K) is split, it is uniquely determined by K and the dimension of (C, K).

NOTATION 3.2. Let $s_C(a, b) = t_C(a^\sigma b)$ for all $a, b \in C$. The form s_C is the bilinear form associated with n_C. In particular, s_C is identically zero if (C, K) is as in Notation 3.1(i) and s_C is non-degenerate otherwise

REMARK 3.3. In case (i) of Notation 3.1, we make no restriction on $\dim_K C$; in particular, this dimension is allowed to be infinite. In the remaining cases, $\dim_K C$ divides 8.

REMARK 3.4. Note that in cases (i) and (iii) of Notation 3.1, K is not uniquely determined by C. Nevertheless, we write, for example, n_C rather than $n_{(C,K)}$. We will always have at most one composition algebra in mind, so this commonly used convention should not cause any confusion.

Since $t_C(e)e = et_C(e)$, we have

$$n_C(e) = e^\sigma e = t_C(e)e - e^2 = et_C(e) - e^2 = ee^\sigma \tag{3.1}$$

for all $e \in C$. It follows that

$$n_C(a^\sigma + b) = (a^\sigma + b)(a + b^\sigma) = (a + b^\sigma)(a^\sigma + b)$$

as well as

$$t_C(ba) = (a^\sigma + b)(a + b^\sigma) - n_C(a) - n_C(b)$$

and

$$t_C(ab) = (a + b^\sigma)(a^\sigma + b) - n_C(a) - n_C(b)$$

for all $a, b \in C$. Therefore

$$t_C(ab) = t_C(ba) \tag{3.2}$$

for all $a, b \in C$.

The *associator* of (C, K) is the map $(a, b, c) \mapsto [a, b, c]$ from $C \times C \times C$ to C given by

$$[a, b, c] = ab \cdot c - a \cdot bc$$

for all $a, b, c \in C$. The Moufang identities [**12**, Definition 9.1(ii) and (iii)] hold in C and the associator is trilinear and, by [**12**, (9.14)], alternating. (Note that he proof of [**12**, (9.14)] does not require (C, K) to be division). It follows that

$$[a^\sigma, b, c] = [a, b^\sigma, c] = [a, b, c^\sigma] = -[a, b, c] \tag{3.3}$$

for all $a, b, c \in C$. Hence

$$\begin{aligned}[a, b, c]^\sigma &= c^\sigma \cdot b^\sigma a^\sigma - c^\sigma b^\sigma \cdot a^\sigma \\ &= -[c^\sigma, b^\sigma, a^\sigma] = [c, b, a] = -[a, b, c]\end{aligned}$$

and thus

$$t_C(ab \cdot c - a \cdot bc) = t_C([a, b, c]) = 0$$

for all $a, b, c \in C$. Therefore

$$t_C(ab \cdot c) = t_C(a \cdot bc)$$

for all $a, b, c \in C$. We will thus, in general, write $t_C(abc)$ rather than $t_C(a \cdot bc)$ or $t_C(ab \cdot c)$ to denote the trace of a product of three terms a, b and c. By (3.2), we have $t_C(abc) = t_C(cab) = t_C(bca)$ for all $a, b, c \in C$.

Proposition 3.5. *Let $a, b, c, e \in C$. Then the following hold*:

(i) $(a \cdot cb^\sigma)e = b^\sigma e \cdot t_C(ac) - t_C(e^\sigma ba) \cdot c^\sigma + c^\sigma e^\sigma \cdot ba$.

(ii) $a(ec \cdot b^\sigma) = ae \cdot t_C(cb^\sigma) - t_C(aeb) \cdot c^\sigma + b^\sigma e^\sigma \cdot a^\sigma c^\sigma$.

PROOF. Replacing a by c^σ, d by a^σ, b by b^σ and c by e in [**12**, (9.16)(ii)] (whose proof does not require (C, K) to be division), we obtain the identity

$$[c^\sigma a^\sigma, b^\sigma, e] - c^\sigma[a^\sigma, b^\sigma, e] = [c^\sigma, a^\sigma b^\sigma, e] - [c^\sigma, a^\sigma, b^\sigma e] + [c^\sigma, a^\sigma, b^\sigma]e,$$

which we can rewrite as

$$-[ac, b^\sigma, e] - c^\sigma[e^\sigma, b, a] = -[c^\sigma, e^\sigma, ba] - [c^\sigma, a^\sigma, b^\sigma e] - [a, c, b^\sigma]e$$

using (3.3) and the fact that the associator is alternating. Expanding each associator in this identity, we obtain (i). We obtain (ii) by applying σ to every term in (i) and then replacing e by a^σ, b by e, c by c^σ and a by b. □

LEMMA 3.6. *Let $f(a, b, c, e) = [ab, c, e] - b[a, c, e] - [b, c, e]a$ for all $a, b, c, e \in C$. Then f is alternating.*

PROOF. This holds by [**12**, (9.20)]. □

PROPOSITION 3.7. *Suppose that* $\mathrm{char}(K) = 2$. *Then*

$$c[b, a, e] + b[c, a, e] + [c, e, ab] + [b, e, ac] = 0$$

for all $a, b, c, e \in C$.

PROOF. Let f be as in Lemma 3.6. Since f and the associator are both alternating and hence both symmetric since $\mathrm{char}(K) = 2$, we have

$$c[b, a, e] + [b, e, ac] + [c, e, b]a = f(a, b, c, e)$$

and

$$b[c, a, e] + [c, e, ab] + [c, e, b]a = f(a, b, c, e)$$

for all $a, b, c, e \in C$. Adding these two identities, we obtain the desired conclusion. □

4. Examples.

Let (C, K), n_C, t_C and σ_C be as in Notation 3.1. We set $N = n_C$, $T = t_C$ and $\sigma = \sigma_C$.

NOTATION 4.1. Let $L_C = K \oplus K \oplus K \oplus K \oplus C$ and let

$$q_C(t_1, t_2, t_3, t_4, e) = t_1t_4 + t_2t_3 + N(e)$$

for all $(t_1, t_2, t_3, t_4, e) \in L_C$. Thus q_C is a quadratic form, the Witt index of q_C is 2 if (C, K) is division and q_C is hyperbolic if (C, K) is not division.

NOTATION 4.2. Let $X = C \oplus C \oplus C \oplus C$ and let $L = L_C$ be as in Notation 4.1. We set

$$(a, b, c, d) \cdot (t_1, t_2, t_3, t_4, e)$$

equal to the the element

$$\big(ae + b^\sigma t_2 + c^\sigma t_1,\ eb - a^\sigma t_3 - d^\sigma t_1,\\ ec - a^\sigma t_4 + d^\sigma t_2,\ de + b^\sigma t_4 - c^\sigma t_3\big)$$

of X for all $(a, b, c, d) \in X$ and all $(t_1, t_2, t_3, t_4, e) \in L$. Thus $(u, v) \mapsto u \cdot v$ is a bilinear map from $X \times L$ to X.

Observation 4.3. Let $q = q_C$, let ε denote the element $(0, 0, 0, 0, 1)$ of L and let $\cdot$ be as in Notation 4.2. Then $q(\varepsilon) = 1$, $u \cdot \varepsilon = u$ for all $u \in X$ and

$$(t_1, t_2, t_3, t_4, e)^\sigma = (-t_1, -t_2, -t_3, -t_4, e^\sigma)$$

for all $(t_1, t_2, t_3, t_4, e) \in L$, where the σ on the left is as in A3 of Definition 2.1 and the σ on the right is σ_C. It follows immediately from the formulas that $uv \cdot v^\sigma = q(v)u$ for all $u \in X$ and all $v \in L$. Note also that

$$f\big((t_1, t_2, t_3, t_4, e), \varepsilon\big) = T(e) \tag{4.1}$$

for all $(t_1, t_2, t_3, t_4, e) \in L$.

Notation 4.4. Let h denote the bilinear map from $X \times X$ to L given by

$$h\big((a, b, c, d), (a', b', c', d')\big) = \big(-T(ab' + a'b),\ T(ac' + a'c),\\ T(bd' + b'd),\ T(cd' + c'd),\ a^\sigma d' - d^\sigma a' - c'b^\sigma + b'c^\sigma\big)$$

for all $(a, b, c, d), (a', b', c', d') \in X$.

Proposition 4.5. *Let h be as in* Notation 4.4*, let $\cdot$ be as in* Notation 4.2*, let ε be as in* Observation 4.3 *and let $f = f_C$ be the bilinear form associated with q_C. Then*

(i) $h(u, u_0v) = h(u_0, uv) + f(h(u, u_0), \varepsilon)v$ *and*

(ii) $f(h(uv, u_0), \varepsilon) = f(h(u, u_0), v)$

for all $u, u_0 \in X$ and all $v \in L$.

Proof. Choose elements $u = (a, b, c, d)$ and $u_0 = (a_0, b_0, c_0, d_0)$ in X and an element $v = (t_1, t_2, t_3, t_4, e)$ in L. Then

$$f(h(u, u_0), \varepsilon) = T(a^\sigma d_0 - d^\sigma a_0 - c_0 b^\sigma + b_0 c^\sigma) \tag{4.2}$$

by (4.1), the first coordinate of $h(u, u_0v)$ is

$$-T\big(a(eb_0 - a_0^\sigma t_3 - d_0^\sigma t_1) + (a_0 e + b_0^\sigma t_2 + c_0^\sigma t_1)b\big)$$

and the first coordinate of $h(u_0, uv)$ is

$$-T\big(a_0(eb - a^\sigma t_3 - d^\sigma t_1) + (ae + b^\sigma t_2 + c^\sigma t_1)b_0\big).$$

Hence the first coordinate of $h(u, u_0v) - h(u_0, uv)$ is

$$t_1T(ad_0^\sigma - c_0^\sigma b) - t_1T(a_0d^\sigma - b_0c^\sigma)$$

and this expression equals $t_1f(h(u, u_0), \varepsilon)$. Thus the first coordinates on both sides of the identity (i) are equal. By similar calculations, also the second, third and fourth coordinates are equal. The last coordinate of $h(u, u_0v)$ is

$$\begin{aligned}a^\sigma(d_0e{+}b_0^\sigma t_4 - c_0^\sigma t_3) &- d^\sigma(a_0e + b_0^\sigma t_2 + c_0^\sigma t_1)\\ &- (ec_0 - a_0^\sigma t_4 + d_0^\sigma t_2)b^\sigma + (eb_0 - a_0^\sigma t_3 - d_0^\sigma t_1)c^\sigma\end{aligned}$$

and the last coordinate of $h(u_0, uv)$ is

$$\begin{aligned}a_0^\sigma(de{+}b^\sigma t_4 - c^\sigma t_3) &- d_0^\sigma(ae + b^\sigma t_2 + c^\sigma t_1)\\ &- (ec - a^\sigma t_4 + d^\sigma t_2)b_0^\sigma + (eb - a^\sigma t_3 - d^\sigma t_1)c_0^\sigma.\end{aligned}$$

It follows that the last coordinate of $h(u, u_0v) - h(u_0, uv)$ is precisely

$$T(a^\sigma d_0 - d^\sigma a_0 - c_0b^\sigma + b_0c^\sigma)e.$$

By (4.2), we conclude that (i) holds.

The expression $f(h(uv, u_0), \varepsilon)$ equals

$$\begin{aligned}T\Big((ae + b^\sigma t_2 + c^\sigma t_1)^\sigma d_0 &- (de + b^\sigma t_4 - c^\sigma t_3)^\sigma a_0\\ &- c_0(eb - a^\sigma t_3 - d^\sigma t_1)^\sigma + b_0(ec - a^\sigma t_4 + d^\sigma t_2)^\sigma\Big).\end{aligned}$$

The expression $f(h(u, u_0), v)$, on the other hand, equals

$$\begin{aligned}-T(ab_0 + a_0b)t_4 + T(cd_0 + c_0d)t_1 + T(ac_0 + a_0c)t_3 + T(bd_0 + b_0d)t_2&\\ + T\big((a^\sigma d_0 - d^\sigma a_0 - c_0b^\sigma + b_0c^\sigma)e^\sigma\big)&.\end{aligned}$$

These two expressions are equal and thus (ii) holds. □

Notation 4.6. Let θ denote the map from $X \times L$ to L given by

$$\begin{aligned}&\theta\big((a, b, c, d), (t_1, t_2, t_3, t_4, e)\big) =\\ &\Big(-T(aeb) + N(a)t_3 - N(b)t_2 + \tfrac{1}{2}t_1T(a^\sigma d - b^\sigma c),\\ &\quad T(aec) + N(c)t_1 - N(a)t_4 + \tfrac{1}{2}t_2T(a^\sigma d + b^\sigma c),\\ &\quad T(deb) + N(b)t_4 - N(d)t_1 - \tfrac{1}{2}t_3T(a^\sigma d + b^\sigma c),\\ &\quad T(dec) + N(d)t_2 - N(c)t_3 - \tfrac{1}{2}t_4T(a^\sigma d - b^\sigma c),\\ &\quad t_4a^\sigma b^\sigma - t_3a^\sigma c^\sigma - t_2d^\sigma b^\sigma - t_1d^\sigma c^\sigma + \tfrac{1}{2}\big(a^\sigma \cdot de - d^\sigma \cdot ae - ec \cdot b^\sigma + eb \cdot c^\sigma\big)\Big)\end{aligned}$$

for all $(a,b,c,d) \in X$ and all $(t_1,t_2,t_3,t_4,e) \in L$ if $\mathrm{char}(K) \neq 2$ and by

$$\begin{aligned}
&\theta\big((a,b,c,d),(t_1,t_2,t_3,t_4,e)\big) = \\
&\quad\Big(T(aeb) + N(a)t_3 + N(b)t_2 + t_1T(a^\sigma d + b^\sigma c), \\
&\qquad\quad T(aec) + N(c)t_1 + N(a)t_4 + t_2T(a^\sigma d + b^\sigma c), \\
&\qquad\qquad T(deb) + N(b)t_4 + N(d)t_1 + t_3T(a^\sigma d + b^\sigma c), \\
&\qquad\qquad\quad T(dec) + N(d)t_2 + N(c)t_3 + t_4T(a^\sigma d + b^\sigma c), \\
&\qquad\qquad\qquad t_4a^\sigma b^\sigma + t_3a^\sigma c^\sigma + t_2d^\sigma b^\sigma + t_1d^\sigma c^\sigma + a^\sigma \cdot de + ec \cdot b^\sigma\Big)
\end{aligned}$$

for all $(a,b,c,d) \in X$ and all $(t_1,t_2,t_3,t_4,e) \in L$ if $\mathrm{char}(K) = 2$.

Proposition 4.7. *$\theta(u,v) = h(u,uv)/2$ for all $u \in X$ and all $v \in L$ if $\mathrm{char}(K) \neq 2$.*

Proof. Choose elements $u = (a,b,c,d)$ in X and $v = (t_1,t_2,t_3,t_4,e)$ in L. Then $h(u,uv)$ equals

$$\begin{aligned}
&\Big(-T\big(a(eb - a^\sigma t_3 - d^\sigma t_1) + (ae + b^\sigma t_2 + c^\sigma t_1)b\big), \\
&\qquad T\big(a(ec - a^\sigma t_4 + d^\sigma t_2) + (ae + b^\sigma t_2 + c^\sigma t_1)c\big), \\
&\qquad\quad T\big(b(de + b^\sigma t_4 - c^\sigma t_3) + (eb - a^\sigma t_3 - d^\sigma t_1)d\big), \\
&\qquad\qquad T\big(c(de + b^\sigma t_4 - c^\sigma t_3) + (ec - a^\sigma t_4 + d^\sigma t_2)d\big), \\
&\qquad\qquad\quad a^\sigma(de + b^\sigma t_4 - c^\sigma t_3) - d^\sigma(ae + b^\sigma t_2 + c^\sigma t_1) \\
&\qquad\qquad\qquad\quad - (ec - a^\sigma t_4 + d^\sigma t_2)b^\sigma + (eb - a^\sigma t_3 - d^\sigma t_1)c^\sigma\Big)
\end{aligned}$$

It is straightforward to check that this expression equals $2\theta(u,v)$. $\square$

Notation 4.8. Let $\pi(u) = \theta(u,\varepsilon)$ for all $u \in X$, where ε is as in Notation 4.3. Then

$$\pi\big(a,b,c,d\big) = \Big(-T(ab),\ T(ac),\ T(bd),\ T(cd),\ \tfrac{1}{2}\big(a^\sigma d - d^\sigma a - cb^\sigma + bc^\sigma\big)\Big)$$

for all $(a,b,c,d) \in X$ if $\mathrm{char}(K) \neq 2$ and

$$\pi\big(a,b,c,d\big) = \big(T(ab),\ T(ac),\ T(bd),\ T(cd),\ a^\sigma d + cb^\sigma\big)$$

for all $(a,b,c,d) \in X$ if $\mathrm{char}(K) = 2$.

Lemma 4.9. *Let $u = (a,b,c,d) \in X$. Then $u\pi(u)$ equals*

$$\begin{aligned}
&\big(aE + b^\sigma T(ac) - c^\sigma T(ab), \\
&\qquad Eb - a^\sigma T(bd) + d^\sigma T(ab), \\
&\qquad\quad Ec - a^\sigma T(cd) + d^\sigma T(ac), \\
&\qquad\qquad dE + b^\sigma T(cd) - c^\sigma T(bd)\big),
\end{aligned}$$

where

$$E = \tfrac{1}{2}(a^\sigma d - d^\sigma a - cb^\sigma + bc^\sigma) \tag{4.3}$$

if $\mathrm{char}(K) \neq 2$ *and*

$$E = a^\sigma d + cb^\sigma \tag{4.4}$$

if $\mathrm{char}(K) = 2$.

Proof. This holds by Notations 4.2 and 4.8. □

Proposition 4.10. $u\pi(u)v = u\theta(u,v)$ *for all* $u \in X$ *and all* $v \in L$, *where* π *is as in* Notation 4.8.

Proof. Choose $u = (a,b,c,d)$ in X, let $v = (1,0,0,0,0)$ in L and let E be as in Lemma 4.9. We assume first that $\mathrm{char}(K) \neq 2$. Then $u\pi(u)v$ equals

$$\big(c^\sigma E^\sigma - aT(cd) + dT(ac),\ -E^\sigma d^\sigma - bT(cd) + cT(bd),\ 0, 0\big)$$

and $\theta(u,v)$ equals

$$\Big(\tfrac{1}{2}T(a^\sigma d - b^\sigma c),\ N(c),\ -N(d),\ 0,\ -d^\sigma c^\sigma\Big).$$

Thus $u\theta(u,v)$ equals

$$\begin{aligned}\Big(-a\cdot d^\sigma c^\sigma &+ b^\sigma N(c) + \tfrac{1}{2}T(a^\sigma d - b^\sigma c)c^\sigma,\\ &-d^\sigma c^\sigma \cdot b + a^\sigma N(d) - \tfrac{1}{2}T(a^\sigma d - b^\sigma c)d^\sigma,\ 0,\ 0\Big).\end{aligned}$$

In the first coordinate of $u\pi(u)v$, we expand $aT(cd)$ as $a\cdot cd + a\cdot d^\sigma c^\sigma$ and $dT(ac)$ as $ac\cdot d + c^\sigma a^\sigma \cdot d$. In the first coordinate of $u\theta(u,v)$, we expand $c^\sigma T(a^\sigma d - b^\sigma c)$ as $c^\sigma \cdot a^\sigma d + c^\sigma \cdot d^\sigma a - b^\sigma c \cdot c^\sigma - c^\sigma b \cdot c^\sigma$. After collecting terms, we find that the difference between these two first coordinates is $[a,c,d] + [c^\sigma, a^\sigma, d] = 0$. Thus the first coordinates of $u\pi(u)v$ and of $u\theta(u,v)$ are equal. Expanding $bT(cd)$ as $d^\sigma c^\sigma \cdot b + cd\cdot b$, $cT(bd)$ as $c\cdot db + c\cdot b^\sigma d^\sigma$, $d^\sigma T(a^\sigma d)$ as $d^\sigma \cdot da^\sigma + d^\sigma \cdot ad^\sigma$ and $d^\sigma T(b^\sigma c)$ as $cb^\sigma \cdot d^\sigma + bc^\sigma \cdot d^\sigma$ in the second coordinates of $u\pi(u)v$ and $u\theta(u,v)$ and collecting terms, we see that they, too, are equal. Thus

$$u\pi(u)v = u\theta(u,v) \tag{4.5}$$

holds for $v = (1,0,0,0,0)$. Once these calculations are carried out, it is straightforward (and, in fact, a trifle easier) to verify that (4.5) also holds for $v = (1,0,0,0,0)$ when $\mathrm{char}(K) = 2$.

By similar calculations, it can be verified that (4.5) holds also for the elements $v = (0,1,0,0,0)$, $(0,0,1,0,0)$ and $(0,0,0,1,0)$, both when $\mathrm{char}(K) \neq 2$ and when $\mathrm{char}(K) = 2$. Since both sides of (4.5) are linear in the variable v, it remains only to show that (4.5) holds for $v = (0, 0, 0, 0, e)$, where e is an arbitrary element of C. The first coordinate of $u\pi(u)v$ is then

$$x := aE \cdot e + b^\sigma eT(ac) - c^\sigma eT(ab),$$

where E is as in (4.3) or (4.4).

Suppose that $\mathrm{char}(K) \neq 2$. Then the first coordinate of $u\theta(u, v)$ equals

$$y := \tfrac{1}{2}a(a^\sigma \cdot de - d^\sigma \cdot ae - ec \cdot b^\sigma + eb \cdot c^\sigma) + b^\sigma T(aec) - c^\sigma T(aeb).$$

Let $\delta = 2(y - x)$. Then δ equals

$$\begin{aligned} &a(eb \cdot c^\sigma) - a(ec \cdot b^\sigma) - (a \cdot bc^\sigma)e + (a \cdot cb^\sigma)e \\ &\qquad + 2b^\sigma T(aec) - 2b^\sigma eT(ac) - 2c^\sigma T(aeb) + 2c^\sigma eT(ab) \end{aligned} \tag{4.6}$$

since $a(d^\sigma \cdot ae) = (a \cdot d^\sigma a)e$ (by [**12**, Definition 9.1(ii)]) and $a(a^\sigma \cdot de) = N(a)de = (a \cdot a^\sigma d)e$. Applying Proposition 3.5(i) twice, we have

$$(a \cdot cb^\sigma)e - b^\sigma eT(ac) + T(e^\sigma ba)c^\sigma = c^\sigma e^\sigma \cdot ba$$

and

$$(a \cdot bc^\sigma)e - c^\sigma eT(ab) + T(e^\sigma ca)b^\sigma = b^\sigma e^\sigma \cdot ca.$$

Applying Proposition 3.5(ii) twice, we have

$$a(eb \cdot c^\sigma) - aeT(cb^\sigma) + b^\sigma T(aec) = c^\sigma e^\sigma \cdot a^\sigma b^\sigma$$

and

$$a(ec \cdot b^\sigma) - aeT(bc^\sigma) + c^\sigma T(aeb) = b^\sigma e^\sigma \cdot a^\sigma c^\sigma.$$

We then observe that

$$c^\sigma e^\sigma \cdot ba + c^\sigma e^\sigma \cdot a^\sigma b^\sigma = c^\sigma e^\sigma T(ab)$$

and

$$b^\sigma e^\sigma \cdot ca + b^\sigma e^\sigma \cdot a^\sigma c^\sigma = b^\sigma e^\sigma T(ac).$$

By (4.6), we conclude that

$$\begin{aligned} \delta = {}& b^\sigma T(eca) + b^\sigma T(e^\sigma ca) + c^\sigma eT(ab) + c^\sigma e^\sigma T(ab) \\ &\qquad - c^\sigma T(eba) - c^\sigma T(e^\sigma ba) - b^\sigma eT(ac) - b^\sigma e^\sigma T(ac). \end{aligned}$$

Since $T(eca) + T(e^\sigma ca) = T(e)T(ac)$ and $T(eba) + T(e^\sigma ba) = T(e)T(ab)$, we conclude that $\delta = 0$. Thus $u\pi(u)v$ and $u\theta(u, v)$ agree in the first coordinate. By similar calculations, we find that $u\pi(u)v$ and $u\theta(u, v)$ agree in the other three coordinates as well and hence

(4.5) holds for $v = (0, 0, 0, 0, e)$.

Suppose, finally, that $\mathrm{char}(K) = 2$. This time the first coordinate of $u\theta(u, v)$ is

$$a(a^\sigma \cdot de + ec \cdot b^\sigma) + b^\sigma T(aec) + c^\sigma T(aeb).$$

and hence the difference δ between the first coordinates of $u\pi(u)v$ and of $u\theta(u, v)$ is

$$a(ec \cdot b^\sigma) + (a \cdot cb^\sigma)e + b^\sigma T(aec) + c^\sigma T(aeb) + b^\sigma eT(ac) + c^\sigma eT(ab).$$

By applying Proposition 3.5(i) and (ii), we turn δ into

$$c^\sigma T(e^\sigma ba) + b^\sigma T(aec) + aeT(cb^\sigma) + c^\sigma eT(ab) + c^\sigma e^\sigma \cdot ba + b^\sigma e^\sigma \cdot a^\sigma c^\sigma.$$

Next we expand $c^\sigma T(e^\sigma ba)$ as $c^\sigma(e^\sigma \cdot ba) + c^\sigma(a^\sigma b^\sigma \cdot e)$, we expand $aeT(cb^\sigma)$ as $b^\sigma c \cdot ae + c^\sigma b \cdot ae$, we expand $b^\sigma T(aec)$ as $b^\sigma(e^\sigma \cdot a^\sigma c^\sigma) + b^\sigma(ca \cdot e)$ and we expand $c^\sigma eT(ab)$ as $c^\sigma e \cdot ba + c^\sigma e \cdot a^\sigma b^\sigma$. Then we replace $b^\sigma(e^\sigma \cdot a^\sigma c^\sigma) + b^\sigma e^\sigma \cdot a^\sigma c^\sigma$ by $[b^\sigma, e^\sigma, a^\sigma c^\sigma] = [b, e, ca]$, we replace $b^\sigma c \cdot ae + b^\sigma(ca \cdot e)$ by $b^\sigma[c, a, e] + [b^\sigma, c, ae] = b^\sigma[c, a, e] + [b, c, ae]$, we replace $c^\sigma e \cdot ba + c^\sigma e^\sigma \cdot ba$ by $c^\sigma \cdot baT(e) = c^\sigma(e^\sigma \cdot ba) + c^\sigma(e \cdot ba)$, we then replace $c^\sigma(e \cdot ba)$ by $[c^\sigma, e, ba] + c^\sigma e \cdot ba = [c, e, ba] + c^\sigma e \cdot ba$, we then replace $c^\sigma e \cdot ba + c^\sigma e \cdot a^\sigma b^\sigma$ by $c^\sigma eT(ab) = c^\sigma(a^\sigma b^\sigma \cdot e) + c^\sigma(ba \cdot e)$ and lastly, we replace $c^\sigma(ba \cdot e) + c^\sigma b \cdot ae$ by $c^\sigma[b, a, e] + [c^\sigma, b, ae] = c^\sigma[b, a, e] + [c, b, ae]$. At this point, we have

$$\delta = b^\sigma[c, a, e] + c^\sigma[b, a, e] + [b, e, ca] + [c, e, ba].$$

Hence $\delta = 0$ by Proposition 3.7. Thus $u\pi(u)v$ and $u\theta(u, v)$ agree in the first coordinate. By similar calculations, we find that $u\pi(u)v$ and $u\theta(u, v)$ agree in the other three coordinates as well and hence (4.5) holds for $v = (0, 0, 0, 0, e)$ also in the case that $\mathrm{char}(K) = 2$. □

THEOREM 4.11. *Let*

$$\Xi = (K, L_C, q_C, f_C, \varepsilon, X, \cdot, h, \theta),$$

where L_C, q_C, f_C, etc., are as in Notation *and* Observations 4.1–4.4 *and* 4.6 *and* Proposition 4.5. *Then Ξ is a quadrangular algebra.*

PROOF. By Notation 4.3 and Propositions 4.5 and 4.10, A1–B3 and D1 hold (see Remark 2.6). Thus by Proposition 4.7 and [**13**, Remark 4.8], Ξ is a quadrangular algebra if $\mathrm{char}(K) \neq 2$. It thus suffices to assume that $\mathrm{char}(K) = 2$. By Notation 4.6, C1 and C2 hold, and by lengthy but straightforward calculations, it can be checked that C3 and C4 hold in this case too. □

NOTATION 4.12. We denote the quadrangular algebra Ξ in Theorem 4.11 by

$$\mathcal{Q}_4(C, K).$$

The subscript refers to the fact that X is the direct sum of four copies of C.

REMARK 4.13. If $\Xi = \mathcal{Q}_4(C, K)$ with $C = K$, then $q(\pi(u)) = 0$ for all $u \in X$ if and only if $\mathrm{char}(K) \neq 2$, where π is as in Notation 4.8.

NOTATION 4.14. Let $\Xi = (K, L_C, q_C, f_C, \varepsilon, X, \cdot, h, \theta)$ be as in Theorem 4.11. Let

$$L_0 = \{(t_1, t_2, t_3, t_4, u) \in L_C \mid t_1 = t_2 = t_3 = t_4 = 0\},$$

let

$$X_0 = \{(a, b, c, d) \in X \mid b = c = 0\},$$

let q_0 denote the restriction of q_C to L_0 and let f_0 denote the bilinear form associated with q_0. (We can, of course, identify X_0 with $C \oplus C$ and L_0 with C so that q_0 is simply n_C and f_0 is the form s_C defined in Notation 3.2.) Note that $X_0 \cdot L_0 \subset X_0$, $h(X_0, X_0) \subset L_0$ and $\theta(X_0, L_0) \subset L_0$. Thus

$$\Xi_0 := (K, L_0, q_0, f_0, \varepsilon, X_0, \cdot_0, h_0, \theta_0)$$

is a quadrangular algebra, where $\cdot_0$, h_0 and θ_0 denote the restrictions of $\cdot$, h and θ to $X_0 \times L_0$, to $X_0 \times X_0$ and to $X_0 \times L_0$. We denote this quadrangular algebra by $\mathcal{Q}_2(C, K)$. The subscript refers to the fact that X_0 is the direct sum of two copies of C.

OBSERVATION 4.15. Let X_0, L_0, h_0 and θ_0 be as in Notation 4.14 and let X_1 denote the subspace $\{(a, b, c, d) \mid b = c = d = 0\}$ of X_0. Then $X_1 L_0 \subset X_1$, but the restriction of θ to $X_1 \times L_0$ and the restriction of h to $X_1 \times X_1$ are both identically zero. See Observation 2.8.

NOTATION 4.16. Let (C, K) be a composition algebra with standard involution $\sigma = \sigma_C$. Suppose that (C, K) is not octonion, i.e. that C is associative, and let X be a right vector space over C. If $\sigma = 1$, we assume that $\mathrm{char}(K) \neq 2$ (in which case $C = K$), that h is a symplectic form on X and that π is the map from X to K that is identically zero. If $\sigma \neq 1$, we assume (in all characteristics) that h is a form on X that is skew-hermitian with respect to (C, σ) and that (C, σ, X, h, π) is a standard pseudo-quadratic space defined in [**13**, Definition 1.16]. (Note that this definition makes sense even though we are now neither requiring that π be anisotropic nor that (C, K) be division.) In both cases, we set $\theta(u, v) = \pi(u)v$ for all $u \in X$ and all $v \in C$, denote the scalar multiplication from $X \times C$ to X by $\cdot$ and let s_C be as in Notation 3.2. Then

$$\Xi = (K, C, n_C, s_C, 1, X, \cdot, h, \theta)$$

is a quadrangular algebra. This claim is clear if $\sigma = 1$ and holds by the proof of [**13**, Proposition 1.18] (which remains valid verbatim without the hypotheses that π is anisotropic and (C, K) is division). We denote this quadrangular algebra by $\mathcal{Q}_s(C, K, X, h, \pi)$. The subscript stands for "special"; see Definition 5.6 below.

The following pseudo-quadratic space will appear in Theorem 10.16.

EXAMPLE 4.17. Suppose that $\mathrm{char}(K) \neq 2$. Let $C = M_2(K)$ (i.e. the split quaternion algebra over K), let q be its determinant map, let ε be the identity matrix of C, let

σ be the classical adjoint of C, let $X = K \oplus K$ viewed as a right C-module in the usual way, let

$$h\big((a,b),(c,d)\big) = \begin{pmatrix} -bc & -bd \\ ac & ad \end{pmatrix}$$

for all $(a,b),(c,d) \in X$ and let

$$\pi(a,b) = \begin{pmatrix} -ab & -b^2 \\ a^2 & ab \end{pmatrix}$$

for all $(a,b) \in X$. Then (C,σ,X,h,π) is a standard pseudo-quadratic space. Note, too, that $q(\pi(u)) = 0$ for all $u \in X$.

5. Statement of the main theorem.

In order to formulate our main result in Theorem 5.10 below, we first need to introduce (or adopt from [**13**]) some additional notation.

DEFINITION 5.1. We apply the notion of *equivalent* quadrangular algebras in [**13**, Definition 1.22] and the notion of an *isomorphism* of quadrangular algebras in [**13**, Definition 1.25] verbatim and observe that [**13**, Remark 1.26] remains valid in the present context.

REMARK 5.2. The results [**13**, Propositions 1.23 and 1.24] remain valid in the present context, but we need to modify the proof of [**13**, Proposition 1.23] where A3 is used to conclude that $au \neq 0$. We choose $a \in X$ and first note that we can assume that $r(a,0) = r(a,\varepsilon)$. We then observe that by C1, $r(a,tu) = r(a,u)$ and $r(a,u+v)(u+v) = r(a,u)u + r(a,v)v$ for all $t \in K$ and all $u,v \in L$. It follows that the $r(a,u)$ is independent of u.

REMARK 5.3. The notion of an *isotope* of a quadrangular algebra defined in [**13**, Definition 8.7] and all the results about isotopic quadrangular algebras in [**13**, Chapter 8] remain valid in the present context. In particular, we note that if $\Xi = (K,L,q,f,\varepsilon,X,\cdot,h,\theta)$ is an arbitrary quadrangular algebra, then for each $u \in L$ such that $q(u) \neq 0$, Ξ has a unique isotope in which K and L remain the same but ε is replaced by u and q by $q/q(u)$.

DEFINITION 5.4. Let $\Xi = (K,L,q,f,\varepsilon,X,\cdot,h,\theta)$ be a quadrangular algebra. As in [**13**, Definition 1.27], we say that Ξ *proper* if the map σ defined in (2.1) is non-trivial. Thus Ξ is proper if and only if ε does not lie in the radical of f. By [**13**, Proposition 9.1], Ξ is isotopic to a proper quadrangular algebra if and only if f is not identically zero.

REMARK 5.5. Let $\mathcal{Q}_s(C,K,X,h,\pi)$ and σ be as in Notation 4.16. This quadrangular algebra is proper if and only if $\mathrm{char}(K) \neq 2$ or $\sigma \neq 1$, i.e. if (C,K) is as in Notation 3.1(ii)–(iv).

Definition 5.6. Let Ξ be a quadrangular algebra. We will say that Ξ is *special* if Ξ is isotopic to $\mathcal{Q}_s(C,K,X,h,\pi)$ for some (C,K,X,h,π) as described in Notation 4.16 and we will say that Ξ is *exceptional* if either Ξ is isotopic to $\mathcal{Q}_4(C,K)$ for some composition algebra (C,K) as defined in Notation 4.12 or Ξ is isotopic to $\mathcal{Q}_2(C,K)$ for some octonion algebra (C,K) as defined in Notation 4.14 (but see Corollary 5.11(ii) below) or Ξ is anisotropic and Ξ is as in [**13**, Theorem 6.42 or Theorem 7.57] up to isotopy. Note that by Corollary 5.11(i) below, $\mathcal{Q}_2(C,K)$ is special if C is associative.

Observation 5.7. Let $\Xi = (K,L,q,f,\varepsilon,X,\cdot,h,\theta)$ be a quadrangular algebra. If Ξ is special, then $\dim_K L \le 4$ or f is identically zero and if Ξ is exceptional, then $\dim_K L \ge 5$ and f is not identically zero. It follows, in particular, that there are no quadrangular algebras that are both special and exceptional.

Observation 5.8. If $\Xi = (K,L,q,f,\varepsilon,X,\cdot,h,\theta)$ is an exceptional quadrangular algebra, then the bilinear map h defined in Notations 4.4 and 4.14 is non-degenerate, i.e. for each $a \in X$, there exists $b \in X$ such that $h(a,b) \neq 0$.

Example 5.9. There exist special quadrangular algebras

$$\Xi = (K,L,q,f,\varepsilon,X,\cdot,h,\theta)$$

with h non-degenerate and others where h is not non-degenerate. Let L and X, for example, each be a copy of $M_2(K)$, let q be the determinant map on L, let ε be the identity matrix in L, let $\cdot$ be matrix multiplication, let γ be an arbitrary element of L, let $\theta(a,v) = a^\sigma\gamma av$ for all $a \in X$ and all v in L and let $h(a,b) = a^\sigma\gamma b$ for all $a,b \in X$. Then $\Xi = (K,L,q,f,\varepsilon,X,\cdot,h,\theta)$ is a special quadrangular algebra and h is non-degenerate if and only if γ is invertible.

We can now state the main result of this paper:

Theorem 5.10. *Let $\Xi = (K,L,q,f,\varepsilon,X,\cdot,h,\theta)$ be a proper isotropic quadrangular algebra as defined in* Definition 5.4. *Suppose that h is non-degenerate as defined in* Observation 5.8 *and that $|K| > 5$. Then Ξ is isotopic to one of the following*:

(i) $\mathcal{Q}_4(C,K)$ *for some composition algebra (C,K) or*

(ii) $\mathcal{Q}_2(C,K)$ *for some octonion division algebra (C,K) or*

(iii) $\mathcal{Q}_s(C,K,X,h,\pi)$ *for some composition algebra (C,K) as in* Notation 3.1(ii)–(iv).

In particular, Ξ is special if $\dim_K L \le 4$ *and exceptional if* $\dim_K L > 4$.

This theorem is the conjunction of Theorems 8.16, 9.8, 10.16 and 11.16. The remainder of this paper is devoted to the proof of these results. In the appendix, we describe a correspondence between the exceptional quadrangular algebras (up to isotopy) and certain forms of exceptional groups.

Note that we make no assumptions on either $\dim_K L$ or $\dim_K X$ in Theorem 5.10, nor do we make any restrictions on the characteristic of K. See Remark 5.13.

Corollary 5.11. *Let (C,K) be a composition algebra, let*

$$\mathcal{Q}_2(C,K)=(K,L,q,f,\varepsilon,X,\cdot,h,\theta)$$

be as in Notation 4.14 *and let π be as in* D1. *Then the following hold*:

(i) *If (C,K) is associative, then X is a free right C-module of rank* 2 *and $\mathcal{Q}_2(C,K)$ is isotopic to $\mathcal{Q}_s(C,K,X,h,\pi)$.*

(ii) *If (C,K) is split octonion, then $\mathcal{Q}_2(C,K)$ is isotopic to $\mathcal{Q}_4(C_0,K)$, where (C_0,K) is the split quaternion algebra over K.*

Proof. This holds by Theorem 5.10. □

Remark 5.12. Let $\Xi=(K,L,q,\varepsilon,X,\cdot,h,\theta)$ be an anisotropic quadrangular algebra such that h is not non-degenerate. By [**13**, Theorems 5.9, 6.42 and 7.57 and Proposition 9.1], either Ξ is special or the bilinear form f associated with q is identically zero (in which case Ξ is not proper). If f is identically zero, then Ξ is as in [**13**, Theorem 9.26 or Theorem 9.33].

Remark 5.13. The dimensions of L and X in the three cases of Theorem 5.10 are as follows. If $\Xi=\mathcal{Q}_4(C,K)$, then $\dim_K L=4+\dim_K C$ and $\dim_K X=4\cdot\dim_K C$. If (C,K) is as in Notation 3.1(ii)–(v), these two dimensions are finite, but if (C,K) is as in Notation 3.1(i), $\dim_K C$ and thus also $\dim_K L$ and $\dim_K X$ could well be infinite. If $\Xi=\mathcal{Q}_2(C,K)$ for some octonion algebra (C,K), then $\dim_K L=8$ and $\dim_K X=16$. If Ξ is special, then $\dim_K L=1$, 2 or 4, but there is no bound on $\dim_K X$. In particular, $\dim_K X$ could be infinite also in this case.

Remark 5.14. Let $\Xi=(K,L,q,f,\varepsilon,X,\cdot,h,\theta)$ be a proper quadrangular algebra and suppose that $\dim_K L=1$. By [**13**, Proposition 1.24 and Definition 1.25], we can assume that π and θ are both identically zero. By Definition 5.4, $\mathrm{char}(K)\neq 2$ and by B2 with $v=\varepsilon$, h is a symplectic form on X. Hence Ξ is isomorphic to $\mathcal{Q}_s(C,K,X,h,\pi)$ with $C=K$. We can therefore assume in the proof of Theorem 5.10 that $\dim_K L\geq 2$.

6. Norm splitting maps.

In this section we assemble a few elementary observations about quadratic forms that we will need. For the most part, they are simple modifications of results in [**13**, Chapter 2].

Lemma 6.1. *Let (K,L,q) be a quadratic space and let f be the bilinear form associated with q. Suppose that* $\dim_K L=2$, *and let $\{u,v\}$ be a basis for L over K such that $q(u)\neq 0$. Suppose, too, that $f(u,v)=0$ but $q(v)\neq 0$ if* $\mathrm{char}(K)\neq 2$ *and $f(u,v)\neq 0$ if* $\mathrm{char}(K)=2$. *Let $p(x)$ denote the polynomial*

$$p(x)=q(u)x^2-f(u,v)x+q(v),$$

let E be the splitting field of p over K if p is irreducible over K and let E be the split *étale quadratic extension $K \oplus K$ of K if it is not. Then (K, L, q) is isomorphic to the quadratic space $(K, E, q(u)N)$, where N denotes the norm of the extension E/K.*

Proof. Replacing q by $q/q(u)^{-1}$, we may assume that $q(u) = 1$. Suppose that $p(x)$ is irreducible over K. Let w and z be the two roots of $p(x)$ in E. Then $w+z = f(u, v)$ and $wz = q(v)$. Hence

$$\begin{aligned} N(t + sw) &= (t + sw)(t + sz) \\ &= t^2 + f(u, v)t + s^2 q(v) = q(tu + sv) \end{aligned}$$

for all $s, t \in K$. Thus the unique K-linear map from L to E that sends u to 1 and v to w is an isomorphism from (K, L, q) to (K, E, N). Suppose now that $p(x)$ is reducible over K. Then there exists $\alpha, \beta \in K$ such that $p(\alpha) = p(\beta) = 0$ and $\alpha + \beta = f(u, v)$. We set $r = 4q(v)$ if $\mathrm{char}(K) \neq 2$ and $r = f(u, v)^2$ if $\mathrm{char}(K) = 2$. We then let $w = \alpha u - v$ and $z = r^{-1}(\beta u - v)$ and observe that $q(w) = q(z) = 0$ and $f(w, z) = 1$. It follows that the map $sw + tz \mapsto (s, t)$ is an isomorphism from (K, L, q) to (K, E, N). □

Definition 6.2. Let (K, L, q) be a quadratic space and let f be the bilinear map associated with q. A *norm splitting map* of (K, L, q), or of q, is a linear automorphism ψ of L such that for some monic quadratic polynomial $p(x) = x^2 - \alpha x + \beta \in K[x]$ with $\alpha = 0$ and $\beta \neq 0$ if $\mathrm{char}(K) \neq 2$ and $\alpha \neq 0$ if $\mathrm{char}(K) = 2$, the following hold:

(i) $q(\psi(u)) = \beta q(u)$,

(ii) $f(u, \psi(u)) = \alpha q(u)$ and

(iii) $p(\psi)(u) = 0$

for all $u \in L$.

Proposition 6.3. *Let (K, L, q), ψ and $p(x)$ be as in* Definition 6.2. *Let E be the splitting field of p over K if $p(x)$ is irreducible over K and let E be the split étale extension $K \oplus K$ of K if it is not. Let N denote the norm of the extension E/K. For each $u \in L$, let L_u denote the subspace $\langle u, \psi(u) \rangle$ and let q_u denote the restriction of q to L_u.*

(i) *L_u is ψ-invariant for each $u \in L$.*

(ii) *$\dim_K L_u = 2$ for all $u \in L$ such that $q(u) \neq 0$.*

(iii) *$L_u = L_v$ for all $u \in L$ and for all $v \in L_u$ such that $q(v) \neq 0$.*

(iv) *If $q(u) \neq 0$ for some $u \in L$, then q_u is isomorphic to $q(u)N$.*

Proof. By Definition 6.2(iii), (i) holds. By Definition 6.2(i) and (ii) and the conditions on α and β, (ii) holds. By (i) and (ii), (iii) holds. Let $u \in L$ and let $v = \psi(u)$. Then

$$q(u)p(x) = q(u)x^2 - f(u, v)x + q(v).$$

By Lemma 6.1, therefore, (iv) holds. □

NOTATION 6.4. Let ψ be a norm splitting map of a quadratic space (K, L, q). Let L_u for each $u \in L$ be as in Proposition 6.3. A subset $\{v_1, \dots, v_m\}$ of L is *ψ-orthogonal* if $q(v_1), \dots, q(v_m)$ are all non-zero and the subspaces $L_{v_1}, \dots, L_{v_m}$ are pairwise orthogonal with respect to the bilinear form associated with q.

NOTATION 6.5. Let ψ and E be as in Proposition 6.3. We call the étale extension E/K the *splitting extension* of ψ.

DEFINITION 6.6. Let (K, L, q) be a quadratic space and let ψ be a norm splitting map of q. We will say that ψ is *reducible* if its splitting extension is the split étale extension of K and we will say that ψ is *irreducible* if its splitting extension is a field extension.

PROPOSITION 6.7. *Let $\Omega = (K, L, q)$ be a finite-dimensional non-degenerate quadratic space. Then the following hold*:

(i) *If Ω has a norm splitting map ψ, then Ω is isomorphic to*

$$(K, E^d, \alpha_1 N + \cdots + \alpha_d N)$$

for some $d \geq 1$, where $\alpha_1, \dots, \alpha_d$ are non-zero elements of K, N is the norm of the splitting extension of ψ and $+$ denotes the orthogonal sum.

(ii) *Suppose that Ω is isomorphic to $(K, E^d, \alpha_1 N + \cdots + \alpha_d N)$, where E/K is an étale quadratic extension, N is its norm and $\alpha_1, \dots, \alpha_d$ are non-zero elements of K, and let T be the trace of the extension E/K. Let a be an element of E such that $N(a) \neq 0$ and $T(a) = 0$ if $\mathrm{char}(K) \neq 2$ and $T(a) \neq 0$ if $\mathrm{char}(K) = 2$. Then left multiplication by a is a norm splitting map of Ω whose splitting extension is E/K.*

(iii) *Ω is hyperbolic if and only if it has a reducible norm splitting map.*

PROOF. The assertion (i) holds by [**13**, Proposition 2.20] and assertion (ii) by [**13**, Proposition 2.17] with only minor changes in the proofs. (In particular, we need to observe in the proof of [**13**, Proposition 2.20] that if $W^\perp \neq 0$, it contains elements u such that $q(u) \neq 0$.) If Ω is hyperbolic, then it has a decomposition into the orthogonal sum of subspaces isomorphic to (K, E, N), where E/K is the split étale quadratic extension of K and hence by (ii), Ω has a reducible norm splitting map. If, conversely, Ω has a reducible norm splitting map ψ, then by (i), Ω is hyperbolic. Thus (iii) holds. □

COROLLARY 6.8. *Let $\Omega = (K, L, q)$ be a finite dimensional quadratic space and let f be the bilinear form associated with q. If Ω has a norm splitting map, then f is non-degenerate.*

PROOF. This holds by Proposition 6.7(i). □

COMMENT 6.9. Let $\Omega = (K, L, q)$ be a hyperbolic quadratic space. By Proposition 6.7(iii), it has a reducible norm splitting map, but it could have irreducible norm splitting maps as well. Suppose, for example, that E/K is an arbitrary separable quadratic field extension with norm N and let $\Omega = (K, E^2, \alpha_1 N + \alpha_2 N)$ with $\alpha_1 = -\alpha_2 \in K^*$.

Then Ω is hyperbolic, but by Proposition 6.7(ii), it has an irreducible norm splitting map with splitting extension E/K.

Notation 6.10. Let E/K be a separable quadratic field extension with norm N and standard involution σ and let

$$\Omega = (K, E^d, \alpha_1 N + \cdots + \alpha_d N)$$

for some $d \geq 1$ and some $\alpha_1, \ldots, \alpha_d \in K^*$. Let H denote the map from $E^d \times E^d$ to E given by

$$H\big((x_1, \ldots, x_d), (y_1, \ldots, y_d)\big) = \sum_{i=1}^{d} x_i^\sigma \alpha_i y_i$$

for all $x_1, \ldots, y_d \in E$. Then H is a non-degenerate hermitian form with respect to (E, σ) and Ω is isomorphic to the quadratic space (K, E^d, q_H), where $q_H(x) = H(x, x)$ for all $x \in E^d$.

Remark 6.11. Let E/K, N, H and q_H be as in Notation 6.10 and suppose that $d = 2$. Suppose, too, that H is hyperbolic, by which we mean that there exists a basis $\{e_1, e_2\}$ of E^2 such that $H(e_i, e_j) = 0$ if $i = j$ and $H(e_i, e_j) = 1$ if $i \neq j$. We have

$$q_H(ae_i) = H(ae_i, ae_i) = N(a)H(e_i, e_i) = 0$$

for $i = 1$ and 2 and for all $a \in E$. Thus the 1-dimensional E-spaces spanned by e_1 and by e_2 are both totally isotropic with respect to q_H. Let f_H denote the bilinear form associated with q_H. Then

$$f_H(ae_1, be_2) = H(ae_1, be_2) + H(be_2, ae_1) = a^\sigma b + b^\sigma a$$

for all $a, b \in E$. Let T be the trace of the extension E/K. If $\mathrm{char}(K) \neq 2$, let W_1 be the K-subspace $\langle e_1, e_2 \rangle$ of E^2 and let W_2 be the K-subspace $\langle \gamma e_1, \gamma e_2 \rangle$, where γ is an element of E^* such that $T(\gamma) = 0$. If $\mathrm{char}(K) = 2$, let W_1 be the K-subspace $\langle e_1, \gamma e_2 \rangle$ and let W_2 be the K-subspace $\langle \gamma e_1, e_2 \rangle$, where γ is an element of E^* such that $T(\gamma) \neq 0$. Then W_1 and W_2 are orthogonal to each other with respect to f_H and the restrictions of q_H to both W_1 and W_2 are hyperbolic. Thus q_H is hyperbolic.

In the proof of the next result, we use a strategy suggested by Holger Petersson.

Proposition 6.12. *Let $\Omega = (K, L, q)$ be a quadratic space with a norm splitting map. Suppose that* $\dim_K L$ *is* 6, 8 *or* 12 *and if* $\dim_K L = 12$*, suppose as well that the Clifford invariant of q is trivial. Then one of the following holds*:

(i) *q is of type E_6, E_7 or E_8 as defined in* [**13**, Definition 2.13].

(ii) *q is similar to the norm of an octonion division algebra.*

(iii) *There exists a composition division algebra (C, K) as in* Notation 3.1(iii)–(v) *such that q is similar to the quadratic space (K, L, q_C) described in* Notation 4.1.

In the first two cases, q is anisotropic. In the third case, q has Witt index 2 if (C,K) is division and q is hyperbolic if (C,K) is split.

Proof. We can assume that q is not hyperbolic. By Proposition 6.7(i) and (iii), therefore, we can identify Ω with the quadratic space

$$(K, E^d, \alpha_1 N + \cdots + \alpha_d N)$$

for some $\alpha_1, \ldots, \alpha_d \in K^*$ and some separable quadratic field extension E/K with norm N, where $d = 3$, 4 or 6. If $d = 6$, then the Clifford invariant of q is trivial and hence $-\alpha_1\alpha_2 \cdots \alpha_6 \in N(E)$ (by [**12**, (12.28)]). Suppose that q is anisotropic. If $d = 3$ or 6, then q is of type E_6 or E_8 (by [**13**, Definition 2.13]). If $d = 4$, then q is of type E_7 if $\alpha_1\alpha_2\alpha_3\alpha_4 \notin N(E)$ (again by [**13**, Definition 2.13]), and q is similar to the norm of an octonion division algebra if $\alpha_1\alpha_2\alpha_3\alpha_4 \in N(E)$ (by, for example, the description of octonion algebras in [**12**, (9.8)]). Thus from now on, we can assume that q is isotropic.

Let H and q_H be as in Notation 6.10. Since q is isotropic, so is q_H. Thus H is isotropic. Since H is non-degenerate, it follows that there is a decomposition of E^d into the direct sum of E-subspaces V_1 and V_2 such that $\dim_E V_1 = 2$, the restriction of H to V_1 is hyperbolic and $H(V_1, V_2) = 0$. Let Q_i denote the restriction of q_H to V_i for $i = 1$ and 2. By Remark 6.11, Q_1 is hyperbolic. Since V_2 is a subspace over E, Q_2 has splitting map with splitting extension E/K. Thus (K, V_2, Q_2) is isomorphic to $(K, E^e, \beta_1 N + \cdots + \beta_e N)$ for some $\beta_1, \ldots, \beta_e \in K^*$, where $e = d - 2$. If $d = 6$, then the Clifford invariant of this restriction is trivial (by [**1**, Lemma 3.8]) and hence $\beta_1\beta_2\beta_3\beta_4 = 1$. We conclude that Q_2 is similar to the norm of a composition algebra (C, K) as in Notation 3.1(iii)–(v). It follows that q_H is similar to q_C. Hence also q is similar to q_C. □

In the next result, K^2 denotes $\{t^2 \mid t \in K\}$ (and not $K \oplus K$) and $F^{1/2}$ denotes the unique field C containing F such that $C^2 = F$.

Proposition 6.13. *Suppose that* $\mathrm{char}(K) = 2$. *Let E/K be an étale quadratic extension, let F be a subfield of K such that $K^2 \subset F$, let (C, K) be the composition algebra of type* Notation 3.1(i) *with $C = F^{1/2}$, let (K, F, q_F) be as in* [**13**, Notation 2.14], *let $V = E \oplus E \oplus F$ and let*

$$Q(a, b, s) = N(a) + \alpha N(b) + \beta q_F(s)$$

for all $(a, b, s) \in V$, where α is an element of F^*, β is an element of K^* and N is the norm of the extension E/K. Suppose that the quadratic space (K, V, Q) is isotropic. Then (K, V, Q) is similar to the quadratic space

$$(K, L_C, q_C)$$

defined in Notation 4.1.

Proof. Let T denote the trace of E/K and let γ be an element of E such that $T(\gamma) = 1$. Thus E is the splitting field of the polynomial $x^2 + x + N(\gamma)$ over K. Let ξ denote the bilinear form associated with Q, let R_0 denote the radical of ξ, let Q_0 denote the restriction of Q_0 to R_0 and let Q_1 denote the restriction of Q to $\{(a,b,0) \mid a,b \in E\}$. Thus $R_0 = \{(0,0,t) \mid t \in F\}$, Q_0 is similar to q_F and q_F is isomorphic to the norm n_C of (C,K). The quadratic form Q_1 is isomorphic to the norm of a quaternion algebra over K. Thus if Q_1 is isotropic, this quaternion algebra is split, hence Q_1 is hyperbolic and thus Q is similar to q_C. We can assume, therefore, that Q_1 is anisotropic. Hence, in particular, $\alpha \notin K^2$.

Let (a,b,r) be a non-zero element of V such that $Q(a,b,r) = 0$. We have $a = s_1 + t_1\gamma$ and $b = s_2 + t_2\gamma$ for some $s_1, t_1, s_2, t_2 \in K$. Suppose first that t_1 and t_2 are not both zero and let $\kappa = t_1^2 + \alpha t_2^2$. Since $\alpha \notin K^2$, we have $\kappa \neq 0$ and since K^2 and α are contained in F, we have $\kappa \in F$. Hence $\kappa^{-1} r \in F$. Let $u_1 = (1,0,0)$, let $v_1 = (\gamma, 0, \kappa^{-1} r)$, let $u_2 = (0,1,0)$, let $v_2 = (0, \gamma, \alpha\kappa^{-1} r)$, let $W_i = \langle u_i, v_i\rangle$ for $i = 1$ and 2, let Q_W denote the restriction of Q to $W := W_1 + W_2$ and let $p(x)$ denote the polynomial $x^2 + x + N(\gamma) + \beta\kappa^{-1} r$ over K. Thus $Q(u_1) = 1$, $\xi(u_1, v_1) = 1$, $Q(v_1) = N(\gamma) + \beta\kappa^{-1} r$, $Q(u_2) = \alpha$, $\xi(u_2, v_2) = \alpha$ and $Q(v_2) = \alpha(N(\gamma) + \beta\kappa^{-1} r)$. It follows by Lemma 6.1 that the restrictions of Q to W_1 and to W_2 are both hyperbolic if $p(x)$ is reducible over K and they are both similar to the norm of the extension $\hat{E}/K$ if $p(x)$ is irreducible over K, where $\hat{E}$ is the splitting field of $p(x)$ over K. Since $\xi(W_1, W_2) = 0$, we conclude that Q_W is isomorphic to the norm of a quaternion algebra over K (whether or not $p(x)$ is irreducible). Furthermore,

$$Q(s_1 u_1 + t_1 v_1 + s_2 u_2 + t_2 v_2) = Q(a,b,r) = 0,$$

so Q_W is isotropic, and $V = W \oplus R_0$. It follows that Q_W is hyperbolic and hence Q is similar to q_C.

It remains to consider the case that $t_1 = t_2 = 0$. In this case, we set $\kappa = s_1^2 + \alpha s_2^2$, $v_1 = (1, 0, \kappa^{-1} r)$, $u_1 = (\gamma, 0, 0)$, $v_2 = (0, 1, \alpha\kappa^{-1} r)$ and $u_2 = (0, \gamma, 0)$, so that $Q(v_1) = 1 + \beta\kappa^{-1} r$, $\xi(u_1, u_1) = 1$, $Q(u_1) = N(\gamma)$, $Q(v_2) = \alpha(1 + \beta\kappa^{-1} r)$, $\xi(u_2, v_2) = \alpha$ and $Q(u_2) = \alpha N(\gamma)$. We again let $W = \langle u_1, v_1, u_2, v_2\rangle$ and let Q_W denote the restriction of Q to W. Then $\xi(W_1, W_2) = 0$ and since $Q(u_1) \neq 0$, we can deduce from Lemma 6.1 exactly as in the previous paragraph that Q_W is isomorphic to the norm of a quaternion algebra over K. Furthermore,

$$Q(s_1 v_1 + s_2 v_2) = Q(a,b,r) = 0,$$

so Q_W is isotropic, and $V = W \oplus R_0$. It follows as before that Q_W is hyperbolic and hence Q is similar to q_C □

In the next result, we give the structure of $C(q,\varepsilon)$ for the quadratic forms that appear in Theorem 5.10. In the proof, we assume that the reader is familiar with the basic structure theory for even Clifford algebras. A good source in arbitrary characteristic is [**4**, Chapter 11, Sections A and B]; see, in particular, [**4**, Theorems 11.1, 11.2 and 11.3] as well as [**1**, Section 5] and [**12**, (12.28)].

Proposition 6.14. *Let (K,L,q) be a quadratic space and let ε be an element of L such that $q(\varepsilon) = 1$. Then the following hold*:

(i) *If q is similar to q_C for some composition algebra (C,K), then $C(q,\varepsilon)$ is isomorphic to*

(a) *$M(4, K)$ if $C = K$,*

(b) *$M(4, K) \oplus M(4, K)$ if C/K is a split étale quadratic extension,*

(c) *$M(4, C)$ if C/K is a separable quadratic extension,*

(d) *$M(8, K) \oplus M(8, K)$ if (C, K) is a split quaternion algebra,*

(e) *$M(4, C) \oplus M(4, C)$ if (C, K) is a quaternion division algebra and*

(f) *$M(32, K) \oplus M(32, K)$ if (C, K) is octonion, whether or not it is split.*

(ii) *If q is similar to the norm of an octonion division algebra, then $C(q, \varepsilon)$ is isomorphic to $M(8, K) \oplus M(8, K)$.*

(iii) *If q is of type E_ℓ, then $C(q, \varepsilon)$ is isomorphic to*

(I) *$M(4, E)$, where E/K is the discriminant extension of q, if $\ell = 6$,*

(II) *$M(4, D) \oplus M(4, D)$, where D is the Clifford invariant of q, if $\ell = 7$ and*

(III) *$M(32, K) \oplus M(32, K)$ if $\ell = 8$.*

In subcases (I) *and* (II)*, neither the étale extension E/K nor the quaternion algebra D is split.*

PROOF. As already mentioned in Observation 2.8, $C(q, \varepsilon)$ is canonically isomorphic to the even Clifford algebra $C_0(q)$ of q. Let E/K be the discriminant extension of q if q is as in (c) or (I) and let $E = K$ in every other case. Let $n = \dim_K L$. Then $\dim_K C(q, \varepsilon) = 2^{n-1}$ and there exists a division algebra D with center E and an integer m such that $C(q, \varepsilon) \cong M_m(D)$ if either n is odd or E/K is quadratic and $n \equiv 2 \pmod 4$ and

$$C(q, \varepsilon) \cong M_m(D) \oplus M_m(D)$$

if either $n \equiv 0 \pmod 4$ or $E = K$ and $n \equiv 2 \pmod 4$. (All the isomorphisms in this proof are isomorphisms of K-algebras.)

Let (C, K) be a composition algebra with norm n_C. Then $C_0(n_C) \cong C$ if $C = K$ or (C, K) is quadratic, $C_0(n_C) \cong C \oplus C$ if (C, K) is quaternion and $C_0(n_C) \cong M_8(K) \oplus M_8(K)$ if (C, K) is octonion. Thus, in particular, (ii) holds.

Suppose that q is similar to q_C for some composition algebra (C, K). Then $C_0(n_C) \cong M_{m-2}(D)$ if $C(q, \varepsilon) \cong M_m(D)$ and $C_0(n_C) \cong M_{m-2}(D) \oplus M_{m-2}(D)$ if $C(q, \varepsilon) \cong M_m(D) \oplus M_m(D)$. By the observations in the previous paragraph, it follows that $D = K$ except when (C, K) is division and either quadratic or quaternion, in which case $D = C$. Thus (i) and (ii) hold. By [**12**, (12.43)], (iii) holds. □

We close this section with three more small observations.

PROPOSITION 6.15. *Suppose* $\operatorname{char}(K) \neq 2$. *Let (K, L, q) be a non-degenerate quadratic space of dimension* 5 *and let ε be an element of L such that $q(\varepsilon) = 1$. Then $C(q, \varepsilon) \cong M(2, D)$ for some quaternion division algebra D over K if the Witt index of q is* 1 *and $C(q, \varepsilon) \cong M(4, K)$ if the Witt index of q is* 2.

PROOF. Suppose that the Witt index of q is 1 and let q_a be the anisotropic part of q. Then q_a is the restriction of n_D to a suitable 3-dimensional subspace of D for some quaternion algebra (D, K). Hence $C_0(q_a) = D$ and thus $C_0(q) = M_2(D)$. The other claim holds by Proposition 6.14(a). $\square$

PROPOSITION 6.16. *Suppose that (K, L, q) is a non-degenerate quadratic space of dimension 4 and let ε be an element of L such that $q(\varepsilon) = 1$. Then the following hold*:

(i) *If q is similar to the norm of a quaternion algebra (C, K), then $C(q, \varepsilon) \cong C \oplus C$.*

(ii) *If $C(q, \varepsilon)$ has a non-trivial right module of dimension 2 over K, then q is isomorphic to the norm of the split quaternion algebra (C, K), i.e. q is hyperbolic.*

PROOF. If q is similar to the norm of a quaternion algebra (C, K), then $C_0(q) \cong C \oplus C$. Thus (i) holds. Suppose that $C(q, \varepsilon)$ has a non-trivial right module of dimension 2 over K and let E/K be the discriminant extension of q. If E/K is quadratic, then $C(q, \varepsilon)$ is isomorphic to a quaternion algebra over E. Since $C(q, \varepsilon)$ has a right module of dimension 2 over K, we must have $E = K$. It follows that q is similar to the norm of a quaternion algebra (C, K) and hence $C(q, \varepsilon) \cong C \oplus C$ by (i). Furthermore, (C, K) is split since $C \oplus C$ does not have a 2-dimensional right module if (C, K) is division. Hence q is, in fact, isomorphic to the norm of (C, K). Thus (ii) holds. $\square$

PROPOSITION 6.17. *Let (K, L, q) be a non-degenerate quadratic space, let ε be an element of L such that $q(\varepsilon) = 1$ and X is a non-trivial right module for $C(q, \varepsilon)$. If $\dim_K X = 1$, then either $\dim_K L = 1$ or $\dim_K L = 2$ and q is hyperbolic.*

PROOF. Let $n = \dim_K L$. Then $\dim_K C(q, \varepsilon) = 2^{n-1}$ and $C(q, \varepsilon)$ is either central simple over an extension of K or the direct sum of two copies of a central simple algebra over an extension of K. It follows that $n \leq 2$. If $\dim_K L = 2$ and q is not hyperbolic, then $C(q, \varepsilon)$ is a field of degree 2 over K and thus has no 1-dimensional non-trivial right modules. $\square$

7. Basic identities.

Most of the results and identities in [**13**, Chapters 3 and 4] hold for quadrangular algebras as defined in Definition 2.1, but some minor modifications are required which we now describe.

We turn first to [**13**, Chapter 3]. The results [**13**, Propositions 3.4 and 3.5] are not valid in the present context and we must pay attention to avoid or repair any results that use them. We discard the result [**13**, Proposition 3.11] (which is not used in [**13**, Chapters 3 or 4]) and replace the proof of [**13**, Proposition 3.13] by Remark 5.14. All the remaining results from [**13**, Proposition 3.6] to [**13**, Proposition 3.22] and their proofs remain valid verbatim. (At various places in [**13**], an asterisk, as in L^* or X^*, is used to denote the set of non-zero elements of a given set. We use this notation here only in the case that the set is a field.)

We now turn to [**13**, Chapter 4]. The following definition generalizes [**13**, Definition 4.1].

Definition 7.1. Let $\Xi = (K, L, q, f, \varepsilon, X, \cdot, h, \theta)$ be a quadrangular algebra, let π be the map that appears in D1 and let $\delta \in L$. Then Ξ is *δ-standard* whenever the following hold:

(i) $\delta = \varepsilon/2$ if $\mathrm{char}(K) \neq 2$,

(ii) $f(\varepsilon, \delta) = 1$ if $\mathrm{char}(K) = 2$ and

(iii) $f(\pi(a), \delta) = 0$ for all $a \in X$ in all characteristics.

(iv) $q(\delta) \neq 0$.

Thus, in particular,

$$f(\pi(a), \varepsilon) = 0 \text{ if } \mathrm{char}(K) \neq 2. \tag{7.1}$$

The following result is an adjustment of the first statement in [**13**, Proposition 4.2] to the generalization of [**13**, Definition 4.1] given in Definition 7.1 (and we ignore the second claim in [**13**, Proposition 4.2], that $\hat{\theta}$ is unique).

Proposition 7.2. *Let $\Xi = (K, L, q, f, \varepsilon, X, \cdot, h, \theta)$ be a quadrangular algebra. Suppose that Ξ is proper as defined in* Definition 5.4 *and that $|K| > 2$. Then Ξ is isomorphic to a quadrangular algebra that is δ-standard for some $\delta \in L$ as defined in* Definition 7.1.

Proof. Suppose that $\mathrm{char}(K) = 2$ and choose $t \in K$ such that t is not a root of $x^2 + x$. Since Ξ is proper, we can choose an element $\delta \in L$ such that $f(\varepsilon, \delta) = 1$. Then $f(\varepsilon, t\varepsilon + \delta) = 1$ and $q(t\varepsilon + \delta) = t^2 + t + q(\delta) \neq q(\delta)$. Replacing δ by $t\varepsilon + \delta$ if necessary, we can thus assume that $f(\varepsilon, \delta) = 1$ and $q(\delta) \neq 0$. By the first claim in [**13**, Proposition 4.2], whose proof holds verbatim, Ξ is isomorphic to a quadrangular algebra that is δ-standard in all characteristics. □

In all the subsequent sections, we assume that $|K| > 2$ and that the quadrangular algebra we are considering is δ-standard for some $\delta \in L$. As a consequence of this assumption, the results and the proofs of all the results in [**13**, Chapter 4] up to and including [**13**, Proposition 4.18] remain valid unchanged. The result [**13**, Propositions 4.19–4.22] remain valid, but the proofs, which depend on [**13**, Proposition 3.4], need to be modified. We describe these modifications now.

Proposition 7.3. *Suppose that $aL \subset \langle a \rangle$ for some $a \in X$. Then $\theta(a, v) = 0$ for all $v \in L$.*

Proof. We can assume that $a \neq 0$. Let $X_0 = \langle a \rangle$ and let

$$\Xi_0 = (K, L, q, f, \varepsilon, X_0, *, h_0, \theta_0)$$

be the quadrangular algebra we obtain by replacing X_0 with $\langle a \rangle$, where $*$, h_0 and θ_0 are the suitable restrictions of $\cdot$, h and θ (see Observation 2.8), and let π_0 be the restriction of π to $\langle a \rangle$. By Proposition 6.17, either $\dim_K L = 1$ or $\dim_K L = 2$ and q is hyperbolic.

Suppose first that $\dim_K L = 1$. We can identify both X_0 and L with K so that $r * s = rs$ and $q(s) = s^2$ for all $r \in X_0$ and $s \in L$ and $\varepsilon = 1$. Since Ξ is proper as defined in Definition 5.4, we have $\mathrm{char}(K) \neq 2$. By (7.1), therefore, π_0 is identically zero. By A2 and D1, therefore, θ_0 is also identically zero. Hence $\theta(a, v) = 0$ for all $v \in L$.

Now suppose that $\dim_K L = 2$. We can identify X_0 with K and L with $K \oplus K$ so that $r * (s, t) = rs$ and $q(s, t) = st$ for all $r \in X_0$ and $(s, t) \in L$ and $\varepsilon = (1, 1)$. Suppose $\mathrm{char}(K) = 2$. By [**13**, Proposition 3.15], $h_0(r, r * (s, t)) = f(\pi_0(r), \varepsilon)(s, t)$ for all $r \in X_0$ and all $(s, t) \in L$. Since $r * (s, t)$ is independent of t, we must have

$$f(\pi_0(r), \varepsilon) = 0 \tag{7.2}$$

for all $r \in X_0$. Since Ξ is δ-standard and L is spanned by ε and δ, it follows that π_0 is identically zero.

Now suppose that $\mathrm{char}(K) \neq 2$. By [**13**, Proposition 4.5(i)], $\theta_0(r, (s, t))$ is independent of t and by [**13**, Proposition 4.9(iii)] and (7.1), we have

$$f(\theta_0(r, (s, t)), \varepsilon) = -f(\pi_0(r), (s, t))$$

for all $r \in X_0$ and all $(s, t) \in L$. Hence the expression $f(\pi_0(r), (s, t))$ is also independent of t. Therefore the first coordinate of $\pi_0(r)$ is 0 for all $r \in X_0$. By another application of (7.1), it follows that π_0 is identically zero also in this case.

Finally, we suppose that $\mathrm{char}(K)$ is arbitrary. Since π_0 is identically zero, it follows by D1 that the first coordinate of $\theta_0(r, (s, t))$ is 0 for all $r \in X_0$ and all $(s, t) \in L$. By [**13**, Proposition 3.19(iii)], (7.1) and (7.2), we have

$$f(\theta_0(r, (0, 1)), (1, 0)) = f(\theta_0(r, (1, 0)), (0, 1)) = 0$$

and hence $\theta_0(r, (0, 1)) = 0$ for all $r \in X_0$. Since L is spanned by $(0, 1)$ and ε, it follows that θ_0 is identically zero. Thus $\theta(a, v) = 0$ for all $v \in L$. $\square$

Corollary 7.4. *Suppose that $aL \subset \langle a \rangle$ for some $a \in X$. Then all the identities in* [**13**, Propositions 4.19–4.22] *hold for this choice of a.*

Proof. This holds by Proposition 7.3. $\square$

Now let $a \in X$ and $v \in L$ be arbitrary. We set

$$\xi(a, v) = f(\theta(a, v), \pi(a)) - q(\pi(a))f(\varepsilon, v). \tag{7.3}$$

By the proof of [**13**, Proposition 4.20] starting in its fourth line (and thus avoiding the application of [**13**, Proposition 4.19]), we have

$$a\theta(a, v)\pi(a) - f(\theta(a, v), \varepsilon)a\pi(a) + q(\pi(a))f(v, \varepsilon)a = q(\pi(a))av. \tag{7.4}$$

Thus

$$\begin{aligned} a\pi(a)^\sigma\theta(a, v) &= -a\theta(a, v)^\sigma\pi(a) + f(\theta(a, v), \pi(a))a && \text{by [13, Proposition 3.8]} \\ &= a\theta(a, v)\pi(a) - f(\theta(a, v), \varepsilon)a\pi(a) \end{aligned}$$

$$
\begin{aligned}
&\quad + q(\pi(a))f(v,\varepsilon)a + \xi(a,v)a && \text{by (2.1) and (7.3)}\\
&= q(\pi(a))av + \xi(a,v)a && \text{by (7.4).}
\end{aligned} \tag{7.5}
$$

Therefore

$$
a\pi(a)\theta(a,v) = -q(\pi(a))av + f(\pi(a),\varepsilon)a\pi(a)v - \xi(a,v)a \tag{7.6}
$$

by (2.1) and D1. Hence

$$
\begin{aligned}
q(\theta(a,v))a &= a\theta(a,v)\theta(a,v)^\sigma && \text{by A3}\\
&= a\pi(a)v\theta(a,v)^\sigma && \text{by D1}\\
&= -a\pi(a)\theta(a,v)v^\sigma + f(\theta(a,v),v)a\pi(a) && \text{by [\textbf{13}, Proposition 3.8]}\\
&= -a\pi(a)\theta(a,v)v^\sigma + f(\pi(a),\varepsilon)q(v)a\pi(a) && \text{by [\textbf{13}, Proposition 4.9(i)]}\\
&= q(\pi(a))avv^\sigma - f(\pi(a),\varepsilon)a\pi(a)vv^\sigma\\
&\qquad + \xi(a,v)av^\sigma + f(\pi(a),\varepsilon)q(v)a\pi(a) && \text{by (7.6)}\\
&= q(\pi(a))q(v)a + \xi(a,v)av^\sigma && \text{by A3.}
\end{aligned}
$$

We conclude that

$$
\zeta(a,v)a = \xi(a,v)av^\sigma \tag{7.7}
$$

for all $a \in X$ and all $v \in L$, where $\zeta(a,v) = q(\theta(a,v)) - q(\pi(a))q(v)$ and $\xi(a,v)$ is as in (7.3).

Now choose $a \in X$ and let $L_0 = \{w \in L \mid aw^\sigma \not\in \langle a\rangle\}$. By Corollary 7.4, we can assume that $L_0 \neq \emptyset$. Choose $w \in L_0$. For each $v \in L$, either $v \in L_0$ or $v + w \in L_0$. Hence L_0 spans L. By (7.7), $\xi(a,v) = 0$ for all $v \in L_0$. Since the map $v \mapsto \xi(a,v)$ is linear, it follows that $\xi(a,v) = 0$ for all $v \in L$. Thus [**13**, Proposition 4.19] holds (since a is arbitrary) and [**13**, Proposition 4.20] holds by (7.5). Another application of (7.7) yields $\zeta(a,v) = 0$ for all $a \in X$ and all $v \in L$. Therefore also [**13**, Proposition 4.22] holds as well and hence

$$
f(\theta(a,u),\theta(a,v)) = q(\pi(a))f(u,v) \tag{7.8}
$$

for all $u, v \in L$.

Again choose $a \in X$ and $v \in L$ and let

$$
w = \theta(a,\theta(a,v)) - f(\pi(a),\varepsilon)\theta(a,v) + q(\pi(a))v.
$$

Now that we know that [**13**, Proposition 4.20] holds, the proof of [**13**, Proposition 4.21] yields $aw = 0$. Thus if we assume that $a \neq 0$, it follows that $q(w) = 0$ (by A3). By [**13**, Proposition 4.9(iii)] and (7.8), we have

$$
\begin{aligned}
f(w,u) &= f(\theta(a,\theta(a,v)),u) - f(\pi(a),\varepsilon)f(\theta(a,v),u) + q(\pi(a))f(u,v)\\
&= -f(\theta(a,u),\theta(a,v)) + q(\pi(a))f(u,v) = 0
\end{aligned} \tag{7.9}
$$

for all $u \in L$. Since q is non-degenerate and $q(w) = 0$, it follows that $w = 0$. Thus [**13**, Proposition 4.21] holds. Finally, we observe that the proof of [**13**, Proposition 4.23] is easily modified to show that

$$\phi(av, v^\sigma) = q(v)\phi(a, v) \tag{7.10}$$

for all $a \in X$ and all $v \in L$, where ϕ is the function defined in C4 of Definition 2.1.

Conclusion 7.5. Subject to the observations we have made in this section and the assumptions that Ξ is δ-standard and $|K| > 2$, we will feel free from now to cite [**13**, Propositions, Corollaries and Remarks 3.6–3.10, 3.12–3.22 and 4.3–4.23] with the understanding that [**13**, Definition 4.1 and Proposition 4.2] are replaced by Definition 7.1 and Proposition 7.2.

Note that we have already applied Conclusion 7.5 several times in this section. Here are two more applications of Conclusion 7.5 (and there will be many more in the next sections):

Proposition 7.6. *Let $\Xi = (K, L, q, f, \varepsilon, X, \cdot, h, \theta)$ be a δ-standard quadrangular algebra for some $\delta \in L$ as defined in* Definition 7.1 *and let*

$$X^\flat := \{a \in X \mid q(\pi(a)) \neq 0\},$$

where π is as in D1. *Suppose that $|K| > 4$ and that the set $X^\flat$ is not empty. Then X is spanned by $X^\flat$.*

Proof. Choose $a \in X^\flat$ and let $b \in X$. By [**13**, Corollary 4.4], the map g in C3 is bilinear. By C2 and C3, it follows that $q(\pi(ta + b))$ is a polynomial in $K[t]$ of degree 4 with highest coefficient $q(\pi(a)) \neq 0$. Since $|K| > 4$, there exists $t \in K$ such that $q(\pi(ta + b)) \neq 0$. Since b is arbitrary, we conclude that X is spanned by $X^\flat$. □

Proposition 7.7. *Let $\Xi = (K, L, q, f, \varepsilon, X, \cdot, h, \theta)$ be a δ-standard quadrangular algebra for some $\delta \in L$ as defined in* Definition 7.1 *and let ϕ be as in* C4. *Then*

$$q\big(\pi(au) + tq(u) + \phi(a, u)\varepsilon\big) = q\big(\pi(a) + t\varepsilon\big)q(u)^2$$

for all $a \in X$ and all $u \in L$.

Proof. The proof of [**14**, Proposition 21.10(ii)] (which uses [**13**, Propositions 4.5(iii), 4.9(iii) and 4.19]) holds verbatim in the present context. □

We close this section with two more observations.

Lemma 7.8. *If $h(X, X)$ does not lie in the radical of f, then $h(X, X)$ spans L.*

Proof. Let $L_0 = \langle h(X, X)\rangle$. Suppose that there exist $e, b \in X$ and $w \in L$ such that $f(h(e, b), w) \neq 0$. By B3, it follows that $f(h(a, b), \varepsilon) \neq 0$ for $a = ew$. Let v be an arbitrary element of L. By B2, we have

$$f(h(a, b), \varepsilon)v = h(a, bv) - h(b, av) \in L_0$$

and thus $v \in L_0$. $\square$

REMARK 7.9. Suppose that π is identically zero, where π is as in D1. By C3, it follows that $h(a,b) \in \langle \varepsilon \rangle$ for all $a, b \in X$. Since Ξ is δ-standard, ε is not in the radical of f and $\dim_K L \geq 2$ if $\mathrm{char}(K) = 2$. By Lemma 7.8, therefore, either h is identically zero or $\dim_K L = 1$, $\mathrm{char}(K) \neq 2$ and after identifying L with K via the map $t\varepsilon \mapsto t$, h is a symplectic form on X and $\Xi = \mathcal{Q}_s(C, K, X, h, \pi)$ with $C = K$ as defined in Notation 4.16.

8. The generic case.

In this section, we make the following assumptions:

HYPOTHESIS 8.1. Let $\Xi = (K, L, q, f, \varepsilon, X, \cdot, h, \theta)$ be a δ-standard quadrangular algebra for some $\delta \in L$ as defined in Definition 7.1. Let ϕ be the map that appears in C4 and let

$$Q(a) = f(\pi(a), \varepsilon) \tag{8.1}$$

for all $a \in X$. We suppose the following:

(i) $q(\pi(a)) \neq 0$ for some $a \in X$ if $\mathrm{char}(K) \neq 2$ and $Q(a) \neq 0$ for some $a \in X$ if $\mathrm{char}(K) = 2$,

(ii) $|K| > 4$ and

(iii) h is non-degenerate as defined in Observation 5.8.

The main result of this section is Theorem 8.16.

PROPOSITION 8.2. *Suppose that a is as in* Hypothesis 8.1(i) *and that $av = 0$ for some $v \in L$. Then $v = 0$.*

PROOF. Suppose first that $\mathrm{char}(K) \neq 2$. Then $\theta(a, v) = h(a, av)/2 = 0$ by [**13**, Proposition 4.5(i)]. By [**13**, Proposition 4.21], therefore, $v = -\theta(a, \theta(a, v))/q(\pi(a)) = 0$. If $\mathrm{char}(K) = 2$, then $Q(a)v = h(a, av) = 0$ by [**13**, Propositions 3.15 and 3.16], so $v = 0$ also in this case. $\square$

PROPOSITION 8.3. *Suppose that a is as in* Hypothesis 8.1(i) *and that $u \in \langle \varepsilon, \pi(a) \rangle^{\perp}$. Then $q(\pi(au)) = q(\pi(a))q(u)^2$.*

PROOF. If $\mathrm{char}(K) \neq 2$, the claim holds by [**13**, Proposition 4.5(iii)] and Proposition 7.7 with $t = 0$. Suppose that $\mathrm{char}(K) = 2$. By [**13**, Proposition 4.10], we have

$$\phi(a, u) = f(\theta(a, u), u) + f(u, \delta) f(\theta(a, u), \varepsilon).$$

By [**13**, Proposition 4.9(i)], $f(\theta(a, u), u) = Q(a)q(u)$ and by [**13**, Proposition 4.9(iii)],

$$f(\theta(a, u), \varepsilon) = f(\pi(a), u) + Q(a) f(u, \varepsilon) = 0.$$

Hence $\phi(a, u) = Q(a)q(u)$. Thus by Proposition 7.7 with $t = Q(a)$, we have

$$q(\pi(au)) = q(\pi(a) + Q(a)\varepsilon)q(u)^2 = q(\pi(a))q(u)^2$$

also in this case. □

PROPOSITION 8.4. *If* $\operatorname{char}(K) = 2$, *then there exists* $e \in X$ *such that both* $Q(e) \neq 0$ *and* $q(\pi(e)) \neq 0$.

PROOF. Suppose that $\operatorname{char}(K) = 2$ and let δ be as in Hypothesis 8.1. Thus $f(\varepsilon, \delta) = 1$ (so, in particular, $\dim_K L \geq 2$), $q(\delta) \neq 0$ and $\pi(b) \in \langle\delta\rangle^\perp$ for all $b \in X$. Suppose that $\dim_K L = 2$. Then $\langle\delta\rangle^\perp = \langle\delta\rangle$. By Hypothesis 8.1(i), π is not identically zero. Since $q(\delta) \neq 0$, it follows that there exists $e \in X$ such that $\pi(e) = t\delta$ for some $t \in K^*$. Hence $Q(e) \neq 0$ and $q(\pi(e)) \neq 0$. We can thus assume from now on that $\dim_K L > 2$.

Suppose now that $q(\pi(e)) = 0$ for all $e \in X$ such that $Q(e) \neq 0$. Let a be as in Hypothesis 8.1(i). Thus $Q(a) \neq 0$. By [**13**, Proposition 3.21], we have

$$Q(av) = Q(a)q(v) \tag{8.2}$$

for all $a \in X$ and all $v \in L$. Hence

$$q(\pi(av)) = 0 \tag{8.3}$$

for all $a \in X$ and all $v \in V$ such that $q(v) \neq 0$. We have

$$q\big(\pi(av) + \phi(a,v)\varepsilon\big) = q(\pi(a))q(v)^2 = 0 \tag{8.4}$$

for all $v \in L$ by Proposition 7.7 with $t = 0$ and

$$\begin{aligned} q\big(\pi(av) + \phi(a,v)\varepsilon\big) &= q(\pi(av)) + Q(av)\phi(a,v) + \phi(a,v)^2 \\ &= q(\pi(av)) + \phi(a,v)\big(Q(a)q(v) + \phi(a,v)\big) \end{aligned}$$

for all $v \in L$ by (8.2). Therefore

$$\phi(a,v)\big(Q(a)q(v) + \phi(a,v)\big) = 0 \tag{8.5}$$

for all $v \in L$ such that $q(v) \neq 0$ by (8.3) and (8.4).

Now let $W = \langle\delta, \theta(a,\delta)\rangle$. We have

$$f(\theta(a,\delta),\delta) = Q(a)q(\delta) \neq 0 \tag{8.6}$$

by [**13**, Proposition 4.9(i)] and Definition 7.1(iv). Thus the restriction of q to W is non-degenerate. Since $\dim_K L > 2$ and q is non-degenerate, it follows that we can choose $u \in W^\perp$ such that $q(u) \neq 0$. By Hypothesis 8.1(ii), we can choose $t \in K$ such that $t^2q(u) + q(\delta)$, $t^2q(u) + tq(\delta)f(u,\varepsilon)$ and $tf(u,\varepsilon) - 1$ are all non-zero. Replacing u by tu, it follows that $q(u) \neq q(\delta)$, $q(u) \neq q(\delta)f(u,\varepsilon)$ and $f(u,\varepsilon) \neq 1$. Let $v = \delta + u$. Thus $q(v) = q(\delta) + q(u) \neq 0$. By [**13**, Propositions 4.9(i) and (iii) and 4.10], we have

$$\phi(a,v) = Q(a)q(v) + f(\pi(a),v)f(v,\delta) + f(\theta(a,\delta),v)f(v,\varepsilon).$$

By the choice of u and v, we have $f(v,\delta)=0$ and $f(\theta(a,\delta),v)=f(\theta(a,\delta),\delta)$ and thus

$$\phi(a,v)=Q(a)q(v)+Q(a)q(\delta)(1+f(u,\varepsilon))=Q(a)\big(q(u)+q(\delta)f(u,\varepsilon)\big)\neq 0$$

by (8.6) and

$$Q(a)q(v)+\phi(a,v)=Q(a)q(\delta)\big(1+f(u,\varepsilon)\big)\neq 0.$$

By (8.5), however, one of these two terms must be 0. With this contradiction, we conclude that there exists $e\in X$ such that both $Q(e)\neq 0$ and $q(\pi(e))\neq 0$. □

PROPOSITION 8.5. *The bilinear form f associated with q is non-degenerate.*

PROOF. By Definition 2.1, q is non-degenerate. By Notation 2.2, therefore, we can assume that $\mathrm{char}(K)=2$. Suppose that $\langle v\rangle^\perp=L$ for $v\in L$. If a is as in Hypothesis 8.1(i), then by [**13**, Proposition 4.9(i)], we have $f(\theta(a,v),v)=Q(a)q(v)$. Since $\langle v\rangle^\perp=L$ and $Q(a)\neq 0$, it follows that $q(v)=0$. Hence $v=0$ by the non-degeneracy of q. □

We now assume, as at the beginning of [**13**, Chapter 6], that

$$\dim_K L>4. \tag{8.7}$$

By Hypothesis 8.1(i) and Proposition 8.4, we can choose the element e in [**13**, Notation 6.4] so that $q(\pi(e))\neq 0$ and, if $\mathrm{char}(K)=2$, also $Q(e)\neq 0$. With Proposition 8.5 and (8.7), we can apply the subsequent results of [**13**, Chapter 6] with only small modifications. We now describe these modifications.

We first observe that by the proof of [**13**, Proposition 6.5], $u\mapsto u^\#:=\theta(e,v)$ is a norm splitting map in the sense of Definition 6.2 (but we do not know whether it is irreducible as defined in Definition 6.6 and we ignore the claim about the minimal polynomial). In place of [**13**, Definition 6.6], we say that a finite subset of L is e-orthogonal if it is ψ-orthogonal as defined in Notation 6.4 with $\psi(e)$ the norm splitting map in [**13**, Proposition 6.5]. In the assertion [**13**, Proposition 6.7(ii)], we require $q(w)\neq 0$ in addition to $w\in W^\perp$. Note that by Hypothesis 8.1(ii), we can apply [**13**, Proposition 3.22]. We observe, too, that [**13**, Proposition 6.8] continues to hold; we only need to observe toward the end of the proof that $q(u)$, $q(v)$ and $q(\pi(e)^\sigma-\pi(e))$ are all non-zero.

We now observe that the results [**13**, Propositions 6.12, 6.13, 6.15, 6.16, 6.21, 6.23 and 6.24] all hold more or less verbatim with only a few small alterations and additions:

(a) In [**13**, Proposition 6.12], we must assume that $q(u)\neq 0$ if $\mathrm{char}(K)\neq 2$ and replace the application of [**13**, Proposition 3.11] in the proof by an application of [**13**, Proposition 4.22].

(b) In the last line of the proof of [**13**, Proposition 6.13], we apply Proposition 8.2 rather than A3 to conclude that $h(e,euv)=0$ and make a similar modification at the end of the proofs of [**13**, Propositions 6.15, 6.16 and 6.21].

(c) At the start of the proof of [**13**, Proposition 6.21], we add the hypothesis that q is non-degenerate to the justification that the e-orthogonal set $1, u, v, w, x, y, z$ exists. A similar remark applies to the beginning of [**13**, Notation 6.24].

(d) Replace the application of [**13**, Proposition 2.18(ii)] at the beginning of the proof of [**13**, Proposition 6.16] by a reference to [**13**, Proposition 4.21].

(e) Let y be as in [**13**, (6.18) or (6.19)] in the proof of [**13**, Proposition 6.16]. We have

$$\begin{aligned} f(h(e, evwx\tilde{u}), \pi(e)^\sigma) &= f(h(e, evwx\tilde{u}\pi(e)), \varepsilon) && \text{by [13, Proposition 3.7]} \\ &= f(h(e, e\pi(e)vwx\tilde{u}), \varepsilon) && \text{by [13, Proposition 3.8]} \\ &= f(h(e, ev^{\#}wx\tilde{u}), \varepsilon) && \text{by D1} \\ &= -f(h(e, ev^{\#}wx), \tilde{u}) && \text{by [13, Proposition 3.7]} \\ &= 0 && \text{by [13, Proposition 6.15],} \end{aligned}$$

and

$$f(h(e, evwx\tilde{u}), \varepsilon) = -f(h(e, evwx), \tilde{u}) = 0$$

by [**13**, Propositions 3.7 and 6.15]. Hence $y \in \langle \varepsilon, \pi(e) \rangle^\perp$. We need to observe that $q(y) \neq 0$. This follows from Proposition 8.3 since we are assuming that $q(u)$, $q(v)$, $q(w)$ and $q(x)$ are all non-zero. Since we do not know whether $q(\tilde{y}) \neq 0$, we cannot say that the set $1, u, v, w, x, \tilde{y}$ is e-orthogonal in the middle of the next page, so we need another argument to prove that

$$h(e, e\tilde{y}u) = h(e, e\tilde{y}v) = \cdots = h(e, e\tilde{y}x) = 0. \tag{8.8}$$

Let Y be the subspace of L spanned by $1, u, v, w, x$ and let $W = Y + \theta(e, Y)$. Then $\tilde{y} \in W^\perp$ and $W^\perp$ is spanned by the set of all $\hat{y}$ such that $1, u, v, w, x, \hat{y}$ is e-orthogonal. By [**13**, Proposition 6.13],

$$h(e, e\hat{y}u) = h(e, e\hat{y}v) = \cdots = h(e, e\hat{y}x) = 0$$

for all such elements $\hat{y}$. It follows that (8.8) does, in fact, hold.

This concludes our list of modifications.

Now let $v_1, \ldots, v_d$ and $\alpha_1, \ldots, \alpha_d$ be as in [**13**, Notation 6.24]. Thus (K, L, q) is isomorphic to

$$(K, E^d, \alpha_1 N + \alpha_2 N + \cdots + \alpha_d N),$$

where E/K is the splitting extension of the norm splitting map $v \mapsto v^{\#}$, N is the norm of E/K, $v_1 = \varepsilon$ and $\alpha_i = q(v_i)$ for all $i = 2, \ldots, d$ (and hence $\alpha_1 = 1$). The result [**13**, Proposition 6.30] and its proof hold verbatim, but we need to replace [**13**, Propositions 6.27 and 6.31] by the following.

PROPOSITION 8.6. *One of the following holds*:

(i) *q is of type E_6, E_7 or E_8 or*

(ii) *q is similar to the norm of an octonion division algebra.*

(iii) *q is similar to q_C for some composition algebra (C, K) as in* Notation 3.1(iii)–(v), *where q_C as as defined in* Notation 4.1.

If q is as in (i), *then Ξ is anisotropic. If q is as in* (ii) *or* (iii), *then Ξ is isotropic.*

PROOF. By Proposition 6.12, q is as in (i), (ii) or (iii). Suppose that q is the norm of an octonion division algebra. It follows from [**13**, Theorem 6.42] (i.e. by the classification of anisotropic quadrangular algebras) that Ξ is isotropic. To reach the same conclusion more directly, we can follow the proof of [**13**, Proposition 6.27], where it is observed at the start that if $\alpha_2\alpha_3\alpha_4 \in N(E)$, then it can be assumed that, in fact, $\alpha_2\alpha_3\alpha_4 = 1$. It is then shown that $\pi(a) \in \langle\varepsilon\rangle$ for $a = e + ev_2v_3v_4$. Jumping ahead, we see that e and $ev_2v_3v_4$ are two elements of the basis of X given in [**13**, Proposition 6.34] (see below), so $a \neq 0$ and thus Ξ is, indeed, isotropic.

Suppose, conversely, that q is anisotropic but that Ξ isotropic. By Definition 2.3, there exists a non-zero element a in X such that $\pi(a) \in \langle\varepsilon\rangle$. By Definition 7.1(iii), we have, in fact, $\pi(a) = 0$. By D1, it follows that $a\theta(a, v) = 0$ for all $v \in L$. By A3 and the assumption that q is anisotropic, it follows that $\theta(a, v) = 0$ for all $v \in L$. By Hypothesis 8.1(iii), we can choose $b \in X$ such that $h(a, b) \neq 0$. By A3 again, the subspaces aL and $ah(a, b)L$ both have dimension $\dim_K L$. By Hypothesis 8.1(ii) and [**13**, Proposition 3.22], we have

$$ah(a, bu)v = ah(a, buv) \tag{8.9}$$

for all $u, v \in L$. Setting $u = \varepsilon$ in (8.9), we obtain $ah(a, b)L = ah(a, bL) \subset aL$ and hence $ah(a, bL) = aL$. By (8.9) with arbitrary u and v, it follows that $aLL = aL$. We conclude that $X = aC(q, \varepsilon) = aL$. By Proposition 6.14, however, $C(q, \varepsilon)$ does not have a right module of dimension equal to $\dim_K L$ if q is of type E_6, E_7 or E_8. Hence q is as in (ii). If q is as in (iii), then q is isotropic, Ξ is also isotropic. $\square$

The result [**13**, Proposition 6.34] holds verbatim as does its proof up to the last line, where the justification given for the inequality

$$\dim_K X \geq |B| \tag{8.10}$$

is no longer valid. Here B is as in [**13**, Notation 6.32] and

$$|B| = \begin{cases} 8 & \text{if } d = 3, \\ 16 & \text{if } d = 4, \\ 32 & \text{if } d = 6. \end{cases} \tag{8.11}$$

We now prove (8.10). By Proposition 8.2, the map $v \mapsto ev$ from L to X is injective. Thus eL is a subspace of X of dimension $\dim_K L$. Thus, in particular,

$$\dim_K X > 4 \quad \text{if } \dim_K L = 6 \tag{8.12}$$

and $\dim_K X \geq 8$ if $\dim_K L = 8$. Suppose that $\dim_K L = 8$ and that $ev_2v_3v_4 = eu$ for some $u \in L$. By Proposition 8.3, $q(\pi(ev_2v_3v_4)) \neq 0$ and if $\mathrm{char}(K) = 2$, then $Q(ev_2v_3v_4) \neq 0$ by (8.2). In particular, $u \neq 0$. Hence $u^\sigma \neq 0$. By Proposition 8.2, it follows that $ev_2v_3v_4u^\sigma \neq 0$. By A3, we deduce that $q(u) \neq 0$. By [**13**, Proposition 6.12], therefore, $h(e, eu) \neq 0$. By [**13**, Proposition 6.15], however, $h(e, ev_2v_3v_4) = 0$. With this contradiction we conclude that

$$\dim_K X > 8 \quad \text{if } \dim_K L = 8. \tag{8.13}$$

Let k be the dimension of an arbitrary irreducible $C(q, \varepsilon)$-module. By Propositions 6.14 and 8.6, either $k = |B|$ or $k = |B|/2$ and in the latter case, $\dim_K L = 6$ or 8. Since X is a direct sum of irreducible $C(q, \varepsilon)$-modules, it follows now by (8.12) and (8.13) that (8.10) does, in fact, hold. Thus [**13**, Proposition 6.34] holds.

Corollary 8.7. *Suppose that X is not irreducible as a $C(q, \varepsilon)$-module. Then X is the sum of two irreducible $C(q, \varepsilon)$-modules and one of the following holds*:

(i) $\dim_K L = 6$, $\dim_K X = 8$ *and q is hyperbolic.*

(ii) $\dim_K L = 8$, $\dim_K X = 16$ *and q the norm of an octonion algebra (C, K) which is either division or split.*

Proof. By [**13**, Proposition 6.34], $\dim_K X = |B|$, where $|B|$ is as in (8.11). The claim holds, therefore, by Propositions 6.14 and 8.6. □

Notation 8.8. Let E be the splitting extension of the norm splitting map $e \mapsto e^\#$ and let τ be the unique non-trivial K-algebra automorphism of E. We identify E with $K[x]/J$, where J is the ideal generated by the polynomial $x^2 - Q(e)x + q(\pi(e))$, we identify K with its natural image in E and we set $\gamma = x + J$. Thus γ is an element of E such that

$$\gamma^\tau \neq \gamma. \tag{8.14}$$

We define a map from $E \times \{v_1, \ldots, v_d\}$ to L by setting

$$(s + t\gamma)v_i = sv_i + t\theta(e, v_i) \tag{8.15}$$

for all $i = 1, 2, \ldots, d$ and all $s, t \in K$. Thus, in particular, $\pi(e) = \gamma v_1 = \gamma\varepsilon$ and $\pi(e)^\sigma = \gamma^\tau\varepsilon$. Now suppose that $d = 3$ or 4 and let I_p as well as e_x and e_x° for all $x \in I_p$ be as in [**13**, Notation 6.32] (so $p = d - 1$). Let $M = \{e_x \mid x \in I_p\}$, let

$$N = \{te_x \mid x \in I_p,\ t \in K^*\}$$

and let ℓ be the map from N to the natural numbers that sends te_x to the cardinality of x. The set N is closed under right multiplication by v_i (by [**13**, Proposition 3.8]) and

$$\ell(e_xv_i) - \ell(e_x) = 1 \text{ or } -1$$

for each $x \in I_p$ and each $i = 1, 2, \ldots, d$. We define a map from $E \times M$ to X by setting

$$(s + t\gamma)e_x = se_x + te_x^\circ \tag{8.16}$$

for all $x \in I_p$ and all $s, t \in K$. By A1–A3, D1 and [**13**, Proposition 3.8], we have

$$Ae_x \cdot Bv_i = \begin{cases} AB^{\tau^{\ell(x)}}x & \text{if } i = 1, \\ ABxv_i & \text{if } i > 1 \text{ and } \ell(e_xv_i) > \ell(e_x), \\ AB^\tau xv_i & \text{if } i > 1 \text{ and } \ell(e_xv_i) < \ell(e_x) \end{cases} \tag{8.17}$$

for all $A, B \in E$, where Bv_i is as defined in (8.15) and Ae_x is as defined in (8.16).

NOTATION 8.9. Let κ be the unique linear automorphism L such that $\kappa(Av_1) = A^\tau v_1$ and $\kappa(Av_i) = Av_i$ for all $A \in E$ and $i = 2, 3, \ldots, d$. Thus, in particular, $\kappa(\pi(e)) = \kappa(\gamma\varepsilon) = \gamma^\tau\varepsilon$. The map κ is a reflection of (K, L, q) fixing ε.

PROPOSITION 8.10. *Let κ be as in* Notation 8.9 *and suppose that q is hyperbolic if $d = 3$. Then the unique extension of κ to an automorphism of $C(q, \varepsilon)$ interchanges the two direct summands of $C(q, \varepsilon)$.*

PROOF. By Proposition 6.14, $C(q, \varepsilon)$ is a sum of two simple subalgebras and the center of $C(q, \varepsilon)$ is a split quadratic étale quadratic extension. The claim follows, therefore, from the observation that the unique extension of κ to an automorphism of $C(q, \varepsilon)$ interchanges the two central elements z and z' in [**12**, (12.41)]. □

The next result and Proposition 8.14(ii) below will not be needed until the proof of Theorem 10.16.

PROPOSITION 8.11. *Let ρ be an arbitrary reflection of q and suppose that q is hyperbolic if $d = 3$. Then the unique extension of ρ to an automorphism of $C(q, \varepsilon)$ interchanges the two direct summands of $C(q, \varepsilon)$.*

PROOF. Let $\bar{K}$ be the algebraic closure of K, let $q_{\bar{K}}$ be the scalar extension of q to $L \otimes_K \bar{K}$ and let $\kappa_{\bar{K}}$ and $\rho_{\bar{K}}$ denote the unique extensions of κ and ρ to reflections of $q_{\bar{K}}$. Then $C(q_{\bar{K}}, \varepsilon)$ is the sum of two matrix rings over $\bar{K}$ and, by Proposition 8.10, the unique extension of $\kappa_{\bar{K}}$ to an automorphism of $C(q_{\bar{K}}, \varepsilon)$ acts non-trivially on the center of $C(q_{\bar{K}}, \varepsilon)$. By [**3**, Theorem 8.3] (Witt's Extension Theorem), $\kappa_{\bar{K}}$ and $\rho_{\bar{K}}$ are conjugate under the isometry group of $q_{\bar{K}}$. Thus the unique extension of $\rho_{\bar{K}}$ to an automorphism also acts non-trivially on the center of $C(q_{\bar{K}}, \varepsilon)$. It follows that the unique extension of ρ to an automorphism of $C(q, \varepsilon)$ acts non-trivially on the center of $C(q, \varepsilon)$ and hence interchanges the two direct summands of $C(q, \varepsilon)$. □

PROPOSITION 8.12. *Let $d = 3$, let κ be as in* Notation 8.9 *and let ψ be the unique automorphism of X such that for all $A \in E$, $\psi(Aev_i) = A^\tau ev_2v_3v_i$ for $i = 1, 2, 3$ and $\psi(Aev_2v_3) = A^\tau ev_2v_3v_2v_3 = \alpha_2\alpha_3A^\tau e$, where $\alpha_2 = q(v_2)$ and $\alpha_3 = q(v_3)$ as in* [**13**, Notation 6.24]. *Then $\psi(av) = \psi(a)\kappa(v)$ for all $a \in X$ and all $v \in L$.*

PROOF. By (8.17) and a bit of calculation, the claim holds for a of the form Ae_x and v of the form Bv_i. Since the map $(a, v) \mapsto a \cdot v$ is bilinear, the claim holds for arbitrary $a \in X$ and $v \in L$. □

PROPOSITION 8.13. *Let $d = 4$ or 6 and let κ be as in* Notation 8.9. *Then there exists no K-linear automorphism ψ of X such that $\psi(av) = \psi(a)\kappa(v)$ for all $a \in X$ and all $v \in L$.*

PROOF. Suppose that $d = 6$. Then $\dim_K X = 32$. By Proposition 6.14, it follows that X is an irreducible module for one of the two direct summands of $C(q, \varepsilon)$ and the other direct summand acts trivially on X. By Proposition 8.10, there is no K-linear automorphism ψ of X such that $\psi(aw) = \psi(a)\hat{\kappa}(w)$ for all $a \in X$ and all $w \in C(q, \varepsilon)$, where $\hat{\kappa}$ denotes the unique extension of κ to an automorphism of $C(q, \varepsilon)$. Hence there is no K-linear automorphism ψ of X such that $\psi(av) = \psi(a)\kappa(v)$ for all $a \in X$ and all $v \in L$.

Now suppose that $d = 4$ and that ψ is an automorphism of X such that $\psi(av) = \psi(a)\kappa(v)$ for all $a \in X$ and all $v \in L$. Let $e_1 = \psi(e)$. Applying ψ to the identity $e\theta(e, v) = e\pi(e)v$, we have

$$e_1\kappa(\theta(e, v)) = e_1\kappa(\pi(e))\kappa(v) \tag{8.18}$$

for all $v \in L$. By (8.15), we have $\theta(e, v_i) = \gamma v_i$ for all i. Thus, in particular,

$$e_1\theta(e, v_i) = e_1 \cdot \gamma^\tau \varepsilon \cdot v_i \tag{8.19}$$

for all $i = 2, 3, 4$. Right multiplication by an element of the form Bv_i with $B \in E$ non-zero permutes the subspaces $\{Ae_x \mid A \in E\}$ of X. Since $e_1 \neq 0$, it follows from (8.19) that there exists $x \in I_p$ and $A \in E$ such that $A \neq 0$ and

$$Ae_x\theta(e, v_i) = Ae_x \cdot \gamma^\tau \varepsilon \cdot v_i \tag{8.20}$$

for all $i = 2, 3, 4$. If $\ell(e_x)$ is odd, choose i such that $\ell(e_xv_i) < \ell(e_x)$. If $\ell(e_x)$ is even, choose i such that $\ell(e_xv_i) > \ell(e_x)$. Applying (8.17), we find that if $\ell(e_x)$ is odd, then $Ae_x\theta(e, v_i) = Ae_x \cdot \gamma v_i = A\gamma^\tau e_xv_i$ but $Ae_x \cdot \gamma^\tau\varepsilon \cdot v_i = A\gamma e_xv_i$ and if $\ell(e_x)$ is even, then $Ae_x\theta(e, v_i) = Ae_x \cdot \gamma v_i = A\gamma e_xv_i$ but $Ae_x \cdot \gamma^\tau\varepsilon \cdot v_i = A\gamma^\tau e_xv_i$. Hence in both cases, (8.20) implies that $A\gamma e_xv_i = A\gamma^\tau e_xv_i$ and hence $A = 0$ by (8.14). With this contradiction, we conclude that there is no non-zero element e_1 such that (8.18) holds for all $v \in L$. $\square$

PROPOSITION 8.14. *Let $d = 4$ or 6. Then the following hold*:

(i) *X is either irreducible as a $C(q, \varepsilon)$-module or the sum of two copies of the same irreducible $C(q, \varepsilon)$-module.*

(ii) *If ρ is an arbitrary reflection of q, then there exists no K-linear automorphism ψ of X such that $\psi(av) = \psi(a)\rho(v)$ for all $a \in X$ and all $v \in L$.*

PROOF. By Proposition 6.14, X is either irreducible as a $C(q, \varepsilon)$-module or the sum of two irreducible $C(q, \varepsilon)$-modules. By Propositions 8.10 and 8.13, the two irreducible $C(q, \varepsilon)$-modules are two copies of the same irreducible $C(q, \varepsilon)$-modules in the second case. Thus (i) holds. Whether or not X is irreducible, one of the two direct summands of $C(q, \varepsilon)$ acts trivially on X and the other does not. By Proposition 8.11, therefore, (ii) holds. $\square$

PROPOSITION 8.15. *Suppose that* $\dim_K L \geq 5$, *let* κ *be as in* Notation 8.9 *and let*

$$\hat{\Xi} = (\hat{K}, \hat{L}, \hat{q}, \hat{f}, \hat{\varepsilon}, \hat{X}, *, \hat{h}, \hat{\theta})$$

be a second quadrangular algebra satisfying the hypotheses of in Hypothesis 8.1. *Suppose that* ξ *is an isomorphism from* (K, L, q) *to* $(\hat{K}, \hat{L}, \hat{q})$ *mapping* ε *to* $\hat{\varepsilon}$. *Then either there exists an isomorphism* (λ, ψ) *from* Ξ *to* $\hat{\Xi}$, *where either* $\lambda = \xi$ *or* $d = 4$ *or* 6 *and* $\lambda = \kappa\xi$.

PROOF. If $C(q, \varepsilon)$ is the direct sum of two central simple summands, then by Proposition 8.12, X is the direct sum of the two distinct irreducible $C(q, \varepsilon)$-modules if $d = 3$ and by Proposition 8.14(i), X is either irreducible as a $C(q, \varepsilon)$-module or the direct sum of two copies of one of the two irreducible $C(q, \varepsilon)$-modules if $d = 4$ or 6. It follows that there exists a pair (λ, ψ), where ψ is an additive bijection from X to $\hat{X}$ and either $\lambda = \xi$ or $\lambda = \kappa\xi$ (and $d = 4$ or 6 in the second case) such that $\psi(a \cdot v) = \psi(a) * \lambda(v)$ for all $a \in X$ and all $v \in L$. We now identify $(K, L, q, f, \varepsilon, X, \cdot)$ with $(\hat{K}, \hat{L}, \hat{q}, \hat{f}, \hat{\varepsilon}, \hat{X}, *)$ via λ and ψ and follow the proof of [**13**, Proposition 6.38]. Where [**13**, Definition 1.17(D2) and Proposition 3.4] are applied, however, we can now only deduce from (8.17) that there exist $\omega, \beta \in K$ such that $\hat{\pi}(e) = \omega\pi(e) + \beta\varepsilon$. Since $\hat{\Xi}$ is δ-standard, we have $\beta = 0$. If $\omega \neq 0$, then the rest of the proof of [**13**, Proposition 6.38] remains valid verbatim.

We can suppose, therefore, that $\omega = 0$ and hence $\hat{\pi}(e) = 0$. By D1 and Proposition 8.2, it follows that $\hat{\theta}(e, v) = 0$ for all $v \in L$. Hence $\hat{h}(e, ev) = 0$ for all $v \in L$ by [**13**, Proposition 4.5(i)] if $\text{char}(K) \neq 2$ and by [**13**, Propositions 3.15 and 3.16] if $\text{char}(K) = 2$. By [**13**, Proposition 3.22], we have $e\hat{h}(e, b)u = e\hat{h}(e, bu) \in eL$ for all $b \in X$ and all $u \in L$. Setting $b = ev$, we conclude using Proposition 8.2 that $h(e, euv) = 0$ for all $u, v \in L$. Repeating this argument, this time with $e = buv$, we conclude that $h(e, euvw) = 0$ for all $u, v, w \in L$. Thus $h(e, b) = 0$ for every element in the set B defined in [**13**, Notation 6.32]. Since B spans X (by [**13**, Proposition 6.34]), we have a contradiction to Hypothesis 8.1(iii). With this contradiction, we conclude that, in fact, $\omega \neq 0$. □

Here now is the main result of this section:

THEOREM 8.16. *Let* $\Xi = (K, L, q, f, \varepsilon, X, \cdot, h, \theta)$ *be a proper quadrangular algebra as defined in as defined in* Definition 5.4 *and let* Q *be as in* (8.1). *Suppose that* $q(\pi(a)) \neq 0$ *for some* $a \in X$ *if* $\text{char}(K) \neq 2$ *and that* $Q(a) \neq 0$ *for some* $a \in X$ *if* $\text{char}(K) = 2$. *Suppose, too, that* $\dim_K L \geq 5$, *that* $|K| > 4$ *and that* h *is non-degenerate as defined in* Observation 5.8. *Then one of the following holds*:

(i) q *is of type* E_6, E_7 *or* E_8 *and* Ξ *is uniquely determined up to isotopy by the similarity class of* q.

(ii) q *is similar to the norm of an octonion division algebra* (C, K) *and* Ξ *is isotopic to* $\mathcal{Q}_2(C, K)$.

(iii) q *is similar to* q_C *for some composition algebra* (C, K) *as in* Notation 3.1(iii)–(v) *and* Ξ *is isotopic to* $\mathcal{Q}_4(C, K)$.

In case (i), Ξ *is anisotropic and in cases* (ii) *and* (iii), Ξ *is isotropic.*

PROOF. This holds by Propositions 7.2, 8.6 and 8.15. $\square$

9. The inseparable F_4-case.

In this section, we make the following assumptions:

HYPOTHESIS 9.1. Let $\Xi = (K, L, q, f, \varepsilon, X, \cdot, h, \theta)$ be a δ-standard quadrangular algebra for some $\delta \in L$ as defined in Definition 7.1. We assume that $\mathrm{char}(K) = 2$, that $|K| > 4$ and that $Q(a) = 0$ for all $a \in X$, where Q is as in (8.1).

The main result of this section is Theorem 9.8.

PROPOSITION 9.2. *Let R denote the radical of f and let g be as in* C3. *Then $h(X, X) \subset R$ and $g(X, X) = 0$.*

PROOF. By [**13**, Proposition 3.16] and Hypothesis 9.1, $g(a, a) = Q(a) = 0$ for all $a \in X$. By [**13**, Proposition 4.3], g is bilinear and hence symmetric. By another application of [**13**, Proposition 4.3], it follows that

$$g(a, b) = f(h(a, b), \delta) = f(h(b, a), \delta) \tag{9.1}$$

for all $a, b \in X$, where δ is as in Hypothesis 9.1. By [**13**, Proposition 3.6], we have $h(b, a) = h(a, b)^\sigma$, so

$$f(h(b, a), \delta) = f(h(a, b)^\sigma, \delta) = f(h(a, b), \delta^\sigma) = f(h(a, b), \delta + \varepsilon)$$

for all $a, b \in X$ by [**13**, (1.4)] and Definition 7.1(ii). Hence $f(h(a, b), \varepsilon) = 0$ for all $a, b \in X$. By B3, therefore,

$$f(h(a, b), v) = f(h(av, b), \varepsilon) = 0$$

for all $a, b \in X$ and all $v \in L$. Hence $h(X, X) \subset R$. By (9.1), it follows that g is identically zero. $\square$

PROPOSITION 9.3. *Let $X^\flat$ be as in* Proposition 7.6. *Then the following hold*:

(i) *$\theta(a, R) \subset R$ for all $a \in X$.*

(ii) *If $\theta(a, u) \in R$ for some $a \in X^\flat$ and some $u \in L$, then $u \in R$.*

PROOF. Let $\rho \in R$. By [**13**, Proposition 3.19(iii)], we have

$$f(\theta(a, \rho), v) = f(\theta(a, v), \rho) = 0$$

for all $v \in L$. Thus (i) holds. By [**13**, Proposition 7.4(iii)], $u = q(\pi(a))^{-1}\theta(a, \theta(a, u))$ for all $a \in X^\flat$ and all $u \in L$. Hence (ii) is a consequence of (i). $\square$

HYPOTHESIS 9.4. We now add the hypothesis that h is non-degenerate as defined in Observation 5.8. Thus, in particular, $h(X, X) \neq 0$, so $R \neq 0$ by Proposition 9.2.

Proposition 9.5. *X is spanned by the set $X^\flat$ defined in* Proposition 7.6.

Proof. By Hypothesis 9.4, we can choose a non-zero element ρ in R. By Proposition 9.2 and C3, we have

$$\theta(a+b,\rho) = \theta(a,\rho) + \theta(b,\rho) + h(a,b\rho) \tag{9.2}$$

for all $a, b \in X$. By [**13**, Proposition 4.22], $q(\theta(a,\rho)) = q(\pi(a))q(\rho)$ for all $a \in X$. Since q is non-degenerate and $\rho \neq 0$, we have $q(\rho) \neq 0$. It follows that $q(\theta(a,\rho)) = 0$ if and only if $q(\pi(a)) = 0$. By Proposition 9.3(i) and the non-degeneracy of q, it follows that $q(\pi(a)) = 0$ if and only if $\theta(a,\rho) = 0$. By Hypothesis 9.4, there exist $a, c \in X$ such that $h(a,c) \neq 0$. By A3, it follows that $h(a,b\rho) = h(a,c)q(\rho) \neq 0$ for $b = c\rho$. By (9.2), we conclude that $X^\flat \neq \emptyset$. By Hypothesis 9.4, we also have $|K| > 4$. By Proposition 7.6, therefore, X is spanned by $X^\flat$. □

By Hypothesis 9.4, $R \neq 0$. Thus from now on we can follow the proof of [**13**, Theorem 7.57] (in which Theorem 9.8 is proved under the hypothesis that Ξ is anisotropic) with a few modifications and comments. We start now to describe these modifications. The result [**13**, Proposition 7.1] holds by Proposition 9.2. The results [**13**, Propositions 7.2–7.6] hold verbatim. In [**13**, Proposition 7.9], where

$$W_a = \langle \varepsilon, \delta, \pi(a), \theta(a,\delta) \rangle,$$

it is necessary only to add the assumption that $a \in X^\flat$. We replace [**13**, Proposition 7.10] (which is needed only for the application of [**13**, Proposition 3.22]) by the assumption $|K| > 4$ in Hypothesis 9.1.

It requires more effort to prove [**13**, Proposition 7.11] in the present context and, in fact, it is easier now to combine [**13**, Proposition 7.11] and [**13**, Proposition 7.12] into one result:

Proposition 9.6. *$W_a^\perp = R$ and $\theta(a,u) = h(a, a\delta u)$ for all $a \in X^\flat$ and all $u \in R$.*

Proof. Let $a \in X^\flat$ and let $u \in W_a^\perp$. The proof of [**13**, Proposition 7.11] yields the conclusion that $aw = 0$ for

$$w = \theta(a,u) + h(a, a\delta u) \tag{9.3}$$

(but we can no longer appeal to [**13**, Proposition 3.4] at this point). By A3, we have $q(w) = 0$. By Proposition 9.2, we have $h(a, a\delta u) \in R$. Suppose that $u \in R$. Then $w \in R$ by Proposition 9.3(i). Hence $w = 0$ since q is non-degenerate. It thus suffices to show that $W_a^\perp = R$.

Suppose that $W_a^\perp \neq R$. We can thus assume that $u \in W_a^\perp$ was chosen so that $u \notin R$. By [**13**, Proposition 7.9], we have $\theta(a,u) \in W_a^\perp$ and by Proposition 9.3(ii), we have $\theta(a,u) \notin R$. Hence $w \in W_a^\perp \backslash R$. By [**13**, Proposition 7.9] and Proposition 9.3(ii) again, we have $\theta(a,w) \in W_a^\perp \backslash R$. Hence we can choose $v \in W_a^\perp$ such that $f(\theta(a,w), v) = 1$. Since $aw = 0$, also $\theta(aw, v) = 0$ (by [**13**, Proposition 3.12]). Let $\alpha = f(v,w)$. The map σ acts trivially on $\langle \varepsilon \rangle^\perp$ and hence on $W_a^\perp$. By [**13**, Proposition 7.5] and C4 with w in place of v and v in place of w, therefore, we have

$$w = \alpha\theta(a, w). \tag{9.4}$$

Hence $\alpha \neq 0$ and

$$\theta(a, w) = \alpha\theta(a, \theta(a, w)) = \alpha q(\pi(a))w = \alpha^2 q(\pi(a))\theta(a, w)$$

by [**13**, Proposition 4.21], so, in fact, $q(\pi(a))\alpha^2 = 1$. By C1, (9.3), (9.4) and another application of [**13**, Proposition 4.21], we also have

$$\begin{aligned}\alpha^{-1}\big(\theta(a, u) + h(a, a\delta u)\big) &= \alpha^{-1}w = \theta(a, w)\\ &= \theta\Big(a, \big(\theta(a, u) + h(a, a\delta u)\big)\Big)\\ &= q(\pi(a))u + \theta\big(a, h(a, a\delta u)\big).\end{aligned}$$

By Propositions 9.2 and 9.3(i), we conclude that

$$\theta(a, u) + q(\pi(a))^{1/2}u \in R \tag{9.5}$$

for all $a \in X^\flat$ and all $u \in W_a^\perp$ (whether or not $u \in R$).

We continue with our choice of $a \in X^\flat$ and $u \in W_a^\perp \backslash R$ and now choose $y \in W_a^\perp$ such that $f(y, u) \neq 0$. We then choose $t \in K^*$ such that $q(z) \neq 0$ for $z = t\varepsilon + y$. By [**13**, Proposition 7.5] and Proposition 7.7, we have

$$q(\pi(az)) = q(\pi(a))q(z)^2, \tag{9.6}$$

so $az \in X^\flat$. We have $f(u, z^\sigma) = f(u^\sigma, z) = f(u, z)$. Hence by [**13**, Proposition 7.5] and C4 with z in place of v and u in place of w, we have

$$\theta(az, u) = \theta(a, u)q(z) + f(u, z)\theta(a, z)^\sigma + f(\theta(a, z), u)z^\sigma.$$

From $az = ta + ay$, it follows that

$$\pi(az) + \pi(ta) + \pi(ay) \in R \tag{9.7}$$

and

$$\theta(az, \delta) + \theta(ta, \delta) + \theta(ay, \delta) \in R \tag{9.8}$$

by C3 and Proposition 9.2. By [**13**, Proposition 7.9], we have $\theta(a, y) \in W_a^\perp$. Thus by [**13**, Proposition 7.5] and C4 with y in place of v and once with ε in place of w and a second time with δ in place of w, we have $\pi(ay) = \pi(a)q(y) \in W_a$ and $\theta(ay, \delta) = \theta(a, \delta^\sigma)^\sigma q(y) \in W_a$. Hence $f(\pi(ay), u) = f(\theta(ay, \delta), u) = 0$. By (9.7) and (9.8), we conclude that $u \in W_{az}^\perp$. Thus by (9.5), both $\theta(a, u)+q(\pi(a))^{1/2}u$ and $\theta(az, u)+q(\pi(az))^{1/2}u$ lie in R. By (9.6), therefore, $\theta(az, u)+\theta(a, u)q(z) \in R$. By [**13**, Proposition 7.5] and C4, this time with z in place of v and u in place of w, it follows that $f(u, z^\sigma)\theta(a, z)^\sigma + f(\theta(a, z), u)z^\sigma \in R$. We have $f(u, z^\sigma) = f(u, y) \neq 0$. By [**13**, Proposition 7.9], we have

$$\theta(a, z) + t\pi(a) = \theta(a, y) \in W_a^\perp$$

and thus $\theta(a, z)^\sigma + t\pi(a) \in W_a^\perp$. Therefore $\pi(a) \in \langle z^\sigma, W_a^\perp\rangle = \langle \varepsilon, W_a^\perp\rangle$. By [**13**, Propositions 7.4(ii) and 7.6], we have $f(\varepsilon, \theta(a, \delta)) = 0$ and hence $\langle \varepsilon, W_a^\perp\rangle \subset \langle\theta(a, \delta)\rangle^\perp$. By [**13**, Proposition 7.6], however, $f(\pi(a), \theta(a, \delta)) \neq 0$. With this contradiction, we conclude that $W_a^\perp = R$. □

Thus also [**13**, Corollary 7.13] holds with the assumption that $a \in X^\flat$ as does [**13**, Proposition 7.17]. The results [**13**, Propositions 7.14–7.16] hold without any modification in their proofs.

The proof of [**13**, Proposition 7.18] remains valid with the additional assumption that $a \in X^\flat$. It is only necessary to recall that if u is a non-zero element of R, then $q(u) \neq 0$. The result [**13**, Proposition 7.26] remains valid. It is only necessary to choose $a \in X^\flat$ at the beginning of the proof and to cite Proposition 9.5 in the last sentence (since at this point, we know that w depends only on u, v and ρ and thus that $au\rho^{-1}v = aw$ for all $a \in X^\flat$).

Nothing needs to be modified in [**13**, Propositions and Notation 7.29–7.35]. In [**13**, Proposition 7.36], X^* and F^* are to be replaced by X and F (and the last sentence of the proof deleted) and in [**13**, Notation 7.37], X^* is to be replaced by $X^\flat$. In [**13**, Proposition 7.40], we need to define E to be the étale quadratic extension $K[\gamma]/K$, where γ is a root of $p(x)$ not in K since we no longer know that $p(x)$ is irreducible over K. The proof of [**13**, Proposition 7.40] remains valid with this modification. Next we replace [**13**, Proposition 7.41] by the following observation:

Proposition 9.7. *The quadratic form q is either anisotropic, in which case it is of type F_4 as defined in* [**13**, Definition 2.15], *or (K, L, q) is similar to (K, L_C, q_C).*

Proof. This holds by [**13**, Proposition 7.40] and Proposition 6.13. □

In [**13**, Proposition 7.42] we insert the hypothesis that the polynomial $p(x)$ is irreducible over K. The result [**13**, Proposition 7.42] is needed only in the proof of [**13**, Proposition 7.56]. In the next paragraph, we give a proof of [**13**, Proposition 7.56] in the case that $p(x)$ is reducible over K that does not depend on [**13**, Proposition 7.42]. (Observe that if $p(x)$ has a root γ in K, then γ^2 is a root of $p_0(x)$ contained in F since $K^2 \subset F$ by [**13**, (7.32)], i.e. if $p(x)$ is reducible over K, then $p_0(x)$ is reducible over F.)

In [**13**, Proposition 7.43], we insert the hypothesis that q is anisotropic. The results [**13**, Propositions 7.44, 7.49 and 7.50] remain valid verbatim (but the Q introduced in [**13**, Proposition 7.44] must not be confused with the Q in Hypothesis 9.1). In [**13**, Proposition 7.55] we need to set D equal to the étale quadratic extension $F[\gamma^2]$, where γ is a root of the polynomial $p(x)$ in [**13**, Proposition 7.40], to allow for the case that $p(x)$ is reducible over K. The proof of [**13**, Proposition 7.55] remains valid with this modification. Thus, in particular, Q is similar to the norm of the split quaternion algebra over F if $p(x)$ is reducible over K. The proof of [**13**, Proposition 7.56] remains valid in the case that $p(x)$ is irreducible over K. If $p(x)$ is reducible over K, then $q(\langle\varepsilon, \delta\rangle^\perp) = K$ and Q is hyperbolic, so its image is all of F and the assertion of [**13**, Proposition 7.56] holds also in this case. Finally, we modify [**13**, Theorem 7.57] to allow the possibility that q is similar to (K, L_C, q_C) as defined in Notation 4.1 for some composition algebra (C, K) as in Notation 3.1(i). The proof of [**13**, Theorem 7.57] remains valid verbatim.

We can now formulate the main theorem of this section:

THEOREM 9.8. *Let* $\Xi = (K, L, q, f, \varepsilon, X, \cdot, h, \theta)$ *be a proper quadrangular algebra as defined in* Definition 5.4. *Suppose that* $\mathrm{char}(K) = 2$*, that* $|K| > 4$ *and that* $Q(a) = 0$ *for all* $a \in X$*, where* Q *is as in* (8.1). *Then either* q *is of type* F_4 *as defined in* [**13**, Definition 2.15] *and* Ξ *is uniquely determined by* q *up to isotopy or* q *is similar to* (K, L_C, q_C) *as defined in* Notation 4.1 *for some composition algebra* (C, K) *as in* Notation 3.1(i) *and* Ξ *is isotopic to* $\mathcal{Q}_4(C, K)$*. In the first case,* Ξ *is anisotropic and in the second, it is isotropic.*

PROOF. This holds by Propositions 7.2 and 9.7 and the uniqueness assertion in [**13**, Theorem 7.57] (as modified above). □

10. The split F_4-case.

In the previous section, we treated the case that $\mathrm{char}(K) = 2$ and Hypothesis 8.1(i) fails to hold. In this section, we turn to the case that $\mathrm{char}(K) \neq 2$ and Hypothesis 8.1(i) fails to hold. Our assumptions are as follows:

HYPOTHESIS 10.1. Let $\Xi = (K, L, q, f, \varepsilon, X, \cdot, h, \theta)$ be a δ-standard quadrangular algebra for some $\delta \in L$ as defined in Definition 7.1, let π be as in D1 and let σ be as in [**13**, (1.2)]. Suppose that

(i) $q(\pi(a)) = 0$ for all $a \in X$,

(ii) h is non-degenerate as defined in Observation 5.8,

(iii) $|K| > 3$ and

(iv) $\mathrm{char}(K) \neq 2$.

The main result of this section is Theorem 10.16. (By Propositions 8.4 and 9.5, Hypothesis 10.1(iv) is superfluous if $|K| > 4$, but this observation is irrelevant for our proof of Theorem 5.10.)

NOTATION 10.2. By Hypothesis 10.1(iv), we can set

$$h_\varepsilon(a, b) = \tfrac{1}{2} f(h(a, b), \varepsilon)\varepsilon$$

and

$$h^\perp(a, b) = h(a, b) - h_\varepsilon(a, b)$$

for all $a, b \in X$. Thus $h^\perp(X, X) \subset \langle\varepsilon\rangle^\perp$, where

$$\langle\varepsilon\rangle^\perp = \{v \in L \mid f(v, \varepsilon) = 0\}, \tag{10.1}$$

and $h_\varepsilon(X,X) \subset \langle\varepsilon\rangle$. By [**13**, Proposition 3.6], $h^\perp$ is symmetric and h_ε is skew-symmetric and by Definition 7.1(i), Hypothesis 10.1(iv) and [**13**, Proposition 4.3], we have

$$g(a,b) = \tfrac{1}{2} f(h(a,b),\varepsilon) \tag{10.2}$$

and thus

$$h_\varepsilon(a,b) = g(a,b)\varepsilon \tag{10.3}$$

for all $a, b \in X$, where g is as in C3.

LEMMA 10.3. *The form g is non-degenerate.*

PROOF. Let b be a non-zero element of X. By Hypothesis 10.1(ii), there exists $a \in X$ such that $h(a,b) \neq 0$. Hence there exists $v \in L$ such that $f(h(a,b),v) \neq 0$. By B3, it follows that $h_\varepsilon(av,b) \neq 0$. By (10.3), therefore, $g(av,b) \neq 0$. □

PROPOSITION 10.4. *$h(a\pi(a),b) = 0$ for all $a, b \in X$.*

PROOF. By Hypothesis 10.1(i), (10.3) and C3, we have

$$\begin{aligned} 0 = q(\pi(a+tb)) &= q\big(\pi(a) + t^2\pi(b) + th^\perp(a,b)\big) \\ &= t^3 f(\pi(b), h^\perp(a,b)) + t^2\big(q(h^\perp(a,b)) + f(\pi(a),\pi(b))\big) \\ &\qquad + tf(\pi(a), h^\perp(a,b)) \end{aligned}$$

for all $a, b \in X$ and all $t \in K$. By Hypothesis 10.1(iii) and [**12**, (2.26)], it follows that

$$f(\pi(a), h^\perp(a,b)) = 0$$

for all $a, b \in X$. Hence $f(\pi(a), h(a,b)) = 0$ for all $a, b \in X$ since $f(\pi(a), \epsilon) = 0$. By B3, therefore, we have

$$f(h(a\pi(a),b),\varepsilon) = 0$$

for all $a, b \in X$. Thus by [**13**, Proposition 3.7],

$$f(h(a\pi(a),b),v) = f(h(a\pi(a),bv^\sigma),\varepsilon) = 0$$

for all $a, b \in X$ and all $v \in L$. The claim follows since by Notation 2.2, f is non-degenerate. □

COROLLARY 10.5. *$a\pi(a) = 0$ for all $a \in X$.*

PROOF. This holds by Hypothesis 10.1(ii) and Proposition 10.4. □

EXAMPLE 10.6. The assertion in Corollary 10.5 need not hold without Hypothesis 10.1(ii). Let $\Xi = (K, L, q, f, \varepsilon, X, \cdot, h, \theta)$ and γ be as in 5.9, for example. If γ is not invertible, then $q(\pi(a)) = 0$ for all $a \in X$ and h is degenerate. If, in addition, $\gamma \neq 0$, then $a\pi(a) \neq 0$ if and only if a is invertible in L.

PROPOSITION 10.7. *$b\pi(a) = -ah^\perp(a,b)$ for all $a, b \in X$.*

PROOF. By C3 and Corollary 10.5, we have

$$\begin{aligned} 0 = (a+tb)\pi(a+tb) &= (a+tb)\big(\pi(a) + t^2\pi(b) + th^\perp(a,b)\big) \\ &= a\pi(a) + t\big(b\pi(a) + ah^\perp(a,b)\big) \\ &\qquad + t^2\big(a\pi(b) + bh^\perp(a,b)\big) + t^3 b\pi(b) \end{aligned}$$

for all $a, b \in X$. The claim holds, therefore, by Hypothesis 10.1(iii) and [**12**, (2.26)]. $\square$

NOTATION 10.8. Let $X_0 = X \oplus X$, let L_0 denote the K-vector space $\langle\varepsilon\rangle^\perp \oplus K \oplus K$, where $\langle\varepsilon\rangle^\perp$ is as in (10.1) (and thus $\dim_K L_0 = 1 + \dim_K L$), let $\varepsilon_0 = (0,1,1) \in L_0$, let q_0 be the quadratic form on L_0 given by

$$q_0(v,s,t) = q(v) + st \tag{10.4}$$

for all $(v,s,t) \in L_0$, let f_0 be the bilinear form associated with q_0, let

$$(v,s,t)^\tau = (-v,t,s)$$

for all $(v,s,t) \in L_0$ and let

$$(a,b) * (v,s,t) = (bv + sa,\ av + tb)$$

for all $(a,b) \in X_0$ and all $(v,s,t) \in L_0$. Then f is non-degenerate (and hence q is non-degenerate). Furthermore,

$$v_0^\tau = f_0(v_0,\varepsilon_0)\varepsilon_0 - v_0,$$

$a_0 * \varepsilon_0 = a_0$ and (since $v^\sigma = -v$ for all $v \in \langle\varepsilon\rangle^\perp$)

$$\big(a_0 * v_0\big) * v_0^\tau = q_0(v_0)a_0$$

for all $v_0 \in L_0$ and all $a_0 \in X_0$. Let

$$h_0\big((a,b),(a',b')\big) = \Big(\tfrac{1}{2}\big(h^\perp(a,a') + h^\perp(b,b')\big), g(b,a'), g(a,b')\Big)$$

for all $(a,b),(a',b') \in X_0$ and let $\theta_0(a_0,v_0) = h_0(a_0, a_0 * v_0)/2$ for all $a_0 \in X_0$ and all $v_0 \in L_0$.

PROPOSITION 10.9. *Let*

$$\Xi_0 := (K, L_0, q_0, f_0, \varepsilon_0, X_0, *, h_0, \theta_0)$$

be as in Notation 10.8. *Then Ξ_0 is a quadrangular algebra.*

PROOF. We begin by observing that

$$\begin{aligned} f(h(a,bv),\varepsilon) &= f(h(a,b),v^\sigma) \\ &= f(h(a,b)^\sigma,v) \\ &= -f(h(b,a),v) = f(h(b,a),v^\sigma) = f(h(b,av),\varepsilon) \end{aligned}$$

(by [**13**, Propositions 3.6 and 3.7]) and hence

$$h_\varepsilon(a,bv) = h_\varepsilon(b,av) \tag{10.5}$$

for all $a,b \in X$ and all $v \in \langle\varepsilon\rangle^\perp$. Note, too, that

$$f_0\Big(h_0\big((a,b),(a',b')\big),\varepsilon_0\Big) = g(b,a') + g(a,b')$$

for all $a,a',b,b' \in X$ and recall that $h^\perp$ is symmetric and h_ε is skew-symmetric. With these observations (and a bit of calculation), verification that Ξ_0 satisfies B2 reduces to showing that

$$h^\perp(a,b'v) + h^\perp(b,a'v) = h^\perp(a',bv) + h^\perp(b',av) + 2\big(g(b,a') + g(a,b')\big)v \tag{10.6}$$

for all $a,a',b,b' \in X$ and all $v \in \langle\varepsilon\rangle^\perp$. By (10.2), (10.5) and B2, we have

$$\begin{aligned} h^\perp(a,b'v) &+ h^\perp(b,a'v) - h^\perp(a',bv) - h^\perp(b',av) \\ &= h(a,b'v) + h(b,a'v) - h(a',bv) - h(b',av) \\ &= \big(f(h(a,b'),\varepsilon) + f(h(b,a'),\varepsilon)\big)v \\ &= 2\big(g(a,b') + g(b,a')\big)v \end{aligned}$$

for all $a,a',b,b' \in X$ and all $v \in \langle\varepsilon\rangle^\perp$. Thus (10.6) does, in fact, hold and thus Ξ_0 satisfies B2. To verify that Ξ_0 satisfies B3, the reader has only to bear in mind that if $(v,s,t) \in L_0$, then $v \in \langle\varepsilon\rangle^\perp$ and hence $f(h^\perp(c,c'),v) = f(h(c,c'),v)$ for all $c,c' \in X$.

We turn now to D1. Choose $a_0 := (a,b) \in X_0$ and $v_0 := (v,s,t) \in L_0$. We need to show that

$$a_0 * \pi_0(a_0) * v_0 = a_0 * \theta_0(a_0,v_0), \tag{10.7}$$

where $\pi_0(a_0) = \theta_0(a_0,\varepsilon_0) = h_0(a_0,a_0)/2$. Since Ξ is δ-standard, we have $f(\pi(c),\varepsilon) = 0$ and thus

$$h^\perp(c,c) = h(c,c) = 2\pi(c) \tag{10.8}$$

for all $c \in X$ by [**13**, Proposition 4.5(i)]. Hence

$$\begin{aligned} \pi_0(a,b) &= \tfrac{1}{2}h_0\big((a,b),(a,b)\big) \\ &= \tfrac{1}{2}\Big(\pi(a) + \pi(b), g(b,a), g(a,b)\Big). \end{aligned} \tag{10.9}$$

By Corollary 10.5, therefore,

$$2(a,b)*\pi_0(a,b) = \big(b\pi(a)+g(b,a)a,\ a\pi(b)+g(a,b)b\big).$$

We also have

$$\begin{aligned}2\theta_0\big(((a,b),v_0\big) &= h_0\big((a,b),(a,b)v_0\big)\\ &= \Big(\tfrac{1}{2}\big(h^\perp(a,bv+sa)+h^\perp(b,av+tb)\big),\\ &\qquad\qquad g(b,bv+sa),\ g(a,av+tb)\Big).\end{aligned}$$

If $v=0$, then by Corollary 10.5, (10.8) and a bit of calculation, (10.7) holds. Suppose, instead, that $s=t=0$. In this case,

$$2a_0*\pi_0(a_0)*v_0 = \big((a\pi(b)+g(a,b)b)v,\ (b\pi(a)+g(b,a)a)v\big)$$

and

$$\begin{aligned}2a_0*\theta_0(a_0,v_0) = \Big(&\tfrac{1}{2}b\big(h^\perp(a,bv)+h^\perp(b,av)\big)+g(b,bv)a,\\ &\qquad\qquad \tfrac{1}{2}a\big(h^\perp(a,bv)+h^\perp(b,av)\big)+g(a,av)b\Big).\end{aligned}$$

Let x denote the first coordinate of $2a_0*\theta_0(a_0,v_0)$. By Notation 10.2, we can substitute $h(a,bv)-h_\varepsilon(a,bv)$ for $h^\perp(a,bv)$ and $h(b,av)-h_\varepsilon(b,av)$ for $h^\perp(b,av)$ in x. By (10.5), we can then replace $h_\varepsilon(a,bv)$ by $h_\varepsilon(b,av)$ and by B2 and (10.2), we can substitute

$$h(b,av)+2g(a,b)v$$

for $h(a,bv)$. By [**13**, Propositions 4.5(i) and 4.9(iii)] and (10.2), we have

$$g(b,bv) = f(\theta(b,v),\varepsilon) = -f(\pi(b),v).$$

By Proposition 10.7 and [**13**, Proposition 3.8], therefore, $x-g(a,b)bv$ equals

$$\begin{aligned}bh(b,av)-g(b,av)b+g(b,bv)a &= bh^\perp(b,av)+g(b,bv)a\\ &= -av\pi(b)-f(\pi(b),v)a\\ &= a\pi(b)^\sigma v^\sigma.\end{aligned}$$

Since $\pi(b)^\sigma=-\pi(b)$ and $v^\sigma=-v$, we conclude that the first coordinates of the elements $a_0*\theta_0(a_0,v_0)$ and $a_0*\pi(a_0)*v_0$ are the same. By a similar calculation, the second coordinates are also the same. Thus (10.7) holds when $s=t=0$. Since θ_0 is linear in its second variable, we conclude that Ξ_0 satisfies D1. By [**13**, Remark 4.8], we do not need to verify that Ξ_0 satisfies C1–C4. □

PROPOSITION 10.10. *Let* $\Xi_0 := (K,L_0,q_0,f_0,\varepsilon_0,X_0,*,h_0,\theta_0)$ *be as in* Proposition 10.9 *and let* π_0 *be as in* D1 *applied to* Ξ_0. *Then the following hold*:

(i) h_0 *is non-degenerate.*

(ii) $q_0(\pi_0(a,b)) = \big(f(\pi(a),\pi(b))-g(a,b)^2\big)/4$ *for all* $(a,b)\in X_0$.

PROOF. Assertion (i) follows from Lemma 10.3 and assertion (ii) from (10.4) and (10.9). □

LEMMA 10.11. *Suppose* $\dim_K L \geq 6$ *and let* $W = \langle \varepsilon, w, z\rangle$ *for some* $w, z \in L$. *Then there exist* $u \in W^\perp$ *such that* $q(u) \neq 0$.

PROOF. The restriction of q to $\langle\varepsilon\rangle^\perp$ is non-degenerate. We can thus replace w and z by their projections to $\langle\varepsilon\rangle^\perp$. If $q(w) \neq 0$, then the restriction of q to $\langle\varepsilon, w\rangle$ is non-degenerate. If $w \neq 0$ but $q(w) = 0$, then there exists $w_1 \in \langle\varepsilon, w\rangle^\perp$ such that $q(w_1) = 0$ but $f(w, w_1) \neq 0$ and the restriction of f to $\langle\varepsilon, w, w_1\rangle$ is non-degenerate. Thus whether or not $q(w) = 0$ (or if $w = 0$), there exists a subspace V_1 of dimension at most 3 containing $\langle\varepsilon, w\rangle$ such that the restriction of f to V_1 is non-degenerate. We can now replace z by its projection to $V_1^\perp$. By a similar argument, there exists a subspace V_2 of dimension at most 5 containing V_1 and z such that the restriction of f to V_2 is non-degenerate. Since f is non-degenerate and $\dim_K L \geq 6$, the restriction of q to $V_2^\perp$ is not identically zero. □

PROPOSITION 10.12. *Suppose that* $\dim_K L \geq 5$. *Then there exists* $a_0 \in X_0$ *such that* $q_0(\pi_0(a_0)) \neq 0$.

PROOF. Suppose first that $\dim_K L \geq 6$. By Lemma 10.3, we can choose $a, b \in X$ such that $g(a, b) \neq 0$. By Proposition 10.10(ii), we can assume that $f(\pi(a), \pi(b)) \neq 0$ and by Lemma 10.11, we can choose $u \in \langle\varepsilon, \pi(b), h(a, b)\rangle^\perp$ such that $q(u) \neq 0$. Then $f(u^\sigma, \varepsilon) = 0$, so by C4 and [**13**, Proposition 4.5(iii)], we have

$$\pi(au) = -\pi(a)q(u) + f(\theta(a, u), \varepsilon)u^\sigma.$$

Since $f(u^\sigma, \pi(b)) = f(u, \pi(b)^\sigma) = 0$, it follows that

$$f(\pi(au), \pi(b)) = -q(u)f(\pi(a), \pi(b)) \neq 0.$$

By B3 and the choice of u, we have $f(h(au, b), \varepsilon) = f(h(a, b), u) = 0$ and thus $g(au, b) = 0$. By Proposition 10.10(ii), it follows that $q_0(\pi_0(au, b)) \neq 0$ in the case that $\dim_K L \geq 6$.

Next we suppose that $\dim_K L = 5$ and $q_0(\pi_0(a_0)) = 0$ for all $a_0 \in X_0$. In this case, Ξ_0 satisfies all the conditions in Hypothesis 10.1 and hence we can apply our construction described in Notation 10.8 to Ξ_0 to obtain a quadrangular algebra

$$\Xi_1 = (K, L_1, q_1, f_1, \varepsilon_1, X_1, \circ, h_1, \theta_1)$$

with $\dim_K L_1 = 1 + \dim_K L_0 = 7$. By Proposition 10.10(i), h_1 is non-degenerate and by the conclusion of the previous paragraph, there exists $a_1 \in X_1$ such that $q_1(\pi_1(a_1)) \neq 0$. By Theorem 8.16, however, there is no such quadrangular algebra. With this contradiction, we conclude that $q_0(\pi_0(a_0)) \neq 0$ for some $a_0 \in X_0$ also in the case that $\dim_K L = 5$. □

NOTATION 10.13. Let $\Xi_1 = (K_1, L_1, q_1, f_1, \varepsilon_1, X_1, \circ, h_1, \theta_1)$ be an arbitrary quadrangular algebra. Suppose that (ψ_1, ψ_2) is a linear automorphism of Ξ_1 (i.e. that $(\mathrm{id}_K, \psi_1, \psi_2)$ is an isomorphism from Ξ to itself as defined in [**13**, Definition 1.25]).

Let $\hat{L}_1$ be the set of fixed points of ψ_1 in L_1 (so $\varepsilon_1 \in \hat{L}_1$) and let $\hat{X}_1$ be the set of fixed points of ψ_2 in X_1. Suppose that the following hold:

(i) $\dim_K \hat{X}_1 > 0$.

(ii) The restriction of f_1 to $\hat{L}_1$ is non-degenerate

(iii) The parameter $\hat{\omega}$ in [**13**, Definition 1.25] is 1, i.e. $\psi_1(h(a,b)) = h(\psi_2(a), \psi_2(b))$ for all $a, b \in X_1$.

By (iii), $h(\hat{X}_1, \hat{X}_1) \subset \hat{L}_1$. By [**13**, Proposition 4.5(i)], therefore,

$$\Xi_1^{(\psi_1,\psi_2)} := (K_1, \hat{L}_1, \hat{q}_1, \hat{f}_1, \varepsilon_1, \hat{X}_1, \hat{\circ}, \hat{h}_1, \hat{\theta}_1)$$

is a quadrangular algebra, where $\hat{q}_1, \hat{f}_1, \dots, \hat{\theta}_1$ denote the appropriate restrictions of $q_1, f_1, \dots, \theta_1$. We call $\Xi_1^{(\psi_1,\psi_2)}$ the *fixed point algebra* of (ψ_1, ψ_2).

Remark 10.14. Let $\Xi_1 = (K_1, L_1, q_1, f_1, \varepsilon_1, X_1, \circ, h_1, \theta_1)$ be an arbitrary quadrangular algebra, let (α, β) be a linear automorphism of Ξ_1 satisfying the conditions in Notation 10.13(i)–(iii), let $u \in L$ be a fixed point of α such that $q_1(u) = 1$ and let $\hat{\Xi}_1$ denote the isotope of Ξ with respect to u as defined in [**13**, Proposition 8.1]. Then (α, β) is also an isomorphism of $\hat{\Xi}_1$ satisfying the conditions in Notation 10.13(i)–(iii) and the fixed point algebra $\hat{\Xi}^{(\alpha,\beta)}$ is an isotope of the fixed point algebra $\Xi^{(\alpha,\beta)}$.

Proposition 10.15. *Let α be the automorphism of L_0 given by $(v, s, t)^\alpha = (v, t, s)$ for all $(v, s, t) \in L_0$, let β be the automorphism of X_0 given by $(a, b)^\beta = (b, a)$ for all $(a, b) \in X_0$ and let ξ be the automorphism of X_0 given by $(a, b)^\xi = (-b, -a)$ for all $(a, b) \in X_0$. Then (α, β) and (α, ξ) are both linear automorphisms of Ξ_0 satisfying* Notation 10.13(i)–(iii) *and Ξ is isomorphic to both $\Xi_0^{(\alpha,\beta)}$ and $\Xi_0^{(\alpha,\xi)}$.*

Proof. By Notation 10.8, we have $(a_0 * v_0)^\beta = a_0^\beta * v_0^\alpha$ and $h_0(a_0^\beta, b_0^\beta) = h_0(a_0, b_0)^\alpha$ as well as $(a_0 * v_0)^\xi = a_0^\xi * v_0^\alpha$ and $h_0(a_0^\xi, b_0^\xi) = h_0(a_0, b_0)^\alpha$ for all $a_0, b_0 \in X_0$ and all $v_0 \in L_0$. The restriction of q_0 to the set of fixed points of α is isomorphic to q, (a, a) is a fixed point of β for all $a \in X$ and $(a, -a)$ is a fixed point of ξ for all $a \in X$. Hence (α, β) and (α, ξ) are both linear automorphisms of Ξ_0 satisfying the conditions Notation 10.13(i)–(iii).

Let

$$\Xi_0^{(\alpha,\beta)} := (K, \hat{L}_0, \hat{q}_0, \hat{f}_0, \varepsilon_0, \hat{X}_0, \hat{*}, \hat{h}_0, \hat{\theta}_0)$$

and let

$$\Xi_0^{(\alpha,\xi)} := (K, \hat{L}_0, \hat{q}_0, \hat{f}_0, \varepsilon_0, \tilde{X}_0, \tilde{*}, \tilde{h}_0, \tilde{\theta}_0).$$

Let β_0 be the map $a \mapsto (a, a)$ from X to $\hat{X}_0$ and let α_0 be the map

$$v + s\varepsilon \mapsto (v, s, s)$$

from L to $\hat{L}_0$. Then (α_0, β_0) is an isomorphism from Ξ to $\Xi_0^{(\alpha,\beta)}$ as defined in [**13**, Definition 1.25] with $\hat{\omega} = 1$. Let β_1 be the map $a \mapsto \big(a, -a\big)$ from X to $\tilde{X}_0$ and let α_1 be the map

$$v + s\varepsilon \mapsto (-v, s, s)$$

from L to $\hat{L}_0$. Then (α_1, β_1) is an isomorphism from Ξ to $X_0^{(\alpha,\xi)}$ as defined in [**13**, Definition 1.25] with $\hat{\omega} = -1$. □

We come now to the main result of this section.

THEOREM 10.16. *Let $\Xi = (K, L, q, f, \varepsilon, X, \cdot, h, \theta)$ be a proper quadrangular algebra and let π be as in* D1. *Suppose that $|K| > 3$, that* $\mathrm{char}(K) \neq 2$, *that $q(\pi(a)) = 0$ for all $a \in X$, that h is non-degenerate and that* $\dim_K L \geq 4$. *Then either*

(i) $\dim_K L = 4$, $\dim_K X = 2$ *and there is a unique multiplication on L giving (L, K) the structure of a split quaternion algebra with norm q and identity ε such that X is a right module over L with respect to the map $\cdot$, (L, K, X, h, π) is a standard pseudo-quadratic space isomorphic to the standard pseudo-quadratic space described in* Example 4.17 *and Ξ is isotopic to $\mathcal{Q}_s(L, K, X, h, \pi)$ or*

(ii) $\dim_K L = 5$ *and Ξ is isomorphic to $\mathcal{Q}_4(C, K)$ for $C = K$.*

PROOF. By Proposition 7.2, we can assume that Ξ is δ-standard for some $\delta \in L$. Let Ξ_0, L_0 and q_0 be as in Notation 10.8 and let (α, β) be the automorphism of Ξ_0 defined in Proposition 10.15. By Proposition 10.10(i), h_0 is non-degenerate. By Remark 7.9, π is not identically zero. Since $q(\pi(a)) = 0$ for all $a \in X$, it follows that q is isotropic. Suppose first that

$$\dim_K L \geq 5. \tag{10.10}$$

By Theorem 8.16 and Proposition 10.12, Ξ_0 is isotopic to $\mathcal{Q}_4(C_1, K)$ for some composition algebra (C_1, K) as in Notation 3.1(ii)–(v). Since $\dim_K L_0 \geq 6$, (C_1, K) is not as in Notation 3.1(ii). We provide the various terms in $\mathcal{Q}_4(C_1, K)$ with the subscript 1, so that

$$\mathcal{Q}_4(C_1, K) = (K, L_1, q_1, f_1, \varepsilon_1, X_1, \cdot_1, h_1, \theta_1),$$

where $q_1 = q_{C_1}$. The automorphism α of L defined in Proposition 10.15 is a reflection of q_0. By Proposition 8.14(ii), therefore, (C_1, K) cannot be as in Notation 3.1(iv) or (v). We conclude that (C_1, K) is as in Notation 3.1(iii). Thus $\dim_K L = 5$ and $\dim_K X = 4$. Hence the Witt index of q is either 1 or 2. By Proposition 6.15, the Witt index of q must, in fact, be 2 and $C(q, \varepsilon) \cong M(4, K)$ since otherwise $C(q, \varepsilon)$ would have no module of dimension 4 over K. Thus the restriction of q to $\langle \varepsilon \rangle^\perp$ is hyperbolic, so by (10.4), also q_0 is hyperbolic. Therefore (C_1, K) is split, so we can identify C_1 with $K \oplus K$ and choose a linear isomorphism ξ from (K, L_0, q_0) to (K, L_1, q_1) mapping $(0, 1, 0)$ to $(0, 0, 0, 0, (1, 0))$ and $(0, 0, 1)$ to $(0, 0, 0, 0, (0, 1))$ (and thus also ε_0 to ε_1). By Proposition 8.15, there exists ψ such that (ξ, ψ) is an isomorphism from Ξ_0 to $\mathcal{Q}_4(C_1, K)$. We identify Ξ_0

with $\mathcal{Q}_4(C_1, K)$ via (ξ, ψ). As a consequence, the automorphism α of L_0 defined in Proposition 10.15 is the map

$$(t_1, t_2, t_3, t_4, e) \mapsto (t_1, t_2, t_3, t_4, e^\sigma), \tag{10.11}$$

where now σ is the standard involution of (C_1, K), and the automorphism β defined in Proposition 10.15 is a linear automorphism of X_0 of order 2 such that $(a_0 v_0)^\beta = a_0^\beta v_0^\alpha$ for all $a_0 \in X_0$ and all $v_0 \in L_0$.

Now let γ be an arbitrary linear involutory automorphism of X_0 such that $(a_0 v_0)^\gamma = a_0^\gamma v_0^\alpha$ for all $a_0 \in X_0$ and all $v_0 \in L_0$. Since $C(q, \varepsilon)$ acts irreducibly on X_0, it follows that for some $w \in K^*$, $a_0^\beta = (wa_0)^\gamma$ for all $a_0 \in X_0$. Since both β and γ are linear involutions, we have $w = 1$ or -1. By Proposition 10.15, therefore, Ξ is isomorphic to the fixed point algebra $\Xi_0^{(\alpha,\gamma)}$. It follows that Ξ is uniquely determined by $\mathcal{Q}_4(C_1, K)$ and the automorphism (10.11) of (K, L_{C_1}, q_{C_1}) and hence by K. Since $\mathcal{Q}_4(K, K)$ as well as every isotope of Ξ satisfy all the conditions in Hypothesis 10.1 and (10.10), we conclude that, in fact,

$$\Xi \text{ and all its isotopes are isomorphic to } \mathcal{Q}_4(K, K). \tag{10.12}$$

Thus, in particular, (10.10) implies that (ii) holds.

Let $\Xi_2 = \mathcal{Q}_4(K, K)$. We provide the various terms in Ξ_2 with the subscript 2, so that

$$\Xi_2 = (K, L_2, q_2, f_2, \varepsilon_2, X_2, \cdot_2, h_2, \theta_2)$$

and choose a reflection α_2 of (K, L_2, q_2) fixing ε_2. By (10.12), the structure group of $\mathrm{Str}(\Xi_2)$ as defined in [**13**, Notation 12.4 and Theorem 12.9] acts transitively on

$$\{\langle v\rangle \mid v \in L_2,\ q_2(v) \neq 0\}.$$

Hence

$$\mathrm{Str}(\Xi_2) \text{ acts transitively on the set of reflections of } (K, L, q_2). \tag{10.13}$$

Now suppose that

$$\dim_K L = 4, \tag{10.14}$$

so $\dim_K L_0 = 5$. By Theorem 8.16, it follows that $q_0(\pi_0(a_0)) = 0$ for all $a_0 \in X_0$. By (10.12), therefore, Ξ_0 is isomorphic to Ξ_2. Thus, in particular, we have $\dim_K X = 2$, so q is isomorphic to the norm of a split quaternion algebra (C, K) by Proposition 6.16(ii). By (10.13), we can choose a linear isomorphism (φ, ψ) from Ξ_0 to an isotope of $\hat{\Xi}_2$ of Ξ_2 with basepoint $\hat{\varepsilon}_2$ such that $\alpha_2 = \varphi^{-1}\alpha\varphi$. Let $\beta_2 = \varphi^{-1}\beta\varphi$. Then α_2 fixes both ε_2 and $\hat{\varepsilon}_2$, (α_2, β_2) is a linear automorphism of $\hat{\Xi}_2$ of order 2 and $\hat{\Xi}_2^{(\alpha_2,\beta_2)}$ is isomorphic to Ξ. Now suppose that δ_2 is an arbitrary linear automorphism of X_2 such that (α_2, δ_2) is an automorphism of $\hat{\Xi}_2$ of order 2. By Remark 10.14 and [**13**, Proposition 8.9(ii)], (α_2, δ_2) is an automorphism of Ξ_2 and $\Xi_2^{(\alpha_2,\delta_2)}$ is an isotope of $\hat{\Xi}_2^{(\alpha_2,\delta_2)}$. By Proposition 6.14(a), X_2

is irreducible as a $C(q_2, \hat{\varepsilon}_2)$-module. It follows as in the previous case that either $\delta_2 = \beta_2$ or δ_2 is the composition of β_2 with the automorphism $a_2 \mapsto -a_2$ of X_2. By Proposition 10.15, therefore, Ξ is isomorphic to $\hat{\Xi}_2^{(\alpha_2,\delta_2)}$. Hence Ξ is isotopic to $\Xi_2^{(\alpha_2,\delta_2)}$. Thus Ξ is uniquely determined up to isotopy by $\Xi_2 = \mathcal{Q}_4(K, K)$ and the choice of α_2. Since α_2 is an arbitrary reflection of (K, L_2, q_2) fixing ε_2, we conclude that Ξ is uniquely determined up to isotopy by K and (10.14). Since the quadrangular algebra $\mathcal{Q}_s(C, K, X, h, \pi)$ with (C, K, X, h, π) as in Example 4.17 also satisfies all the conditions in Hypothesis 10.1 and (10.14), we conclude that that Ξ is isotopic to $\mathcal{Q}_s(C, K, X, \pi, h)$. $\square$

11. The special case.

In this section, we make the following assumptions:

HYPOTHESIS 11.1. Let $\Xi = (K, L, q, f, \varepsilon, X, \cdot, h, \theta)$ be a δ-standard quadrangular algebra for some $\delta \in L$ as defined in Definition 7.1 and let π be as in D1. Suppose that

(i) $2 \le \dim_K L \le 4$,

(ii) h is non-degenerate as defined in Observation 5.8 and

(iii) $|K| > 5$.

The main result of this section is Theorem 11.16.

PROPOSITION 11.2. *If* $\mathrm{char}(K) = 2$, *then* $q(\pi(a)) \neq 0$ *for some* $a \in X$.

PROOF. This holds by Proposition 8.4 if Q is not identically zero and Proposition 9.5 if Q is identically zero, where Q is as in (8.1). $\square$

LEMMA 11.3. *π is not identically zero.*

PROOF. This holds by Remark 7.9 and Hypothesis 11.1(i)–(ii). $\square$

PROPOSITION 11.4. $\dim_K L \neq 3$.

PROOF. Suppose that $\dim_K L \ge 3$. Assume first that there exists $a \in X$ such that $q(\pi(a)) \neq 0$. Assume, too, that if $\mathrm{char}(K) = 2$, also $Q(a) \neq 0$. Thus the restriction of f to $\langle \varepsilon, \pi(a) \rangle$ is non-degenerate in all characteristics. Hence we can choose $w \in \langle \varepsilon, \pi(a) \rangle^\perp$ such that $q(w) \neq 0$. By [**13**, Proposition 4.9(i)], $f(\theta(a, w), w) = 0$ if and only if $\mathrm{char}(K) \neq 2$, and by [**13**, Proposition 4.22], $q(\theta(a, w)) \neq 0$. Thus $\langle w, \theta(a, w) \rangle$ is 2-dimensional. We have

$$f(\theta(a, w), \varepsilon) = -f(\pi(a), w) + f(\pi(a), \varepsilon) f(w, \varepsilon) = 0$$

and

$$f(\theta(a, w), \pi(a)) = q(\pi(a)) f(\varepsilon, w) = 0$$

by [**13**, Propositions 4.9.(iii) and 4.19], so

$$\langle w, \theta(a, w)\rangle \subset \langle \varepsilon, \pi(a)\rangle^{\perp}$$

and hence $\dim_K L > 3$. In the case that $q(\pi(a)) \neq 0$ and $Q(a) = 0$ for some $a \in X$ and $\mathrm{char}(K) = 2$, we argue exactly as in the second half of the proof of [**13**, Proposition 5.2] to conclude again that $\dim_K L > 3$. It thus suffices to assume from now on that $\dim_K L = 3$ and that $q(\pi(a)) = 0$ for all $a \in X$.

By Proposition 11.2, we have $\mathrm{char}(K) \neq 2$. Hence f is non-degenerate. By Lemma 11.3, we can choose $a_0 \in X$ such that $\pi(a_0) \neq 0$. Let $u = \pi(a_0)$. Since $f(\pi(a_0), \varepsilon) = 0$, the restriction of q to $\langle \varepsilon\rangle^{\perp}$ is isotropic. Since f is non-degenerate, we can thus choose $v \in \langle \varepsilon\rangle^{\perp}$ such that $f(u, v) = 1$ and $q(v) = 0$. Let

$$X_u = \{a \in X \mid \pi(a) \in \langle u\rangle\} \text{ and } X_v = \{a \in X \mid \pi(a) \in \langle v\rangle\}.$$

Since $q(\pi(a)) = 0$ and $\pi(a) \in \langle \epsilon\rangle^{\perp}$ for all $a \in X$, it follows that $X = X_u \cup X_v$. By C3, we have $h(a, b) \in \langle \varepsilon, u\rangle$ for all $a, b \in X$ if $X = X_u$. By Lemma 7.8, it follows that $X_u \neq X$. Similarly, $X_v \neq X$.

Suppose that X_u is not closed under addition. Choose $a, b \in X_u$ such that $\pi(a+b) \notin \langle u\rangle$. In particular, $\pi(a + b) \neq 0$. By C3 again, we have

$$h^{\perp}(a, b) = \pi(a + b) - \pi(a) - \pi(b)$$

and

$$\begin{aligned}\pi(a + tb) &= \pi(a) + t^2\pi(b) + th^{\perp}(a, b)\\ &= \pi(a) + t^2\pi(b) - t(\pi(a) + \pi(b)) + t\pi(a + b)\end{aligned}$$

for all $t \in K$. By Hypothesis 11.1(iii), there exists t such that

$$t^2\pi(b) - t(\pi(a) + \pi(b)) + \pi(a)$$

is a non-zero element of $\langle u\rangle$ and $t\pi(a + b)$ is a non-zero element of $\langle v\rangle$. This contradicts the fact that $\pi(a + tb)$ must lie in X_u or X_v. It follows that X_u is closed under addition. By a similar argument, X_v is closed under addition. A group is, however, never the union of two proper subgroups. With this contradiction, we conclude that $\dim_K L \neq 3$. □

PROPOSITION 11.5. *Suppose that* $\dim_K L = 4$ *and let* $\circ$ *and* $*$ *be two* K*-bilinear multiplications on* L *such that both* $(L, \circ)$ *and* $(L, *)$ *are quaternion algebras over* K *with norm* q *and identity* ε*. Then* $\circ$ *and* $*$ *are either the same or opposites.*

PROOF. Choose $v \in L$ such that the restriction of f to $B := \langle \varepsilon, v\rangle$ is non-degenerate, let q_B denote the restriction of q to B, let E be the étale quadratic algebra $K[\gamma]$, where $\gamma \in E\backslash K$ is a root of $p(x) := x^2 - f(\varepsilon, v)x + q(v)$, let σ denote the unique non-trivial K-algebra automorphism of E, let N denote the norm of E/K, let κ denote the map $s\varepsilon + tv \mapsto s + t\gamma$ from B to E and let λ denote the map $s\varepsilon + tv \mapsto s + t\gamma^{\sigma}$ from B to E. Then κ and λ are the only two K-linear maps from B to E mapping ε to 1 and q_B to N.

Since $(L, \circ)$ and $(L, *)$ are quaternion algebras with norm q and identity ε, we have

$v \circ (f(\varepsilon, v)\varepsilon - v) = q(v)$ and $v * (f(\varepsilon, v)\varepsilon - v) = q(v)$. It follows that B is a subalgebra of both $(L, \circ)$ and $(L, *)$, that v is a root of the polynomial $p(x)$ in both subalgebras and that κ and λ are isomorphisms from $(B, \circ)$ and from $(B, *)$ to E.

Next we note that [**12**, (20.17)] holds also when q is isotropic. It is only necessary to add the assumption that $e \in B^\perp$ is chosen so that $q(e) \neq 0$, to delete the first sentence of the proof and to insert at the end the observation that $eB \subset B^\perp$ implies that $B \cap eB = 0$. As a consequence of this result, there are exactly two multiplications on L endowing L with the K-algebra structure of a quaternion algebra with identity ε and norm q, one obtained by identifying B with E via κ and one obtained by identifying B with E via λ, and the two multiplications are opposites. $\square$

PROPOSITION 11.6. *Suppose that (C, K) is a quaternion algebra with norm n_C. Then there exist $a, b \in C$ such that $n_C(ab - ba) \neq 0$.*

PROOF. We can assume that (C, K) is split and hence that $C \cong M(2, K)$. We leave it to the reader to verify the claim in this case. $\square$

PROPOSITION 11.7. *Let $X^\flat = \{a \in X \mid q(\pi(a)) \neq 0\}$. Then either $X^\flat$ is empty or X is spanned by $X^\flat$.*

PROOF. This holds by Proposition 7.6 and Hypothesis 11.1(iii). $\square$

PROPOSITION 11.8. *Suppose that $X^\flat = \emptyset$. Then* $\mathrm{char}(K) \neq 2$ *and* $\dim_K L = 4$.

PROOF. By Proposition 11.2, we have $\mathrm{char}(K) \neq 2$. Suppose that $\dim_K L = 2$. Since q is non-degenerate, $\langle\varepsilon\rangle^\perp$ is spanned by an element v such that $q(v) \neq 0$. Since $\pi(a) \in \langle\varepsilon\rangle^\perp$, it follows that $\pi(a) = 0$ for all $a \in X$. By Lemma 11.3, however, this is impossible. Thus $\dim_K L \neq 2$. The claim holds, therefore, by Hypothesis 11.1(i) and Proposition 11.4. $\square$

Next we replace [**13**, Proposition 5.3 and Lemma 5.4] by the following two results:

PROPOSITION 11.9. *Suppose that* $\dim_K L = 2$. *Then there exists a unique bilinear multiplication $\times$ on L making (L, K) into an étale quadratic extension with norm q and identity ε. Furthermore, the following hold*:

(i) *$auv = a(u \times v)$ for all $a \in X$ and all $u, v \in L$ and*

(ii) *$\theta(a, v) = \pi(a) \times v$ for all $a \in X^\flat$ and all $v \in L$.*

PROOF. By Proposition 11.8, $X^\flat \neq \emptyset$. Choose $a \in X^\flat$. We have $L = \langle\varepsilon, \pi(a)\rangle$. As in the proof of [**13**, Proposition 5.3], we endow L with the unique bilinear multiplication $\times$ on L with identity ε such that

$$\pi(a) \times \pi(a) = f(\pi(a), \varepsilon)\pi(a) - q(\pi(a))\varepsilon. \tag{11.1}$$

Then $\pi(a) \times \pi(a)^\sigma = q(\pi(a))\varepsilon$. Thus $\times$ is the unique bilinear multiplication on L with identity ε that makes L/K into an étale quadratic extension with norm q and by [**13**, Proposition 3.10], (i) holds for $u = v = \pi(a)$. Hence (i) holds for our choice of a and for all $u, v \in L$. By [**13**, Proposition 4.21] and (11.1), we have $\theta(a, \pi(a)) = \pi(a) \times \pi(a)$. Therefore (ii) holds for all $v \in L$. By Proposition 11.7, X is spanned by $X^\flat$, so (i) holds for arbitrary $a \in X$. $\square$

LEMMA 11.10. *Suppose that* $\dim_K L = 4$ *and let* $\circ$ *and* $*$ *be as in* Proposition 11.5. *Let* $a \in X^\flat$ *and let*

$$X_a = \{b \in X \mid b\theta(a, v) = b\pi(a)v \textit{ for all } v \in L\}.$$

Then for $\times = \circ$ *or* $*$, *the following hold*:

(i) $buv = b(u \times v)$ *for all* $b \in X_a$ *and all* $u, v \in L$,

(ii) $bu \in X_a$ *for all* $b \in X_a$ *and all* $u \in L$ *and*

(iii) $\theta(a, v) = \pi(a) \times v$

for all $b \in X_a$ *and all* $u, v \in L$.

PROOF. The proof of [**13**, Lemma 5.4] consists of two parts, the first under the hypothesis that the map Q defined in (8.1) is not identically zero if $\mathrm{char}(K) = 2$ and the second under the hypothesis that $\mathrm{char}(K) = 2$ and Q is identically zero. In both parts a multiplication on L is produced satisfying the hypotheses of Proposition 11.5. In the proof of [**13**, Lemma 5.4] this multiplication is denoted by $\cdot$ or by juxtaposition; we denote it now by $\times$. In both cases, it is shown that (i) holds and it can be verified using [**13**, Proposition 4.21] that (iii) holds. It follows that

$$bu\pi(a)v = b(u \times \pi(a) \times v) = b(u \times \theta(a, v)) = bu\theta(a, v)$$

for all $b \in X_a$ and all $u, v \in L$. Thus (ii) holds. $\square$

We now replace [**13**, Proposition 5.8] by the following:

PROPOSITION 11.11. *Suppose that* $\dim_K L = 4$ *and* $X^\flat \neq \emptyset$ *and let* $\circ$ *and* $*$ *be as in* Proposition 11.5. *Then for* $\times = \circ$ *or* $*$, *the following hold*:

(i) $auv = a(u \times v)$ *for all* $a \in X$ *and all* $u, v \in L$ *and*

(ii) $\theta(a, v) = \pi(a) \times v$ *for all* $a \in X^\flat$ *and all* $u, v \in L$.

PROOF. Let X_a and $\times_a = \times$ be as in Lemma 11.10 for each $a \in X^\flat$. Now let $a, b \in X^\flat$. We set $c_v = b\theta(a, v) - b\pi(a)v$ and let $d_v = ah(a, b)v - ah(a, bv)$ for all $v \in L$. By Lemma 11.10(ii), $c_v \in X_b$ and $d_v \in X_a$ and by [**13**, Proposition 3.22] and Hypothesis 11.1(iii), $c_v = d_v$ and thus $c_v \in X_a \cap X_b$ for all v. If $c_v = 0$ for all v, then $b \in X_a$. In this case, we set $e = b$. If $c_{v_0} \neq 0$ for some $v_0 \in L$, we set $e = c_{v_0}$. Thus e is a non-zero element of $X_a \cap X_b$ in both cases. By Proposition 11.6, we can choose $w, z \in L$ such that $q(w \times_a z - z \times_a w) \neq 0$. By Proposition 11.5, $\times_a$ and $\times_b$ are either the same or opposites. Since $e \in X_a \cap X_b$, we have

$$ewz = e(w \times_a z) \quad \text{and} \quad ezw = e(z \times_a w) = e(z \times_b w) \tag{11.2}$$

by Lemma 11.10(i). Thus $ewz - ezw = e(w \times_a z - z \times_a w)$ and $q(w \times_a z - z \times_a w) \neq 0$, so $ewz \neq ezw$ by A3. By (11.2), it follows that $w \times_a z \neq z \times_b w$. Hence the multiplications $\times_a$ and $\times_b$ are not opposites. By Proposition 11.5, therefore, they are equal. We conclude that $\times_a$ is independent of the choice of $a \in X^\flat$, so we can set $\times = \times_a$ for some $a \in X^\flat$ and observe that (i) holds for all $a \in X^\flat$. By Proposition 11.7, it follows that (i) holds for all $a \in X$ and by Lemma 11.10(iii), (ii) holds. □

PROPOSITION 11.12. *Suppose that either* $\dim_K L = 2$ *or* $\dim_K L = 4$ *and* $X^\flat \neq \emptyset$ *and let* $\times$ *be as in* Proposition 11.9 *or* 11.11. *Then* $\theta(a, v) = \pi(a) \times v$ *for all* $a \in X$ *and all* $v \in L$.

PROOF. Let $v \in L$. By Propositions 11.9(ii) and 11.11(ii), we have $\theta(a, v) = \pi(a) \times v$ for all $a \in X^\flat$ and by Propositions 11.7 and 11.8, X is spanned by $X^\flat$. Let $a, b \in X^\flat$. Then $q(\pi(ta + b))$ is a polynomial of degree 4 in t (as was observed in the proof of Proposition 7.6). By Hypothesis 11.1(iii), therefore, there exists $t \in K^*$ such that $ta + b \in X^\flat$. Thus $\theta(ta + b, v) = \pi(ta + b) \times v$. Therefore

$$t^2\theta(a, v) + \theta(b, v) + th(a, bv) = t^2\pi(a) \times v + \pi(b) \times v + th(a, b) \times v$$

by C3 and [**13**, Proposition 7.2]. Hence $h(a, bv) = h(a, b) \times v$. Since h is bilinear, it follows that $h(a, bv) = h(a, b) \times v$ for all $a, b \in X$.

Now let $X^\sharp = \{a \in X \mid \theta(a, v) = \pi(a) \times v\}$. Then $X^\sharp$ is closed under scalar multiplication. If $a, b \in X^\sharp$, then

$$\theta(a + b, v) - \pi(a + b) \times v = h(a, bv) - h(a, b) \times v = 0$$

by C3, [**13**, Proposition 7.2] and the conclusion of the previous paragraph. Hence $X^\sharp$ is closed under addition. Thus $X^\sharp = X$ since $X^\flat \subset X^\sharp$ and $X^\flat$ spans X. □

PROPOSITION 11.13. *Suppose that* $X^\flat \neq \emptyset$ *and* $\dim_K L = 2$ *or* 4 *and let* L *be endowed with the multiplication* $\times$ *in* Proposition 11.12. *Then* (L, σ, X, h, π) *is a standard pseudo-quadratic space defined in* [**13**, Definition 1.16] *and* Ξ *is isotopic to* $\mathcal{Q}_s(L, K, X, h, \pi)$ *as defined in* Notation 4.16.

PROOF. The claims hold by [**13**, Theorem 5.9]. The only change required in the proof is to cite Proposition 11.12 rather than [**13**, Proposition 3.4] at the start of the second paragraph. □

We pause now in our proof of Theorem 11.16 to make some related observations in Propositions 11.14 and 11.15.

PROPOSITION 11.14. *Suppose that one of the following holds*:

(a) $\dim_K L = 2$ *and* C *is* L *endowed with the multiplication* $\times$ *in* Proposition 11.9 *or*

(b) $\dim_K L = 4$, $X^\flat = \emptyset$ *and* C *is* L *endowed with the multiplication on* L *in* Theorem 10.16(i) *or*

(c) $\dim_K L = 4$, $X^\flat \neq \emptyset$ *and* C *is* L *endowed with the multiplication* $\times$ *in* Proposition 11.11.

Then $L = C = C(q,\varepsilon)$ *in case* (a) *and* $C(q,\varepsilon)$ *is the direct sum of two copies of* C *and one of them acts trivially on* X *in cases* (b) *and* (c).

Proof. If $\dim_K L = 2$, then $\dim_K C(q,\varepsilon) = 2$ and hence $L = C = C(q,\varepsilon)$. Suppose that $\dim_L = 4$. In this case, $C(q,\varepsilon) \cong C \oplus C$ by Proposition 6.16(i). Choose $u, v \in L = C$ such that $\{\varepsilon, u, v, u \times v\}$ is a basis of L over K, where $\times$ is multiplication in C. By [**12**, (12.51)], $u \otimes v - u \times v$ is a non-zero element of $C(q,\varepsilon)$ that acts trivially on X. $\square$

Proposition 11.15. *Suppose that* $\dim_K L = 2$ *or* 4 *and* $\dim_K X < \infty$ *and let* C *be as in* Proposition 11.14. *Then either* X *is a free* C*-module or* $\dim_K L = 4$, $\mathrm{char}(K) \neq 2$, C *is split and* X *has a decomposition* $X_0 \oplus X_1$ *into sub-*C*-modules* X_0 *and* X_1, *where* $\dim_K X_0 = 2$ *and* X_1 *is free.*

Proof. Suppose that if $\mathrm{char}(K) \neq 2$ and $\dim_K L = 4$, then $\dim_K X \neq 2$. Then by Proposition 11.8 if $\mathrm{char}(K) \neq 2$ and $\dim_K L = 2$ and by Theorem 10.16 if $\mathrm{char}(K) \neq 2$ and $\dim_K L = 4$, we can choose $e \in X$ such that $q(\pi(e)) \neq 0$. Since h is non-zero and f is non-degenerate, it follows from Proposition 9.2 that we can choose $e \in X$ such that $Q(e) \neq 0$ if $\mathrm{char}(K) = 2$. By Proposition 8.2 and the choice of e, the map $v \mapsto ev$ from L to X is injective in all characteristics.

Let $F(a,b) = f(h(a,b),\varepsilon)$ for all $a, b \in X$. By [**13**, Proposition 3.6], F is a symplectic form on X. By [**13**, Proposition 3.15], therefore,

$$F(a,a) = 0 \tag{11.3}$$

for all $a \in X$ in all characteristics. Since f is non-degenerate, it follows from B3 and Hypothesis 11.1(ii) that F is also non-degenerate. By B3 and [**13**, Propositions 3.15, 3.16 and 4.5(i)], we have

$$F(eu, ev) = f(h(e, ev), u) = 2f(\theta(e,v), u)$$

if $\mathrm{char}(K) \neq 2$ and

$$F(eu, ev) = f(h(e, ev), u) = Q(e) f(u,v)$$

if $\mathrm{char}(K) = 2$ for all $u, v \in L$. If $\mathrm{char}(K) \neq 2$ and $w \in L$, then $w = \theta(e,v)$ for $v = -\theta(e,w)/q(\pi(e))$ by [**13**, Proposition 4.21]. Since f is non-degenerate, it follows that the restriction of F to eL is non-degenerate in all characteristics.

We call a subset B of X an *F-set* if $q(\pi(a)) \neq 0$ for all $a \in B$ in the case that $\mathrm{char}(K) \neq 2$, $Q(a) \neq 0$ for all $a \in B$ in the case that $\mathrm{char}(K) = 2$ and $F(aL, a'L) = 0$ for all distinct $a, a' \in B$ in all characteristics. Let B be a maximal F-set, let X_1 be the submodule of X spanned by B and let

$$X_0 = \{a \in X \mid F(a, X_1) = 0\}.$$

If $a \in X_0$ and $u \in L$, then by B3 and [**13**, Proposition 3.7],

$$F(au, X_1) = f(h(au, X_1), \varepsilon) \subset f(h(a, X_1), \varepsilon) = 0$$

and hence $au \in X_0$. Thus X_0 is a submodule of X. By the last sentence in the first paragraph above and the conclusion of the previous paragraph, X_1 is a free C-module with basis B, the restriction of F to X_1 is non-degenerate and $q(\pi(a)) = 0$ for all $a \in X_0$ if $\mathrm{char}(K) \neq 2$ and $Q(a) = 0$ for all $a \in X_0$ if $\mathrm{char}(K) = 2$ (by the choice of B). Suppose that $X_0 \neq 0$. Since F is non-degenerate, the restriction of h to X_0 is non-degenerate. Replacing X by X_0, we obtain a new quadrangular algebra Ξ_0 (see Observation 2.8). By the observations in the first paragraph applied to Ξ_0, we have $\mathrm{char}(K) \neq 2$, $\dim_K L = 4$ and $\dim_K X_0 = 2$. By Proposition 6.16(ii), C is split. Since the restriction of F to X_1 is non-degenerate, we can apply (11.3) to deduce the existence of a symplectic basis for X_1 that extends to a symplectic basis of X. Hence $X = X_0 \oplus X_1$. □

Here now is the main result of this section.

THEOREM 11.16. *Let $\Xi = (K, L, q, f, \varepsilon, X, \cdot, h, \theta)$ be a proper quadrangular algebra, let π be as in* D1 *and let $X^\flat$ be as in* Proposition 11.7. *Suppose that $2 \leq \dim_K L \leq 4$, that h is non-degenerate as defined in* Observation 5.8 *and that $|K| > 5$. Then one of the following holds*:

(i) $X^\flat \neq \emptyset$, $\dim_K L = 2$ *or* 4 *and Ξ is as in* Proposition 11.13.

(ii) $X^\flat = \emptyset$, $\mathrm{char}(K) \neq 2$, $\dim_K L = 4$ *and Ξ is isotopic to the special quadrangular algebra $\mathcal{Q}_2(C, K, h, \pi)$, where (C, K, h, π) is as in* Example 4.17.

PROOF. By Proposition 7.2, we can assume that Ξ is δ-standard for some $\delta \in L$. All the claims hold, therefore, by Theorem 10.16 and Propositions 11.4, 11.8 and 11.13. □

With Remark 5.14 and Theorems 8.16, 9.8, 10.16(ii) and 11.16, we have now completed the proof of Theorem 5.10.

12. Appendix.

In this appendix, we indicate the connection between quadrangular algebras, buildings, Tits indices and the exceptional groups.

In [**7**], we introduced the notion of a *Tits polygon*, a generalization of the notion of a Moufang polygon. A Tits n-gon is a bipartite graph Γ endowed with a distinguished class of $2n$-circuits called apartments and an opposition relation on Γ_v for each vertex v, where Γ_v denotes the set of vertices adjacent to v, satisfying certain axioms. A Tits polygon is a Moufang polygon exactly when these opposition relations are all trivial, i.e. when in each Γ_v, all distinct pairs are opposite.

We indicate now one way in which Tits polygons arise "in nature."

DEFINITION 12.1. A *Tits index*, as defined in [**6**, Definition 20.1], is a triple

$$(\Pi, \Theta, A),$$

where Π is a Coxeter diagram with vertex set S, Θ is a subgroup of $\mathrm{Aut}(\Pi)$ and A is a Θ-invariant subset of S such that for each Θ-orbit Z disjoint from A, the subdiagram of Π spanned by the subset $A \cup Z$ of S is spherical and A is invariant under the opposite map of this subdiagram (i.e. the map called σ in [**6**, Notation 19.25]). The Coxeter diagram Π is called the *absolute type* of a Tits index $T := (\Pi, \Theta, A)$ and $|S|$ is called the *absolute rank* of T. A Tits index $T = (\Pi, \Theta, A)$ is called *spherical* if its absolute type is spherical and *split* if Θ is trivial and A is empty. All the Tits indices considered in this appendix are spherical.

Definition 12.2. Let $T = (\Pi, \Theta, A)$ be a Tits index. For each subset J of the vertex set S of Π, let w_J denote the longest element in the Coxeter system (W_J, J). Let M be the set of all Θ-orbits that are disjoint from A. For each $Z \in M$, let $w_Z = w_A \cdot w_{Z \cup A}$. Finally, we set $\tilde{S} = \{w_Z \mid Z \in M\}$ and $\tilde{W} = \langle \tilde{S} \rangle$. By [**6**, Theorem 20.32], $(\tilde{W}, \tilde{S})$ is a Coxeter system. We refer to $(\tilde{W}, \tilde{S})$ (or the corresponding Coxeter diagram) as the *relative type* of T and to $|\tilde{S}|$ as the *relative rank* of T.

Notation 12.3. Let $T = (\Pi, \Theta, A)$ be a Tits index of relative rank 2, let Z_1 and Z_2 be the two Θ-orbits disjoint from A, let $J_i = Z_i \cup A$ for $i = 1$ and 2 and let Δ be a Moufang building of type Π. Let $\Gamma_{\Delta,T}$ denote the bipartite graph with vertex set the union of the set of all J_1-residues of Δ and the set of all J_2-residues of Δ, where two of these residues are adjacent in $\Gamma_{\Delta,T}$ whenever their intersection is an A-residue of Δ.

In [**7**], we show that the graph $\Gamma_{\Delta,T}$ for every pair as described in Notation 12.3 has, canonically, the structure of a Tits n-gon, where $\bullet\overset{n}{\rule{2em}{0.4pt}}\bullet$ is the relative type of T.

Now let Δ be one of the buildings in third column of Table 1. Here we are using the notation described in [**14**, Notation 30.15] with the following modifications (to make everything fit better into the table):

Notation 12.4. For each anisotropic quadrangular algebra

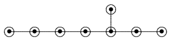
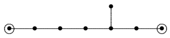
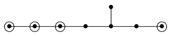
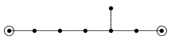
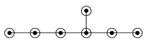
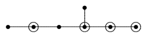
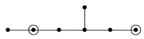
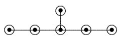
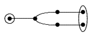
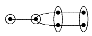
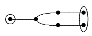
$$\Xi = (K, L, q, f, \varepsilon, X, \cdot, h, \theta),$$

we set $C_2(\Xi) = C_2^{\mathcal{E}}(K, L, q)$ if (K, L, q) is of type E_ℓ for $\ell = 6$, 7 or 8 and we set $C_2(\Xi) = C^{\mathcal{F}}(K, L, q)$ if (K, L, q) is of type F_4. We also set $C_3(C, K) = C_2^{\mathcal{I}}(C, K, \sigma)$ for each octonion division algebra (C, K), where σ is the standard involution of (C, K).

Next we let T be the corresponding Tits index in the second column of Table 1 which we have drawn using [**6**, Conventions 34.2]. Notice that in each case, the absolute type of T is the same as the type of Δ.

In every row of Table 1 except the last, Δ is the spherical building associated with the group G of K-points of an exceptional group. As described in [**10**], there is a Tits index corresponding to each of these groups. This Tits index is given in the first column of Table 1; its relative type coincides with the absolute type of the Tits index in the second column. Alternatively, Δ is the fixed point building (in the sense defined in [**6**, Definition 22.22]) of a descent group (as defined in [**6**, Definition 22.19]) of the spherical

Table 1. The exceptional Tits quadrangles.

		$E_8(K)$	$\mathcal{Q}_4(C,K)$, (C,K) octonion split
		$F_4(C,K)$	$\mathcal{Q}_4(C,K)$, (C,K) octonion div.
		$C_2(\Xi)$	Ξ anisotropic, q of type E_8
		$E_7(K)$	$\mathcal{Q}_4(C,K)$, (C,K) quaternion split
		$F_4(C,K)$	$\mathcal{Q}_4(C,K)$, (C,K) quaternion div.
		$C_2(\Xi)$	Ξ anisotropic, q of type E_7
		$E_6(K)$	$\mathcal{Q}_4(C,K)$, (C,K) étale quadr. split
		$F_4(C,K)$	$\mathcal{Q}_4(C,K)$, (C,K) étale quadr. div.
		$C_3(C,K)$	$\mathcal{Q}_2(C,K)$, (C,K) octonion div.
		$C_2(\Xi)$	Ξ anisotropic, q of type E_6
		$F_4(C,K)$	$\mathcal{Q}_4(C,K)$, $C=K$, $\mathrm{char}(K)\neq 2$
		$F_4(C,K)$	$\mathcal{Q}_4(C,K)$, $C^2\subset K$, $\mathrm{char}(K)=2$
		$C_2(\Xi)$	Ξ anisotropic, q of type F_4

building corresponding to an exceptional group that is either split or mixed of type F_4 (as described in [**11**, 10.3.2]) and T is the Tits index attached to this descent group (as described in [**6**, Theorem 22.25]). This second description includes also the last row of Table 1.

The graphs $\Gamma_{\Delta,T}$ for (Δ,T) in Table 1 all have the structure of a Tits quadrangle (i.e. a Tits polygon with $n=4$), and $\Gamma_{\Delta,T}$ is a Moufang quadrangle if and only if the absolute rank of T is 2. We will say that a Tits quadrangle is *exceptional* if it is isomorphic to one of these quadrangles.

Note that the Tits indices in the first column of Table 1 arise in [**6**, Theorem 22.25] through the choice of a Galois involution, whereas the Tits indices in the second column are being applied in Notation 12.3 to give rise to the corresponding exceptional Tits quadrangle, a much simpler process.

Every Tits polygon has an associated "root group sequence" defined exactly as in [**12**, Definition 8.10]. For an exceptional Tits quadrangle, this root group sequence can be obtained, up to isomorphism, by applying the recipe in [**12**, Example 16.6] to the exceptional quadrangular algebra Ξ given in the fourth column of Table 1. (In [**12**], this recipe is meant to be applied only to anisotropic quadrangular algebras, but there is no reason for this restriction.) This quadrangular algebra is an invariant of the quadrangle up to isotopy. Note that all the exceptional quadrangular algebras occur in the last column of Table 1. This reflects the fact that there is a one-to-one correspondence between isomorphism classes of exceptional Tits quadrangles and isotopy classes of exceptional quadrangular algebras with respect to which the exceptional Moufang quadrangles correspond to the anisotropic quadrangular algebras.

We conjecture that under suitable hypotheses, every Tits polygon is isomorphic to some $\Gamma_{\Delta,T}$ as described in Notation 12.3. This conjecture is supported by a characterization in [**7**] of the hexagons whose root group sequence is parametrized by a non-degenerate cubic norm structure (as defined in [**9**]) and by a characterization in [**8**] of the Tits quadrangles whose root group sequence is parametrized by a quadrangular algebra satisfying Hypothesis 8.1(iii), where Theorem 5.10 plays an essential role.

In [**7**] we also showed that there is a correspondence via the recipe in [**12**, Example 16.8] between isotopy classes of non-degenerate cubic norm structures and isomorphism classes of exceptional Tits hexagons with respect to which the Moufang hexagons correspond to the anisotropic cubic norm structures.

We mention, too, that it was shown in [**7**] that under a certain natural hypothesis (which is satisfied by all the Tits quadrangles and hexagon we have been discussing), Tits n-gons exist only for $n = 3$, 4, 6 and 8.

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