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Solomon–Terao algebra of hyperplane arrangements

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Abstract. We introduce a new algebra associated with a hyperplane arrangement $\mathcal{A}$, called the Solomon–Terao algebra $ST(\mathcal{A},\eta)$, where η is a homogeneous polynomial. It is shown by Solomon and Terao that $ST(\mathcal{A},\eta)$ is Artinian when η is generic. This algebra can be considered as a generalization of coinvariant algebras in the setting of hyperplane arrangements. The class of Solomon–Terao algebras contains cohomology rings of regular nilpotent Hessenberg varieties. We show that $ST(\mathcal{A},\eta)$ is a complete intersection if and only if $\mathcal{A}$ is free. We also give a factorization formula of the Hilbert polynomials of $ST(\mathcal{A},\eta)$ when $\mathcal{A}$ is free, and pose several related questions, problems and conjectures.

1. Introduction.

The aim of this article is to introduce a new algebra, called the *Solomon–Terao algebra* and the *Solomon–Terao complex* associated with hyperplane arrangements. The classical and well-studied algebra of hyperplane arrangement is the logarithmic derivation module, and our Solomon–Terao algebra is defined by using logarithmic derivation modules. The Solomon–Terao algebra has two remarkable aspects. The first one is that it corresponds to one specialization of the Solomon–Terao polynomial defined in [**17**], while the famous Orlik–Solomon algebra in [**11**] reflects the other specialization of the Solomon–Terao polynomial. Hence the Solomon–Terao algebra is considered to be comparable with the Orlik–Solomon algebra, which is isomorphic to the cohomology ring of the complement of the hyperplane arrangement when the base field is $\mathbb{C}$. We study the algebraic structure of the Solomon–Terao algebra in Section 1.1.

The second aspect is a geometric feature of the Solomon–Terao algebra, which gives some supports of the suitability of our definition. The Solomon–Terao algebra happens to be isomorphic to the cohomology ring of some varieties, analogously to the Orlik–Solomon algebra. When the arrangement consists of all reflecting hyperplanes of reflections belonging to the Weyl group, the Solomon–Terao algebra is isomorphic to the cohomology ring of the flag variety or the coinvariant algebra of the reflection group. This part is described in Section 1.2.

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1.1. Solomon–Terao algebra and main results.

Let us introduce several definitions. Let $\mathbb{K}$ be an algebraically closed field, $V = \mathbb{K}^\ell$ and $S := \mathrm{Sym}(V^*)$ its coordinate ring. Let us fix a coordinate system $x_1, \dots, x_\ell$ for V^* such that $S = \mathbb{K}[x_1, \dots, x_\ell]$. The $\mathbb{K}$-linear S-derivation module $\mathrm{Der}\, S$ is an S-graded free module of rank ℓ defined by

$$\mathrm{Der}\, S := \bigoplus_{i=1}^{\ell} S\partial_{x_i}.$$

Also, let $\mathrm{Der}^p\, S := \bigwedge^p \mathrm{Der}\, S$ ($p \geq 0$), agreeing that $\mathrm{Der}^0\, S = S$. Let $\mathcal{A}$ be an arrangement of linear hyperplanes, i.e., a finite set of linear hyperplanes in V. For each $H \in \mathcal{A}$ fix a linear form $\alpha_H \in V^*$ such that $\ker \alpha_H = H$. Let $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H$. Now we can define the logarithmic derivation modules $D^p(\mathcal{A})$ for $\mathcal{A}$ as follows:

Definition 1.1. For $p \geq 0$, define

$$D^p(\mathcal{A}) := \{\theta \in \mathrm{Der}^p\, S \mid \theta(\alpha_H, f_2, \dots, f_p) \in S\alpha_H \ (\forall H \in \mathcal{A},\ \forall f_2, \dots, f_p \in S)\}.$$

The logarithmic derivation module was introduced by K. Saito for the study of the universal unfolding of the isolated hypersurface singularity, see [**14**] for example. The logarithmic derivation module has been studied mainly for $p = 1$ particularly in case of hyperplane arrangements. On the other hand, by using $D^p(\mathcal{A})$ for all p, Solomon and Terao defined the following interesting series.

Definition 1.2 ([**17**], Section One). Define the *Solomon–Terao polynomial* $\Psi(\mathcal{A}; x, t)$ by

$$\Psi(\mathcal{A}; x, t) := t^\ell \sum_{p=0}^{\ell} \mathrm{Hilb}(D^p(\mathcal{A}); x) \left(\frac{1-x}{t} - 1\right)^p.$$

Here for a graded S-module $M = \bigoplus_{i=0}^{\infty} M_i$,

$$\mathrm{Hilb}(M; x) := \sum_{i=0}^{\infty} (\dim_{\mathbb{K}} M_i) x^i$$

is the Hilbert series of M. In the definition above, the Solomon–Terao polynomial seems to be introduced as a series. However, in fact it is a polynomial.

Theorem 1.3 ([**17**], Proposition 5.3). *The series $\Psi(\mathcal{A}; x, t)$ is contained in the polynomial ring $\mathbb{Q}[x, t]$.*

Moreover, we have the following astonishing result by [**17**].

Theorem 1.4 ([**17**], Theorem 1.2). *Let $\mathbb{K} = \mathbb{C}$ and $\pi(\mathcal{A}; t)$ the topological Poincaré polynomial of $M(\mathcal{A}) := V \setminus \bigcup_{H \in \mathcal{A}} H$. Then*

$$\Psi(\mathcal{A}; 1, t) = \pi(\mathcal{A}; t).$$

In fact, $\pi(\mathcal{A}; t)$ can be defined over an arbitrary field $\mathbb{K}$ by using combinatorial data of $\mathcal{A}$, see Section 2. Hence Theorem 1.4 connects algebra, topology and combinatorics of $\mathcal{A}$. By [**11**], we know that there is the algebra $A(\mathcal{A})$ called the *Orlik–Solomon algebra* depending only on the intersection lattice $L(\mathcal{A})$ such that

$$A(\mathcal{A}) \simeq H^*(M(\mathcal{A}), \mathbb{Z})$$

when $\mathbb{K} = \mathbb{C}$, see [**12**] Section 3 and Section 5 for details. In particular, Theorem 1.4 implies that

$$\Psi(\mathcal{A}; 1, t) = \pi(\mathcal{A}; t) = \mathrm{Hilb}(A(\mathcal{A}) \otimes \mathbb{Q}; x). \tag{1.1}$$

As we see that the specialization $\Psi(\mathcal{A}; 1, t)$ has a geometric meaning in Theorem 1.4, it is natural to ask whether the specialization with respect to the t-variable has a nice interpretation. For example, can we understand $\Psi(\mathcal{A}; x, 1)$ by using algebra, geometry of arrangements or other geometric objects? In this subsection we give an answer to this problem from algebraic point of view. Let us introduce algebraic counterpart of $\Psi(\mathcal{A}; x, 1)$ in the following.

Theorem 1.5 ([**17**]). *Let d be a non-negative integer and S_d the set of all homogeneous polynomials of degree d. Fix $\eta \in S_d$. Also, define the boundary map $\partial_p : D^p(\mathcal{A}) \to D^{p-1}(\mathcal{A})$ $(p = 1, \ldots, \ell)$ by*

$$\partial_p(\theta)(f_2, \ldots, f_p) := \theta(\eta, f_2, \ldots, f_p)$$

for all $f_2, \ldots, f_p \in S$. We call the complex $(D^*(\mathcal{A}), \partial_*)$ the Solomon–Terao complex *of degree d with respect to $\eta \in S_d$. Define their cohomology group*

$$H^p(D^*(\mathcal{A}), \partial_*) := \frac{\ker \partial_p}{\mathrm{Im}\, \partial_{p+1}}.$$

Then

(1) *there is a non-empty Zariski open set $U_d = U_d(\mathcal{A}) \subset S_d$ such that every cohomology of the Solomon–Terao complex with respect to $\eta \in U_d$ is of finite dimension over $\mathbb{K}$. For the details of U_d, see Section 2.*

(2) *If $\mathrm{pd}_S\, D^p(\mathcal{A}) \le \ell - p$ for all $p = 1, 2, \ldots, \ell$ (such an arrangement is called* tame*), then $H^i(D^*(\mathcal{A}), \partial_*) = 0$ for $i \neq 0$.*

Definition 1.6 (Solomon–Terao algebra). In the notation of Theorem 1.5, define $ST(\mathcal{A}, \eta) := H^0(D^*(\mathcal{A}), \partial_*)$ and let us call $ST(\mathcal{A}, \eta)$ the *Solomon–Terao algebra* of $\mathcal{A}$ with respect to η. We call $\mathfrak{a}(\mathcal{A}, \eta) := \{\theta(\eta) \in S \mid \theta \in D(\mathcal{A})\} = \mathrm{Im}\, \partial_1$ the *Solomon–Terao ideal* of $\mathcal{A}$ with respect to η, i.e., $S/\mathfrak{a}(\mathcal{A}, \eta) = ST(\mathcal{A}, \eta)$.

Remark 1.7. By definition, the structure of the Solomon–Terao algebra depends on the choice of the polynomial $\eta \in U_d(\mathcal{A})$. See Example 5.9 for details.

The Solomon–Terao algebra can be defined for all arrangements, but the most useful case is when $\mathcal{A}$ is tame. In fact, we can show that the Solomon–Terao algebra is the algebraic counterpart of $\Psi(\mathcal{A}; x, 1)$ when $\mathcal{A}$ is tame.

Theorem 1.8. *Let $\mathcal{A}$ be tame and $\eta \in U_2(\mathcal{A})$. Then we have*

$$\mathrm{Hilb}(ST(\mathcal{A}, \eta); x) = \Psi(\mathcal{A}; x, 1).$$

In particular, $\mathrm{Hilb}(ST(\mathcal{A}, \eta); 1) = \pi(\mathcal{A}; 1)$ *coincides with the number of chambers when* $\mathbb{K} = \mathbb{R}$*, and with the total Betti numbers of* $M(\mathcal{A})$ *when* $\mathbb{K} = \mathbb{C}$.

Theorem 1.8 is essentially proved in [**17**]. We have reformulated the result as in Theorem 1.8 to explain a reason to consider the Solomon–Terao algebra. Theorem 1.8 affords a good motivation to study. In other words, the Solomon–Terao algebra is closely related to one specialization of the Solomon–Terao polynomial from algebraic point of view when it is tame. Though tameness is a generic property, to check the tameness for a given arrangement is very hard. Fortunately, one of the most famous classes of hyperplane arrangements is known to be tame.

Definition 1.9. An arrangement $\mathcal{A}$ is *free* with $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$ if $D(\mathcal{A})$ is a free S-module of rank ℓ with homogeneous basis $\theta_1, \ldots, \theta_\ell$, $\deg \theta_i = d_i$ $(i = 1, \ldots, \ell)$.

When $\mathcal{A}$ is free, $D^p(\mathcal{A})$ is also free (see [**12**] or [**17**]). Thus the freeness implies the tameness. Since the freeness is a very strong property of arrangements, it is worth studying $ST(\mathcal{A}, \eta)$ when $\mathcal{A}$ is free, which is our second main result. To state it, let us recall some fundamental definitions on commutative ring theory. Let M be a graded S-module. Let M_n denote the homogeneous degree n-part of M. Then the *socle* $\mathrm{soc}(M)$ of M is defined as

$$\mathrm{soc}(M) := 0 :_M S_+,$$

where $S_+ = (x_1, \ldots, x_\ell)$. When S/I is an Artinian Gorenstein $\mathbb{K}$-algebra for an ideal $I \subset S$, $\dim_{\mathbb{K}} \mathrm{soc}(S/I) = 1$, hence there is an integer r such that $\mathrm{soc}(S/I) = (S/I)_r$. We call r the *socle degree* of S/I and denote it by $\mathrm{socdeg}(S/I)$. Now we can state the second main theorem in this article.

Theorem 1.10 (Freeness and C.I.). *For an arrangement $\mathcal{A}$ and a non-negative integer d, let $U_d := U_d(\mathcal{A})$. Let $\eta \in U_d$.*

(1) *Assume that $\mathcal{A}$ is free with* $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$*. Then $ST(\mathcal{A}, \eta)$ is a complete intersection with*

$$\mathrm{Hilb}(ST(\mathcal{A}, \eta); x) = \prod_{i=1}^{\ell} (1 + x + \cdots + x^{d_i + d - 2}).$$

Hence $\mathrm{socdeg}\, ST(\mathcal{A}, \eta) = |\mathcal{A}| + \ell(d - 2)$*, and*

(2) *conversely, if $ST(\mathcal{A}, \eta)$ is a complete intersection, then $\mathcal{A}$ is free. In this case,*

$$\mathrm{Hilb}(ST(\mathcal{A}, \eta); x) = \prod_{i=1}^{\ell} (1 + x + \cdots + x^{e_i}),$$

implies that $\exp(\mathcal{A}) = (e_1 - d + 2, \ldots, e_\ell - d + 2)$.

REMARK 1.11. When $d = 2$, Theorem 1.10 is also proved by Epure and Schulze in [**8**] independently. They actually proved the same result more generally, i.e., not only for hyperplane arrangements but also for hypersurface singularities.

Theorems 1.8 and 1.10 help us to investigate the algebraic structure of $ST(\mathcal{A}, \eta)$ in terms of commutative ring theory. Then the next question is whether we have a nice geometric understanding of $ST(\mathcal{A}, \eta)$ and $\Psi(\mathcal{A}; x, 1) = \mathrm{Hilb}(ST(\mathcal{A}, \eta); x)$ when $\mathcal{A}$ is tame. Let us give an answer from classical result by Borel in [**5**], and the recent results on Hessenberg varieties in [**3**] in the next subsection.

1.2. Geometry of the Solomon–Terao algebra.

In this subsection, we show the relation between $ST(\mathcal{A}, \eta)$ and the cohomology ring of some variety, which is analogous to the one between the Orlik–Solomon algebra $A(\mathcal{A})$ and the open manifold $M(\mathcal{A})$ when $\mathbb{K} = \mathbb{C}$. In this subsection let $\mathbb{K} = \mathbb{C}$.

First let W be the irreducible crystallographic Weyl group acting on V. Let G be the corresponding complex semisimple linear algebraic group, and B the fixed Borel subgroup. Let $\mathcal{A} = \mathcal{A}_W$ be a set of reflecting hyperplanes of all reflections of the Weyl group W (so called the *Weyl arrangement*). By the result of Saito (see [**14**] for example), $\mathcal{A}_W$ is free with exponents $(d_1^W, \ldots, d_\ell^W)$ coinciding with the exponents of W. Combining the results in [**5**] and [**17**], we have

$$\Psi(\mathcal{A}_W; x, 1) = \mathrm{Poin}(G/B; \sqrt{x}) = \prod_{i=1}^{\ell} (1 + x + \cdots + x^{d_i^W}). \tag{1.2}$$

Here G/B is the flag variety corresponding to W. Hence $\Psi(\mathcal{A}_W; x, t)$ has two important specializations for $x = 1$ and $t = 1$ in the geometric point of view. Also, let S^W denote the W-invariant part of the polynomial ring with the W-action, and let $\mathrm{coinv}(W) := S/(S_+^W)$ the coinvariant algebra. Then it is well-known that

$$\mathrm{Hilb}(\mathrm{coinv}(W); x) = \mathrm{Poin}(G/B; \sqrt{x}) = \Psi(\mathcal{A}_W; x, 1). \tag{1.3}$$

Hence when $\mathcal{A} = \mathcal{A}_W$, the algebraic counterpart of $t = 1$ is the coinvariant algebra, which is also known to be isomorphic to the cohomology ring of the flag variety G/B by Borel in [**5**]. In fact, we get a natural interpretation of the Solomon–Terao algebra for $\mathcal{A} = \mathcal{A}_W$. Let P_1 be the lowest degree basic invariant of S^W. Then Theorem 3.9 in [**3**] shows that

$$ST(\mathcal{A}_W, P_1) \simeq \mathrm{coinv}(W) \simeq H^*(G/B, \mathbb{C}).$$

Thus we can understand the Solomon–Terao algebra from geometric point of view in a way suggestive of the Orlik–Solomon algebra when $\mathcal{A} = \mathcal{A}_W$. The above isomorphism is now extended to a wider class. We refer to the results in [**3**].

DEFINITION 1.12. Let Φ be the root system with respect to W and fix a positive system Φ^+. Let $I \subset \Phi^+$ be a *lower ideal*, i.e., the set satisfying that, if $\beta \in I$, $\gamma \in \Phi^+$

and $\beta - \gamma \in \sum_{i=1}^{\ell} \mathbb{Z}_{\geq 0}\alpha_i$ for the simple system $\alpha_1, \ldots, \alpha_\ell$ of Φ^+, then $\gamma \in I$. Then $\mathcal{A}_I := \{\alpha = 0 \mid \alpha \in I\}$ is called the *ideal arrangement.*

The freeness of the ideal arrangements are known as follows:

Theorem 1.13 (Theorem 1.1, [**2**]). *Let $I \subset \Phi^+$ be a lower ideal. Then $\mathcal{A}_I$ is free with exponents $(d_1^I, \ldots, d_\ell^I)$ which coincides with the dual partition of the height distribution of the positive roots in I.*

Hence Theorem 1.10 is applicable to the algebra $ST(\mathcal{A}_I, P_1)$. On the other hand, we can also associate a variety with the lower ideal, so called the regular nilpotent Hessenberg variety, see [**3**] or [**7**] for details. For their cohomology rings, the following is known.

Theorem 1.14 (Theorem 1.1, [**3**]). *Let $X(N, I)$ be the regular nilpotent Hessenberg variety determined by a lower ideal I and a regular nilpotent element $N \in \mathfrak{g} = Lie(G)$. Then*

$$ST(\mathcal{A}_I, P_1) \simeq H^*(X(N, I)).$$

In particular,

$$\mathrm{Poin}(X(N, I), \sqrt{x}) = \prod_{i=1}^{\ell}(1 + x + \cdots + x^{d_i^I}).$$

In [**3**], there are no terminology "Solomon–Terao algebras". Here we state the main result in [**3**] in terms of the Solomon–Terao algebra. Theorem 1.14 shows that the Solomon–Terao algebra $ST(\mathcal{A}_I, P_1)$ is realized as the cohomology ring of the variety $X(N, I)$, which reminds us of that the Orlik–Solomon algebra is isomorphic to the cohomology ring of $M(\mathcal{A}_I)$. Note that $X(N, \Phi^+) = G/B$. Hence we can say that *the Solomon–Terao algebra generalizes the coinvariant algebra of the Weyl groups in the setting of hyperplane arrangements.*

Remark 1.15. From now on, when $\mathbb{K} = \mathbb{C}$, $\mathcal{A} \subset \mathcal{A}_W$ and P_1 is the same as in Theorem 1.14, let $ST(\mathcal{A}) := ST(\mathcal{A}, P_1)$ and $\mathfrak{a}(\mathcal{A}) := \mathfrak{a}(\mathcal{A}, P_1)$. It is clear that $P_1 \in U_2$ for any $\mathcal{A}$.

The organization of this article is as follows. In Section 2 we recall several results on arrangements, mainly from [**17**]. In Section 3, we prove Theorem 1.10. In Section 4 we investigate the Solomon–Terao algebra for the inversion arrangements, and the relation to the Schubert varieties. In Section 5 we pose several questions related to the Solomon–Terao algebras.

2. Preliminaries.

In this section we collect several definitions and results, mainly from [**12**] and [**17**]. First, let us recall definitions on combinatorics of arrangements.

Definition 2.1. Define the *intersection lattice* $L(\mathcal{A})$ of $\mathcal{A}$ by

$$L(\mathcal{A}) := \Big\{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subset \mathcal{A} \Big\}.$$

The *Möbius function* μ on $L(\mathcal{A})$ is defined by, $\mu(V) = 1$, and by

$$\mu(X) := - \sum_{Y \in L(\mathcal{A}),\ V \supset Y \supsetneq X} \mu(Y).$$

Then define the *Poincaré polynomial* of $\mathcal{A}$ by

$$\pi(\mathcal{A}; t) := \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{codim} X},$$

and define the *characteristic polynomial* of $\mathcal{A}$ by

$$\chi(\mathcal{A}; t) := \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}.$$

By [**11**], $\pi(\mathcal{A}; t)$ coincides with the topological Poincaré polynomial of $M(\mathcal{A}) := V \setminus \bigcup_{H \in \mathcal{A}} H$ when $\mathbb{K} = \mathbb{C}$. Moreover, $H^*(M(\mathcal{A}), \mathbb{Z})$ has a presentation depending only on $L(\mathcal{A})$, see [**11**] for details.

Now let us recall several properties and results on $D(\mathcal{A})$. For $\theta \in \operatorname{Der} S$, we say that θ is homogeneous of degree d if $\deg \theta(\alpha) = d$ for all $\alpha \in V^*$ with $\theta(\alpha) \neq 0$. Also, let us introduce a criterion for the freeness of $\mathcal{A}$.

Theorem 2.2 (Saito's criterion, [**14**]). *Let $\theta_1, \dots, \theta_\ell \in D(\mathcal{A})$ be homogeneous elements. Then they form a basis for $D(\mathcal{A})$ if and only if* (1) *they are S-linearly independent, and* (2) $\sum_{i=1}^{\ell} \deg \theta_i = |\mathcal{A}|$.

The following is the most important consequence of the freeness.

Theorem 2.3 (Terao's factorization, [**20**]). *Let $\mathcal{A}$ be free with* $\exp(\mathcal{A}) = (d_1, \dots, d_\ell)$. *Then*

$$\pi(\mathcal{A}; t) = \prod_{i=1}^{\ell} (1 + d_i t).$$

The following plays a key role in the proof of Theorem 1.10.

Proposition 2.4 (e.g., [**12**], Proposition 4.12). *Let $\theta_1, \dots, \theta_\ell \in D(\mathcal{A})$. Then* $\det(\theta_i(x_j)) \in Q(\mathcal{A})S$.

Let us introduce two sufficient conditions to check the freeness of $\mathcal{A}$ for our purpose.

Theorem 2.5 (Terao's addition-deletion theorem, [**19**]). *Let $H \in \mathcal{A}$, $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ and $\mathcal{A}'' := \mathcal{A}^H = \{H \cap L \mid L \in \mathcal{A}'\}$. Then two of the following three imply the third*:

(1) *$\mathcal{A}$ is free with* $\exp(\mathcal{A}) = (d_1, \dots, d_{\ell-1}, d_\ell)$.

(2) $\mathcal{A}'$ *is free with* $\exp(\mathcal{A}') = (d_1, \ldots, d_{\ell-1}, d_\ell - 1)$.

(3) $\mathcal{A}''$ *is free with* $\exp(\mathcal{A}'') = (d_1, \ldots, d_{\ell-1})$.

In particular, (1), (2) *and* (3) *are all true if* $\mathcal{A}$ *and* $\mathcal{A}'$ *are both free.*

Theorem 2.6 (Division theorem, [**1**], Theorem 1.1). *Let* $H \in \mathcal{A}$. *If* $\mathcal{A}^H$ *is free and* $\pi(\mathcal{A}^H; t) \mid \pi(\mathcal{A}; t)$, *then* $\mathcal{A}$ *is free.*

The results in this article rely on those in [**17**]. To prove Theorem 1.3, Solomon and Terao introduced the Solomon–Terao complex as in Theorem 1.5. At the same time, the structure of their complex in itself deserves our attention. We summarize some definitions and results from [**17**] below.

Definition 2.7 ([**17**], Definition 4.5). Let d be a non-negative integer and $\mathcal{A}$ an arrangement in V. For $X \in L(\mathcal{A})$ let S^X be the coordinate ring of X. We say that $h \in S^X$ is *non-degenerate on* X if the zero-locus of all polynomials in $\mathrm{Jac}(h) := \{\theta(h) \in S^X \mid \theta \in \mathrm{Der}\, S^X\}$ is contained in the origin of X. Define

$$U_d^X(\mathcal{A}) := \{f \in S_d \mid f|_X \text{ is non-degenerate on } X\}.$$

Proposition 2.8 ([**17**], Section 4 and Corollary 3.6). *Let* $\mathcal{A}$ *be an arrangement in* $V = \mathbb{K}^\ell$. *Then*

(1) *for each* $d > 0$, *the open set*

$$U_d(\mathcal{A}) := \bigcap_{X \in L(\mathcal{A})} U_d^X(\mathcal{A}) \subset S_d$$

is non-empty, and the Solomon–Terao complex has a finite dimensional cohomology group $H^i(D^*(\mathcal{A}), \eta)$ $(i = 0, \ldots, \ell)$ *for all* $\eta \in U_d(\mathcal{A})$.

(2) *If* $\mathcal{A}$ *is free with* $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$, *then*

$$\Psi(\mathcal{A}; x, t) = \prod_{i=1}^{\ell} (t(1 + x + \cdots + x^{d_i - 1}) + x^{d_i}).$$

By definition, the following is clear.

Lemma 2.9. *Let* $\eta \in U_d(\mathcal{A})$. *Then* $(\partial_{x_1}(\eta), \ldots, \partial_{x_\ell}(\eta))$ *is an* S*-regular sequence.*

Proof. Apply the definition of $U_d^X(\mathcal{A})$ when $X = V$. $\square$

In arrangement theory, for $H \in \mathcal{A}$, we often consider the deletion $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ and the restriction $\mathcal{A}'' := \mathcal{A}^H := \{L \cap H \mid L \in \mathcal{A}'\}$ together to obtain the information of $\mathcal{A}$. In our setup, $U_d(\mathcal{A})$, $U_d(\mathcal{A}')$ and $U_d(\mathcal{A}'')$ behave as follows:

Lemma 2.10. *Let* $\eta \in U_d(\mathcal{A})$ *and* $H \in \mathcal{A}$. *Then* $\eta \in U_d(\mathcal{A} \setminus \{H\})$, *and* $\eta|_H \in U_d(\mathcal{A}^H)$.

PROOF. If $X \in L(\mathcal{A} \setminus \{H\})$, then $X \in L(\mathcal{A})$ since $L(\mathcal{A} \setminus \{H\}) \subset L(\mathcal{A})$. If $X \in L(\mathcal{A}^H)$, then so is $X \in L(\mathcal{A})$ since $L(\mathcal{A}^H) \subset L(\mathcal{A})$. $\square$

Hence for $\eta \in U(\mathcal{A})$, we have the following two maps:

$$F_1 : ST(\mathcal{A} \setminus \{H\}, \eta) \overset{\cdot \alpha_H}{\to} ST(\mathcal{A}, \eta), \tag{2.1}$$

$$F_2 : ST(\mathcal{A}, \eta) \to ST(\mathcal{A}^H, \eta|_H). \tag{2.2}$$

Also, it is clear that $F_2 \circ F_1 = 0$ and F_2 is surjective. To investigate a general property of $\Psi(\mathcal{A}; x, t)$, we use the following.

PROPOSITION 2.11 ([**17**], Proposition 4.4). *For $H \in \mathcal{A}$, consider the boundary map $\partial_p^H : D^p(\mathcal{A}) \to D^{p-1}(\mathcal{A})$ $(p = 1, \ldots, \ell)$ defined by*

$$\partial_p^H(\theta)(f_2, \ldots, f_p) := \frac{\theta(\alpha_H, f_2, \ldots, f_p)}{\alpha_H}$$

for $\theta \in D^p(\mathcal{A})$. Then the complex $(D^*(\mathcal{A}), \partial_p^H)$ is exact. In particular,

$$\sum_{p=0}^{\ell} \mathrm{Hilb}(D^p(\mathcal{A}); x)(-x)^{\ell-p} = 0.$$

PROPOSITION 2.12. *Assume that $\mathcal{A}$ is tame, i.e.,*

$$\mathrm{pd}_S\, D(\mathcal{A})^p \le \ell - p \ (p = 0, \ldots, \ell).$$

Then $H^i(D^*(\mathcal{A}), \eta) = 0$ for $i \neq 0$.

PROOF. Theorem 5.8 in [**13**] states that for the complex

$$0 \to C^0 \to \cdots \to C^\ell \to 0$$

of $S := \mathbb{K}[x_1, \ldots, x_\ell]$-modules, the cohomology group H^k of this complex vanishes if

$$\mathrm{pd}_S\, C^p < \ell + p - k \ (\forall p).$$

Now $C^p = D^{\ell-p}(\mathcal{A})$. Hence

$$\mathrm{pd}_S\, C^p \le p = \ell + (p - \ell) < \ell + (p - k)$$

for all p and $k \neq \ell$. Hence $H^k(D^*(\mathcal{A}), \partial_*) = 0$ for $k \neq \ell$. $\square$

The following results in commutative ring theory play the key roles in the proof of Theorem 1.10.

THEOREM 2.13 (e.g., [**16**], Theorem 6.5.1). *Let $S = \mathbb{K}[x_1, \ldots, x_\ell]$ and let $t_1, \ldots, t_\ell$ be an S-regular sequence. Let $I = (t_1, \ldots, t_\ell) \subset S$ be an ideal, and assume that $t_i = \sum_{j=1}^{\ell} a_{ij} x_j$ for $i = 1, \ldots, \ell$. Then $\Delta := \det(a_{ij})$ is a $\mathbb{K}$-basis for $\mathrm{soc}(S/I)$.*

Theorem 2.14 (e.g., [**16**], Theorem 6.7.6). *Let $t_1, \ldots, t_\ell$ be homogeneous polynomials with* $\deg t_i = d_i$ $(i = 1, \ldots, \ell)$, *and let $R = S/I$ for an ideal $I = (t_1, \ldots, t_\ell) \subset S$. If* $\dim_{\mathbb{K}} R < \infty$, *then $t_1, \ldots, t_\ell$ is an S-regular sequence, and*

$$\mathrm{Hilb}(R; x) = \prod_{i=1}^{\ell} (1 + x + \cdots + x^{d_i - 1}).$$

3. Proof of Theorems 1.8 and 1.10.

From now on let us fix $\eta \in U_d(\mathcal{A})$ unless otherwise specified. First we prove Theorem 1.8. For that, let us show the following proposition essentially proved in [**17**].

Proposition 3.1. *For $\eta \in U_2(\mathcal{A})$ and an arbitrary arrangement $\mathcal{A}$,*

$$\sum_{p=0}^{\ell} (-x)^p \,\mathrm{Hilb}(H^p(D^*(\mathcal{A}), \eta); x) = \Psi(\mathcal{A}; x, 1).$$

Proof. By the results in Section 2, we know that

$$\begin{aligned} \Psi(\mathcal{A}; x, 1) &= \sum_{p=0}^{\ell} \mathrm{Hilb}(D^p(\mathcal{A}); x)((1 - x) - 1)^p \\ &= \sum_{p=0}^{\ell} \mathrm{Hilb}(D^p(\mathcal{A}); x)(-x)^p \\ &= \sum_{p=0}^{\ell} \mathrm{Hilb}(H^p(D^*(\mathcal{A})); x)(-x)^p. \end{aligned}$$

Here we used the fact that ∂_p is of degree one since $\eta \in U_2(\mathcal{A})$, and the finite dimensionality of $H^p(D^*(\mathcal{A}))$ by Theorem 1.5. □

Proof of Theorem 1.8. Combine Proposition 3.1 with Proposition 2.12 and the properties of $\pi(\mathcal{A}; t)$. □

Next let us prove Theorem 1.10.

Proof of Theorem 1.10. (1) Let us show that $ST(\mathcal{A}, \eta)$ is a complete intersection. Let $\theta_1, \ldots, \theta_\ell$ be a homogeneous basis for $D(\mathcal{A})$, and let $\theta_i(\eta) =: f_i$. By Theorem 1.5, we know that $ST(\mathcal{A}, \eta) = S/\mathfrak{a}(\mathcal{A}, \eta)$ is a finite dimensional $\mathbb{K}$-algebra, and $\mathfrak{a}(\mathcal{A}, \eta)$ is generated by ℓ homogeneous polynomials $f_1, \ldots, f_\ell$. Then $f_1, \ldots, f_\ell$ form a regular sequence by Theorem 2.14, and hence $ST(\mathcal{A}, \eta)$ is a complete intersection. On the Hilbert series, apply Theorem 2.14.

(2) To prove (2), let us prove the following proposition. □

Proposition 3.2. *Let $\eta \in U_d(\mathcal{A})$ and let $\eta_{ij} = \partial_{x_i} \partial_{x_j} \eta$ for $1 \leq i, j \leq \ell$. The element $Q(\mathcal{A}) \det(\eta_{ij})$ is contained in* $\mathrm{soc}(ST(\mathcal{A}, \eta))$. *Moreover, if $ST(\mathcal{A}, \eta)$ is a complete intersection, then $Q(\mathcal{A}) \det(\eta_{ij})$ is a $\mathbb{K}$-basis for* $\mathrm{soc}(ST(\mathcal{A}, \eta))$.

Proof. By Lemma 2.9, $\eta \in U_d(\mathcal{A})$ implies that $\partial_{x_1}(\eta), \dots, \partial_{x_\ell}(\eta)$ form an S-regular sequence. Let $\eta_i := \partial_{x_i}(\eta)$. Since η_i is homogeneous, it holds that

$$\eta_i = \sum_{j=1}^{\ell} \eta_{ij} x_j$$

up to non-zero scalar. Hence Theorem 2.13 implies that $\zeta := \det(\eta_{ij})$ is a $\mathbb{K}$-basis for $\mathrm{soc}(ST(\emptyset, \eta))$. Since $\emptyset \subset \mathcal{A}$, we have an S-morphism

$$F : ST(\emptyset, \eta) \to ST(\mathcal{A}, \eta)$$

induced by the multiplication of $Q(\mathcal{A})$, which is well-defined by definition of the Solomon–Terao algebra. Since

$$xQ(\mathcal{A})\zeta = xF(\zeta) = F(x\zeta) = F(0) = 0$$

for any $x \in S_+$, it holds that $Q(\mathcal{A})\zeta \in \mathrm{soc}(ST(\mathcal{A}, \eta))$.

Now, we assume that $ST(\mathcal{A}, \eta)$ is a complete intersection, and let us show that $Q(\mathcal{A})\zeta \neq 0$ in $ST(\mathcal{A}, \eta)$. Let $\theta_1, \dots, \theta_\ell \in D(\mathcal{A})$ such that $f_1 := \theta_1(\eta), \dots, f_\ell := \theta_\ell(\eta)$ is an S-regular sequence belonging to $\mathfrak{a}(\mathcal{A}, \eta)$. Let $\theta_i = \sum_{j=1}^{\ell} f_{ij} \partial_{x_j}$. Then we have

$$f_i = \theta_i(\eta) = \sum_{k=1}^{\ell} f_{ik}\eta_k = \sum_{k=1}^{\ell} \left(\sum_{j=1}^{\ell} f_{ik}\eta_{kj} x_j \right) = \sum_{j=1}^{\ell} \left(\sum_{k=1}^{\ell} f_{ik}\eta_{kj} \right) x_j$$

up to nonzero scalar. Hence Theorem 2.13 implies that

$$\det \left(\sum_{k=1}^{\ell} f_{ik}\eta_{kj} \right) = \det(f_{ij}) \det(\eta_{ij}) = \det(f_{ij})\zeta$$

is a $\mathbb{K}$-basis for $\mathrm{soc}(ST(\mathcal{A}, \eta))$. In particular, $\det(f_{ij})\zeta$ is not zero in $ST(\mathcal{A}, \eta)$. By Proposition 2.4, we know that $\det(f_{ij})$ is divisible by $Q(\mathcal{A})$. In other words, there is $g \in S$ such that $\det(f_{ij})\zeta = gQ(\mathcal{A})\zeta$ in $ST(\mathcal{A}, \eta)$. Since both are elements of $\mathrm{soc}(ST(\mathcal{A}, \eta))$, g is a nonzero-scalar. □

Proof of Theorem 1.10 (2), continued. Now assume that $\eta \in U_d(\mathcal{A})$ and $ST(\mathcal{A}, \eta)$ is a complete intersection. Let $\theta_1, \dots, \theta_\ell \in D(\mathcal{A})$ be derivations such that $\theta_1(\eta), \dots, \theta_\ell(\eta)$ form an S-regular sequence. Let $\zeta := \det(\eta_{ij})$. We show that $\theta_1, \dots, \theta_\ell$ form a basis for $D(\mathcal{A})$ by Saito's criterion. For $\theta_i = \sum_{j=1}^{\ell} f_{ij}\partial_{x_j} \in D(\mathcal{A})$, by Proposition 3.2, we know that $Q(\mathcal{A})\zeta = \det(f_{ij})\zeta$ is a $\mathbb{K}$-basis for $\mathrm{soc}(ST(\mathcal{A}, \eta))$. Thus $\theta_1, \dots, \theta_\ell$ are S-independent. Moreover, $\deg \det(f_{ij}) = \sum_{i=1}^{\ell} \deg \theta_i = \deg Q(\mathcal{A}) = |\mathcal{A}|$. Hence Saito's criterion implies that $\mathcal{A}$ is free. On the Hilbert series, apply Theorem 2.14. □

In the following low-dimensional cases, we can always apply some of results above.

Proposition 3.3 (Two and three dimensional case). (1) *Assume that $\ell = 2$. Then all non-empty arrangements are free and hence tame. In particular,*

$$\mathrm{Hilb}(ST(\mathcal{A},\eta);x)=(1+x+\cdots+x^{d-1})(1+x+\cdots+x^{d+|\mathcal{A}|-3})$$

for $\eta\in U_d(\mathcal{A})$.

(2) *Assume that* $\ell=3$. *Then all arrangements are tame. Hence* $\mathrm{Hilb}(ST(\mathcal{A},\eta);x)=\Psi(\mathcal{A};x,1)$ *for* $\eta\in U_2(\mathcal{A})$.

Proof. (1) It is famous that $\mathcal{A}$ is free with $\exp(\mathcal{A})=(1,|\mathcal{A}|-1)$ when $\ell=2$. Hence Theorem 1.10 completes the proof.

(2) Since $D^p(\mathcal{A})$ is reflexive, their projective dimension is at most 1. Hence the only case we have to check the tameness is whether $\mathrm{pd}_S D^3(\mathcal{A})\le 0$. This is true since $D^3(\mathcal{A})\simeq S$. For the rest, apply Theorem 1.8. □

In general, it is not easy to compute Solomon–Terao polynomials and $\mathrm{Hilb}(ST(\mathcal{A},\eta);x)$ when $\mathcal{A}$ is not free.

Example 3.4. Let $\mathcal{A}$ be defined as $xyz(x+y+z)=0$. Then we may compute $\mathrm{Hilb}(ST(\mathcal{A});x)=1+3x+5x^2+4x^3+x^4=(1+x)(1+2x+3x^2+x^3)$.

It is known that $\mathcal{A}$ is not necessarily free even if $\pi(\mathcal{A};t)=\prod_{i=1}^{\ell}(1+d_it)$ with $d_1,\ldots,d_\ell\in\mathbb{Z}$. For the Solomon–Terao algebras, we do not know any such examples. Based on several computations, we pose the following conjectures.

Conjecture 3.5. (1) *$\mathcal{A}$ is free if and only if*

$$\mathrm{Hilb}(ST(\mathcal{A},\eta);x)=\prod_{i=1}^{\ell}(1+x+\cdots+x^{d_i})$$

for some integers $d_1,\ldots,d_\ell$.

(2) *$\mathcal{A}$ is free if and only if* $\mathrm{Hilb}(ST(\mathcal{A},\eta);x)$ *is palindromic, i.e., for* $\mathrm{Hilb}(ST(\mathcal{A},\eta);x)=\sum_{i=0}^{n}a_ix^i$ *with* $a_n\neq 0$, *it holds that* $a_i=a_{n-i}$ *for all* i.

The "only if" parts of Conjecture 3.5 (1) and (2) are surely true by Theorem 1.10. Let us check Conjecture 3.5 when $\mathcal{A}$ is not free but $\pi(\mathcal{A};t)$ splits over $\mathbb{Z}$ for the following case.

Example 3.6. Let

$$\mathcal{A}:=\{x(x^2-y^2)(x^2-2y^2)(y-z)z=0\}.$$

It is easy to check that $\pi(\mathcal{A};t)=(1+t)(1+3t)^2$, but $\mathcal{A}$ is not free (hence the factorization of $\pi(\mathcal{A};t)$ is not the sufficient condition for the freeness). In this case, let us compute $\mathrm{Hilb}(ST(\mathcal{A},\eta);x)$. Let $\eta:=x^2+y^2+z^2$. Then we can compute that

$$\mathfrak{a}(\mathcal{A},\eta)=(x^2+y^2+z^2,\ z^3-yz^2,\ y^6-y^5z,\ y^6+3y^4z^2).$$

Hence

$$\mathrm{Hilb}(ST(\mathcal{A},\eta);x)=1+3x+5x^2+6x^3+6x^4+6x^5+4x^6+x^7$$

$$= (1+x)^2(1+x+2x^2+x^3+2x^4+x^5),$$

which does not split into the form $\prod(1+x+\cdots+x^{d_i-1})$. Thus this example does not give a counter example to Conjecture 3.5.

We give one result related to Conjecture 3.5 as follows.

PROPOSITION 3.7. *Let $\mathcal{A}$ be an arrangement in $\mathbb{K}^3$ and $\eta \in U_2(\mathcal{A})$. Assume that*

$$\mathrm{Hilb}(ST(\mathcal{A},\eta);x) = \prod_{i=1}^{3}(1+x+\cdots+x^{d_i})$$

for $d_1 = 1$ and some $d_2, d_3 \in \mathbb{Z}$. If $|\mathcal{A}| = 1+d_2+d_3$, and there is $H \in \mathcal{A}$ such that

$$\mathrm{Hilb}(ST(\mathcal{A}^H,\eta|_H);x) = \prod_{i=1}^{2}(1+x+\cdots+x^{d_i}),$$

then $ST(\mathcal{A},\eta)$ is a complete intersection.

PROOF. By Lemma 2.10, $\eta|_H \in U_2(\mathcal{A}^H)$. By Proposition 3.3 (1), $ST(\mathcal{A}^H,\eta|_H)$ is always a complete intersection, and $\mathcal{A}$ is tame by Proposition 3.3 (2). Let us compute the coefficients of $\pi(\mathcal{A};t) = 1+|\mathcal{A}|t+b_2t^2+b_3t^3$ in terms of $\mathrm{Hilb}(ST(\mathcal{A},\eta);x)$. By Theorem 1.8,

$$\Psi(\mathcal{A};1,1) = \pi(\mathcal{A};1) = \mathrm{Hilb}(ST(\mathcal{A},\eta);1) = 1+|\mathcal{A}|+b_2+b_3 = 2(d_2+1)(d_3+1).$$

Also, since $\mathcal{A}$ is central, $\pi(\mathcal{A};t)$ is divisible by $1+t$, see [**12**] for example. Hence

$$\pi(\mathcal{A};-1) = 1-|\mathcal{A}|+b_2-b_3 = 0.$$

By these two equations, we can compute $b_2 = d_2+d_3+d_2d_3$. Since $|\mathcal{A}| = 1+d_2+d_3$, it holds that $b_3 = d_2d_3$. Thus $\pi(\mathcal{A};t) = (1+t)(1+d_2t)(1+d_3t)$. Again by Theorems 1.10 and 2.3, we have $\pi(\mathcal{A}^H;t) = (1+t)(1+d_2t)$. Hence Theorem 2.6 implies that $\mathcal{A}$ is free. Thus Theorem 1.10 shows that $ST(\mathcal{A},\eta)$ is a complete intersection. $\square$

4. Inversion arrangements and Schubert varieties.

In this section we use the notation in Section 1.2, i.e., $\mathbb{K} = \mathbb{C}$ and $\eta = P_1$.

DEFINITION 4.1. Let Φ be the root system with respect to the Weyl group W and fix a positive system Φ^+. For $w \in W$, define $\mathcal{A}_w := \{\alpha = 0 \mid \alpha \in \Phi^+,$ $w\alpha$ is a negative root$\}$, which is called the *inversion arrangement*.

Also for $w \in W$, we have the Schubert variety $Y_w := \overline{BwB}$. Now let us check the freeness of inversion arrangements. For details of them, see [**15**].

THEOREM 4.2 (Theorem 3.3, [**15**]). *Let Y_w be the Schubert variety determined by $w \in W$. Then Y_w is rationally smooth if and only if $\mathcal{A}_w$ is free with* $\exp(\mathcal{A}_w) =$

$(d_1^w, \ldots, d_\ell^w)$, *and* $\prod_{i=1}^{\ell}(1+d_i^w) = |[e,w]|$, *the number of elements between* e *and* w *in the Bruhat order. Moreover,*

$$\mathrm{Poin}(Y_w; \sqrt{x}) = \prod_{i=1}^{\ell}(1 + x + \cdots + x^{d_i^w}).$$

Hence by Theorem 1.10, we have the following:

Corollary 4.3. *Assume that* Y_w *is rationally smooth and* $|[e,w]| = \prod_{i=1}^{\ell}(1+d_i^w)$ *in the notation of Theorem* 4.2. *Then*

$$\mathrm{Poin}(Y_w; \sqrt{x}) = \mathrm{Hilb}(ST(\mathcal{A}_w, P_1); x).$$

The claim above suggests a correspondence between inversion arrangements and Schubert varieties. We have the Solomon–Terao algebra on one side and the cohomology ring on the other. However, they are not isomorphic as algebras in general.

Proposition 4.4. *Let* $w = (4123) \in S_4$. *Then*

$$H^*(Y_w, \mathbb{C}) \not\simeq ST(\mathcal{A}_w, P_1)$$

as rings.

Proof. By the computation of Schubert polynomials, it holds that

$$H^*(Y_w, \mathbb{C}) \simeq \mathbb{C}[x_1, x_2, x_3, x_4]/(f_1, f_2, f_3, f_4),$$

where

$$\begin{aligned} f_1 &= x_1 + x_2 + x_3 + x_4, \\ f_2 &= (x_1 + x_2 + x_3)^2, \\ f_3 &= x_2 x_3 + x_1 x_3, \\ f_4 &= x_1 x_2. \end{aligned}$$

Hence $H^*(Y_w, \mathbb{C})$ has a non-zero element $x_1 + x_2 + x_3$ of degree one such that $(x_1 + x_2 + x_3)^2 = 0$. On the other hand, we can check by the direct computation that there are no such elements of degree one in $ST(\mathcal{A}_w, P_1)$. Indeed, $\mathcal{A}_w$ is defined by

$$(x_1 - x_2)(x_1 - x_3)(x_1 - x_4) = 0$$

in $\mathbb{C}^4$, and

$$\begin{aligned} ST(\mathcal{A}_w, P_1) &= \mathbb{C}[x_1, \ldots, x_4]/(x_1 + \cdots + x_4, (x_1 - x_2)x_2, (x_1 - x_3)x_3, (x_1 - x_4)x_4) \\ &\cong \mathbb{C}[x_2, x_3, x_4]/((2x_2 + x_3 + x_4)x_2, (x_2 + 2x_3 + x_4)x_3, (x_2 + x_3 + 2x_4)x_4). \end{aligned}$$

The degree 2 homogeneous component of this ring has a $\mathbb{C}$-basis $\{x_2x_3, x_2x_4, x_3x_4\}$, and for any $\alpha, \beta, \gamma \in \mathbb{C}$, we have

$$(\alpha x_2+\beta x_3+\gamma x_4)^2 = \frac{1}{2}\{(4\alpha\beta-\alpha^2-\beta^2)x_2x_3+(4\alpha\gamma-\alpha^2-\gamma^2)x_2x_4+(4\beta\gamma-\beta^2-\gamma^2)x_3x_4)$$

in $ST(\mathcal{A}_w, P_1)$. The above element is zero only when $\alpha=\beta=\gamma=0$. □

So the statement of Theorem 1.14 for Hessenberg varieties does not hold for Schubert varieties in general, though we have Corollary 4.3. However, since the Solomon–Terao algebra depends on the choice of $\eta \in U_2(\mathcal{A})$, we may pose the following problem.

PROBLEM 4.5. Is there any $\eta \in U_2(\mathcal{A}_w)$ such that

$$H^*(Y_w) \simeq ST(\mathcal{A}_w, \eta)$$

as rings?

By Theorem 1.14, Problem 4.5 has a positive answer for $\eta = P_1$ when w is the longest element in W, and for some special w by Theorem 1.14.

5. Questions, problems and conjectures.

In this section we collect several problems and conjectures related to Solomon–Terao algebras. We assume that $\eta \in U_2(\mathcal{A})$ unless otherwise specified. The most important problem is the following.

QUESTION 5.1. Is there any topological meaning of $\Psi(\mathcal{A}; x, t)$ and $\mathrm{Hilb}(ST(\mathcal{A},\eta); x)$? Also, is there any common property among them?

For geometric meaning, by Theorem 1.14, we can say that $\Psi(\mathcal{A}; x, 1) = \mathrm{Hilb}(ST(\mathcal{A}, P_1); x)$ is the Poincaré polynomial of a regular nilpotent Hessenberg variety when $\mathcal{A}$ is an ideal arrangement. For general properties, we can say the following, which is essentially proved in Proposition 5.4, [**17**].

PROPOSITION 5.2. *If $\mathcal{A} \neq \emptyset$, then $x+t$ divides $\Psi(\mathcal{A}; x, t)$. Therefore, $1+x$ divides $\mathrm{Hilb}(ST(\mathcal{A},\eta); x)$ when $\mathcal{A}$ is tame.*

PROOF. By Proposition 2.11,

$$\sum_{p=0}^{\ell} \mathrm{Hilb}(D^p(\mathcal{A}); x)(-x)^{\ell-p} = 0.$$

On the other hand,

$$\sum_{p=0}^{\ell} \mathrm{Hilb}(D^p(\mathcal{A}); x)(-x)^{\ell-p} = \Psi(\mathcal{A}; x, -x) = 0.$$

Since $\Psi(\mathcal{A}; x, -x)$ is a polynomial in x, we complete the proof. □

Another question is to ask whether we can compute $\mathrm{Hilb}(ST(\mathcal{A},\eta); x)$ inductively as for $\pi(\mathcal{A}; t)$ or not. For $\pi(\mathcal{A}; t)$, let $H \in \mathcal{A}$, $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ and $\mathcal{A}'' := \mathcal{A}^H$. Then it holds that

$$\pi(\mathcal{A};t) = \pi(\mathcal{A}';t) + t\pi(\mathcal{A}'';t),$$

which is called the deletion-restriction formula.

QUESTION 5.3. Is there a deletion-restriction type formula for $\mathrm{Hilb}(ST(\mathcal{A},\eta);x)$? i.e., for the triple $(\mathcal{A},\mathcal{A}',\mathcal{A}'')$, is there a formula between $\mathrm{Hilb}(ST(\mathcal{A},\eta);x)$, $\mathrm{Hilb}(ST(\mathcal{A}',\eta);x)$, and $\mathrm{Hilb}(ST(\mathcal{A}^H,\overline{\eta});x)$? Also, what about the same question for the Solomon–Terao polynomials $\Psi(\mathcal{A};x,t)$?

The naive generalization of the deletion-restriction formula does not work well in general as follows.

EXAMPLE 5.4. The simplest idea for generalization is as follows:

$$\mathrm{Hilb}(ST(\mathcal{A},\eta);x) = \mathrm{Hilb}(ST(\mathcal{A}',\eta);x) + x^{|\mathcal{A}|-|\mathcal{A}^H|}\,\mathrm{Hilb}(ST(\mathcal{A}^H,\eta|_H);x). \tag{5.1}$$

This does not hold true in general. Let $\mathcal{A}$ be an arrangement defined by $xyz(x+y+z)=0$. Then we have

$$\Psi(\mathcal{A};x,1) = 1+3x+5x^2+4x^3+x^4.$$

Let $H=\{x+y+z=0\}\in\mathcal{A}$. Then it is easy to check that $\mathcal{A}' := \mathcal{A}\setminus\{H\}$ and $\mathcal{A}^H$ are both free with exponents $(1,1,1)$ and $(1,1)$. Thus by Theorem 1.10, it holds that

$$\Psi(\mathcal{A}';x,1) = (1+x)^3,$$
$$\Psi(\mathcal{A}^H;x,1) = (1+x)(1+x+x^2).$$

Hence the deletion-restriction does not hold for $\Psi(\mathcal{A};x,1)$, neither does $\Psi(\mathcal{A};x,t)$.

By Theorems 1.10, 2.5 and 2.6, the formula (5.1) holds true either when (a) $\mathcal{A}$ and $\mathcal{A}'$ are free, or (b) $\mathcal{A}^H$ is free and $\pi(\mathcal{A}^H;t)$ divides $\pi(\mathcal{A};t)$.

QUESTION 5.5. Are there any similarity between the Solomon–Terao polynomials or $\mathrm{Hilb}(ST(\mathcal{A},\eta);x)$, and Poincaré polynomials of arrangements for the triple $(\mathcal{A},\mathcal{A}',\mathcal{A}^H)$? For example, the division

$$\pi(\mathcal{A}^H;t) \mid \pi(\mathcal{A};t)$$

implies

$$\mathrm{Hilb}(ST(\mathcal{A}^H,\eta|_H);x) \mid \mathrm{Hilb}(ST(\mathcal{A},\eta);x)$$

and vice versa?

Again Question 5.5 is true for free cases.

PROPOSITION 5.6. *Assume that, either* (a) $\mathcal{A}$ *and* $\mathcal{A}'$ *are free, or* (b) $\mathcal{A}^H$ *is free and* $\pi(\mathcal{A}^H;t)$ *divides* $\pi(\mathcal{A};t)$. *Then* $\mathrm{Hilb}(ST(\mathcal{A}^H);x) \mid \mathrm{Hilb}(ST(\mathcal{A});x)$.

PROOF. Apply Theorems 1.10, 2.3, 2.5 and 2.6. $\square$

By the definition of $ST(\mathcal{A})$, we can treat some finite-dimensional algebras associated with tame arrangements. Hence it will be meaningful to give names to arrangements depending on the properties as follows:

DEFINITION 5.7. Let $\mathcal{A}$ be tame. Then we say that

(1) $(\mathcal{A}, \eta)$ is *of complete intersection* if $ST(\mathcal{A}, \eta)$ is a complete intersection ring,

(2) $(\mathcal{A}, \eta)$ is *Gorenstein* if $ST(\mathcal{A}, \eta)$ is a Gorenstein ring, or equivalently, $ST(\mathcal{A}, \eta)$ is a Poincaré duality algebra,

(3) $(\mathcal{A}, \eta)$ is *SLP* if $ST(\mathcal{A}, \eta)$ has a strong Lefschetz element (see [**10**] for details), and

(4) $\mathcal{A}$ is *ST-finite* if $H^i(D^*(\mathcal{A}), \eta) = 0$ for $i \neq 0$.

REMARK 5.8. Note that the properties in Definition 5.7 depend on the choice of $\eta \in U_2(\mathcal{A})$ in general. See the following example. So to investigate which properties depends only on $\mathcal{A}$ may be an interesting question. For example, when $\eta = Q(\mathcal{A})$, it follows by definition of $D(\mathcal{A})$ that $\mathfrak{a}(\mathcal{A}, Q(\mathcal{A})) = SQ(\mathcal{A})$ for a non-empty $\mathcal{A}$. Hence ST-finiteness depends on the choice of $\eta \in S_d$.

EXAMPLE 5.9. Let $\mathcal{A} := \{xy(x+y) = 0\}$. Then $D(\mathcal{A})$ has a basis $\theta_E, y(x+y)\partial_y$. Let $\eta_0 := x^4 + y^4 \in U_4(\mathcal{A})$ and $\eta := \sum_{i=0}^4 a_i x^i y^{4-i}$ for $a_i \in \mathbb{R}$. We show that for generic $a_0, \ldots, a_4$, two Solomon–Terao algebras $ST_0 := ST(\mathcal{A}, \eta_0)$ and $ST := ST(\mathcal{A}, \eta)$ are not isomorphic as graded rings. Note that $\mathfrak{a}(\mathcal{A}, \eta_0)_{\leq 1} = \mathfrak{a}(\mathcal{A}, \eta)_{\leq 1} = 0$. Hence if a graded ring isomorphism $\psi : ST_0 \to ST$ exists, then it is induced from a graded ring isomorphism $\varphi : S \to S$ since they are generated by degree one part. Let $\varphi(x) = \alpha x + \beta y$, $\varphi(y) = \gamma x + \delta y$. Since ψ is induced from φ, it holds that $\varphi(\eta_0) \in (\eta)$. We can see that this can be expressed as a closed condition. Hence generically, there are no φ which induces ψ. Hence $ST_0 \not\simeq ST$.

By Theorem 1.10, a complete intersection property is same as the freeness independent of the choice of $\eta \in U_d(\mathcal{A})$. Now we may pose the following natural problems:

QUESTION 5.10. Give a sufficient, or equivalence condition for the Gorenstein or ST-finite arrangement.

Also, not all arrangements are Gorenstein.

PROPOSITION 5.11. *The arrangement $xyz(x+y+z) = 0$ is not Gorenstein.*

PROOF. As seen in Example 3.4, $\mathrm{Hilb}(ST(\mathcal{A}); x) = 1 + 3x + 5x^2 + 4x^3 + x^4$, which is not palindromic. Hence this arrangement is not Gorenstein. $\square$

To ask the top degree of the nonzero part of the Solomon–Terao algebra is a natural question. By some computation, we conjecture the following:

CONJECTURE 5.12. *Let $\eta \in U_d(\mathcal{A})$ and define $r := \max\{n \mid ST(\mathcal{A}, \eta)_n \neq (0)\}$. Then*

(1) $r = |\mathcal{A}| + \ell(d-2)$, *and*
(2) $\dim_{\mathbb{K}} ST(\mathcal{A},\eta)_r = 1$.

Under some generic condition, we can give a partial affirmative answer to Conjecture 5.12.

Theorem 5.13. *Let $\mathcal{A}$ be an arrangement of linear hyperplanes. For a generic $\eta \in U_d(\mathcal{A})$, the element $Q(\mathcal{A})\det(\eta_{ij})$ is a nonzero element of $\mathrm{soc}(ST(\mathcal{A},\eta))$, in particular, $\dim_{\mathbb{K}} ST(\mathcal{A},\eta)_{|\mathcal{A}|+\ell(d-2)} \geq 1$, where $\eta_{ij} = \partial_{x_i}\partial_{x_j}\eta$ for all $1 \leq i,j \leq \ell$.*

Proof. It is known that there exists a supersolvable arrangement $\mathcal{B}$ containing $\mathcal{A}$. See the proof of Proposition 3.5 in [**4**] for example. By the genericity, we may assume $\eta \in U_d(\mathcal{A}) \cap U_d(\mathcal{B})$. We already see in the Proposition 3.2 that $Q(\mathcal{A})\det(\eta_{ij})$ is an element of $\mathrm{soc}(ST(\mathcal{A},\eta))$. We claim that this is non-zero. By Theorem 1.10, the Solomon–Terao algebra of $\mathcal{B}$ is a complete intersection. Consider the S-morphism $F : ST(\mathcal{A},\eta) \to ST(\mathcal{B},\eta)$ sending α to $(Q(\mathcal{B})/Q(\mathcal{A}))\alpha$, which is well-defined since $(Q(\mathcal{B})/Q(\mathcal{A}))D(\mathcal{A}) \subset D(\mathcal{B})$. Since $\mathcal{B}$ is free, $F(Q(\mathcal{A})\det(\eta_{ij})) = Q(\mathcal{B})\det(\eta_{ij})$ is a non-zero element in $ST(\mathcal{B},\eta)$ by Proposition 3.2. Hence $Q(\mathcal{A})\det(\eta_{ij})$ is also a non-zero element in $ST(\mathcal{A},\eta)$ as desired. □

Problem 5.14. Assume that $\mathrm{char}(\mathbb{K}) = 0$ and $(\mathcal{A},\eta)$ is Gorenstein. Then by Lemma 3.74 in [**10**], there is a homogeneous polynomial $h_{\mathcal{A}} \in \mathbb{K}[y_1,\ldots,y_\ell]$ such that

$$ST(\mathcal{A}) \simeq Q/\mathrm{Ann}_Q(h_{\mathcal{A}}).$$

Here $Q = \mathbb{K}[\partial_{y_1},\ldots,\partial_{y_\ell}]$ and

$$\mathrm{Ann}_Q(h_{\mathcal{A}}) := \{\epsilon \in Q \mid \epsilon(h_{\mathcal{A}}) = 0\}.$$

See [**10**] for details. Then determine $h_{\mathcal{A}}$.

For Hessenberg varieties, Theorem 11.3 in [**3**] explicitly determined $h_{\mathcal{A}}$. Based on Theorem 5.13 and results in Problem 5.14, we can show some genericity result on Gorenstein arrangement.

Theorem 5.15. *Let $\mathrm{char}(\mathbb{K}) = 0$, $\mathcal{A}$ be tame with a free arrangement $\mathcal{B}$ containing $\mathcal{A}$.*

(1) Assume that $ST(\mathcal{A},\eta)$ is Gorenstein for $\eta \in U := U_2(\mathcal{A})$. Then there is an open set $V \subset U_2(\mathcal{A})$ containing η such that $ST(\mathcal{A},\eta')$ is Gorenstein for all $\eta' \in V$.

(2) Assume that $(\mathcal{A},\eta)$ is SLP for $\eta \in U = U_2(\mathcal{A})$. Then there is an open set $V \subset U_2(\mathcal{A})$ containing η such that $ST(\mathcal{A},\eta')$ is SLP for all $\eta' \in V$.

Proof. (1) Let $\zeta \in U$ and denote $ST_\zeta := ST(\mathcal{A},\zeta)$. By Theorem 1.8, it holds that

$$\mathrm{Hilb}(ST(\mathcal{A},\eta);x) = \mathrm{Hilb}(ST(\mathcal{A},\zeta);x).$$

Hence for the top degree r of ST_ζ, it holds that $\dim_{\mathbb{K}}(ST_\zeta)_r = 1$. Fix an isomorphism

$[\quad]:(ST_\zeta)_r \to \mathbb{K}$. Define a polynomial $F_\zeta(y_1,\dots,y_\ell)$ by

$$F_\zeta := \frac{1}{r!}\left[\left(\sum_{i=1}^{\ell} x_i y_i\right)^r\right] \in \mathbb{K}[y_1,\dots,y_\ell].$$

Let $STG_\zeta := Q/\mathrm{Ann}_Q(F_\zeta)$ be the Gorenstein algebra. We show that there is a surjection $ST_\zeta \to STG_\zeta$ by sending x_i to ∂_{y_i}. To show it, it is sufficient to show that $\mathfrak{a}_\zeta := \mathfrak{a}(\mathcal{A},\zeta) \subset \mathrm{Ann}_Q(F_\zeta)$ regarding $S = Q$.

Define

$$E := \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{i=1}^{\ell} x_i y_i\right)^n.$$

By definition, $[E] = F_\zeta$ in ST_ζ. Also, note that

$$f(\partial_{y_1},\dots,\partial_{y_\ell})E = f(x_1,\dots,x_\ell)E.$$

Now let $f(x_1,\dots,x_\ell) \in \mathfrak{a}_\zeta$. Then

$$\begin{aligned} f(\partial_{y_1},\dots,\partial_{y_\ell})F_\zeta &= f(\partial_{y_1},\dots,\partial_{y_\ell})[E] \\ &= [f(\partial_{y_1},\dots,\partial_{y_\ell})E] \\ &= [f(x_1,\dots,x_\ell)E]. \end{aligned}$$

Thus $f(\partial_{y_1},\dots,\partial_{y_\ell})F_\zeta = 0$ implies that $f \in \mathrm{Ann}_Q(F_\zeta)$. Thus we have the surjection $g_\zeta : ST_\zeta \to STG_\zeta$.

Now let us show that g_ζ is injective too for generic ζ. Let K_ζ be the kernel of g_ζ. By the assumption, g_η is injective, equivalently, $K_\eta = (0)$. Thus so is $K_\zeta = (0)$ at the neighborhood V of η.

(2) When $\mathcal{A}$ is essential, it holds that $ST(\mathcal{A},\eta)_1 = S_1$. If there exists a strong Lefschetz element α for $ST(\mathcal{A},\eta)$, then we can define a global morphism

$$\cdot\alpha : ST(\mathcal{A},\zeta) \to ST(\mathcal{A},\zeta)$$

for generic ζ. By the assumption, either the kernel or cokernel of the map

$$\cdot\alpha^{r-2k} : ST(\mathcal{A},\eta)_k \to ST(\mathcal{A},\eta)_{r-k}$$

is zero for $0 \le k \le r/2$ with $r := \max\{i \mid ST(\mathcal{A},\eta)_i \neq (0)\}$. Note that $r = \max\{i \mid ST(\mathcal{A},\zeta)_i \neq (0)\}$ for generic ζ by Theorem 1.8. Hence we have a desired open set V. $\square$

Remark 5.16. Right now we have no example of arrangements with a polynomial such that its Solomon–Terao algebra does not have the strong Lefschetz property. For example, the arrangement in Example 3.4 is not Gorenstein, but we can check that it has the strong Lefschetz property with the strong Lefschetz element $x + y + z$.

Also, we have the following simple but important problem.

Problem 5.17. Consider $ST(\mathcal{A}, \eta)$ for non-tame $\mathcal{A}$. We may define it without the assumption of the tameness, but as far as we know, it seems difficult to say properties of $ST(\mathcal{A}, \eta)$ in that case.

References

[1] T. Abe, Divisionally free arrangements of hyperplanes, Invent. Math., **204** (2016), 317–346.

[2] T. Abe, M. Barakat, M. Cuntz, T. Hoge and H. Terao, The freeness of ideal subarrangements of Weyl arrangements, J. European Math. Soc., **18** (2016), 1339–1348.

[3] T. Abe, T. Horiguchi, M. Masuda, S. Murai and T. Sato, Hessenberg varieties and hyperplane arrangements, J. Reine Angew. Math., to appear. DOI:https://doi.org/10.1515/crelle-2018-0039.

[4] T. Abe and L. Kühne, Heavy hyperplane in multiarrangements and their freeness, J. Algebraic Combin., **48** (2018), 581–606.

[5] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogénes de groupes de Lie compacts, Ann. of Math. (2), **57** (1953), 115–207.

[6] C. Chevalley, Invariants of finite groups generated by reflections, Amer. J. Math., **77** (1955), 778–782.

[7] F. De Mari, C. Procesi and M. A. Shayman, Hessenberg varieties, Trans. Amer. Math. Soc., **332** (1992), 529–534.

[8] R. Epure and M. Schulze, A Saito criterion for holonomic divisors, Manuscr. Math., to appear. arXiv:1711:10259.

[9] M. Harada and J. Tymoczko, Poset pinball, GKM-compatible subspaces, and Hessenberg varieties, J. Math. Soc. Japan, **69** (2017), 945–994.

[10] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi and J. Watanabe, The Lefschetz Properties, Lecture Notes in Math, **2080**, Springer, Heidelberg, 2013.

[11] P. Orlik and L. Solomon, Combinatorics and topology of complements of hyperplane arrangements, Invent. Math., **56** (1980), 167–189.

[12] P. Orlik and H. Terao, Arrangements of hyperplanes, Grundlehren der Mathematischen Wissenschaften, **300**, Springer-Verlag, Berlin, 1992.

[13] P. Orlik and H. Terao, Arrangements and Milnor fibers, Math. Ann., **301** (1995), 211–235.

[14] K. Saito, On the uniformization of complements of discriminant loci, AMS Summer Institute, Williams college, 1975, RIMS Kōkyūroku, **287** (1977), 117–137.

[15] W. Slofstra, Rationally smooth Schubert varieties and inversion hyperplane arrangements, Adv. Math., **285** (2015), 709–736.

[16] L. Smith, Polynomial invariants of finite groups, Res. Notes Math., A K Peters, Wellesley, MA, 1995.

[17] L. Solomon and H. Terao, A formula for the characteristic polynomial of an arrangement, Adv. Math., **64** (1987), 305–325.

[18] E. Sommers and J. Tymoczko, Exponents of B-stable ideals, Trans. Amer. Math. Soc., **358** (2006), 3493–3509.

[19] H. Terao, Arrangements of hyperplanes and their freeness I, II, J. Fac. Sci. Univ. Tokyo, **27** (1980), 293–320.

[20] H. Terao, Generalized exponents of a free arrangement of hyperplanes and Shepherd–Todd–Brieskorn formula, Invent. Math., **63** (1981), 159–179.

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