J. Math. Soc. Japan
Vol. 71, No. 3 (2019) pp. 831–860
doi: 10.2969/jmsj/79877987

Discriminants of classical quasi-orthogonal polynomials with application to Diophantine equations

By Masanori SAWA and Yukihiro UCHIDA

(Received Feb. 14, 2018)

Abstract. We derive explicit formulas for the discriminants of classical quasi-orthogonal polynomials, as a full generalization of the result of Dilcher and Stolarsky (2005). We consider a certain system of Diophantine equations, originally designed by Hausdorff (1909) as a simplification of Hilbert's solution (1909) of Waring's problem, and then create the relationship to quadrature formulas and quasi-Hermite polynomials. We reduce these equations to the existence problem of rational points on a hyperelliptic curve associated with discriminants of quasi-Hermite polynomials, and show a nonexistence theorem for solutions of Hausdorff-type equations by applying our discriminant formula.

1. Introduction.

The Jacobi, Laguerre, and Hermite polynomials are the classical orthogonal polynomials, which, as we see in Szegő's book *Orthogonal Polynomials*, provide a great deal of interesting topics in broad areas of mathematics. In this paper we are particularly concerned with a compact elegant formula for the discriminant.

Stieltjes [**27**], [**28**] and Hilbert [**12**] computed the discriminants of all classical orthogonal polynomials. An order-one quasi-orthogonal polynomial is a polynomial of a sum of two orthogonal polynomials of consecutive degrees [**32**]; some authors use the same term 'order' in a bit different meaning [**26**]. Dilcher and Stolarsky [**7**, Theorem 4] derived a compact elegant formula for the discriminant of a quasi-Chebyshev polynomial of the second kind. Their proof is based on algebraic properties of resultants and the Euclidean algorithm for polynomials.

In this paper we derive explicit formulas for the discriminants of classical quasi-orthogonal polynomials, as a full generalization of the result of Dilcher and Stolarsky. Our proof uses Schur's method based on the three-term relations of polynomials [**23**] (see [**31**, Section 6.71]). We create a surprizing connection with Hausdorff's work on Waring's problem. For this purpose of exploring this connection, we also consider a certain system of Diophantine equations, originally designed by Hausdorff [**11**] as a simplification of Hilbert's solution [**13**] of Waring's problem. Interest was revived by Nestarenko [**16**, p.4700], who modified Hausdorff's arguments to simplify Hilbert's solution. The problem of finding a solution of such Hausdorff-type equations was posed

2010 *Mathematics Subject Classification.* Primary 33C45, 05E99, 65D32; Secondary 12E10, 11E76.
Key Words and Phrases. classical quasi-orthogonal polynomial, compact formula, discriminant, Gaussian design, Hausdorff-type equation, quadrature formula.
The first author was supported in part by Grant-in-Aid for Young Scientists (B) 26870259 and Grant-in-Aid for Scientific Research (B) 15H03636 by the Japan Society for the Promotion of Science (JSPS). The second author was also supported by Grant-in-Aid for Young Scientists (B) 25800023 by JSPS.

again in [**21**], where an example of nonexistence of solutions was reported. We elucidate the advantages of examining these equations through the discriminants of quasi-Hermite polynomials. Remarkably, a solution of such Diophantine equations not only establishes a constructive proof of Hilbert's solution but also provides constructions of various objects such as Gaussian designs in algebraic combinatorics [**2**] and a certain class of polynomials identities called Hilbert identities [**17**]; for the details, see [**21**, Section 3].

The paper is organized as follows. Section 2 gives preliminaries, where we review some basic results on discriminants, quasi-orthogonal polynomials, and quadrature formulas. Sections 3 and 4 are the main body of this paper. In Subsection 3.1 we prove explicit formulas for the discriminants of quasi-Jacobi polynomials. In Subsection 3.2, as a limit case of quasi-Jacobi polynomials, we derive explicit formulas for the discriminants of quasi-Laguerre and quasi-Hermite polynomials. In Subsection 4.1, we introduce Hausdorff-type equations and then show the relationship to quasi-Hermite polynomials and quadrature formulas for Gaussian integration. In Subsection 4.2, as a generalization of the above-mentioned report in [**21**, p.32], we show a nonexistence theorem for solutions of such Hausdorff-type equations. To do this, we reduce the problem to the existence of $\mathbb{Q}_2$-rational points on a hyperelliptic curve associated with the discriminants of quasi-Hermite polynomials. By applying our discriminant formula, we then show a necessary and sufficient condition for the existence of $\mathbb{Q}_2$-rational points. Section 5 is the conclusion, where further remarks will be made.

2. Preliminary.

In this section we introduce various notions such as discriminants, quasi-orthogonal polynomials, quadrature formulas and so on. We also review some basic properties and prove lemmas for further arguments in Sections 3 and 4.

2.1. Discriminants and Schur's method.

Let

$$f(x) = a_0 x^n + \cdots + a_n$$

be a polynomial of degree n. Let $\alpha_1, \ldots, \alpha_n$ be the zeros of $f(x)$. The *discriminant* of $f(x)$ is defined by

$$\operatorname{disc}(f) = a_0^{2n-2} \prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)^2. \tag{2.1}$$

Let

$$f(x) = a_0 x^n + \cdots + a_n, \quad g(x) = b_0 x^m + \cdots + b_m$$

be polynomials of degree n and m respectively. The *resultant* of f and g is defined by

$$\operatorname{Res}(f,g) = \begin{vmatrix} a_0 & a_1 & \dots & a_n & & & \\ & \ddots & & & \ddots & & \\ & & a_0 & a_1 & \dots & a_n \\ b_0 & b_1 & \dots & b_m & & & \\ & \ddots & & & \ddots & & \\ & & b_0 & b_1 & \dots & b_m \end{vmatrix}, \tag{2.2}$$

where the determinant is of order $(m+n)$. The discriminant of f is represented in terms of a resultant as follows.

$$\operatorname{disc}(f) = \frac{(-1)^{n(n-1)/2}}{a_0} \operatorname{Res}(f, f'). \tag{2.3}$$

REMARK 2.1. The sign in the right-hand side differs in some literature. For example, the sign $(-1)^{n(n-1)/2}$ does not appear in [**8**] because the definition of discriminants differs by sign.

The following proposition follows from (2.2) and (2.3).

PROPOSITION 2.2. *Let $f(x) = a_0x^n + \cdots + a_n$. Then $\operatorname{disc}(f)$ is a homogeneous polynomial in $a_0, \dots, a_n$ of degree $2n-2$ with integer coefficients.*

By Proposition 2.2, we may substitute a polynomial of degree less than n for f in $\operatorname{disc}(f)$. If necessary, we use the notation $\operatorname{disc}_n(f)$ to emphasize the dependence on n.

PROPOSITION 2.3. *Let $f(x) = a_0x^n + \cdots + a_n$. Then we have*

$$\operatorname{disc}_{n+1}(f) = a_0^2 \operatorname{disc}_n(f).$$

PROOF. See [**8**, Chapter 12, (1.41)]. Note that the definition of discriminants in [**8**] differs by sign from ours. □

PROPOSITION 2.4. *Let $f(x) = a_0x^n + \cdots + a_n$ and let a, b, c be constants. Then we have*

$$\operatorname{disc}(f(ax+b)) = a^{n(n-1)} \operatorname{disc}(f(x)),$$
$$\operatorname{disc}(cf(x)) = c^{2(n-1)} \operatorname{disc}(f(x)).$$

PROOF. The proposition follows from (2.1). See also [**7**, Lemma 4.3]. □

PROPOSITION 2.5. *Let $p(x)$ and $q(x)$ be polynomials of degree n and $n-1$ respectively. Let c be a constant.*

(i) *The discriminant $\operatorname{disc}(p+cq)$ is a polynomial in c and*

$$\deg \operatorname{disc}(p+cq) \leq 2(n-1).$$

The equality holds if and only if q has no multiple zeros.

(ii) *If* $p(-x) = (-1)^n p(x)$ *and* $q(-x) = (-1)^{n-1} q(x)$, *then* $\operatorname{disc}(p + cq)$ *is an even polynomial in* c.

Proof. (i) By Proposition 2.2, $\operatorname{disc}(p + cq)$ is a polynomial in c. By Proposition 2.4, we have

$$\frac{\operatorname{disc}(p + cq)}{c^{2(n-1)}} = \operatorname{disc}_n\left(\frac{1}{c}p + q\right).$$

By Proposition 2.3, we have

$$\lim_{c\to\infty} \frac{\operatorname{disc}(p + cq)}{c^{2(n-1)}} = \operatorname{disc}_n(q) = l^2 \operatorname{disc}_{n-1}(q),$$

where l is the leading coefficient of q. This completes the proof.
(ii) By assumption,

$$p(-x) + cq(-x) = (-1)^n (p(x) - cq(x)).$$

Therefore, by Proposition 2.4,

$$\begin{aligned}
\operatorname{disc}(p - cq) &= \operatorname{disc}((-1)^n (p(-x) + cq(-x))) \\
&= (-1)^{n\cdot 2(n-1)} (-1)^{n(n-1)} \operatorname{disc}(p(x) + cq(x)) \\
&= \operatorname{disc}(p + cq).
\end{aligned}$$

□

The following lemma, due to Schur [**23**] (see [**31**, Section 6.71]), plays a role in the proof of the main theorems of Section 3.

Lemma 2.6 (Schur's method). *Let* $\{\rho_m\}$ *be a sequence of polynomials satisfying*

$$\begin{aligned}
&\rho_m(x) = (a_m x + b_m)\rho_{m-1}(x) - c_m \rho_{m-2}(x), \\
&\rho_0(x) = 1, \quad \rho_1(x) = a_1 x + b_1,
\end{aligned} \tag{2.4}$$

where a_m, b_m, c_m *are constants with* $a_m c_m \neq 0$. *Let* $y_1, \ldots, y_n$ *be the zeros of* $\rho_n(x)$. *Then we have*

$$\prod_{k=1}^{n} \rho_{n-1}(y_k) = (-1)^{n(n-1)/2} \prod_{k=1}^{n} a_k^{n-2k+1} c_k^{k-1}.$$

2.2. Quasi-orthogonal polynomials and Riesz–Shohat Theorem.

Let μ be a positive Borel measure on an interval (a, b) with finite moments. For convenience, we assume that $\int_a^b d\mu = 1$. Let $\{\Phi_n(x)\}$ be a sequence of *orthogonal polynomials* with respect to μ, namely

$$\int_a^b \Phi_m(x)\Phi_n(x) d\mu = 0 \quad \text{for every } m \neq n.$$

Bochner [**3**] completely classified all polynomials which are solutions of a second-order Sturm–Liouville type differential equation. Among such polynomial solutions, the only orthogonal polynomials with respect to a positive definite linear functional are *Jacobi polynomials*, *Laguerre polynomials* and *Hermite polynomials*; for example see [**15**]. These polynomials are often called *classical orthogonal polynomials*; without assuming positive definiteness, some authors also consider Bessel polynomials as a class of classical polynomials.

Jacobi polynomial. For $\alpha, \beta > -1$, the n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is defined by the Rodrigues' formula as follows:

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right). \tag{2.5}$$

The polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal with respect to $(1-x)^\alpha(1+x)^\beta$ on $(-1,1)$.

Laguerre polynomial. For $\alpha > -1$, the n-th Laguerre polynomial $L_n^{(\alpha)}(x)$ is defined by the Rodrigues' formula as follows:

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}). \tag{2.6}$$

The polynomials $L_n^{(\alpha)}(x)$ are orthogonal with respect to $e^{-x}x^\alpha$ on $(0,\infty)$.

Hermite polynomial. The n-th Hermite polynomial $H_n(x)$ is defined by the Rodrigues' formula as follows:

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}. \tag{2.7}$$

The polynomials $H_n(x)$ are orthogonal with respect to e^{-x^2} on $\mathbb{R}$.

Some of the basic properties on classical orthogonal polynomials, used in Sections 3 and 4, are summarized in Appendix A. For the general theory, we refer the readers to Szegő's book *Orthogonal Polynomials* [**31**, Chapter IV and Section 5.1 and Section 5.5].

A *quasi-orthogonal polynomial of degree n and order r* is a polynomial of the form

$$\Phi_{n,r}(x) = \Phi_n(x) + b_1\Phi_{n-1}(x) + \cdots + b_r\Phi_{n-r}(x)$$

in which $b_1, \ldots, b_r \in \mathbb{R}$ and in particular $b_r \neq 0$ [**32**]. Some authors use the term 'order' in a bit different meaning; for example, see [**26**].

For convenience, we set $\Phi_{n,0}(x) = \Phi_n(x)$. The polynomial $\Phi_{n,r}(x)$ is orthogonal to all polynomials of degree at most $n-r-1$.

We now look at two important facts which will be used many times throughout this paper.

Proposition 2.7. *Let b_1 be a real constant. Then the polynomial $\Phi_{n+1,1}(x) = \Phi_{n+1}(x) + b_1\Phi_n(x)$ has $n+1$ distinct real roots.*

PROOF. See [**31**, Theorem 3.3.4]. □

The following result was first obtained by Riesz [**20**, p.23] for $k = 2$, and generalized by Shohat [**25**, Theorem I] for $k \geq 3$.

THEOREM 2.8 (Riesz–Shohat Theorem). *Let $c_1, \dots, c_n$ be distinct real numbers, $\omega_n(x) = \prod_{i=1}^{n}(x - c_i)$, and*

$$\gamma_i = \int_a^b \frac{\omega_n(x)}{(x - c_i)\omega_n'(c_i)} d\mu.$$

Let k be an integer with $1 \leq k \leq n + 1$. The following are equivalent.

(i) *The equation*

$$\sum_{i=1}^{n} \gamma_i f(c_i) = \int_a^b f(x) d\mu, \tag{2.8}$$

holds for all polynomials $f(x)$ of degree at most $2n - k$.

(ii) *For all polynomials $g(x)$ of degree at most $n - k$,*

$$\int_a^b \omega_n(x) g(x) d\mu = 0.$$

(iii) *The polynomial $\omega_n(x)$ is a quasi-orthogonal polynomial of degree n and order $k-1$, that is, there exists real numbers $b_1, \dots, b_{k-1}$ such that*

$$\omega_n(x) = \Phi_n(x) + b_1 \Phi_{n-1}(x) + \cdots + b_{k-1} \Phi_{n-k+1}(x).$$

Integration formulas of type (2.8) are called *quadrature formulas*. Quadrature formulas with positive weights γ_i are important as integration formula, which, by a theorem of Xu [**32**, Theorem 4.1], have an elegant characterization in terms of tri-diagonal matrices. A class of positive quadrature formulas was also implicitly used in Hausdorff's work [**13**] on Waring's problem; the details will be clear in the next subsection.

2.3. Hilbert identities, quadrature formulas, and Hausdorff's theorem.

A *real Hilbert identity* is a polynomial identity of the form

$$(x_1^2 + \cdots + x_n^2)^r = \sum_{i=1}^{M} c_i (a_{i1} x_1 + \cdots + a_{in} x_n)^{2r} \tag{2.9}$$

where $0 < c_i$ and $a_{ij} \in \mathbb{R}$. Clearly, it is always possible to absorb the coefficients c_i's into the linear forms. A *rational Hilbert identity* is an identity of type (2.9) in which $0 < c_i \in \mathbb{Q}$ and $a_{ij} \in \mathbb{Q}$. In this case scaling is no longer simple.

Waring's problem in number theory asks whether every positive integer can be expressed as a sum of r-th powers of integers. The case $r = 2$ had been stated by Fermat in 1640 and was solved by Lagrange in 1770. The first advance for $r \geq 3$ was made by

Liouville in 1859, who proved that every natural integer is a sum of at most 53 fourth powers of integers. In doing so, Liouville used the rational identity

$$6(x_1^2+x_2^2+x_3^2+x_4^2)^2 = \sum_{1\le i<j\le 4} \left\{(x_i+x_j)^4+(x_i-x_j)^4\right\}.$$

Mathematicians in the rest of the 19th century gave similar identities and settled Waring's problem in the small-degree cases. For a good introduction to the early histories on Waring's problem, we refer the readers to Dickson's book *History of the Theory of Numbers, II* [**6**, pp.717–725].

It was Hilbert [**13**] who finally solved Waring's problem in general; namely, for every positive integer r, there exists some positive integer $g(r)$ so that for each $n \in \mathbb{N}$ there exist $x_k \in \mathbb{Z}$ so that

$$n = \sum_{k=1}^{g(r)} x_k^r.$$

We are concerned here only with the first part of Hilbert's proof, which involved the construction of rational Hilbert identities.

The first key step of Hilbert's proof is Theorem 2.9 below, which was stated for $n = 5$; it is obvious that Hilbert's argument applies to general values of n.

Theorem 2.9 (Hilbert's lemma). *It holds that for every positive integers n and r,*

$$(x_1^2+\cdots+x_n^2)^r = \sum_{i=1}^{M} c_i(a_{i1}x_1+\cdots+a_{in}x_n)^{2r}$$

in which $M = (2r+n-1)\cdots(2r+1)/(n-1)!$, $0 < c_i \in \mathbb{Q}$, and $a_{ij} \in \mathbb{Q}$.

Hilbert found his identities in two steps. First, he showed that if $d\mu$ is a suitably-normalized surface measure on S^{n-1} and x_i's are taken parameters, then

$$\int\cdots\int_{u\in S^{n-1}} (x_1u_1+\cdots+x_nu_n)^{2r}d\mu = (x_1^2+\cdots+x_n^2)^r. \tag{2.10}$$

By approximating the integral with a Riemann sum and then using some elementary arguments, he derived the existence of real Hilbert identities. Then by a standard continuity argument, Hilbert found his rational identities. There have been some expository works which, while mainly concerned with Waring's problem, described Hilbert's theorem; for example, see Pollack [**18**].

The first simplification of Hilbert's result was made by Hausdorff [**11**], who replaced the integral on the left of (2.10) by the Gaussian integral

$$\int\cdots\int_{u\in\mathbb{R}^n} e^{-(u_1^2+\cdots+u_n^2)}(x_1u_1+\cdots+x_nu_n)^{2r}\ du_1\cdots du_n,$$

and showed that, up to a constant, the value is $(\sum x_i^2)^r$ again. Then he constructed an iterated sum which leads to explicit real Hilbert identities in any number of variables, by using the roots of the Hermite polynomial H_{2r} and then showing the following key lemma:

LEMMA 2.10 (Hausdorff's lemma). *There exist rationals $x_1, \ldots, x_{2r+1}, y_1, \ldots, y_{2r+1}$ such that*

$$\sum_{i=1}^{2r+1} x_i y_i^j = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^j \ e^{-t^2} dt, \qquad j = 0, 1, \ldots, 2r. \tag{2.11}$$

Hausdorff then quickly argued that the real coefficients may be replaced by rational ones. Interest was revived by Nestarenko [**16**, p.4700], who modified Hausdorff's arguments to simplify Hilbert's result. The problem of finding a solution of such equations was also stated in [**21**, p.32].

Diophantine equations of type (2.11) are significant in the theory of quadrature formulas. Let ξ be a positive Borel measure on an interval (a, b). Let $x_1, \ldots, x_m \in \mathbb{R}$ and $y_1, \ldots, y_m \in (a, b)$. A *quadrature formula of degree t* is an integration formula of the form

$$\sum_{i=1}^{m} x_i f(y_i) = \int_a^b f(x) \ d\xi \tag{2.12}$$

in which f ranges over all polynomials of degree at most t. The points y_i's are called *nodes* and x_i's are called *weights*. A quadrature formula is *positive* if all weights are positive. This is also called a *Gaussian t-design* in algebraic combinatorics [**2**]. We see that the equations (2.11) are equivalent to a *rational Gaussian design* or a *rational quadrature*, meaning a quadrature formula of degree $2r$ for Gaussian integration $(1/\sqrt{\pi}) \int_{-\infty}^{\infty} e^{-t^2} dt$ with rational nodes and weights. In Subsection 4.1, we formulate Diophantine equations of type (2.11) in a more general setting.

The concept of quadrature formula is simply generalized to higher dimensions and integrands may be also replaced by the homogeneous polynomials. A *cubature formula of index t* is an integration formula of type (2.12) in which f ranges over all homogeneous polynomials of degree t. The relationship of Hilbert identities to index-type cubature formulas for $\int_{S^{n-1}} d\rho$, where ρ is a surface measure on S^{n-1}, goes back to the 19th century at least [**19**]. Interest was revived in the development of spherical designs by Delsarte, Goethals and Seidel in the 1970s [**5**]. By a suitable scaling of weights and nodes, cubature formulas for $\int_{S^{n-1}} d\rho$ and $\int \cdots \int_{\mathbb{R}^n} e^{-(u_1^2+\cdots+u_n^2)} du_1 \cdots du_n$ can be transformed each other (cf. [**2**], [**17**]). We can easily construct a cubature formula for Gaussian integration by taking copies of a quadrature formula for $(1/\sqrt{\pi}) \int_{-\infty}^{\infty} e^{-t^2} dt$ and then taking their convolutions. This is an example of the widely-used method in the study of cubature formulas, called *product construction* [**29**], and explains why Hausdorff's simplification works well.

3. Compact formulas for discriminants of classical quasi-orthogonal polynomials.

In this section we derive explicit formulas for the discriminants of quasi-Jacobi, quasi-Laguerre, quasi-Hermite polynomials by using Schur's method (Lemma 2.6).

3.1. Quasi-Jacobi polynomials.

The discriminants of quasi-Jacobi polynomials are computed as follows.

THEOREM 3.1. *Let c be a constant and let $P_{n;c}^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x) + cP_{n-1}^{(\alpha,\beta)}(x)$. Then*

$$\begin{aligned}\operatorname{disc}(P_{n;c}^{(\alpha,\beta)}) &= \frac{(2n+\alpha+\beta)^{2n-1}}{2^{n(n-1)}}\prod_{k=1}^{n} k^{k-2n+3}\\ &\cdot \prod_{k=1}^{n-1}(k+\alpha)^{k-1}(k+\beta)^{k-1}(n+k+\alpha+\beta)^{n-k-1}\\ &\cdot \frac{(-c)^n P_{n;c}^{(\alpha,\beta)}\big(-(2n(n+\alpha+\beta)c^2+(\alpha^2-\beta^2)c+2(n+\alpha)(n+\beta))/(2n+\alpha+\beta)^2c\big)}{(n+\alpha+cn)(n+\beta-cn)}.\end{aligned} \tag{3.1}$$

Furthermore, $\operatorname{disc}(P_{n;c}^{(\alpha,\beta)})$ *is a polynomial in c of degree* $2(n-1)$*.*

REMARK 3.2. Taking the limit as $c \to 0$, we have

$$\operatorname{disc}(P_n^{(\alpha,\beta)}) = 2^{-n(n-1)}\prod_{k=1}^{n} k^{k-2n+2}(k+\alpha)^{k-1}(k+\beta)^{k-1}(n+k+\alpha+\beta)^{n-k}.$$

This formula coincides with Stieltjes's formula [**31**, (6.71.5)].

PROOF OF THEOREM 3.1. Let $y_1,\dots,y_n$ be the zeros of $P_{n;c}^{(\alpha,\beta)}(x)$ and $l_n^{(\alpha,\beta)}$ be the leading coefficient of $P_{n;c}^{(\alpha,\beta)}(x)$. Then we have

$$\begin{aligned}\operatorname{disc}(P_{n;c}^{(\alpha,\beta)}) &= (l_n^{(\alpha,\beta)})^{2n-2}\prod_{1\le i<j\le n}(y_i-y_j)^2\\ &= (-1)^{n(n-1)/2}(l_n^{(\alpha,\beta)})^{n-2}\prod_{k=1}^{n}\frac{d}{dx}P_{n;c}^{(\alpha,\beta)}(y_k).\end{aligned}$$

By (A.3) and (A.4),

$$\begin{aligned}&\frac{d}{dx}P_{n;c}^{(\alpha,\beta)}(x)\\ &\quad= \frac{d}{dx}P_n^{(\alpha,\beta)}(x) + c\frac{d}{dx}P_{n-1}^{(\alpha,\beta)}(x)\\ &\quad= \big((2n+\alpha+\beta)(1-x^2)\big)^{-1}\\ &\qquad\cdot\Big(-n\big((2n+\alpha+\beta)x+\beta-\alpha+2c(n+\alpha+\beta)\big)P_n^{(\alpha,\beta)}(x)\end{aligned}$$

$$+\Big(c(n+\alpha+\beta)\left((2n+\alpha+\beta)x+\alpha-\beta\right)+2(n+\alpha)(n+\beta)\Big)P_{n-1}^{(\alpha,\beta)}(x)\Big).$$

Since $P_{n;c}^{(\alpha,\beta)}(y_k)=P_n^{(\alpha,\beta)}(y_k)+cP_{n-1}^{(\alpha,\beta)}(y_k)=0$, we have

$$\begin{aligned}\frac{d}{dx}P_{n;c}^{(\alpha,\beta)}(y_k)=&\left((2n+\alpha+\beta)(1-y_k^2)\right)^{-1}\\&\cdot\Big(2n(n+\alpha+\beta)c^2+((2n+\alpha+\beta)^2y_k+\alpha^2-\beta^2)c\\&+2(n+\alpha)(n+\beta)\Big)P_{n-1}^{(\alpha,\beta)}(y_k).\end{aligned}$$

Let

$$\xi_{n;c}^{(\alpha,\beta)}=-\frac{2n(n+\alpha+\beta)c^2+(\alpha^2-\beta^2)c+2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2c}.$$

Then we have

$$\begin{aligned}\prod_{k=1}^n\frac{d}{dx}P_{n;c}^{(\alpha,\beta)}(y_k)&=\prod_{k=1}^n\frac{(2n+\alpha+\beta)c}{y_k^2-1}(\xi_{n;c}^{(\alpha,\beta)}-y_k)P_{n-1}^{(\alpha,\beta)}(y_k)\\&=\frac{l_n^{(\alpha,\beta)}(2n+\alpha+\beta)^nc^n}{P_{n;c}^{(\alpha,\beta)}(1)P_{n;c}^{(\alpha,\beta)}(-1)}P_{n;c}^{(\alpha,\beta)}(\xi_{n;c}^{(\alpha,\beta)})\prod_{k=1}^nP_{n-1}^{(\alpha,\beta)}(y_k).\end{aligned}$$

For $k=1,2,\dots,n$, let

$$a_k=\frac{(2k+\alpha+\beta-1)(2k+\alpha+\beta)}{2k(k+\alpha+\beta)},\quad c_k=\frac{(k+\alpha-1)(k+\beta-1)(2k+\alpha+\beta)}{k(k+\alpha+\beta)(2k+\alpha+\beta-2)}.$$

Let $\rho_n(x)=P_{n;c}^{(\alpha,\beta)}(x)$ and $\rho_k(x)=P_k^{(\alpha,\beta)}(x)$ for $k=0,1,\dots,n-1$. By (A.2), there exist $b_1,b_2,\dots,b_n$ such that $\rho_0(x),\rho_1(x),\dots,\rho_n(x)$ satisfy (2.4). By Lemma 2.6, we have

$$\begin{aligned}\prod_{k=1}^nP_{n-1}^{(\alpha,\beta)}(y_k)&=(-1)^{n(n-1)/2}\prod_{k=1}^na_k^{n-2k+1}c_k^{k-1}\\&=(-1)^{n(n-1)/2}\prod_{k=1}^nk^{k-n}(k+\alpha+\beta)^{k-n}(2k+\alpha+\beta-1)^{n-2k+1}\\&\quad\cdot\prod_{k=1}^{n-1}(k+\alpha)^k(k+\beta)^k(2k+\alpha+\beta)^{n-2k}\\&=(-1)^{n(n-1)/2}\prod_{k=1}^{n-1}k^{k-n}(k+\alpha)^k(k+\beta)^k(n+k+\alpha+\beta)^{-k}.\end{aligned}$$

The leading coefficient $l_n^{(\alpha,\beta)}$ is computed by (A.2) as follows.

$$l_n^{(\alpha,\beta)}=\frac{1}{2^n}\binom{2n+\alpha+\beta}{n}.$$

By (A.1), we have

$$P_{n;c}^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} + c\binom{n-1+\alpha}{n-1},$$
$$P_{n;c}^{(\alpha,\beta)}(-1) = (-1)^n\left(\binom{n+\beta}{n} - c\binom{n-1+\beta}{n-1}\right).$$

Therefore

$$\begin{aligned}\operatorname{disc}(P_{n;c}^{(\alpha,\beta)}) &= \frac{(l_n^{(\alpha,\beta)})^{n-1}(2n+\alpha+\beta)^n(-c)^n}{\left(\binom{n+\alpha}{n}+c\binom{n-1+\alpha}{n-1}\right)\left(\binom{n+\beta}{n}-c\binom{n-1+\beta}{n-1}\right)}P_{n;c}^{(\alpha,\beta)}(\xi_{n;c}^{(\alpha,\beta)})\\ &\quad\cdot\prod_{k=1}^{n-1}k^{k-n}(k+\alpha)^k(k+\beta)^k(n+k+\alpha+\beta)^{-k}\\ &= \frac{(2n+\alpha+\beta)^{2n-1}}{2^{n(n-1)}}\prod_{k=1}^{n}k^{k-2n+3}\\ &\quad\cdot\prod_{k=1}^{n-1}(k+\alpha)^{k-1}(k+\beta)^{k-1}(n+k+\alpha+\beta)^{n-k-1}\\ &\quad\cdot\frac{(-c)^n}{(n+\alpha+cn)(n+\beta-cn)}P_{n;c}^{(\alpha,\beta)}(\xi_{n;c}^{(\alpha,\beta)}).\end{aligned}$$

The latter part of the theorem follows from Propositions 2.5 and 2.7. □

We now describe some specializations of Theorem 3.6.
For $\lambda \in \mathbb{R}$ and $0 < n \in \mathbb{Z}$, we define

$$(\lambda)_0 = 1, \quad (\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1).$$

The n-th *Gegenbauer polynomial* is defined by

$$C_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{(\lambda+1/2)_n}P_n^{(\lambda-1/2,\lambda-1/2)}(x).$$

These polynomials often appear in the study of spherical designs (cf. [**5**]).

COROLLARY 3.3. *Let c be a constant and let $C_{n;c}^{(\lambda)}(x) = C_n^{(\lambda)}(x) + cC_{n-1}^{(\lambda)}(x)$. Then*

$$\begin{aligned}\operatorname{disc}(C_{n;c}^{(\lambda)}) &= 2^{n(n-1)}(2n+2\lambda-1)^n\prod_{k=1}^{n}k^{k-2n+3}(k+\lambda-1)^{2n-2k}\\ &\cdot\prod_{k=1}^{n-1}(k+2\lambda-1)^{k-2}\cdot\frac{(-c)^nC_{n;c}^{(\lambda)}\left(-(nc^2+n+2\lambda-1)/(2n+2\lambda-1)c\right)}{(n+2\lambda-1)^2-(cn)^2}.\end{aligned} \tag{3.2}$$

Furthermore, $\operatorname{disc}(C_{n;c}^{(\lambda)})$ *is an even polynomial in c of degree $2(n-1)$.*

PROOF. By definition,

$$C_{n;c}^{(\lambda)}(x) = \frac{(2\lambda)_n}{(\lambda+1/2)_n} P_n^{(\lambda-1/2,\lambda-1/2)}(x) + c\frac{(2\lambda)_{n-1}}{(\lambda+1/2)_{n-1}} P_{n-1}^{(\lambda-1/2,\lambda-1/2)}(x)$$
$$= \frac{(2\lambda)_n}{(\lambda+1/2)_n} P_{n;c'}^{(\lambda-1/2,\lambda-1/2)}(x),$$

where $c' = c(\lambda + n - (1/2))/(2\lambda + n - 1)$. By Proposition 2.4 and Theorem 3.6,

$$\operatorname{disc}(C_{n;c}^{(\lambda)}) = \left(\frac{(2\lambda)_n}{(\lambda+1/2)_n}\right)^{2(n-1)} \frac{(2n+2\lambda-1)^{2n-1}}{2^{n(n-1)}} \prod_{k=1}^{n} k^{k-2n+3}$$
$$\cdot \prod_{k=1}^{n-1} \left(k+\lambda-\frac{1}{2}\right)^{2k-2} (n+k+2\lambda-1)^{n-k-1}$$
$$\cdot \frac{(-c')^n P_{n;c'}^{(\lambda-1/2,\lambda-1/2)}\left(-(2n(n+2\lambda-1)(c')^2 + 2(n+\lambda-1/2)^2)/(2n+2\lambda-1)^2 c'\right)}{(n+\lambda-(1/2)+c'n)(n+\lambda-(1/2)-c'n)}$$
$$= \left(\frac{(2\lambda)_n}{(\lambda+1/2)_n}\right)^{2n-3} \frac{(2n+2\lambda-1)^{2n-1}}{2^{n(n-1)}} \prod_{k=1}^{n} k^{k-2n+3}$$
$$\cdot \prod_{k=1}^{n-1} \left(k+\lambda-\frac{1}{2}\right)^{2k-2} (n+k+2\lambda-1)^{n-k-1}$$
$$\cdot \frac{(n+\lambda-1/2)^{n-2}(-c)^n C_{n;c}^{(\lambda)}\left(-(nc^2+n+2\lambda-1)/(2n+2\lambda-1)c\right)}{(n+2\lambda-1)^{n-2}(n+2\lambda-1+cn)(n+2\lambda-1-cn)}$$
$$= \frac{(2n+2\lambda-1)^n}{2^{(n-1)^2}(n+2\lambda-1)^{n-2}} \prod_{k=1}^{n} k^{k-2n+3}(k+2\lambda-1)^{2n-3}$$
$$\cdot \prod_{k=1}^{n-1} \left(k+\lambda-\frac{1}{2}\right)^{2k-2n+1} (n+k+2\lambda-1)^{n-k-1}$$
$$\cdot \frac{(-c)^n C_{n;c}^{(\lambda)}\left(-(nc^2+n+2\lambda-1)/(2n+2\lambda-1)c\right)}{(n+2\lambda-1)^2-(cn)^2}.$$

The constant factor is computed as follows.

$$\frac{(2n+2\lambda-1)^n}{2^{(n-1)^2}(n+2\lambda-1)^{n-2}} \prod_{k=1}^{n} k^{k-2n+3}(k+2\lambda-1)^{2n-3}$$
$$\cdot \prod_{k=1}^{n-1} \left(k+\lambda-\frac{1}{2}\right)^{2k-2n+1} (n+k+2\lambda-1)^{n-k-1}$$
$$= \frac{(2n+2\lambda-1)^n}{(n+2\lambda-1)^{n-2}} \prod_{k=1}^{n} k^{k-2n+3}(k+2\lambda-1)^{2n-3}$$
$$\cdot \prod_{k=1}^{n-1} (2k+2\lambda-1)^{2k-2n+1} \cdot \prod_{k=n+1}^{2n-1} (k+2\lambda-1)^{2n-k-1}$$

$$= \frac{(2n+2\lambda-1)^n}{(n+2\lambda-1)^{n-2}} \prod_{k=1}^{n} k^{k-2n+3}(k+2\lambda-1)^{2n-3} \cdot \prod_{k=1}^{2n-1} (k+2\lambda-1)^{k-2n+1}$$

$$\cdot \prod_{k=1}^{n} (2k-1+2\lambda-1)^{2n-2k} \cdot \prod_{k=n+1}^{2n-1} (k+2\lambda-1)^{2n-k-1}$$

$$= 2^{n(n-1)}(2n+2\lambda-1)^n \prod_{k=1}^{n} k^{k-2n+3}(k+\lambda-1)^{2n-2k} \cdot \prod_{k=1}^{n-1} (k+2\lambda-1)^{k-2}.$$

The latter part of the corollary follows from Propositions 2.5 and 2.7. □

We describe another specialization of Theorem 3.6. The n-th *Chebyshev polynomial of the first kind* is defined by

$$T_n(x) = \frac{P_n^{(-1/2,-1/2)}(x)}{P_n^{(-1/2,-1/2)}(1)} = \binom{n-1/2}{n}^{-1} P_n^{(-1/2,-1/2)}(x).$$

When $n \geq 1$, we have

$$T_n(x) = \lim_{\lambda \to 0} \frac{n}{2\lambda} C_n^{(\lambda)}(x). \tag{3.3}$$

The n-th *Chebyshev polynomial of the second kind* is defined by

$$U_n(x) = (n+1)\frac{P_n^{(1/2,1/2)}(x)}{P_n^{(1/2,1/2)}(1)} = C_n^{(1)}(x).$$

Corollary 3.4 ([**7**]). *Let c be a constant and let $T_{n;c}(x) = T_n(x) + cT_{n-1}(x)$ and $U_{n;c}(x) = U_n(x) + cU_{n-1}(x)$. Then we have*

$$\mathrm{disc}(T_{n;c}) = \frac{2^{(n-1)(n-2)}(2n-1)^n(-c)^n}{1-c^2} T_{n;c}\left(-\frac{(n-1)c^2+n}{(2n-1)c}\right). \tag{3.4}$$

$$\mathrm{disc}(U_{n;c}) = \frac{2^{n(n-1)}(2n+1)^n(-c)^n}{(n+1)^2-(cn)^2} U_{n;c}\left(-\frac{nc^2+n+1}{(2n+1)c}\right). \tag{3.5}$$

Furthermore, $\mathrm{disc}(T_{n;c})$ *and* $\mathrm{disc}(U_{n;c})$ *are even polynomials in c of degree* $2(n-1)$.

Proof. We first consider $\mathrm{disc}(T_{n;c})$. When $n = 1$, it is easy to verify (3.4). We assume that $n \geq 2$. By (3.3),

$$T_{n;c}(x) = \lim_{\lambda\to 0} \frac{n}{2\lambda} C_n^{(\lambda)}(x) + c \lim_{\lambda \to 0} \frac{n-1}{2\lambda} C_{n-1}^{(\lambda)}(x) = \lim_{\lambda \to 0} \frac{n}{2\lambda} C_{n;c'}^{(\lambda)}(x),$$

where $c' = (n-1)c/n$. By Proposition 2.4 and Corollary 3.3,

$$\mathrm{disc}(T_{n;c}^{(\lambda)}) = \lim_{\lambda\to 0}\left(\frac{n}{2\lambda}\right)^{2n-2} 2^{n(n-1)}(2n+2\lambda-1)^n \prod_{k=1}^{n} k^{k-2n+3}(k+\lambda-1)^{2n-2k}$$

$$
\begin{aligned}
&\cdot \prod_{k=1}^{n-1}(k+2\lambda-1)^{k-2} \cdot \frac{(-c')^n C_{n;c'}^{(\lambda)}\bigl(-(n(c')^2+n+2\lambda-1)/(2n+2\lambda-1)c'\bigr)}{(n+2\lambda-1)^2-(c'n)^2} \\
&= \lim_{\lambda\to 0} 2^{(n-1)(n-2)}(n-1)^n n^{n-3}(2n+2\lambda-1)^n \\
&\cdot \prod_{k=2}^{n} k^{k-2n+3}(k+\lambda-1)^{2n-2k} \\
&\cdot \prod_{k=2}^{n-1}(k+2\lambda-1)^{k-2} \cdot \frac{(-c)^n (n/2\lambda) C_{n;c'}^{(\lambda)}\bigl(-(n(c')^2+n+2\lambda-1)/(2n+2\lambda-1)c'\bigr)}{(n+2\lambda-1)^2-c^2(n-1)^2} \\
&= 2^{(n-1)(n-2)}(n-1)^n n^{n-3}(2n-1)^n \prod_{k=2}^{n} k^{k-2n+3}(k-1)^{2n-2k} \\
&\cdot \prod_{k=2}^{n-1}(k-1)^{k-2} \cdot \frac{(-c)^n T_{n;c}\bigl(-((n-1)c^2+n)/(2n-1)c\bigr)}{(n-1)^2-c^2(n-1)^2} \\
&= \frac{2^{(n-1)(n-2)}(2n-1)^n(-c)^n}{1-c^2} T_{n;c}\left(-\frac{(n-1)c^2+n}{(2n-1)c}\right).
\end{aligned}
$$

Next, we consider $\operatorname{disc}(U_{n;c})$. Since $U_{n;c}(x) = C_{n;c}^{(1)}(x)$, by Corollary 3.3,

$$
\begin{aligned}
\operatorname{disc}(U_{n;c}) &= 2^{n(n-1)}(2n+1)^n \prod_{k=1}^{n} k^{3-k} \\
&\quad \cdot \prod_{k=1}^{n-1}(k+1)^{k-2} \cdot \frac{(-c)^n U_{n;c}\bigl(-(nc^2+n+1)/(2n+1)c\bigr)}{(n+1)^2-(cn)^2} \\
&= \frac{2^{n(n-1)}(2n+1)^n(-c)^n}{(n+1)^2-(cn)^2} U_{n;c}\left(-\frac{nc^2+n+1}{(2n+1)c}\right). \qquad \square
\end{aligned}
$$

REMARK 3.5. Dilcher and Stolarsky [**7**, Theorem 4] derived the compact formula (3.5) by using algebraic properties of resultants and the Euclidean algorithm. They also obtained similar results on the resultant of two quasi-Chebyshev polynomials of the second kind. Based on Schur's method, Gishe and Ismail [**9**] also found similar resultant formulas concerning quasi-Chebyshev polynomials.

3.2. Quasi-Laguerre and quasi-Hermite polynomials.

In this subsection, as a limit case of quasi-Jacobi polynomials, we derive explicit formulas for the discriminants of quasi-Laguerre and quasi-Hermite polynomials, respectively.

We first consider quasi-Laguerre polynomials.

THEOREM 3.6. *Let c be a constant and let $L_{n;c}^{(\alpha)}(x) = L_n^{(\alpha)}(x) + cL_{n-1}^{(\alpha)}(x)$. Then*

$$\operatorname{disc}(L_{n;c}^{(\alpha)}) = \frac{1}{n+\alpha+cn} \prod_{k=1}^{n} k^{k-2n+3} \prod_{k=1}^{n-1} (k+\alpha)^{k-1} \cdot (-c)^n L_{n;c}^{(\alpha)} \left(\frac{nc^2 + (2n+\alpha)c + n + \alpha}{c} \right). \tag{3.6}$$

Furthermore, $\operatorname{disc}(L_{n;c}^{(\alpha)})$ *is a polynomial in* c *of degree* $2(n-1)$.

We now give a proof by using the fact that Laguerre polynomials can be expressed as a limit case of Jacobi polynomials (see [**31**, (5.3.4)]):

$$L_n^{(\alpha)}(x) = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1 - 2\beta^{-1}x). \tag{3.7}$$

Proof of Theorem 3.6. By (3.7), we have

$$L_{n;c}^{(\alpha)}(x) = \lim_{\beta\to\infty} P_{n;c}^{(\alpha,\beta)}(1 - 2\beta^{-1}x). \tag{3.8}$$

The discriminant $\operatorname{disc}(f)$ is continuous with respect to the coefficients of f by Proposition 2.2. Hence, by Proposition 2.4, (3.8), and Theorem 3.1, we have

$$\begin{aligned}
&\operatorname{disc}(L_{n;c}^{(\alpha)}) = \lim_{\beta\to\infty} \operatorname{disc}\left(P_{n;c}^{(\alpha,\beta)}(1-2\beta^{-1}x)\right) \\
&= \lim_{\beta\to\infty} (-2\beta^{-1})^{n(n-1)} \operatorname{disc}(P_{n;c}^{(\alpha,\beta)}) \\
&= \lim_{\beta\to\infty} \frac{2^{n(n-1)}}{\beta^{n(n-1)}} \frac{(2n+\alpha+\beta)^{2n-1}}{2^{n(n-1)}} \prod_{k=1}^{n} k^{k-2n+3} \\
&\quad \cdot \prod_{k=1}^{n-1} (k+\alpha)^{k-1}(k+\beta)^{k-1}(n+k+\alpha+\beta)^{n-k-1} \\
&\quad \cdot \frac{(-c)^n P_{n;c}^{(\alpha,\beta)}\left(-(2n(n+\alpha+\beta)c^2 + (\alpha^2-\beta^2)c + 2(n+\alpha)(n+\beta))/(2n+\alpha+\beta)^2 c\right)}{(n+\alpha+cn)(n+\beta-cn)} \\
&= \prod_{k=1}^{n} k^{k-2n+3} \prod_{k=1}^{n-1} (k+\alpha)^{k-1} \frac{(-c)^n}{n+\alpha+cn} \\
&\quad \cdot \lim_{\beta\to\infty} P_{n;c}^{(\alpha,\beta)} \left(-\frac{2n(n+\alpha+\beta)c^2 + (\alpha^2-\beta^2)c + 2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2 c} \right).
\end{aligned}$$

Let

$$1 - 2\beta^{-1}x_\beta = -\frac{2n(n+\alpha+\beta)c^2 + (\alpha^2-\beta^2)c + 2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2 c}.$$

Then we have

$$\lim_{\beta\to\infty} x_\beta = \frac{nc^2 + (2n+\alpha)c + n + \alpha}{c}.$$

Therefore, by (3.8), we have

$$\lim_{\beta\to\infty} P_{n;c}^{(\alpha,\beta)}\left(-\frac{2n(n+\alpha+\beta)c^2+(\alpha^2-\beta^2)c+2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2 c}\right)$$
$$= L_n^{(\alpha)}\left(\frac{nc^2+(2n+\alpha)c+n+\alpha}{c}\right),$$

which completes the proof. □

Remark 3.7. Taking the limit as $c \to 0$, we have

$$\mathrm{disc}(L_n^{(\alpha)}) = \prod_{k=1}^{n} k^{k-2n+2}(k+\alpha)^{k-1}.$$

This formula coincides with Stieltjes's formula [**31**, (6.71.6)].

Next, we derive an explicit formula for the discriminants of quasi-Hermite polynomials.

Theorem 3.8. *Let c be a constant and let $H_{n;c}(x) = H_n(x) + cH_{n-1}(x)$. Then*

$$\mathrm{disc}(H_{n;c}) = 2^{n(3n-5)/2}\prod_{k=1}^{n-1} k^k \cdot (-c)^n H_{n;c}\left(-\frac{c^2+2n}{2c}\right). \tag{3.9}$$

Furthermore, $\mathrm{disc}(H_{n;c})$ *is an even polynomial in c of degree* $2(n-1)$.

We give a proof by using the limiting property, namely the fact that

$$\frac{H_n(x)}{n!} = \lim_{\lambda\to\infty} \lambda^{-n/2} C_n^{(\lambda)}(\lambda^{-1/2}x); \tag{3.10}$$

for example, see [**31**, (5.6.3)].

Proof of Theorem 3.8. By (3.10), we have

$$\begin{aligned} H_{n;c}(x) &= H_n(x) + cH_{n-1}(x) \\ &= n! \lim_{\lambda\to\infty} \lambda^{-n/2} C_n^{(\lambda)}(\lambda^{-1/2}x) + c(n-1)! \lim_{\lambda\to\infty} \lambda^{-(n-1)/2} C_{n-1}^{(\lambda)}(\lambda^{-1/2}x) \\ &= n! \lim_{\lambda\to\infty} \lambda^{-n/2} C_{n;c\sqrt{\lambda}/n}^{(\lambda)}(\lambda^{-1/2}x). \end{aligned}$$

By Proposition 2.4 and Corollary 3.3, we have

$$\begin{aligned} &\mathrm{disc}\left(\lambda^{-n/2} C_{n;c\sqrt{\lambda}/n}^{(\lambda)}(\lambda^{-1/2}x)\right) \\ &= \lambda^{-n(n-1)-n(n-1)/2}\,\mathrm{disc}\left(C_{n;c\sqrt{\lambda}/n}^{(\lambda)}(x)\right) \\ &= \lambda^{-3n(n-1)/2} 2^{n(n-1)}(2n+2\lambda-1)^n \prod_{k=1}^{n} k^{k-2n+3}(k+\lambda-1)^{2n-2k} \end{aligned}$$

$$\cdot \prod_{k=1}^{n-1}(k+2\lambda-1)^{k-2}$$
$$\cdot \frac{\left(-c\sqrt{\lambda}/n\right)^n C^{(\lambda)}_{n;c\sqrt{\lambda}/n}\left(-((c^2+2n)\lambda+n^2-n)/\sqrt{\lambda}(2n+2\lambda-1)c\right)}{(n+2\lambda-1)^2-c^2\lambda}.$$

Therefore, taking the limit as $\lambda \to \infty$, we have

$$\begin{aligned}\mathrm{disc}(H_{n;c}) &= (n!)^{2(n-1)} \lim_{\lambda\to\infty} \lambda^{-3n(n-1)/2} 2^{n(n-1)}(2n+2\lambda-1)^n \\ &\quad \cdot \prod_{k=1}^{n} k^{k-2n+3}(k+\lambda-1)^{2n-2k} \cdot \prod_{k=1}^{n-1}(k+2\lambda-1)^{k-2} \\ &\quad \cdot \frac{\left(-c\sqrt{\lambda}/n\right)^n C^{(\lambda)}_{n;c\sqrt{\lambda}/n}\left(-((c^2+2n)\lambda+n^2-n)/\sqrt{\lambda}(2n+2\lambda-1)c\right)}{(n+2\lambda-1)^2-c^2\lambda} \\ &= \frac{2^{n(3n-5)/2}}{n^n}\prod_{k=1}^{n} k^{k+1}\cdot(-c)^n \\ &\quad \cdot \lim_{\lambda\to\infty}\lambda^{-n/2} C^{(\lambda)}_{n;c\sqrt{\lambda}/n}\left(-\frac{(c^2+2n)\lambda+n^2-n}{\sqrt{\lambda}(2n+2\lambda-1)c}\right) \\ &= 2^{n(3n-5)/2}\prod_{k=1}^{n-1} k^k\cdot(-c)^n H_{n;c}\left(-\frac{c^2+2n}{2c}\right).\end{aligned}$$ □

Remark 3.9. Taking the limit as $c \to 0$, we have

$$\mathrm{disc}(H_n) = 2^{3n(n-1)/2}\prod_{k=1}^{n} k^k.$$

This formula coincides with Hilbert's formula [**31**, (6.71.7)].

Remark 3.10. Hermite polynomials are expressed as a limit case of Laguerre polynomials (see [**31**, Problem 80, p.389]):

$$\lim_{\alpha\to\infty}\alpha^{-n/2}L_n^{(\alpha)}(\alpha^{1/2}x+\alpha) = (-1)^n 2^{-n/2}(n!)^{-1}H_n(2^{-1/2}x).$$

By this, together with Proposition 2.4 and Theorem 3.6, we can give another proof of Theorem 3.8. As in the proof of Theorem 3.1, we can directly show Theorems 3.6 and 3.8 by using Schur's method and the elementary properties of Hermite and Laguerre polynomials. The proof we present above will be a quicker way of getting the same formulas as Theorems 3.6 and 3.8.

4. Applications.

In this section we give a generalization of Hausdorff's equations (2.11) and then examine solutions for such equations. We use the explicit formula for discriminants of quasi-Hermite polynomials given in Theorem 3.8.

Throughout this section, let

$$a_k = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^k \, e^{-t^2} dt, \qquad k = 0, 1, \ldots \tag{4.1}$$

It is then obvious that

$$a_{2k} = \frac{(2k)!}{2^{2k} k!}, \qquad a_{2k+1} = 0. \tag{4.2}$$

4.1. Hausdorff-type equations.

The following question originally goes back to Hausdorff's equations (2.11) and are considered in [**16**, p.4700] and [**21**, p.32].

Problem 4.1 ([**11**], [**16**], [**21**]). Let $m, n \geq 0$ be integers. Do the Diophantine equations

$$\sum_{i=1}^{m} x_i y_i^j = a_j, \qquad j = 0, 1, \ldots, n \tag{4.3}$$

have a solution $(x_1, \ldots, x_m, y_1, \ldots, y_m) \in \mathbb{Q}^{2m}$?

The following proposition makes the relationship of Problem 4.1 to quadrature formulas for Gaussian integration.

Proposition 4.2. *The following are equivalent*:

(i) *The equations* (4.3) *have a solution* $(x_1, \ldots, x_m, y_1, \ldots, y_m) \in \mathbb{Q}^{2m}$;

(ii) *The formula*

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(t) \, e^{-t^2} dt = \sum_{i=1}^{m} x_i f(y_i) \tag{4.4}$$

is a rational quadrature of degree n.

Proof. We remark that $1, x, x^2, \ldots, x^n$ form a basis of the vector space of all polynomials of degree at most n. □

The following proposition gives a slight generalization of *Stroud-type bound* for positive quadrature formulas [**29**] or *Fisher-type bound* for Gaussian designs [**2**].

Proposition 4.3. *If there exists a rational solution of* (4.3)*, then* $n \leq 2m - 1$.

Proof. Suppose contrary. Let f be a polynomial which vanishes at all y_i's. Then

$$0 < \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f^2(t) \, e^{-t^2} dt = \sum_{i=1}^{m} x_i (f(y_i))^2 = 0,$$

which is clearly a contradiction. □

The first pair (m, n) to consider is that $n = 2m - 1$. Formulas of type (4.4) are then called *Gaussian quadrature* and the nodes y_i's are the zeros of the Hermite polynomial H_m (cf. [**31**]). By a classical result by Schur [**22**] (see also [**30**]), the polynomials $H_{2r}(x)$ and $H_{2r+1}(x)/x$ are irreducible over $\mathbb{Q}$. So in this case, the equations (4.3) have no rational solutions.

The next case to consider is the 'almost extremal' situation.

PROPOSITION 4.4. *Assume that $n = 2m - 2$. Let $y_1, \ldots, y_m$ be distinct rationals. The following are equivalent*:

(i) *The equations* (4.3) *have a solution* $(x_1, \ldots, x_m, y_1, \ldots, y_m) \in \mathbb{Q}^{2m}$;

(ii) *There exists $c \in \mathbb{Q}$ such that $y_1, \ldots, y_m$ are the zeros of the quasi-Hermite polynomial $H_{m;c}(x) = H_m(x) + cH_{m-1}(x)$.*

PROOF. By (A.8), we remark that $H_m(x)$ is a polynomial with rational coefficients. The result then follows by Theorem 2.8 and Proposition 4.2. □

In [**21**], the nonexistence of solutions was reported only for $m = 3$. In this paper, we prove a more general nonexistence theorem for $n = 2m - 2$.

We work with the 2-adic numbers $\mathbb{Q}_2$ rather than $\mathbb{Q}$. Let $v_2 \colon \mathbb{Q}_2^\times \to \mathbb{Z}$ be the normalized valuation, where $\mathbb{Q}_2^\times$ is the set of units in $\mathbb{Q}_2$. We use the convention that $v_2(0) = \infty$. We denote by $\mathbb{Z}_2$ and $\mathbb{Z}_2^\times$ the ring of 2-adic integers and the set of units in $\mathbb{Z}_2$, respectively. We remark that

$$\mathbb{Z}_2 = \{x \in \mathbb{Q}_2 \mid v_2(x) \geq 0\}, \quad \mathbb{Z}_2^\times = \{x \in \mathbb{Q}_2 \mid v_2(x) = 0\}.$$

The following lemma is used in the proof of the main theorem in Subsection 4.2.

LEMMA 4.5. *Let $x = 2^n u$ be an element in $\mathbb{Q}_2^\times$ with $n \in \mathbb{Z}$ and $u \in \mathbb{Z}_2^\times$. For x to be a square in $\mathbb{Q}_2$ it is necessary and sufficient that n is even and $u \equiv 1 \pmod 8$.*

PROOF. See [**24**, Chapter II, Theorem 4]. □

4.2. Nonexistence theorem.

The following is the main result in this subsection.

THEOREM 4.6. *If $n \equiv 3, 4, 5, 6, 7 \pmod 8$, then $\operatorname{disc}(H_{n;c})$ is not a square in $\mathbb{Q}_2$ for any $c \in \mathbb{Q}_2$.*

COROLLARY 4.7. *If $r \equiv 2, 3, 4, 5, 6 \pmod 8$, then there exist no rationals $x_1, \ldots, x_{r+1}, y_1, \ldots, y_{r+1}$ such that*

$$\sum_{i=1}^{r+1} x_i y_i^k = a_k, \quad k = 0, 1, \ldots, 2r. \tag{4.5}$$

PROOF. Assume that $x_1, \ldots, x_{r+1}, y_1, \ldots, y_{r+1}$ are a rational solution of (4.5). Then by Proposition 4.3, $y_1, \ldots, y_{r+1}$ are distinct each other. By Proposition 4.4 there exists $c \in \mathbb{Q}$ such that the zeros of $H_{r+1;c}(x)$ are $y_1, \ldots, y_{r+1}$. Therefore $\operatorname{disc}(H_{r+1;c})$ is a square in the rationals by (2.1), which however contradicts Theorem 4.6. □

Proof of Theorem 4.6. Let

$$D_n(c) = (-c)^n H_{n;c}\left(-\frac{c^2+2n}{2c}\right).$$

By Theorem 3.8,

$$\operatorname{disc}(H_{n;c}) = 2^{n(3n-5)/2} \prod_{k=1}^{n-1} k^k \cdot D_n(c). \tag{4.6}$$

It is easily seen that

$$\frac{n(3n-5)}{2} \equiv \begin{cases} 0 \pmod 2 & \text{if } n \equiv 0,3 \pmod 4, \\ 1 \pmod 2 & \text{if } n \equiv 1,2 \pmod 4. \end{cases} \tag{4.7}$$

Let $t = 2 \cdot 3 \cdot 4^2 \cdot 5^2 \cdot \cdots \cdot (n-1)^{\lfloor (n-1)/2 \rfloor}$. Then we have

$$\prod_{k=1}^{n-1} k^k = \begin{cases} t^2 \cdot (n-1)!! & \text{if } n \text{ is even}, \\ t^2 \cdot (n-2)!! & \text{if } n \text{ is odd}. \end{cases}$$

By Lemma 4.5, we have $2^{-2v_2(t)} t^2 \equiv 1 \pmod 8$. Since $1 \cdot 3 \cdot 5 \cdot 7 \equiv 1 \pmod 8$,

$$2^{-2v_2(t)} \prod_{k=1}^{n-1} k^k \equiv \begin{cases} 1 \pmod 8 & \text{if } n \equiv 0,1,2,3 \pmod 8, \\ 3 \pmod 8 & \text{if } n \equiv 4,5 \pmod 8, \\ 7 \pmod 8 & \text{if } n \equiv 6,7 \pmod 8. \end{cases} \tag{4.8}$$

By (A.8),

$$D_n(c) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_k - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} b_k, \tag{4.9}$$

where

$$\begin{aligned} a_k &= (-1)^k \frac{n!}{k!(n-2k)!} c^{2k} (c^2+2n)^{n-2k}, \\ b_k &= (-1)^k \frac{(n-1)!}{k!(n-1-2k)!} c^{2k+2} (c^2+2n)^{n-1-2k}. \end{aligned} \tag{4.10}$$

$$\frac{n!}{k!(n-2k)!} = \frac{n!}{(2k)!(n-2k)!} \cdot \frac{(2k)!}{k!} = \binom{n}{2k} 2^k (2k-1)!!, \tag{4.11}$$

we have

$$\begin{aligned} v_2(a_k) &= v_2\left(\binom{n}{2k}\right) + m_n(c)k + nv_2(c^2+2n), \\ v_2(b_k) &= v_2\left(\binom{n-1}{2k}\right) + m_n(c)k + 2v_2(c) + (n-1)v_2(c^2+2n), \end{aligned} \tag{4.12}$$

where $m_n(c) = 1 + 2v_2(c) - 2v_2(c^2+2n)$.

By (4.10),

$$a_0 - b_0 = 2n(c^2+2n)^{n-1}, \quad a_1 - b_1 = -2(n-1)c^2(c^2+n^2)(c^2+2n)^{n-3}. \tag{4.13}$$

By expanding the right-hand sides,

$$a_0 - b_0 = 2nc^{2n-2} + 4p_0(c), \quad a_1 - b_1 = -2(n-1)c^{2n-2} + 4p_1(c),$$

where $p_0(c)$ and $p_1(c)$ are polynomials in c of degree $2n-4$ with integer coefficients. By (4.11), we have $a_k, b_k \in 4\mathbb{Z}[c]$ for $k \geq 2$. The degrees of a_k and b_k in c are $2n-2k$ by definition. Therefore, by (4.9),

$$D_n(c) = 2c^{2n-2} + 4s_{n-2}c^{2n-4} + \cdots + 4s_1c^2 + 4s_0, \tag{4.14}$$

where $s_i \in \mathbb{Z}$; if $v_2(c) \leq 0$, then $v_2(D_n(c)) = v_2(2c^{2n-2}) = 2(n-1)v_2(c) + 1$.

We divide the proof into four cases.

The case $n \equiv 3, 7 \pmod 8$.

If $v_2(c) \leq 0$, then $v_2(D_n(c)) = 2(n-1)v_2(c) + 1$ by (4.14). By (4.6), (4.7), and (4.8), we have $v_2(\mathrm{disc}(H_{n;c})) \equiv 1 \pmod 2$. Therefore $\mathrm{disc}(H_{n;c})$ is not a square in $\mathbb{Q}_2$ by Lemma 4.5.

If $v_2(c) \geq 1$, then $v_2(c^2+2n) = 1$ and $m_n(c) = 2v_2(c) - 1 \geq 1$. By (4.12), we have $v_2(a_k) \geq v_2(a_0)+1$ and $v_2(b_k) \geq v_2(b_0)+1$ for $k \geq 1$. Since $v_2(b_0) - v_2(a_0) = 2v_2(c) - 1 \geq 1$, we have $v_2(D_n(c)) = v_2(a_0) = n$. Since $n \equiv 3, 7 \pmod 8$, we have $v_2(\mathrm{disc}(H_{n;c})) \equiv 1 \pmod 2$ by (4.6) through (4.8). Therefore $\mathrm{disc}(H_{n;c})$ is not a square in $\mathbb{Q}_2$ by Lemma 4.5.

The case $n \equiv 5 \pmod 8$.

If $v_2(c) \leq -1$, then by (4.14)

$$\frac{D_n(c)}{2c^{2n-2}} = 1 + 2s_{n-2}c^{-2} + \cdots + 2s_1c^{-2n} + 2s_0c^{-2n+2} \equiv 1 \pmod 8.$$

By Lemma 4.5, we have $c^{2n-2}/2^{2(n-1)v_2(c)} \equiv 1 \pmod 8$ and so $D_n(c)/2^{2(n-1)v_2(c)+1} \equiv 1 \pmod 8$. By (4.6) through (4.8), we have

$$2^{2e}\,\mathrm{disc}(H_{n;c}) \equiv 3 \cdot 1 \equiv 3 \pmod 8,$$

where e is an integer. Therefore $\mathrm{disc}(H_{n;c})$ is not a square in $\mathbb{Q}_2$ by Lemma 4.5.

If $v_2(c) = 0$, then $v_2(c^2+2n) = 0$ and $m_n(c) = 1$. Since $n-1$ is even, we have $(c^2+2n)^{n-1} \equiv 1 \pmod 8$ by Lemma 4.5. By (4.13),

$$\frac{a_0 - b_0}{2} = n(c^2+2n)^{n-1} \equiv 5 \cdot 1 \equiv 5 \pmod 8.$$

Since $n \equiv 5 \pmod 8$, and by (4.12), we have

$$v_2(a_1) = v_2(a_2) = 2, \quad v_2(a_k) = v_2\left(\binom{n}{2k}\right) + k \geq 3,$$
$$v_2(b_1) = v_2(b_2) = 2, \quad v_2(b_k) = v_2\left(\binom{n-1}{2k}\right) + k \geq 3$$

for $k \geq 3$. Therefore, by (4.9),

$$D_n(c) \equiv a_0 - b_0 + a_1 - b_1 + a_2 - b_2 \equiv 2 \cdot 5 + 4 - 4 + 4 - 4 \equiv 2 \pmod 8,$$

and so $D_n(c)/2 \equiv 1 \pmod 4$. By (4.6) through (4.8), we have

$$2^{2e} \operatorname{disc}(H_{n;c}) \equiv 3 \cdot 1 \equiv 3 \pmod 4,$$

where e is an integer. Therefore $\operatorname{disc}(H_{n;c})$ is not a square in $\mathbb{Q}_2$ by Lemma 4.5.

If $v_2(c) \geq 1$, then $v_2(c^2 + 2n) = 1$ and $m_n(c) = 2v_2(c) - 1 \geq 1$. Since $n \equiv 5 \pmod 8$, by (4.12), we have

$$v_2(a_k) = v_2\left(\binom{n}{2k}\right) + m_n(c)k + n \geq n + 2,$$
$$v_2(b_k) = v_2\left(\binom{n-1}{2k}\right) + m_n(c)k + 2v_2(c) + n - 1 \geq n + 3$$

for $k \geq 1$. Hence we have $a_k/2^n \equiv b_k/2^n \equiv 0 \pmod 4$ for $k \geq 1$. Since $n - 1$ is even, $(c^2 + 2n)^{n-1}/2^{n-1} \equiv 1 \pmod 8$ by Lemma 4.5. By (4.13),

$$\frac{a_0 - b_0}{2^n} = n \cdot \frac{(c^2 + 2n)^{n-1}}{2^{n-1}} \equiv 5 \pmod 8.$$

Therefore, by (4.9),

$$\frac{D_n(c)}{2^n} \equiv \frac{a_0 - b_0}{2^n} \equiv 1 \pmod 4.$$

By (4.6) through (4.8), we have

$$2^{2e} \operatorname{disc}(H_{n;c}) \equiv 3 \cdot 1 \equiv 3 \pmod 4,$$

where e is an integer. Therefore $\operatorname{disc}(H_{n;c})$ is not a square in $\mathbb{Q}_2$ by Lemma 4.5.

The case $n \equiv 4 \pmod 8$.

If $v_2(c) \leq 0$, then $v_2(D_n(c)) = 2(n-1)v_2(c) + 1$ by (4.14). By (4.6) through (4.8), we have $v_2(\operatorname{disc}(H_{n;c})) \equiv 1 \pmod 2$. Therefore $\operatorname{disc}(H_{n;c})$ is not a square in $\mathbb{Q}_2$ by Lemma 4.5.

If $v_2(c) = 1$, then $v_2(c^2 + 2n) = 2$ and $m_n(c) = -1$. Since $n \equiv 4 \pmod 8$, and by (4.12),

$$v_2(a_{n/2}) = \frac{3}{2}n, \quad v_2(a_{n/2-1}) = \frac{3}{2}n + 2, \quad v_2(a_k) \geq \frac{3}{2}n + 2,$$

$$v_2(b_{n/2-1}) = \frac{3}{2}n + 1, \quad v_2(b_k) \geq \frac{3}{2}n + 2$$

for $k \leq n/2 - 2$.

By (4.10) and (4.11),

$$a_{n/2} = (-1)^{n/2} \frac{n!}{(n/2)!0!} c^n = 2^{n/2}(n-1)!!c^n. \tag{4.15}$$

Since $n \equiv 4 \pmod 8$, we have $(n-1)!! \equiv 3 \pmod 8$ and $c^n/2^n \equiv 1 \pmod 8$. Hence $a_{n/2}/2^{3n/2} \equiv 3 \pmod 8$. Therefore, by (4.9),

$$\frac{D_n(c)}{2^{3n/2}} \equiv \frac{a_{n/2} - b_{n/2-1}}{2^{3n/2}} \equiv 3 - 2 \equiv 1 \pmod 4.$$

By (4.6) through (4.8),

$$2^{2e} \operatorname{disc}(H_{n;c}) \equiv 3 \cdot 1 \equiv 3 \pmod 4,$$

where e is an integer. Therefore $\operatorname{disc}(H_{n;c})$ is not a square in $\mathbb{Q}_2$ by Lemma 4.5.

If $v_2(c) = 2$, then $v_2(c^2 + 2n) = 3$ and $m_n(c) = -1$. By (4.12),

$$v_2(a_{n/2}) = \frac{5}{2}n, \quad v_2(a_{n/2-1}) = v_2(a_{n/2-2}) = \frac{5}{2}n + 2, \quad v_2(a_k) \geq \frac{5}{2}n + 3,$$
$$v_2(b_{n/2-1}) = \frac{5}{2}n + 2, \quad v_2(b_{n/2-2}) = \frac{5}{2}n + 3, \quad v_2(b_k) \geq \frac{5}{2}n + 4$$

for $k \leq n/2 - 3$. Since $v_2(c) = 2$ and n is even, $c^n/2^{2n} \equiv 1 \pmod 8$. By (4.15), we have $a_{n/2}/2^{5n/2} \equiv 3 \pmod 8$. Hence, by (4.9) and (4.15),

$$\frac{D_n(c)}{2^{5n/2}} \equiv \frac{a_{n/2} + a_{n/2-1} + a_{n/2-2} - b_{n/2-1}}{2^{5n/2}} \equiv 3 + 4 + 4 - 4 \equiv 7 \pmod 8.$$

By (4.6) through (4.8), we have

$$2^{2e} \operatorname{disc}(H_{n;c}) \equiv 3 \cdot 7 \equiv 5 \pmod 8,$$

where e is an integer. Therefore $\operatorname{disc}(H_{n;c})$ is not a square in $\mathbb{Q}_2$ by Lemma 4.5.

If $v_2(c) \geq 3$, then $v_2(c^2 + 2n) = 3$ and $m_n(c) = 2v_2(c) - 5 \geq 1$. By (4.12),

$$v_2(a_0) = 3n, \quad v_2(a_k) = v_2\left(\binom{n}{2k}\right) + m_n(c)k + 3n \geq 3n + 2 \quad \text{for } k \geq 1,$$
$$v_2(b_k) = v_2\left(\binom{n-1}{2k}\right) + m_n(c)k + 2v_2(c) + 3(n-1) \geq 3n + 3 \quad \text{for } k \geq 0.$$

Since n is even, $(c^2 + 2n)^n/2^{3n} \equiv 1 \pmod 8$ by Lemma 4.5. Hence, by (4.9),

$$\frac{D_n(c)}{2^{3n}} \equiv \frac{a_0}{2^{3n}} \equiv \frac{(c^2 + 2n)^n}{2^{3n}} \equiv 1 \pmod 4.$$

By (4.6) through (4.8), we have

$$2^{2e}\,\mathrm{disc}(H_{n;c}) \equiv 3\cdot 1 \equiv 3 \pmod 4,$$

where e is an integer. Therefore $\mathrm{disc}(H_{n;c})$ is not a square in $\mathbb{Q}_2$ by Lemma 4.5.

The case $n \equiv 6 \pmod 8$.

If $v_2(c) \le -1$, then we have

$$\frac{D_n(c)}{2^{2(n-1)v_2(c)+1}} \equiv 1 \pmod 8$$

as in the case where $n \equiv 5 \pmod 8$. By (4.6) through (4.8), we have

$$2^{2e}\,\mathrm{disc}(H_{n;c}) \equiv 7\cdot 1 \equiv 7 \pmod 8,$$

where e is an integer. Therefore $\mathrm{disc}(H_{n;c})$ is not a square in $\mathbb{Q}_2$ by Lemma 4.5.

If $v_2(c) = 0$, then $v_2(c^2+2n) = 0$ and $m_n(c) = 1$. By (4.12), we have

$$\begin{gathered} v_2(a_0) = 0, \quad v_2(a_1) = 1, \quad v_2(a_2) = 2, \quad v_2(a_k) \ge 3, \\ v_2(b_0) = 0, \quad v_2(b_1) = v_2(b_2) = 2, \quad v_2(b_k) \ge 3 \end{gathered}$$

for $k \ge 3$. Since $c^2 \equiv 1 \pmod 8$ and $c^2 + 2n \equiv 5 \pmod 8$, by (4.13),

$$\begin{aligned} a_0 - b_0 &= 2n(c^2+2n)^{n-1} \equiv 2\cdot 6\cdot 5 \equiv 4 \pmod 8, \\ a_1 &= -n(n-1)c^2(c^2+2n)^{n-2} \equiv -6\cdot 5\cdot 1\cdot 1 \equiv 2 \pmod 8. \end{aligned}$$

Hence, by (4.9),

$$D_n(c) \equiv a_0 - b_0 + a_1 - b_1 + a_2 - b_2 \equiv 4 + 2 - 4 + 4 - 4 \equiv 2 \pmod 8.$$

Therefore $D_n(c)/2 \equiv 1 \pmod 4$. By (4.6) through (4.8), we have

$$2^{2e}\,\mathrm{disc}(H_{n;c}) \equiv 7\cdot 1 \equiv 3 \pmod 4,$$

where e is an integer. Therefore $\mathrm{disc}(H_{n;c})$ is not a square in $\mathbb{Q}_2$ by Lemma 4.5.

If $v_2(c) = 1$, then $c^2 + 2n \equiv 4 + 12 \equiv 0 \pmod{16}$. Hence we have $v_2(c^2+2n) \ge 4$ and $m_n(c) \le -5$. By (4.12),

$$\begin{gathered} v_2(a_{n/2}) = \frac{3}{2}n, \quad v_2(a_k) = 3k + (n-2k)v_2(c^2+2n) \ge \frac{3}{2}n + 5, \\ v_2(b_k) = 3k + 2 + (n-1-2k)v_2(c^2+2n) \ge \frac{3}{2}n + 3 \end{gathered}$$

for $k \le n/2 - 1$. By (4.10) and (4.11),

$$a_{n/2} = (-1)^{n/2}\frac{n!}{(n/2)!0!}c^n = -2^{n/2}(n-1)!!c^n.$$

Since $(n-1)!! \equiv 7 \pmod 8$, and by (4.9),

$$\frac{D_n(c)}{2^{3n/2}} \equiv \frac{a_{n/2}}{2^{3n/2}} = -(n-1)!!\frac{c^n}{2^n} \equiv 1 \pmod 8.$$

By (4.6) through (4.8), we have

$$2^{2e}\,\mathrm{disc}(H_{n;c}) \equiv 7 \cdot 1 \equiv 7 \pmod 8,$$

where e is an integer. Therefore $\mathrm{disc}(H_{n;c})$ is not a square in $\mathbb{Q}_2$ by Lemma 4.5.

If $v_2(c) \geq 2$, then $v_2(c^2 + 2n) = 2$ and $m_n(c) = 2v_2(c) - 3 \geq 1$. By (4.12),

$$v_2(a_0) = 2n, \quad v_2(a_k) \geq 2n + 1 \ \text{ for } k \geq 1, \ v_2(b_k) \geq 2n + 2 \ \text{ for } k \geq 0.$$

Hence $v_2(D_n(c)) = v_2(a_0) = 2n$. Since $n \equiv 6 \pmod 8$, we have $v_2(\mathrm{disc}(H_{n;c})) \equiv 1 \pmod 2$ by (4.6) through (4.8). So $\mathrm{disc}(H_{n;c})$ is not a square in $\mathbb{Q}_2$ by Lemma 4.5. □

We now translate Theorem 4.6 in terms of rational points on curves. Let

$$f_r(c) = \mathrm{disc}(H_{r+1;c}). \tag{4.16}$$

Then $f_r(c)$ is a polynomial in c of degree $2r$ with integer coefficients. Let C_r be the hyperelliptic curve defined by $y^2 = f_r(x)$.

THEOREM 4.8. *The curve C_r has no $\mathbb{Q}_2$-rational points if and only if $r \equiv 2, 3, 4, 5, 6 \pmod 8$.*

PROOF. Assume that $r \equiv 2, 3, 4, 5, 6 \pmod 8$. By Theorem 4.6, it is sufficient to prove that the points at infinity of C_r are not $\mathbb{Q}_2$-rational. By the proof of Theorem 4.6, the leading coefficient of $f_r(x)$ is equal to

$$2^{(r+1)(3r-2)/2+1}\prod_{k=1}^{r} k^k. \tag{4.17}$$

It is not a square in $\mathbb{Q}_2$ by Lemma 4.5, (4.7), and (4.8). Therefore the points at infinity of C_r are not $\mathbb{Q}_2$-rational.

Assume that $r \equiv 0, 7 \pmod 8$. By Remark 3.9,

$$f_r(0) = \mathrm{disc}(H_{r+1}) = 2^{3r(r+1)/2}\prod_{k=1}^{r+1} k^k.$$

If $r \equiv 0, 7 \pmod 8$, then $3r(r+1)/2 \equiv 0 \pmod 2$. By (4.8), we have

$$2^{2e}\prod_{k=1}^{r+1} k^k \equiv 1 \pmod 8,$$

where e is an integer. Hence $f_r(0)$ is a square in $\mathbb{Q}_2$ by Lemma 4.5. Therefore C_r has a $\mathbb{Q}_2$-rational point.

Finally, assume that $r \equiv 1 \pmod 8$. Then the leading coefficient of $f_r(x)$ is a square in $\mathbb{Q}_2$ by Lemma 4.5, (4.8), and (4.17). Therefore the points at infinity of C_r are $\mathbb{Q}_2$-rational. □

Remark 4.9. In fact, if $r \equiv 1 \pmod 8$, then $f_r(x)$ is a square in $\mathbb{Q}_2$ when $v_2(x)$ is sufficiently small. Therefore C_r has a $\mathbb{Q}_2$-rational point in the affine part.

5. Conclusion and further remarks.

We have derived explicit formulas for the discriminants of all classical quasi-orthogonal polynomials, as a full generalization of the result of Dilcher and Stolarsky [**7**]. Their proof is based on algebraic properties of resultants and the Euclidean algorithm for polynomials, whereas our proof uses Schur's method [**23**] (see [**31**, Section 6.71]). A natural question then asks whether we can generalize Theorem 3.1 to larger classes of orthogonal polynomials. For this purpose, we may use some recent results concerning the question of when a family of quasi-orthogonal polynomials is actually an orthogonal polynomial sequence. For example, Alfaro et al. [**1**] characterize the families of orthogonal polynomials, say $\{\Phi_n\}_n$, such that quasi-orthogonal polynomials $\Phi_{n,2}(x) = \Phi_n(x) + a_1\Phi_{n-1}(x) + a_2\Phi_{n-2}(x)$ are also orthogonal. By computing the recurrence coefficients and then using Schur's method, we may find an explicit formula for the discriminants even in the order-two case. Also, it is well known (cf. [**10**]) that the Bernstein–Szegő polynomials can be expressed as linear combinations of Chebyshev polynomials. By combining this with Schur's method, we may prove the Dilcher–Stolarsky formula.

We have also dealt with Hausdorff-type equations and created the relationship to quasi-Hermite polynomials and quadrature formulas for Gaussian integration. We have then proved a necessary and sufficient condition for the hyperelliptic curve $C_r : y^2 = \mathrm{disc}(H_{r+1;x})$ to have $\mathbb{Q}_2$-rational points. This not only provides a general nonexistence theorem for solutions of Hausdorff-type equations, but also gives opportunities to use discriminants in the study of quadrature formulas and quasi-Hermite polynomials.

The hyperelliptic curve C_r may possibly have $\mathbb{Q}_p$-rational points for prime numbers $p \geq 3$. For example by using the function IsLocallySolvable in Magma [**4**], we have examined $r \leq 40$ and p such that the curve C_r has no $\mathbb{Q}_p$-rational points; see Table 1. Accordingly, by the same argument as in Corollary 4.7, the equations (4.3) for $(m, n) = (r+1, 2r)$ have no rational solutions for $r \leq 40$. To improve Theorem 4.6 is again left for future work.

It may be also interesting to consider analogues of Theorem 4.6 for other classical quasi-orthogonal polynomials. For example, by Corollary 3.4, we have

$$\mathrm{disc}(U_{r+1}) = \lim_{c \to 0} \mathrm{disc}(U_{r+1;c}) = 2^{(r+1)^2}(r+2)^{r-1}.$$

This is a square in the rationals if r is odd. Therefore, for any odd integer r and any prime number p, the hyperelliptic curve C_r has a $\mathbb{Q}_p$-rational point, which does not give informations on solutions of (4.3). Another interesting case will be the Legendre polynomials which correspond to the integration $(1/2)\int_{-1}^{1} dx$. In this case, by a classical

Table 1. The prime numbers p such that $C_r(\mathbb{Q}_p) = \emptyset$.

r	p	r	p	r	p	r	p
1		11	2,3	21	2,3,11,13	31	11,31
2	2,3	12	2,5,7	22	2,11,13,17,19	32	23,31
3	2	13	2,5,7,11,13	23	3,17,23	33	3,17,23,29
4	2	14	2,7,11,13	24	13,23	34	2,13,19,23,29,31
5	2,3,5	15	7	25	11,19,23	35	2,5,7,13,23,29,31
6	2,3,5	16	11	26	2,13	36	2,5,7,13,17,19,23,29,31
7	7	17	7,11	27	2,11	37	2,13,19,23,29,37
8	5,7	18	2,3,11	28	2,7,11,17,19,23	38	2,3,5,7,19,23,37
9	3	19	2,17	29	2,11,29	39	17,37
10	2,5,7	20	2,5,11,13	30	2,3,17,19,23,29	40	5,7,17,31

result of Holt [**14**], we see that there exist no rational solutions of (4.3) for $n = 2m - 1$. We are again naturally interested in the case when $n = 2m - 2$.

Acknowledgements. The first author would like to thank Bruce Reznick who kindly told Hausdorff's paper through private email conversation. The first author would also like to thank Eiichi Bannai and Satoshi Tsujimoto for valuable informations about Gaussian designs and classical orthogonal polynomials. The authors would like to thank the referee for helpful suggestions.

A. The classical orthogonal polynomials and some basic properties.

We here describe some basic properties on Jacobi polynomials, Laguerre polynomials, Hermite polynomials, which are used in the proof of our results.

A.1. Jacobi polynomials.

The following informations can be found in [**31**, Chapter IV].

Explicit expression.

$$P_n^{(\alpha,\beta)}(x) = \sum_{m=0}^{n} \binom{n+\alpha}{m}\binom{n+\beta}{n-m}\left(\frac{x-1}{2}\right)^{n-m}\left(\frac{x+1}{2}\right)^{m}. \tag{A.1}$$

Three-term relation.

$$\begin{aligned} &2n(n+\alpha+\beta)(2n+\alpha+\beta-2)P_n^{(\alpha,\beta)}(x) \\ &\quad = (2n+\alpha+\beta-1)\left((2n+\alpha+\beta)(2n+\alpha+\beta-2)x+\alpha^2-\beta^2\right)P_{n-1}^{(\alpha,\beta)}(x) \\ &\qquad - 2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)P_{n-2}^{(\alpha,\beta)}(x). \end{aligned} \tag{A.2}$$

Derivative formulas.

$$\begin{aligned} &(2n+\alpha+\beta)(1-x^2)\frac{d}{dx}P_n^{(\alpha,\beta)}(x) \\ &\quad = -n\left((2n+\alpha+\beta)x+\beta-\alpha\right)P_n^{(\alpha,\beta)}(x) + 2(n+\alpha)(n+\beta)P_{n-1}^{(\alpha,\beta)}(x), \end{aligned} \tag{A.3}$$

$$
\begin{aligned}
(2n+\alpha+\beta+2)(1-x^2)\frac{d}{dx}P_n^{(\alpha,\beta)}(x) \\
&= (n+\alpha+\beta+1)\left((2n+\alpha+\beta+2)x+\alpha-\beta\right)P_n^{(\alpha,\beta)}(x) \\
&\quad - 2(n+1)(n+\alpha+\beta+1)P_{n+1}^{(\alpha,\beta)}(x). \qquad \text{(A.4)}
\end{aligned}
$$

A.2. Laguerre polynomials.

The following informations can be found in [**31**, Section 5.1].

Explicit expression.

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}. \qquad \text{(A.5)}$$

Three-term relation.

$$nL_n^{(\alpha)}(x) = (-x+2n+\alpha-1)L_{n-1}^{(\alpha)}(x) - (n+\alpha-1)L_{n-2}^{(\alpha)}(x). \qquad \text{(A.6)}$$

Derivative formulas.

$$\frac{d}{dx}L_n^{(\alpha)}(x) = x^{-1}\left(nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x)\right). \qquad \text{(A.7)}$$

A.3. Hermite polynomials.

The following informations can be found in [**31**, Section 5.5].

Explicit expression.

$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n!}{k!(n-2k)!}(2x)^{n-2k}. \qquad \text{(A.8)}$$

Three-term relation.

$$H_n(x) - 2xH_{n-1}(x) + 2(n-1)H_{n-2}(x) = 0. \qquad \text{(A.9)}$$

Derivative formulas.

$$H_n'(x) = 2nH_{n-1}(x). \qquad \text{(A.10)}$$

References

[1] M. Alfaro, F. Marcellán, A. Peña and M. L. Rezola, When do linear combinations of orthogonal polynomials yield new sequences of orthogonal polynomials?, J. Comput. Appl. Math., **233** (2010), 1446–1452.

[2] E. Bannai and E. Bannai, Tight Gaussian 4-designs, J. Algebr. Comb., **22** (2005), 39–63.

[3] S. Bochner, Über Sturm–Liouvillesche Polynomsysteme, Math. Z., **29** (1929), 730–736.

[4] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system, I, The user language, J. Symbolic Comput., **24** (1997), 235–265.

[5] P. Delsarte, J. M. Goethals and J. J. Seidel, Spherical codes and designs, Geom. Dedicata, **6** (1977), 363–388.

[6] L. E. Dickson, History of the Theory of Numbers, II, Carnegie Institution of Washington, 1923.

[7] K. Dilcher and K. B. Stolarsky, Resultants and discriminants of Chebyshev and related polynomials, Trans. Amer. Math. Soc., **357** (2005), 965–981.

[8] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, Discriminants, Resultants, and Multidimensional Determinants, Mathematics: Theory & Applications, Birkhäuser, Boston, 1994.

[9] J. E. Gishe and M. E. H. Ismail, Resultants of Chebyshev polynomials, Z. Anal. Anwend., **27** (2008), 499–508.

[10] Z. Grinshpun, Special linear combinations of orthogonal polynomials, J. Math. Anal. Appl., **299** (2004), 1–18.

[11] F. Hausdorff, Zur Hilbertschen Lösung des Waringschen Problems, Math. Ann., **67** (1909), 301–305.

[12] D. Hilbert, Ueber die Discriminante der im Endlichen abbrechenden hypergeometrischen Reihe, J. Reine Angew. Math., **103** (1888), 337–345.

[13] D. Hilbert, Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n^{ter} Potenzen (Waringsches Problem), Math. Ann., **67** (1909), 281–300.

[14] J. B. Holt, The irreducibility of Legendre's polynomials, Proc. London Math. Soc. (2), **11** (1913), 351–356.

[15] P. Lesky, Die Charakterisierung der klassischen orthogonalen Polynome durch Sturm–Liouvillesche Differentialgleichungen, Arch. Rat. Mech. Anal., **10** (1962), 341–351.

[16] Y. V. Nesterenko, On Waring's problem (elementary methods), (Russian), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), **322** (2005), Trudy po Teorii Chisel, 149–175, 254; translation in J. Math. Sci. (N. Y.), **137** (2006), 4699–4715.

[17] H. Nozaki and M. Sawa, Remarks on Hilbert identities, isometric embeddings, and invariant cubature, St. Petersburg Math. J., **25** (2014), 615–646.

[18] P. Pollack, Not Always Buried Deep: A Second Course in Elementary Number Theory, Amer. Math. Soc., 2009.

[19] T. Reye, Trägheits- und höhere Momente eines Massensystemes in Bezug auf Ebenen, J. Reine Angew. Math., **72** (1870), 293–326.

[20] M. Riesz, Sur le problème des moments, III, Ark. Math. Fys., **17** (1923), 1–52.

[21] M. Sawa, The theory of cubature formulas, Sugaku (in Japanese), **68** (2016), 24–52.

[22] I. Schur, Einige Sätze über Primzahlen mit Anwendungen auf Irreduzibilitätsfragen, II, Sitzungsber. Preuss. Akad. Wiss. Berlin Phys.-Math. Kl., **14** (1929), 370–391.

[23] I. Schur, Affektlose Gleichungen in der Theorie der Laguerreschen und Hermiteschen Polynome, J. Reine Angew. Math., **165** (1931), 52–58.

[24] J.-P. Serre, A Course in Arithmetic, Graduate Texts in Mathematics, **7**, Springer-Verlag, New York, 1973.

[25] J. Shohat, On mechanical quadratures, in particular, with positive coefficients, Trans. Amer. Math. Soc., **42** (1937), 461–496.

[26] J. A. Shohat and J. D. Tamarkin, The Problem of Moments, American Mathematical Society Mathematical Surveys and Monographs, I, American Mathematical Society, Providence, 1970.

[27] T. J. Stieltjes, Sur quelques théorèmes d'algèbre, C. R. Acad. Sci. Paris, **100** (1885), 439–440; Œuvres Complètes/Collected Papers, **I**, Springer-Verlag, Berlin, 1993, 528–529.

[28] T. J. Stieltjes, Sur les polynômes de Jacobi, C. R. Acad. Sci. Paris, **100** (1885), 620–622; Œuvres Complètes/Collected Papers, **I**, Springer-Verlag, Berlin, 1993, 530–532.

[29] A. H. Stroud, Approximate Calculation of Multiple Integrals, Prentice-Hall, Englewood Cliffs, N.J., 1971.

[30] P. R. Subramanian, Nonzero zeros of the Hermite polynomials are irrational, Fibonacci Quart., **33** (1995), 131–134.

[31] G. Szegő, Orthogonal Polynomials (4th ed.), American Mathematical Society Colloquium Publications, **23**, American Mathematical Society, Providence, 1975.

[32] Y. Xu, A characterization of positive quadrature formulae, Math. Comp., **62** (1994), 703–718.

Masanori SAWA
Kobe University
1-1 Rokkodai, Nada, Kobe
Hyogo 657-8501
Japan
E-mail: sawa@people.kobe-u.ac.jp

Yukihiro UCHIDA
Department of Mathematical Sciences
Graduate School of Science
Tokyo Metropolitan University
1-1 Minami-Osawa, Hachioji
Tokyo 192-0397
Japan
E-mail: yuchida@tmu.ac.jp