

Weak subsolutions to complex Monge–Ampère equations

By Vincent GUEDJ, Chinh H. LU and Ahmed ZERIAHI

(Received Jan. 25, 2018)

Abstract. We compare various notions of weak subsolutions to degenerate complex Monge–Ampère equations, showing that they all coincide. This allows us to give an alternative proof of mixed Monge–Ampère inequalities due to Kolodziej and Dinew.

1. Introduction.

Let Ω be a domain of \mathbb{C}^n . We consider in this article the notion of subsolution for degenerate complex Monge–Ampère equations in Ω . These are bounded plurisubharmonic functions which satisfy

$$(dd^c \varphi)^n \geq f dV,$$

where dV denotes the Lebesgue measure and $0 \leq f \in L^1(\Omega)$.

This inequality can be interpreted in various senses (pluripotential sense [BT76], viscosity sense [EGZ11], distribution sense [HL13]) and the goal of this note is to show that all point of views do coincide.

MAIN THEOREM. *Assume φ is plurisubharmonic and bounded. The following are equivalent:*

- (i) $(dd^c \varphi)^n \geq f dV$ in the pluripotential sense;
- (ii) $(dd^c(\varphi \star \chi_\varepsilon))^n \geq (f^{1/n} \star \chi_\varepsilon)^n dV$ in the classical sense, for all $\varepsilon > 0$;
- (iii) $\Delta_H \varphi \geq f^{1/n}$ in the sense of distributions, for all $H \in \mathcal{H}$.

In a particular case when φ and f are continuous, our main theorem was proved by Błocki (see [B196, Theorem 3.10]).

The operator $dd^c = ai\partial\bar{\partial}$ is here normalized so that $dV = (dd^c|z|^2)^n$ is the Euclidean volume form on \mathbb{C}^n . Thus for a smooth function φ ,

$$(dd^c \varphi)^n = \det \left(\frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} \right) dV.$$

We let \mathcal{H} denote the space of Hermitian positive definite matrix H that are normalized by $\det H = 1$, and let Δ_H denote the Laplace operator

2010 *Mathematics Subject Classification.* Primary 32W20; Secondary 35D30.
Key Words and Phrases. complex Monge–Ampère equation, weak subsolutions, pluripotential theory.

The authors are partially supported by the ANR project GRACK.

$$\Delta_H \varphi := \frac{1}{n} \sum_{j,k=1}^n h_{jk} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}.$$

The functions χ_ε are standard mollifiers, i.e. radial smooth non-negative functions with compact support in the ε -ball centered at the origin, and such that $\int \chi_\varepsilon dV = 1$. It is then classical that the convolutions $\varphi \star \chi_\varepsilon$ are smooth, plurisubharmonic, and decrease to φ as ε decreases to 0.

When f is moreover continuous, one can also interpret the inequality $(dd^c \varphi)^n \geq f dV$ in the viscosity sense, as shown in [EGZ11, Proposition 1.5].

Our main theorem easily implies the following result of Kołodziej [Kol03, Lemma 1.2] (see also [Din09], [DL15]):

COROLLARY. *Assume $\varphi_1, \dots, \varphi_n$ are bounded plurisubharmonic functions in Ω , such that $(dd^c \varphi_i)^n \geq f_i dV$, where $0 \leq f_i \in L^1(\Omega)$. Then*

$$dd^c \varphi_1 \wedge \dots \wedge dd^c \varphi_n \geq f_1^{1/n} \dots f_n^{1/n} dV.$$

The note is organized as follows. We start by extending Kołodziej’s subsolution theorem (see Theorem 2.1), providing a solution to special Monge–Ampère equations that we are going to use in the sequel. We prove our main result in Section 3.1. The starting point is an identification of viscosity subsolutions and pluripotential subsolutions obtained in [EGZ11]. We connect these identifications to mixed Monge–Ampère inequalities in Section 3.2 and propose some generalizations in Section 3.3.

2. The subsolution theorem.

Let $\Omega \Subset \mathbb{C}^n$ be a bounded hyperconvex domain (in the sequel we only need to deal with the case when Ω is a ball). Let μ be a Borel measure on Ω . If there exists a function $v \in PSH(\Omega) \cap L^\infty(\Omega)$ such that

$$\mu \leq (dd^c v)^n \text{ in } \Omega, \text{ with } \lim_{\Omega \ni z \rightarrow \zeta} v(z) = 0, \forall \zeta \in \partial\Omega,$$

then it was proved by S. Kołodziej [Kol95] that there exists a unique solution $\psi \in PSH(\Omega) \cap L^\infty(\Omega)$ to the equation

$$(dd^c \psi)^n = \mu,$$

such that $\lim_{\Omega \ni z \rightarrow \zeta} \psi(z) = 0$. We need the following generalization:

THEOREM 2.1. *Assume μ is a non pluripolar Borel measure on Ω which has finite total mass. Then there exists a unique function $\varphi \in \mathcal{F}^1(\Omega)$ satisfying*

$$(dd^c \varphi)^n = e^\varphi \mu \text{ in } \Omega. \tag{2.1}$$

Moreover if μ satisfies $\mu \leq (dd^c u)^n$ in Ω , for some bounded negative psh function u , then the solution $\varphi \in PSH(\Omega)$ is bounded with $u \leq \varphi$. In particular, if $\limsup_{z \rightarrow \zeta} u(z) = 0$ for every $\zeta \in \partial\Omega$ then the same property holds for φ .

Before entering into the proof let us recall the definition of Cegrell’s finite energy classes. We refer the reader to [Ce98], [Ce04] for more details.

A domain Ω is called hyperconvex if there exists a continuous plurisubharmonic exhaustion function $\rho : \Omega \rightarrow [-\infty, 0)$ such that the sublevel sets $\{\rho < -c\}$ are relatively compact in Ω , for all constants $c > 0$.

Let u be a negative plurisubharmonic function in Ω . We recall the following definitions:

- $u \in \mathcal{E}_0(\Omega)$ if u is bounded in Ω , u vanishes on the boundary, i.e. $\lim_{z \rightarrow \partial\Omega} u(z) = 0$, and $\int_{\Omega} (dd^c u)^n < +\infty$.
- $u \in \mathcal{E}(\Omega)$ if for each $z_0 \in \Omega$ there exists an open neighborhood $z_0 \in V_{z_0} \Subset \Omega$ and a decreasing sequence $(u_j) \subset \mathcal{E}_0(\Omega)$ such that u_j converges to u in V_{z_0} and $\sup_j \int_{\Omega} (dd^c u_j)^n < +\infty$.
- $u \in \mathcal{E}^p(\Omega), p > 0$ if there exists a sequence (u_j) in $\mathcal{E}_0(\Omega)$ decreasing to u and satisfying

$$\sup_{j \in \mathbb{N}} \int_{\Omega} (-u_j)^p (dd^c u_j)^n < +\infty.$$

If we ask additionally that $\int_{\Omega} (dd^c u_j)^n$ is uniformly bounded then by definition $u \in \mathcal{F}^p(\Omega)$.

It was proved in [Ce98], [Ce04] that the Monge–Ampère operator $(dd^c)^n$ is well-defined for functions in $\mathcal{E}(\Omega)$. Moreover, it was shown in [BGZ09, Theorem A] that if $u \in \mathcal{E}(\Omega)$ then $(dd^c \max(u, -j))^n$ converges in the strong sense of Borel measures in $\Omega \cap \{u > -\infty\}$ to $(dd^c u)^n$.

Theorem 2.1 was proved in [CK06] using a fixed point argument. We provide in this note an alternative proof using the variational method, adapting the techniques developed in Kähler geometry in [BBGZ13] (similar ideas have been used in [ACC12], [Lu15]).

PROOF OF THEOREM 2.1. Consider

$$F_{\mu}(\phi) := E_1(\phi) - \int_{\Omega} e^{\phi} d\mu, \quad \phi \in \mathcal{E}^1(\Omega),$$

where

$$E_1(\phi) := \frac{1}{n+1} \int_{\Omega} \phi (dd^c \phi)^n.$$

The Euler-Lagrange equation of F_{μ} can be computed as follows. Fix $\phi \in \mathcal{E}^1(\Omega)$ and assume $(\phi(t))$ is a smooth path in $\mathcal{E}^1(\Omega)$ starting at $\phi(0) = \phi$ with $\dot{\phi}(0) = v \in C(X)$. It follows from Stokes theorem that

$$\frac{d}{dt} E_1(\phi(t))|_{t=0} = \int_{\Omega} v (dd^c \phi)^n,$$

hence

$$\frac{d}{dt}F_\mu(\phi(t))|_{t=0} = \int_\Omega v(dd^c\phi)^n - \int_\Omega ve^\phi d\mu.$$

Thus ϕ is a critical point of the functional F_μ if it is a solution to the complex Monge–Ampère equation (2.1). It is thus natural to try and extremize F_μ in order to solve (2.1). We proceed in three steps:

Step 1: *Upper semi-continuity of F_μ .* Observe first that the functional $J(\phi) := -E_1(\phi) = |E_1(\phi)|$ is a positive proper functional on the space $\mathcal{E}^1(\Omega)$ i.e. its sublevel subsets

$$\mathcal{E}_B^1(\Omega) := \{\phi \in \mathcal{E}^1(\Omega); 0 \leq J(\phi) \leq B\}, \quad B > 0$$

are compact for the L^1 -topology. Moreover the functional E_1 is upper semi-continuous on each compact subset $\mathcal{E}_B^1(\Omega)$ for the L^1 -topology.

The continuity of the functional $L_\mu : \phi \mapsto \int_\Omega e^\phi d\mu$ on each compact subset $\mathcal{E}_B^1(\Omega)$ follows from the following fact due to Cegrell [Ce98], [ACC12, Lemma 4.1]: if $\phi_j \rightarrow \phi$ in $\mathcal{E}_B^1(\Omega)$ then $\phi_j \rightarrow \phi$ μ -a.e., hence by Lebesgue’s convergence theorem, $\lim_j L_\mu(\phi_j) = L_\mu(\phi)$ (we use here the fact that μ is non-pluripolar).

This proves that F_μ is upper semi-continuous on each $\mathcal{E}_B^1(\Omega)$.

Step 2: *Coercivity of F_μ .* Observe that $0 \leq e^\varphi \leq 1$ for $\varphi \in \mathcal{E}^1(\Omega)$, hence

$$F_\mu(\phi) \leq E_1(\phi).$$

We infer that F_μ is J -proper on $\mathcal{E}^1(\Omega)$, i.e.

$$\lim_{J(\phi) \rightarrow +\infty} F_\mu(\phi) = -\infty.$$

This implies that the maximum of F_μ on $\mathcal{E}^1(\Omega)$ is localized at a finite level of energy, i.e. there exists a constant $B > 0$ such that

$$\sup\{F_\mu(\phi); \phi \in \mathcal{E}^1(\Omega)\} = \sup\{F_\mu(\phi); \phi \in \mathcal{E}_B^1(\Omega)\},$$

Since F_μ is upper semi-continuous on the compact set $\mathcal{E}_B^1(\Omega)$, there exists $\phi \in \mathcal{E}_B^1(\Omega)$ which maximizes F_μ on $\mathcal{E}_B^1(\Omega)$ i.e.

$$F_\mu(\phi) = \inf\{F_\mu(\psi); \psi \in \mathcal{E}_B^1(\Omega)\}.$$

Step 3: *ϕ is a critical point of F_μ .* Fix a continuous test function χ with compact support in Ω and set $\phi(t) := P_\Omega(\phi + t\chi)$ for $t \in \mathbb{R}$, where $P_\Omega(u)$ denotes the plurisubharmonic envelope of u in Ω .

Observe that $\phi(t) \in \mathcal{E}^1(\Omega)$. Indeed let ρ be a continuous psh exhaustion for Ω such that $\rho < -|\chi|$ on the support of χ . Then $\phi + |t|\rho \leq \phi(t)$ for $t \in \mathbb{R}$. Since $\phi + |t|\rho \in \mathcal{E}^1(\Omega)$, it follows that $\phi(t) \in \mathcal{E}^1(\Omega)$. Set

$$h(t) := E_1(\phi(t)) - \int_{\Omega} e^{\phi+t\chi} d\mu.$$

Then since $\phi(t) \leq \phi + t\chi$, it follows that $h(t) \leq F_{\mu}(\phi(t)) \leq F_{\mu}(\phi)$ which means that h achieves its maximum at the point 0.

On the other hand we know by [BBGZ13], [ACC12], [Lu15] that

$$\frac{d}{dt}h(t) \Big|_{t=0} = \int_{\Omega} \chi(dd^c\phi)^n - \int_{\Omega} \chi e^{\phi} d\mu.$$

Since h achieves its maximum at the point 0, we have $h'(0) = 0$, hence

$$\int_{\Omega} \chi(dd^c\phi)^n = \int_{\Omega} \chi e^{\phi} d\mu.$$

As the test function χ was arbitrary, this means that the function ϕ is a solution of the equation (2.1). As μ has finite total mass we actually have that $\varphi \in \mathcal{F}^1(\Omega)$.

We now prove the uniqueness. If $\psi \in \mathcal{F}^1(\Omega)$ is another solution to (2.1) then it follows from the comparison principle [Ce98, Lemma 4.4] that

$$\begin{aligned} \int_{\{\varphi < \psi\}} e^{\psi} d\mu &= \int_{\{\varphi < \psi\}} (dd^c\psi)^n \leq \int_{\{\varphi < \psi\}} (dd^c\varphi)^n \\ &= \int_{\{\varphi < \psi\}} e^{\varphi} d\mu \leq \int_{\{\varphi < \psi\}} e^{\psi} d\mu. \end{aligned}$$

We infer $\int_{\{\varphi < \psi\}} (e^{\psi} - e^{\varphi})d\mu = 0$ hence $\psi \leq \varphi$, μ -almost everywhere and $(dd^c\varphi)^n$ -almost everywhere in Ω . For each $\varepsilon > 0$, since $(dd^c\varphi)^n$ vanishes in $\{\varphi \leq \psi - \varepsilon\} \subset \{\varphi < \psi\}$, it follows from [BGZ09, Theorem 2.2] that

$$(dd^c \max(\varphi, \psi - \varepsilon))^n \geq \mathbf{1}_{\{\varphi > \psi - \varepsilon\}}(dd^c\varphi)^n = (dd^c\varphi)^n.$$

It then follows from the comparison principle [Ce98, Theorem 4.5] that $\max(\varphi, \psi - \varepsilon) \leq \varphi$, for all $\varepsilon > 0$, hence $\psi \leq \varphi$. Reversing the role of φ and ψ in the above argument gives $\varphi = \psi$, proving the uniqueness.

Assume finally that $\mu \leq (dd^c u)^n$, where u is a bounded negative psh function in Ω . Then since $\varphi \leq 0$ we have $(dd^c\varphi)^n = e^{\varphi}\mu \leq \mu \leq (dd^c u)^n$. Since u is bounded (in particular it belongs to the domain of definition of the complex Monge–Ampère operator) and $\varphi \in \mathcal{F}^1(\Omega)$ with $(dd^c\varphi)^n$ putting no mass on pluripolar sets, it follows from [BGZ09, Corollary 2.4] that $u \leq \varphi$. In particular φ vanishes on the boundary $\partial\Omega$ if u does so. \square

3. The main result.

3.1. Proof of the main result.

Given φ a plurisubharmonic function in a domain Ω , we let

$$\varphi_{\varepsilon}(z) = \varphi \star \chi_{\varepsilon}(z) := \int_{\mathbb{C}^n} \varphi(z - \varepsilon w)\chi(w)dV(w) = \int_{\mathbb{C}^n} \varphi(w)\chi_{\varepsilon}(z - w)dV(w)$$

denote the standard regularizations of φ defined in Ω_ε for $\varepsilon > 0$ small enough, where $\Omega_\varepsilon = \{z \in \Omega, \text{dist}(z, \partial\Omega) > \varepsilon\}$.

Here χ_ε are non-negative radial functions with compact support in the ball $\mathbb{B}(\varepsilon)$ of radius ε and such that $\int_{\mathbb{C}^n} \chi_\varepsilon dV = 1$, where dV denotes the Euclidean volume form. The first expression shows that φ_ε is a (positive) sum of plurisubharmonic functions (hence itself plurisubharmonic) in Ω_ε , while the second expression shows that φ_ε is smooth in Ω .

3.1.1. The implication (ii) \Rightarrow (i).

The mean value property shows that the φ_ε 's decrease to φ as ε decreases to zero. It follows therefore from Bedford–Taylor’s continuity results [BT76], [BT82] that (ii) \Rightarrow (i) holds.

3.1.2. The equivalence (ii) \Leftrightarrow (iii).

The starting point of (iii) is the classical interpretation of the determinant as an infimum of traces:

LEMMA 3.1.

$$(\det Q)^{1/n} = \inf\{n^{-1}\text{tr}(HQ); H \in \mathcal{H}\}.$$

We first show that (iii) \Rightarrow (ii). Indeed assume that

$$\Delta_H \varphi \geq f^{1/n}$$

for all positive definite Hermitian matrix H normalized by $\det H = 1$. Since Δ_H is a linear operator, we infer

$$\Delta_H(\varphi \star \chi_\varepsilon) \geq f^{1/n} \star \chi_\varepsilon.$$

This inequality holds for all normalized H , hence Lemma 3.1 yields

$$(dd^c(\varphi \star \chi_\varepsilon))^n \geq (f^{1/n} \star \chi_\varepsilon)^n dV,$$

where this inequality holds in the classical (pointwise, differential) sense.

We conversely check that (ii) \Rightarrow (iii). Since $\varphi \star \chi_\varepsilon$ is smooth, Lemma 3.1 shows indeed that

$$\Delta_H(\varphi \star \chi_\varepsilon) \geq f^{1/n} \star \chi_\varepsilon$$

for all normalized $H \in \mathcal{H}$. Letting $\varepsilon \rightarrow 0$ and taking limits in the sense of distributions yields (iii).

3.1.3. The implication (i) \Rightarrow (ii).

We finally focus on the most delicate implication.

Step 1: Assume first that $(dd^c \varphi)^n \geq f dV$, with f continuous. This inequality can be here interpreted equivalently in the pluripotential or in the viscosity sense, as shown in [EGZ11, Proposition 1.5], whose proof moreover shows the equivalence with the property

that

$$\Delta_H \varphi \geq f^{1/n} \text{ for all } H \in \mathcal{H}.$$

Thus (i) \Leftrightarrow (iii) in our main theorem, when f is continuous. Since any lower semi-continuous function is the increasing limit of continuous functions, the implication (i) \Rightarrow (iii) immediately extends to the case when f is lower semi-continuous.

It remains to get rid of this extra continuity assumption. We are going to approximate f by continuous densities f_k , use the previous result and stability estimates to conclude. The approximation process, inspired by [Ber13], is somehow delicate, so we proceed in several steps.

Step 2: Note first that we can assume that f is *bounded*: we can replace f by $\min(f, A) \in L^\infty(\Omega)$ and let eventually A increase to $+\infty$. Since the problem is local, we can work on fixed balls $B' \Subset B$ and use a max construction to modify φ in a neighborhood of the boundary ∂B , making it equal to the defining function of B .

We fix $0 < \delta < 1$ and $j \in \mathbb{N}^*$. Since $f \in L^2(B) \supset L^\infty(B)$, it follows from [CP92], [Kol95] that there exists $U_f \in PSH(B) \cap C^0(\bar{B})$ such that

$$(dd^c U_f)^n = f dV, \quad U_f = 0 \text{ in } \partial B.$$

Set $C_j := \sup e^{-j\varphi/n}$ and observe that

$$e^{-j\varphi} \{f dV + \delta(dd^c \varphi)^n\} \leq (dd^c v)^n$$

where $v := C_j(U_f + \varphi)$ is bounded, plurisubharmonic, with $v = 0$ on ∂B .

By Theorem 2.1 there exists a unique bounded plurisubharmonic solution $\varphi_{j,\delta}$ to the Dirichlet problem

$$(dd^c \varphi_{j,\delta})^n = e^{j(\varphi_{j,\delta} - \varphi)} \{f dV + \delta(dd^c \varphi)^n\} \tag{3.1}$$

in B with boundary values 0.

We now observe that $\varphi_{j,\delta}$ uniformly converges to φ , as $j \rightarrow +\infty$, independently of the value of $\delta > 0$:

LEMMA 3.2. *For all $j \geq 1, \delta \in (0, 1), z \in B$,*

$$\varphi(z) - \frac{\log(1 + \delta)}{j} \leq \varphi_{j,\delta}(z) \leq \varphi(z) + \frac{(-\log \delta)}{j}.$$

PROOF. It follows from the comparison principle that $\varphi_{j,\delta}$ is the envelope of subsolutions. It thus suffices to find good sub/supersolutions to insure that $\varphi_{j,\delta}$ converges to φ , as $j \rightarrow +\infty$.

Observe that $u = \varphi - (\log(1 + \delta))/j \leq \varphi$ is plurisubharmonic in B , with boundary values $u|_{\partial B} \leq 0$. Moreover

$$(dd^c u)^n = (dd^c \varphi)^n = e^{j(u - \varphi)}(1 + \delta)(dd^c \varphi)^n \geq e^{j(u - \varphi)} \{f dV + \delta(dd^c \varphi)^n\},$$

since $(dd^c\varphi)^n \geq fdV$. Thus u is a subsolution to the Dirichlet problem, showing that $u \leq \varphi_{j,\delta}$.

Set now $v = \varphi + (-\log \delta)/j$. This is a plurisubharmonic function in B such that $v \geq 0$ on ∂B and

$$(dd^c v)^n = (dd^c \varphi)^n = e^{j(v-\varphi)} \delta(dd^c \varphi)^n \leq e^{j(v-\varphi)} \{fdV + \delta(dd^c \varphi)^n\}.$$

Thus v is a supersolution of the Dirichlet problem hence $\varphi_{j,\delta} \leq v$. □

Step 3: We now approximate f in L^2 by continuous densities $0 \leq f_k$, with $\|f_k - f\|_{L^2} \rightarrow 0$ as $k \rightarrow +\infty$. Extracting a subsequence and relabelling, we can assume that there exists $g \in L^2(B)$ such that $f_k \leq g$ for all $k \in \mathbb{N}$ and f_k converges almost everywhere to f . Arguing as above we obtain

$$e^{-j\varphi} \{f_k dV + \delta(dd^c \varphi)^n\} \leq (dd^c v)^n,$$

where $v := C_j(U_g + \varphi)$ is bounded, plurisubharmonic, with $v = 0$ in ∂B . By Theorem 2.1, there exists a unique bounded plurisubharmonic solution $\varphi_{j,\delta,k}$ to the Dirichlet problem

$$(dd^c \varphi_{j,\delta,k})^n = e^{j(\varphi_{j,\delta,k} - \varphi)} \{f_k dV + \delta(dd^c \varphi)^n\}$$

in B , with zero boundary values.

The comparison principle shows that for all $k \in \mathbb{N}$,

$$C_j(U_g + \varphi) \leq \varphi_{j,\delta,k} \leq 0. \tag{3.2}$$

Thus $k \mapsto \varphi_{j,\delta,k}$ is uniformly bounded in B . Extracting and relabelling, we can assume that it converges to a plurisubharmonic function $\psi = \psi_{j,\delta}$ in $L^1(\Omega)$ such that

$$C_j(U_g + \varphi) \leq \psi_{j,\delta} \leq 0. \tag{3.3}$$

We claim that $\psi_{j,\delta} = \varphi_{j,\delta}$ in B . To simplify notations we write $u_k := \varphi_{j,\delta,k}$ and $u := \psi_{j,\delta}$. From (3.2) and (3.3), it follows that $u_k = u = 0$ in ∂B . On the other hand let $\tilde{u}_\ell := (\sup_{k \geq \ell} u_k)^*$ for $\ell \in \mathbb{N}$. This is a decreasing sequence of bounded plurisubharmonic functions converging to u in B . We infer for all ℓ ,

$$(dd^c \tilde{u}_\ell)^n \geq e^{\inf_{k \geq \ell} j(u_k - \varphi)} \inf_{k \geq \ell} (f_k dV + \delta(dd^c \varphi)^n).$$

Letting $\ell \rightarrow +\infty$ yields

$$(dd^c u)^n \geq e^{j(u-\varphi)} \{fdV + \delta(dd^c \varphi)^n\},$$

which implies that $u = \psi_{j,\delta}$ is a subsolution to the Dirichlet problem for the equation (3.1). Hence $\psi_{j,\delta} \leq \varphi_{j,\delta}$.

By [Kol95] there exists a bounded plurisubharmonic function ρ_k in B , solution to the Dirichlet problem

$$(dd^c \rho_k)^n = e^{j(\varphi_{j,\delta} - \varphi)} |f - f_k| dV, \text{ with } \rho_k|_{\partial B} = 0,$$

with the uniform bound

$$\|\rho_k\|_{L^\infty(B)} \leq C\|f - f_k\|_{L^2(B)}^{1/n},$$

where $C > 0$ is independent of k . In particular $\rho_k \rightarrow 0$ uniformly in B .

Since $f_k \leq f + |f - f_k|$ and $\rho_k \leq 0$ it follows that $w := \varphi_{j,\delta} + \rho_k$ satisfies

$$(dd^c w)^n = (dd^c(\varphi_{j,\delta} + \rho_k))^n \geq e^{j(w-\varphi)}(f_k dV + \delta(dd^c \varphi)^n).$$

The comparison principle insures $w \leq \varphi_{j,\delta,k}$ hence $\varphi_{j,\delta} \leq \psi_{j,\delta}$ since $\rho_k \rightarrow 0$.

Conclusion. We have thus shown that $\psi_{j,\delta} = \varphi_{j,\delta}$ and $\varphi_{j,\delta,k} \geq \varphi_{j,\delta} + \rho_k$. Lemma 3.2 yields

$$j(\varphi_{j,\delta,k} - \varphi) \geq -\log(1 + \delta) - j\eta_k,$$

where $\eta_k := \|\rho_k\|_{L^\infty(B)} \xrightarrow{k \rightarrow +\infty} 0$.

Since f_k is continuous we can apply *Step 1* to insure that

$$(dd^c \varphi_{j,\delta,k} \star \chi_\varepsilon)^n \geq \frac{e^{-j\eta_k}}{\delta + 1} (f_k^{1/n} \star \chi_\varepsilon)^n dV.$$

We know that $\limsup_{k \rightarrow +\infty} \varphi_{j,\delta,k} \leq \varphi_{j,\delta}$ since $\varphi_{j,\delta,k} \rightarrow \varphi_{j,\delta}$ in $L^1(B)$ as $k \rightarrow +\infty$. Since $\varphi_{j,\delta,k} \geq \varphi_{j,\delta} - \eta_k$ and $\lim_{k \rightarrow +\infty} \eta_k = 0$, it follows from Hartogs lemma that $\varphi_{j,\delta,k} \rightarrow \varphi_{j,\delta}$ in capacity. Letting $k \rightarrow +\infty$ we obtain

$$(dd^c \varphi_{j,\delta} \star \chi_\varepsilon)^n \geq \frac{1}{\delta + 1} (f^{1/n} \star \chi_\varepsilon)^n dV.$$

By Lemma 3.2 $(\varphi_{j,\delta})$ uniformly converges to φ as $j \rightarrow +\infty$, hence

$$(dd^c \varphi \star \chi_\varepsilon)^n \geq \frac{1}{\delta + 1} (f^{1/n} \star \chi_\varepsilon)^n dV.$$

We let finally δ decrease to zero to obtain the desired lower bound (ii).

3.1.4. An extension of the main result.

By approximating a given function φ in the Cegrell class $\mathcal{E}(\Omega)$ by the decreasing sequence $\varphi_j := \max(\varphi, -j)$ of bounded plurisubharmonic functions, we let the reader check that the main theorem holds when φ merely belongs to $\mathcal{E}(\Omega)$.

3.2. Mixed inequalities.

We now prove the Corollary of the introduction on mixed Monge–Ampère measures of subsolutions, providing an alternative proof of [Kol03, Lemma 1.2]:

PROPOSITION 3.3. *Assume $\varphi_1, \dots, \varphi_n$ are bounded plurisubharmonic functions in Ω , such that $(dd^c \varphi_i)^n \geq f_i dV$, where $0 \leq f_i \in L^1(\Omega)$. Then*

$$dd^c \varphi_1 \wedge \dots \wedge dd^c \varphi_n \geq f_1^{1/n} \dots f_n^{1/n} dV.$$

PROOF. The inequality is classical when the functions φ_i are smooth, and follows from the concavity of $H \mapsto \log \det H$ (see [HJ85, Corollary 7.6.9]).

To treat the general case we replace each φ_i by its convolutions $\varphi_i \star \chi_\varepsilon$. We can always assume that $f_i \in L^\infty(\Omega)$. Our main result insures that

$$(dd^c(\varphi_i \star \chi_\varepsilon))^n \geq (f_i^{1/n} \star \chi_\varepsilon)^n dV,$$

hence

$$dd^c(\varphi_1 \star \chi_\varepsilon) \wedge \cdots \wedge dd^c(\varphi_n \star \chi_\varepsilon) \geq (f_1^{1/n} \star \chi_\varepsilon) \cdots (f_n^{1/n} \star \chi_\varepsilon) dV.$$

The left hand side converges weakly to $dd^c\varphi_1 \wedge \cdots \wedge dd^c\varphi_n$ by Bedford–Taylor’s continuity results [BT76], [BT82], while $(f_i^{1/n} \star \chi_\varepsilon)$ converges to $(f_i)^{1/n}$ in L^n by Lebesgue convergence Theorem. Hence

$$(f_1 \star \chi_\varepsilon)^{1/n} \cdots (f_n \star \chi_\varepsilon)^{1/n} \text{ converges to } (f_1)^{1/n} \cdots (f_n)^{1/n}$$

in L^1 . The conclusion follows. □

We note conversely that these mixed inequalities yield an important implication in our main result. Assume indeed that $(dd^c\varphi)^n \geq f dV$ in the pluripotential sense. Fix $f_1 = f$ and $\varphi_2 = \dots = \varphi_n = \rho_H$, where $\rho_H = \sum h_{jk} z_j \bar{z}_k$ with $H \in \mathcal{H}$, so that $(dd^c\varphi_i)^n \geq f_i dV$ with $f_2 = \dots = f_n = 1$. It follows from the mixed inequalities above that

$$\Delta_H \varphi = dd^c \varphi \wedge dd^c \varphi_2 \wedge \cdots \wedge dd^c \varphi_n \geq f^{1/n}.$$

One can alternatively proceed as follows: observe that

$$\begin{aligned} & (dd^c \varphi \star \chi_\varepsilon)^n(z) \\ &= \int dd^c \varphi(z - w_1) \wedge \cdots \wedge dd^c \varphi(z - w_n) \chi_\varepsilon(w_1) \cdots \chi_\varepsilon(w_n) dV(w_1, \dots, w_n) \\ &\geq \int f^{1/n}(z - w_1) \cdots f^{1/n}(z - w_n) dV(z) \chi_\varepsilon(w_1) \cdots \chi_\varepsilon(w_n) dV(w_1, \dots, w_n) \\ &= (f^{1/n} \star \chi_\varepsilon)^n(z) dV(z). \end{aligned}$$

3.3. More general right hand side.

There are several ways one can extend our main observation. We note here the following:

THEOREM 3.4. *Assume φ is plurisubharmonic and bounded. Fix $g \in L^1$ and $h : \mathbb{R} \rightarrow \mathbb{R}$ convex. The following are equivalent:*

- (i) $(dd^c\varphi)^n \geq e^{h(\varphi)+g} dV$ in the pluripotential sense;
- (ii) $(dd^c(\varphi \star \chi_\varepsilon))^n \geq e^{h(\varphi \star \chi_\varepsilon)+g \star \chi_\varepsilon} dV$ in the classical sense, for all $\varepsilon > 0$;
- (iii) $\Delta_H \varphi \geq e^{h(\varphi)/n+g/n}$ in the sense of distributions, for all $H \in \mathcal{H}$.

The proof is very close to what we have done above, using the convexity of \exp and h through Jensen’s inequality. We leave the details to the reader.

ACKNOWLEDGEMENTS. We thank the referee for useful comments.

References

- [ACC12] P. Ahag, U. Cegrell and R. Czyz, On Dirichlet’s principle and problem, *Math. Scand.*, **110** (2012), 235–250.
- [BT76] E. Bedford and B. A. Taylor, The Dirichlet problem for a complex Monge–Ampère equation, *Invent. Math.*, **37** (1976), 1–44.
- [BT82] E. Bedford and B. A. Taylor, A new capacity for plurisubharmonic functions, *Acta Math.*, **149** (1982), 1–40.
- [BGZ09] S. Benelkourchi, V. Guedj and A. Zeriahi, Plurisubharmonic functions with weak singularities, Complex analysis and digital geometry, Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist., 86, Uppsala Universitet, Uppsala, 2009, 57–74.
- [Ber13] R. J. Berman, From Monge–Ampère equations to envelopes and geodesic rays in the zero temperature limit, *Mathematische Zeitschrift*, online, 2018.
- [BBGZ13] R. Berman, S. Boucksom, V. Guedj and A. Zeriahi, A variational approach to complex Monge–Ampère equations, *Publ. Math. I.H.E.S.*, **117** (2013), 179–245.
- [Bl96] Z. Błocki, The complex Monge–Ampère operator in hyperconvex domains, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **23** (1996), 721–747.
- [Ce98] U. Cegrell, Pluricomplex energy, *Acta Math.*, **180** (1998), 187–217.
- [Ce04] U. Cegrell, The general definition of the complex Monge–Ampère operator, *Ann. Inst. Fourier (Grenoble)*, **54** (2004), 159–179.
- [CK06] U. Cegrell and S. Kołodziej, The equation of complex Monge–Ampère type and stability of solutions, *Math. Ann.*, **334** (2006), 713–729.
- [CP92] U. Cegrell and L. Persson, The Dirichlet problem for the complex Monge–Ampère operator: stability in L^2 , *Michigan Math. J.*, **39** (1992), 145–151.
- [Din09] S. Dinew, An inequality for mixed Monge–Ampère measures, *Math. Z.*, **262** (2009), 1–15.
- [DL15] S. Dinew and C. H. Lu, Mixed Hessian inequalities and uniqueness in the class $\mathcal{E}(X, \omega, m)$, *Math. Z.*, **279** (2015), 753–766.
- [EGZ11] P. Eyssidieux, V. Guedj and A. Zeriahi, Viscosity solutions to Degenerate Complex Monge–Ampère Equations, *Comm. Pure Appl. Math.*, **64** (2011), 1059–1094.
- [HL13] R. Harvey and B. Lawson, The equivalence of viscosity and distributional subsolutions for convex subequations, a strong Bellman principle, *Bull. Braz. Math. Soc.*, **44** (2013), 621–652.
- [HJ85] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge Univ. Press, 1985.
- [Kol95] S. Kołodziej, The range of the complex Monge–Ampère operator II, *Indiana Univ. Math. J.*, **44** (1995), 765–782.
- [Kol03] S. Kołodziej, The Monge–Ampère equation on compact Kähler manifolds, *Indiana Univ. Math. J.*, **52** (2003), 667–686.
- [Lu15] C. H. Lu, A variational approach to complex Hessian equations in \mathbb{C}^n , *J. Math. Anal. Appl.*, **431** (2015), 228–259.

Vincent GUEDJ

Institut de Mathématiques de Toulouse
 Université de Toulouse, CNRS
 118 route de Narbonne
 31400 Toulouse, France
 E-mail: vincent.guedj@math.univ-toulouse.fr

Chinh H. LU

Laboratoire de Mathématiques d’Orsay
 Univ. Paris-Sud
 CNRS, Université Paris-Saclay
 91405 Orsay, France
 E-mail: hoang-chinh.lu@math.u-psud.fr

Ahmed ZERIAHI

Institut de Mathématiques de Toulouse

Université de Toulouse, CNRS

118 route de Narbonne

31400 Toulouse, France

E-mail: ahmed.zeriahi@math.univ-toulouse.fr