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General formal solutions for a unified family of P_{J}-hierarchies (J=I, II, IV, 34)

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Abstract. A unified family of P_{J}-hierarchies (J=I, II, IV, 34) with a large parameter is introduced and we construct general formal solutions which are called instanton-type solutions for the system.

1. Introduction.

In the pioneering works of [**6**]–[**14**], [**16**]–[**18**], some of Painlevé hierarchies and the Noumi–Yamada system with a large parameter have been studied by the exact WKB analysis and important results have been established. The results established in these papers suggest that there are common structures in analyzing Painlevé hierarchies and the exact WKB analysis is an effective method to study these Painlevé hierarchies and completely integrable systems. Motivated by these works, the aim of this article is to study the common structures between P_{J}-hierarchies (J=I, II, IV, 34). For that purpose, we introduce a unified family of P_{J}-hierarchies (J=I, II, IV, 34) with a large parameter. A key idea for the formulation of the unified family of the hierarchies is to use generating functions. The papers [**2**], [**20**] and [**21**] introduce an additional variable θ to the m-th member of P_{J}-hierarchies (J=I, II, IV, 34), which will be denoted by $(P_{\mathrm{J}})_m$ ($m = 1, 2, \cdots$), and $(P_{\mathrm{J}})_m$ is written in terms of generating functions of unknown functions. Our system (13) given in this article is of a naturally extended form by the common structures of $(P_{\mathrm{J}})_m$ described in [**2**], [**20**] and [**21**]. Note that, although we have succeeded in construction of the unified family of P_{J}-hierarchies, it is still unknown if it contains other known Painlevé hierarchies or essentially new equations. It is our future problem to answer this question.

Once we have obtained the unified family of systems, the next important step of our study is to extend the results of [**2**], [**20**] and [**21**] to the unified family. When we discuss the connection problem for our system by the exact WKB analysis based on Borel resummation, we need a usual formal solution (which is called a 0-parameter solution) and a formal solution of instanton-type with sufficiently many free parameters which can describe the Stokes phenomenon of the Borel sum of a 0-parameter solution. In general, for a non-linear differential equation, the existence of a formal solution of instanton-type is non-trivial. In this paper, we show the existence of instanton-type formal solutions for non-linear differential equations derived from the unified family of P_{J}-hierarchies by

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using the method given in [**2**], [**20**] and [**21**]. As a matter of fact, our system (13) given in Section 3 is expressed by a proper generating function with arbitrary coefficients (see (14)). Because of the arbitrariness of coefficients, many systems of non-linear differential equations can be derived from our system. One can see in Section 4 that the method of [**2**] is applicable to the systems which are derived from (13).

The paper is organized in the following way. In Section 2 we recall expressions of $(P_{\mathrm{J}})_m$ (J=I, II, IV, 34) by generating functions of their unknown functions and we rewrite $(P_{\mathrm{J}})_m$ in the unified form (11). In Section 3 we introduce the explicit form of a unified family of $(P_{\mathrm{J}})_m$ (J=I, II, IV, 34) with a large parameter. As one can see in (16), the coefficients f_1 and f_2 in (13) have special forms, and the reason why they take such forms is explained in Section 4.1. In Section 4.2, we have the concrete form of a 0-parameter solution. In Section 4.3 and Section 4.4, we investigate the algebraic structures associated with our system. Firstly, we derive the linearized equation along the leading term of a 0-parameter solution, and then, we derive a suitable partial differential equation (33) for the construction of instanton-type solutions. In Section 4.4, we define the map Q which is a key to describe the system (33). The structure of (33) varies depending on Case I and Case II which are explained in Section 4.4.1 and Section 4.4.2. Note that, in subsequent discussions, we consider Case I and Case II separately. The most interesting result here is Lemma 4.1 showing that eigenvectors of Q have important multiplicative relations. At the end of Section 4, we give the main theorem, and successive Sections 5, 6 and 7 are devoted to its proof. In the procedure of the construction of instanton-type formal solutions, we need to solve the non-secularity conditions. The solvability of the non-secularity conditions for our system is also studied in Sections 5, 6 and 7.

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2. P_{J}-hierarchies (J=I, II, IV, 34) by generating functions.

In this section, we rewrite the general members $(P_{\mathrm{J}})_m$ $(m = 1, 2, \dots)$ of P_{J}-hierarchies (J=I, II, IV, 34) with a large parameter η in a unified form. The objects discussed here are essentially the same as the P_{J}-hierarchies studied by Kudryashov [**15**], Gordoa–Joshi–Pickering [**5**] and Clarkson–Joshi–Pickering [**3**].

In what follows, θ denotes an independent variable and the notation $A \equiv B$ means that $A - B$ is zero module θ^{m+2}. For any formal power series x of θ, we define $\sigma_i^{\theta}(x)$ by the coefficient of θ^i in x.

(i) The m-th member $(P_{\mathrm{J}})_m$ of P_{J}-hierarchy with a large parameter η (J=I, 34):
We define generating functions of u_k, v_k and c_{k-1} $(k = 1, 2, \dots)$ by

$$U(\theta) := \sum_{k=1}^{\infty} u_k \theta^k, \quad V(\theta) := \sum_{k=1}^{\infty} v_k \theta^k \quad \text{and} \quad C(\theta) := \sum_{k=1}^{\infty} c_{k-1} \theta^k, \tag{1}$$

respectively. Here u_k, v_k are unknown functions of the variable t and c_{k-1} is a constant.

(a) $(P_{\mathrm{I}})_m$ is written in the following form (see [**2**]).

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv \begin{pmatrix} 2V\theta \\ -(1+2u_1\theta)(1-U)+2t\theta^{m+1}+\dfrac{1+2C-V^2\theta}{1-U} \end{pmatrix} \tag{2}$$

with $\sigma^\theta_{m+1}(U)=0$, $\sigma^\theta_{m+1}(V)=0$ and $c_0=0$.

Let us define H and f_i by

$$H(U,\,V):=1+2C-V^2\theta \quad \text{and} \quad f_1:=0, \quad f_2:=-(1+2u_1\theta)+2t\theta^{m+1}, \tag{3}$$

respectively. Using (3), we rewrite (2) as follows.

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\times(1-U)+\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \partial H/\partial U \\ \partial H/\partial V \end{pmatrix}+\begin{pmatrix} 0 \\ H(U,\,V)/(1-U) \end{pmatrix} \tag{4}$$

with $\sigma^\theta_{m+1}(U)=0$, $\sigma^\theta_{m+1}(V)=0$ and $c_0=0$.

(b) Let $\gamma(\neq 0)$ and κ be constants. The paper [**21**] showed that $(P_{34})_m$ studied by [**3**] and [**14**] is essentially equivalent to the system

$$\begin{aligned}\eta^{-1}\frac{d}{dt}\begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv& \begin{pmatrix} 2V\theta \\ -(1+2(u_1+c_0)\theta)(1-U)+\dfrac{1+2C-V^2\theta}{1-U} \end{pmatrix}\\ &+\begin{pmatrix} 0 \\ 2\gamma t\theta^m(1+(u_1+2c_0)\theta) \end{pmatrix}\end{aligned} \tag{5}$$

with

$$\sigma^\theta_{m+1}(U)=-\sigma^\theta_{m+1}(W)+c_0\sigma^\theta_m(U)+\frac{(\sigma^\theta_m(V))^2-\kappa^2}{2\sigma^\theta_m(U)}, \quad \sigma^\theta_{m+1}(V)=0. \tag{6}$$

Here W is defined by

$$W\equiv\frac{U^2-\theta V^2+2C}{2(1-U)}-c_0\theta(1-U)+\gamma t\theta^m(1+(u_1+2c_0)\theta). \tag{7}$$

Set H and f_i by

$$H(U,\,V):=1+2C-V^2\theta \quad \text{and} \quad f_1:=0, \quad f_2:=-\left(1+2(u_1+c_0)\theta\right)\left(1-2\gamma t\theta^m\right),$$

respectively. Then the system (5) is transformed into the same form as (4) with conditions (6) for $(P_{34})_m$.

(ii) The m-th member $(P_{\mathrm{J}})_m$ of P_{J}-hierarchy with a large parameter η (J=II, IV):
Let us define generating functions of u_k, v_k and c_k $(k=1,\,2,\,\dots)$ by

$$U(\theta):=\sum_{k=1}^\infty u_k\theta^k, \quad V(\theta):=\sum_{k=1}^\infty v_k\theta^k \quad \text{and} \quad C(\theta):=\sum_{k=1}^\infty c_k\theta^k,$$

respectively. Here u_k, v_k are unknown functions of the variable t and c_k is a constant.

(c) $(P_{\mathrm{II}})_m$ introduced by [**5**] is equivalent to the following system (see [**13**], [**20**]):

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix}U\theta\\V\theta\end{pmatrix}\equiv 2\begin{pmatrix}u_1(1-U+C)\theta-U-V\theta\\-v_1(1-U+C)\theta+\dfrac{2UV+V^2\theta}{2(1-U+C)}+V\end{pmatrix}\tag{8}$$

with $\sigma^{\theta}_{m+1}(U)=\gamma t$ and $\sigma^{\theta}_{m+1}(V)=\kappa$. Here $\gamma(\neq 0)$ and κ are constants.

(d) $(P_{\mathrm{IV}})_m$ introduced by [**5**] is equivalent to the following system (see [**13**], [**20**]):

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix}U\theta\\V\theta\end{pmatrix}\equiv 2\begin{pmatrix}u_1(1-U+C)\theta-U-V\theta-\gamma t\theta^m\\-v_1(1-U+C)\theta+\dfrac{2UV+V^2\theta}{2(1-U+C)}+V+\gamma tv_1\theta^{m+1}\end{pmatrix}\tag{9}$$

with $\sigma^{\theta}_{m+1}(U)=-\alpha_1$ and $\sigma^{\theta}_{m+1}(V)=-\sigma^{\theta}_{m+1}(W)-\big(((\sigma^{\theta}_m(V)-\alpha_1)^2-\alpha_2^2)/(2(\sigma^{\theta}_m(U)-\sigma^{\theta}_m(C)))\big)$.
Here W is defined by $W\equiv((2UV+\theta V^2)/2(1-U+C))+\gamma tv_1\theta^{m+1}$ and $\gamma(\neq 0),\alpha_1,\alpha_2$ are constants. In parallel with (i), we can rewrite (8) and (9) as follows.

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix}U\theta\\V\theta\end{pmatrix}\equiv\begin{pmatrix}f_1\\f_2\end{pmatrix}\times(1-U+C)+\begin{pmatrix}0\\H(U,V)/(1-U+C)\end{pmatrix}+\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\partial H/\partial U\\\partial H/\partial V\end{pmatrix},\tag{10}$$

where H, f_1 and f_2 are defined by the table below. Note that the conditions of $\sigma^{\theta}_{m+1}(U)$ and $\sigma^{\theta}_{m+1}(V)$ are given in (8) and (9) respectively.

$(P_{\mathrm{II}})_m$	$(P_{\mathrm{IV}})_m$
$H(U,V):=2UV+V^2\theta$ $f_1:=2u_1\theta$ $f_2:=-2v_1\theta$	$H(U,V):=2UV+V^2\theta$ $f_1:=2u_1\theta-2\gamma t\theta^m(1+(u_1-c_1)\theta)$ $f_2:=-2v_1\theta+2\gamma tv_1\theta^{m+1}$

Moreover, by replacing $1-U+C$ in (10) with $1-U$, we can transform (10) into the form of (4).

Summing up, each $(P_{\mathrm{J}})_m$ (J=I, II, IV, 34) is reduced to the following form

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix}U\theta\\V\theta\end{pmatrix}\equiv\begin{pmatrix}f_1\\f_2\end{pmatrix}\times(1-U)+\begin{pmatrix}0\\H(U,V)/1-U\end{pmatrix}+\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\partial H/\partial U\\\partial H/\partial V\end{pmatrix}\tag{11}$$

with the conditions below.

$(P_{\mathrm{I}})_m$	$(P_{34})_m$
$H(U,V):=1+2C-V^2\theta$	$H(U,V):=1+2C-V^2\theta$
$f_1:=0$	$f_1:=0$
$f_2:=-(1+2u_1\theta)+2t\theta^{m+1}$	$f_2:=-\,(1+2(u_1+c_0)\theta)\,(1-2\gamma t\theta^m)$
$\sigma^{\theta}_{m+1}(U)=0$	$\sigma^{\theta}_{m+1}(U)=-\sigma^{\theta}_{m+1}(W)+c_0\sigma^{\theta}_m(U)+\dfrac{(\sigma^{\theta}_m(V))^2-\kappa^2}{2\sigma^{\theta}_m(U)}$
$\sigma^{\theta}_{m+1}(V)=0$	$\sigma^{\theta}_{m+1}(V)=0$ Here W is defined by (7).

$(P_{\mathrm{II}})_m$	$(P_{\mathrm{IV}})_m$
$H(U, V) := 2UV + 2CV + V^2\theta$	$H(U, V) := 2UV + 2CV + V^2\theta$
$f_1 := 2(u_1 + c_1)\theta$	$f_1 := 2(u_1 + c_1)\theta - 2\gamma t\theta^m(1 + u_1\theta)$
$f_2 := -2v_1\theta$	$f_2 := -2v_1\theta + 2\gamma t v_1\theta^{m+1}$
$\sigma^\theta_{m+1}(U) = \gamma t$	$\sigma^\theta_{m+1}(U) = -\alpha_1$
$\sigma^\theta_{m+1}(V) = \kappa$	$\sigma^\theta_{m+1}(V) = -\sigma^\theta_{m+1}(W) - \dfrac{(\sigma^\theta_m(V) - \alpha_1)^2 - \alpha_2^2}{2\sigma^\theta_m(U)}.$
	Here $W \equiv \dfrac{2(U + C)V + \theta V^2}{2(1 - U)} + \gamma t v_1\theta^{m+1}.$

3. A unified family of P_J-hierarchies.

We extend the results presented by [**2**] and [**20**] to our unified family of P_J-hierarchies. Let u_k and v_k be unknown functions of the variable t, and let c_k be a constant. Such as $(P_J)_m$ (J=I, II, IV, 34), we consider a system with $2m$ unknown functions u_k, v_k $(1 \le k \le m)$. Define U, V and C by

$$U(\theta) := \sum_{k=1}^{\infty} u_k\theta^k, \quad V(\theta) := \sum_{k=1}^{\infty} v_k\theta^k \quad \text{and} \quad C(\theta) := \sum_{k=1}^{\infty} c_k\theta^k, \tag{12}$$

respectively. Here θ denotes an independent variable and u_{m+1}, v_{m+1} are arbitrary polynomials of the variables $(u_1, \dots, u_m, v_1, \dots, v_m)$ with coefficients in holomorphic functions of t, i.e., $u_{m+1}, v_{m+1} \in \mathcal{O}(t)[u_1, \dots, u_m, v_1, \dots, v_m]$. Note that u_{m+1} and v_{m+1} are independent of η, and $c_{m+1} = 0$. By $A \equiv B$ we mean that $A - B$ is zero modulo θ^{m+2} and consider

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \times (1 - U) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \partial H/\partial U \\ \partial H/\partial V \end{pmatrix} + \begin{pmatrix} 0 \\ H(U, V)/(1 - U) \end{pmatrix}, \tag{13}$$

where $H(U, V)$ is a polynomial in U and V of degree at most 2 with arbitrary complex constants p_i of the following form

$$H(U, V) := (p_1U^2 + p_2V^2)\theta + p_3UV + p_4CU + p_5CV + p_6U + p_7V + p_8C + p_9 \tag{14}$$

and f_1 and f_2 are defined by

$$f_1 := x_{1,0} + x_{1,1}\theta + x_{1,m}\theta^m + x_{1,m+1}\theta^{m+1}, \quad f_2 := x_{2,0} + x_{2,1}\theta + x_{2,m}\theta^m + x_{2,m+1}\theta^{m+1}. \tag{15}$$

Here the coefficients $x_{1,i}$ and $x_{2,i}$ of f_1 and f_2 are determined later in Section 4. As its consequence, the explicit forms of f_1 and f_2 become the following.

$$\begin{aligned} f_1 &= p_7 + (\alpha u_1 + p_5c_1)\theta + y_1\theta^m + (y_1u_1 + y_2)\theta^{m+1}, \\ f_2 &= -\beta - (2\beta u_1 + \alpha v_1 + \varepsilon c_1)\theta + z_1\theta^m + (2z_1u_1 - y_1v_1 + z_2)\theta^{m+1}, \end{aligned} \tag{16}$$

where y_i, z_i are arbitrary holomorphic functions of t and α, β, ε are given by

$$\alpha := p_3 + p_7, \quad \beta := p_6 + p_9 \quad \text{and} \quad \varepsilon := p_4 + p_8, \tag{17}$$

respectively.

Remark that (13) is a unified family of $(P_{\mathrm{J}})_m$ (J=I, II, IV, 34). Indeed,

- if $p_2 = -1$, $p_8 = 2$, $p_9 = 1$, $z_2 = 2t$ and the others are zero, we have $(P_{\mathrm{I}})_m$.
- if $p_2 = 1$, $p_3 = p_5 = 2$ and the others are zero, we have $(P_{\mathrm{II}})_m$.
- if $p_2 = 1$, $p_3 = p_5 = 2$, $y_1 = -2\gamma t$ $(\gamma \neq 0)$ and the others are zero, we have $(P_{\mathrm{IV}})_m$.
- if $p_2 = -1$, $p_8 = 2$, $p_9 = 1$, $z_1 = 2\gamma t$, $z_2 = 4\gamma t c_0$ $(\gamma \neq 0)$ and the others are zero, we have $(P_{34})_m$.

For the cases of $(P_{\mathrm{I}})_m$ and $(P_{34})_m$, the constants c_k $(k \geq 1)$ in $C(\theta)$ of (12) is replaced with c_{k-1} (see (1)). Note that $\alpha \neq 0$ when J=II, IV, while $\alpha = 0$ and $\beta \neq 0$ when J=I, 34. Roughly speaking, the main result of this paper is the following.

Main result. We have general formal solutions (called instanton-type solutions) with $2m$ free parameters for (13) in the cases I, II:

Case I: $\alpha = p_3 + p_7 \neq 0, \quad p_2 \neq 0.$

Case II: $\alpha = p_3 + p_7 = 0, \quad \beta = p_6 + p_9 \neq 0, \quad p_2 \neq 0.$

See Theorem 4.2 in Section 4 for more details. In the rest of this article, we prove the main result by the method given in [**2**], [**20**].

4. A construction of instanton-type solutions by multiple-scale analysis.

The purpose of Section 4 is to obtain a system of partial differential equations associated with (13) in order to apply the multiple-scale method to (13). In the procedure of getting the system, the explicit forms of f_1 and f_2 will be also determined.

4.1. The forms of f_1 and f_2.

Firstly, we set f_1 and f_2 by (15) so that (13) becomes $(P_{\mathrm{J}})_m$ (J=I, II, IV, 34) as a special case. Substitute (15) for (13). Then the constant terms (with respect to θ) and the terms which contain θ^1 in the right-hand side of (13) become

$$\begin{pmatrix} x_{1,0} + (x_{1,1} - x_{1,0}u_1)\theta - p_7 - (p_3 u_1 + p_5 c_1)\theta \\ x_{2,0} + (x_{2,1} - x_{2,0}u_1)\theta + p_9 + ((p_6 + p_9)u_1 + p_7 v_1 + p_8 c_1)\,\theta + p_6 + (p_3 v_1 + p_4 c_1)\theta \end{pmatrix}.$$

Note that the coefficients of θ^0 and θ^1 in the left-hand side of (13) are zero. Hence we have

$$x_{1,0} = p_7, \quad x_{1,1} = \alpha u_1 + p_5 c_1, \quad x_{2,0} = -\beta, \quad x_{2,1} = -\left(2\beta u_1 + \alpha v_1 + \varepsilon c_1\right),$$

where α, β and ε are defined by (17) respectively.

Next let us consider $x_{1,j}$, $x_{2,j}$ $(j = m, m+1)$ of (15). Noticing that introduction of $x_{1,j}$ and $x_{2,j}$ $(j = m, m+1)$ to f_1 affects the terms of θ^{m+1} of (13), we adjust $x_{1,j}$ and $x_{2,j}$ so that (13) becomes equivalent to $(P_{\mathrm{J}})_m$ (J=II, IV) as follows. The coefficients of θ^{m+1} in the second and third terms of the right-hand side of (13) have the following relations between u_1, v_1, $x_{1,m}$, $x_{2,m}$:

$$\frac{c_{1,m}\theta^m}{1-U} \equiv c_{1,m}(1+u_1\theta)\theta^m, \quad \frac{c_{1,m}\theta^m(p_7V+p_8C)}{1-U} \equiv c_{1,m}(p_7v_1+p_8c_1)\theta^{m+1}$$

for a constant $c_{1,m}$. Thus we set f_1 and f_2 by

$$\begin{aligned} f_1 &= p_7 + (\alpha u_1 + p_5 c_1)\theta + y_1\theta^m + (au_1 + bv_1 + y_2)\theta^{m+1}, \\ f_2 &= -\beta - (2\beta u_1 + \alpha v_1 + \varepsilon c_1)\theta + z_1\theta^m + (du_1 + ev_1 + z_2)\theta^{m+1}. \end{aligned} \tag{18}$$

Here a, b, d, e, y_i, z_i $(i = 1, 2)$ are arbitrary holomorphic functions of t and they are independent of unknown functions u_j, v_j. Remark that a, b, d and e will be determined later by assuming further conditions (32) below.

4.2. Generating functions of 0-parameter solutions for (13).

The first step of the construction of instanton-type solutions is to consider a linearized equation of (13) along a special solution $(\hat{u}_0, \hat{v}_0)$. The solution $(\hat{u}_0, \hat{v}_0)$ takes the following form

$$\hat{u}_0(\theta) := \sum_{i=1}^{\infty} \hat{u}_{i,0}(t)\theta^i \quad \text{and} \quad \hat{v}_0(\theta) := \sum_{i=1}^{\infty} \hat{v}_{i,0}(t)\theta^i. \tag{19}$$

Here $\hat{u}_0$, $\hat{v}_0$ are generating functions of the leading term $\hat{u}_{i,0}$ and $\hat{v}_{i,0}$ of a formal solution (called a "0-parameter solution") in η^{-1}. Since $(\hat{u}_0, \hat{v}_0)$ does not contain η, we see that $(\hat{u}_0, \hat{v}_0)$ satisfies

$$\begin{pmatrix} \hat{f}_1 \\ \hat{f}_2 \end{pmatrix} \times (1-\hat{u}_0) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial U(\hat{u}_0, \hat{v}_0) \\ \partial H/\partial V(\hat{u}_0, \hat{v}_0) \end{pmatrix} + \begin{pmatrix} 0 \\ H(\hat{u}_0, \hat{v}_0)/(1-\hat{u}_0) \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{20}$$

Here $\hat{f}_1$ and $\hat{f}_2$ are defined by

$$\begin{aligned} \hat{f}_1 &:= p_7 + (\alpha\hat{u}_{1,0} + p_5c_1)\,\theta + y_1\theta^m + (a\hat{u}_{1,0} + b\hat{v}_{1,0} + y_2)\theta^{m+1}, \\ \hat{f}_2 &:= -\beta - (2\beta\hat{u}_{1,0} + \alpha\hat{v}_{1,0} + \varepsilon c_1)\,\theta + z_1\theta^m + (d\hat{u}_{1,0} + e\hat{v}_{1,0} + z_2)\theta^{m+1}, \end{aligned}$$

where α, β and ε are given by (17).

Now we explain the existence of $(\hat{u}_0, \hat{v}_0)$. By (20), $(\hat{u}_0, \hat{v}_0)$ satisfies

$$2p_2\hat{v}_0\theta \equiv \hat{f}_1(1-\hat{u}_0) - (p_3\hat{u}_0 + p_5C + p_7), \tag{21}$$

$$\hat{f}_2(1-\hat{u}_0) + \frac{H(\hat{u}_0, \hat{v}_0)}{1-\hat{u}_0} + \frac{\partial H}{\partial U}(\hat{u}_0, \hat{v}_0) \equiv 0. \tag{22}$$

In what follows we always assume $p_2 \neq 0$. Using (21), we eliminate $\hat{v}_0$ in (22) and obtain

$$(1-\hat{u}_0)^2 L(\theta) \equiv K(\theta),$$

where

$$\begin{aligned} L(\theta) &:= (\hat{f}_1 + p_3)^2 + 4p_2\hat{f}_2\theta - 4p_1p_2\theta^2, \\ K(\theta) &:= (\alpha + p_5C)^2 - 4p_2(\varepsilon C + \beta)\theta - 4p_1p_2\theta^2. \end{aligned}$$

Here $L(\theta)$ and $K(\theta)$ have expansions with respect to θ in the forms

$$L(\theta) = \alpha^2 + (2\alpha(\alpha\hat{u}_{1,0} + p_5 c_1) - 4p_2\beta)\,\theta + \cdots\,,$$
$$K(\theta) = \alpha^2 + (2\alpha p_5 c_1 - 4p_2\beta)\,\theta + \cdots\,.$$

Therefore, in both cases of $\alpha \neq 0$ and $\alpha = 0, \beta \neq 0$, we have

$$\hat{u}_0 \equiv 1 - \sqrt{\frac{K(\theta)}{L(\theta)}}, \quad \hat{v}_0\theta \equiv \frac{1}{2p_2}\left(-\alpha - p_5 C + \left(\hat{f}_1 + p_3\right)\sqrt{\frac{K(\theta)}{L(\theta)}}\right). \tag{23}$$

Remark that the right-hand sides of (23) are expressed by $\hat{u}_{1,0}$ and $\hat{v}_{1,0}$. Hence $\hat{u}_{j,0}$, $\hat{v}_{j,0}$ ($j \geq 2$) are written only by $\hat{u}_{1,0}$ and $\hat{v}_{1,0}$. Let $\hat{u}_{m+1}$ (resp. $\hat{v}_{m+1}$) denote u_{m+1} (resp. v_{m+1}) with u_j and v_j being replaced by $\hat{u}_{j,0}$ and $\hat{v}_{j,0}$. Note that u_{m+1} and v_{m+1} are independent of η. The conditions that the coefficients of θ^{m+1} in $\hat{u}_0$ and $\hat{v}_0$ of (23) are equal to $\hat{u}_{m+1}$ and $\hat{v}_{m+1}$, that is $\sigma^\theta_{m+1}(\hat{u}_0) = \hat{u}_{m+1}$ and $\sigma^\theta_{m+1}(\hat{v}_0) = \hat{v}_{m+1}$, become the equations for unknown functions $\hat{u}_{1,0}$, $\hat{v}_{1,0}$. Hence we determine $\hat{u}_{1,0}$ and $\hat{v}_{1,0}$ by the conditions.

Moreover, in the case of $\alpha = \beta = 0$, if leading terms $((p_5c_1)^2 - 4p_2\varepsilon c_1 - 4p_1p_2)\theta^2$ of $L(\theta)$ and $K(\theta)$ do not vanish, then $\hat{u}_0$ and $\hat{v}_0$ are also determined.

4.3. A linearized equation of (13) along ($\hat{u}_0$, $\hat{v}_0$).

We look for a solution to (13) of the form

$$(U,\, V) = \big(\hat{u}_0 + (1-\hat{u}_0)u,\, \hat{v}_0 + (1-\hat{u}_0)v\big) \tag{24}$$

with

$$u = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} u_{i,\,j\ell}(t)\,\theta^i\,\eta^{j\ell}, \qquad v = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} v_{i,\,j\ell}(t)\,\theta^i\,\eta^{j\ell}. \tag{25}$$

Here $\ell = -1/k$ with an integer $k \geq 2$ (see Lemma 2.3 in [**19**]), $u_{i,\,j\ell}$ and $v_{i,\,j\ell}$ ($i,\, j \geq 1$) denote new unknown functions of t, and ($\hat{u}_0$, $\hat{v}_0$) is given by (19) (also (23)).

Under the conditions $\sigma^\theta_{m+1}((1-\hat{u}_0)u) = \sigma^\theta_{m+1}((1-\hat{u}_0)v) = 0$, let us construct u, v. Put (24) into (13). By an argument similar to that employed in deriving (18) in [**20**], we obtain a system for u, v:

$$\begin{aligned}
&\eta^{-1}\left(-\begin{pmatrix}\varrho\\ \delta\end{pmatrix} + \varrho\begin{pmatrix}u\\ v\end{pmatrix} + \frac{d}{dt}\begin{pmatrix}u\\ v\end{pmatrix}\right)\theta \\
&\quad\equiv -u\begin{pmatrix}\hat{f}_1\\ \hat{f}_2\end{pmatrix} + \begin{pmatrix}\alpha\sigma^\theta_1(u)\\ -(2\beta\sigma^\theta_1(u) + \alpha\sigma^\theta_1(v))\end{pmatrix}(1-u)\theta \\
&\qquad + \begin{pmatrix}a\sigma^\theta_1(u) + b\sigma^\theta_1(v)\\ d\sigma^\theta_1(u) + e\sigma^\theta_1(v)\end{pmatrix}\theta^{m+1} + \begin{pmatrix}0\\ \dfrac{uH(\hat{u}_0,\, \hat{v}_0)}{(1-\hat{u}_0)^2(1-u)} + \dfrac{I}{(1-u)(1-\hat{u}_0)}\end{pmatrix} \\
&\qquad + \begin{pmatrix}-\partial_V\partial_U H(\hat{u}_0,\, \hat{v}_0) & -\partial_V\partial_V H(\hat{u}_0,\, \hat{v}_0)\\ \partial_U\partial_U H(\hat{u}_0,\, \hat{v}_0) & \partial_V\partial_U H(\hat{u}_0,\, \hat{v}_0)\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}
\end{aligned} \tag{26}$$

with $\partial_V = \partial/\partial V$ and $\partial_U = \partial/\partial U$. Here ϱ and δ are defined by

$$\varrho := \frac{d}{dt}(\log(1-\hat{u}_0)) \quad \text{and} \quad \delta := \frac{d/dt(-\hat{v}_0)}{1-\hat{u}_0}, \tag{27}$$

respectively. The equation (20) implies

$$\frac{\partial_V H(\hat{u}_0,\, \hat{v}_0)v + \partial_U H(\hat{u}_0,\, \hat{v}_0)u}{(1-\hat{u}_0)} + \frac{uH(\hat{u}_0,\, \hat{v}_0)}{(1-\hat{u}_0)^2} \equiv \hat{f}_1 v - \hat{f}_2 u.$$

Therefore the fourth term in the right-hand side of (26) is equal to

$$\begin{aligned}&\begin{pmatrix} 0 \\ \hat{f}_1 v - \hat{f}_2 u \end{pmatrix} + \frac{u}{1-u}\begin{pmatrix} 0 \\ \hat{f}_1 v - \hat{f}_2 u \end{pmatrix} \\ &\quad + \frac{1}{1-u}\begin{pmatrix} 0 \\ (1/2)\partial_V\partial_V H(\hat{u}_0,\, \hat{v}_0)v^2 + (1/2)\partial_U\partial_U H(\hat{u}_0,\, \hat{v}_0)u^2 + \partial_V\partial_U H(\hat{u}_0,\, \hat{v}_0)uv \end{pmatrix}.\end{aligned}$$

Putting the above equation into the right-hand side of (26) and taking the first-order terms with respect to the variables u and v, we define the map $Q : (\Theta\theta)^2 \longrightarrow \Theta^2$ by

$$\begin{aligned}Q\begin{pmatrix} x\,\theta \\ y\,\theta \end{pmatrix} &= \begin{pmatrix} -\left(\partial^2 H/\partial V\partial U(\hat{u}_0,\, \hat{v}_0) + \hat{f}_1\right) & -\partial^2 H/\partial V^2(\hat{u}_0,\, \hat{v}_0) \\ \left(\partial^2 H/\partial U^2(\hat{u}_0,\, \hat{v}_0) - 2\hat{f}_2\right) & \left(\partial^2 H/\partial V\partial U(\hat{u}_0,\, \hat{v}_0) + \hat{f}_1\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &\quad + \begin{pmatrix} \alpha\sigma_1^\theta(x) \\ -2\beta\sigma_1^\theta(x) - \alpha\sigma_1^\theta(y) \end{pmatrix}\theta + \begin{pmatrix} a\sigma_1^\theta(x) + b\sigma_1^\theta(y) \\ d\sigma_1^\theta(x) + e\sigma_1^\theta(y) \end{pmatrix}\theta^{m+1}\end{aligned} \tag{28}$$

for any $x,\, y \in \Theta$. Here Θ denotes the set of formal power series of θ without constant terms. By the definition of Q, (26) is transformed into

$$\begin{aligned}&\eta^{-1}\left(-\begin{pmatrix} \varrho \\ \delta \end{pmatrix} + \varrho\begin{pmatrix} u \\ v \end{pmatrix} + \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix}\right)\theta \\ &\quad \equiv Q\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} + \begin{pmatrix} -\alpha\sigma_1^\theta(u)u \\ 2\beta\sigma_1^\theta(u)u + \alpha\sigma_1^\theta(v)u \end{pmatrix}\theta \\ &\quad + \frac{1}{2(1-u)}\begin{pmatrix} 0 \\ (-v,\, u)Q\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} + (\alpha\sigma_1^\theta(u)v + 2\beta\sigma_1^\theta(u)u + \alpha\sigma_1^\theta(v)u)\theta \end{pmatrix}.\end{aligned} \tag{29}$$

Moreover, multiplying both sides of (29) by $(1-u)$, we obtain the system for u, v. Before describing the system, we shall consider the definition of map Q. By (28), we have

$$\begin{aligned}Q\begin{pmatrix} x\,\theta \\ y\,\theta \end{pmatrix} &\equiv \begin{pmatrix} -\left(\partial^2 H/\partial V\partial U(\hat{u}_0,\, \hat{v}_0) + \tilde{f}_1\right) & -\partial^2 H/\partial V^2(\hat{u}_0,\, \hat{v}_0) \\ \left(\partial^2 H/\partial U^2(\hat{u}_0,\, \hat{v}_0) - 2\tilde{f}_2\right) & \left(\partial^2 H/\partial V\partial U(\hat{u}_0,\, \hat{v}_0) + \tilde{f}_1\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &\quad + \begin{pmatrix} \alpha\sigma_1^\theta(x) \\ -2\beta\sigma_1^\theta(x) - \alpha\sigma_1^\theta(y) \end{pmatrix}\theta + \begin{pmatrix} a - y_1 & b \\ d - 2z_1 & e + y_1 \end{pmatrix}\begin{pmatrix} \sigma_1^\theta(x) \\ \sigma_1^\theta(y) \end{pmatrix}\theta^{m+1},\end{aligned} \tag{30}$$

where

$$\begin{aligned}\tilde{f}_1 &:= \hat{f}_1 - y_1\theta^m - (a\hat{u}_{1,0} + b\hat{v}_{1,0} + y_2)\theta^{m+1} = p_7 + (\alpha\hat{u}_{1,0} + p_5c_1)\,\theta,\\ \tilde{f}_2 &:= \hat{f}_2 - z_1\theta^m - (d\hat{u}_{1,0} + e\hat{v}_{1,0} + z_2)\theta^{m+1} = -\beta - (2\beta\hat{u}_{1,0} + \alpha\hat{v}_{1,0} + \varepsilon c_1)\,\theta.\end{aligned} \tag{31}$$

Here, in order that the key lemma (Lemma 4.1 below) for the construction of instanton-type solutions may hold, we assume that the coefficients of θ^{m+1} in the third term of the right-hand side of (30) vanish, that is,

$$a = y_1, \quad b = 0, \quad d = 2z_1, \quad e = -y_1. \tag{32}$$

Remark that, substituting (32) for (18), we see that f_1 and f_2 are given explicitly by (16).

We now summarize the results obtained so far. Let $Q : (\Theta\theta)^2 \longrightarrow \Theta^2$ denote the map defined by

$$\begin{aligned}Q\begin{pmatrix} x\,\theta \\ y\,\theta\end{pmatrix} := &\begin{pmatrix} -\left(\partial^2 H/\partial V\partial U(\hat{u}_0,\,\hat{v}_0) + \tilde{f}_1\right) & -\partial^2 H/\partial V^2(\hat{u}_0,\,\hat{v}_0) \\ \left(\partial^2 H/\partial U^2(\hat{u}_0,\,\hat{v}_0) - 2\tilde{f}_2\right) & \left(\partial^2 H/\partial V\partial U(\hat{u}_0,\,\hat{v}_0) + \tilde{f}_1\right)\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}\\ &+ \begin{pmatrix} \alpha\sigma_1^\theta(x) \\ -2\beta\sigma_1^\theta(x) - \alpha\sigma_1^\theta(y)\end{pmatrix}\theta\end{aligned}$$

for any $x,\, y \in \Theta$. Here $\tilde{f}_j$ $(j = 1,\,2)$ are defined by (31). Then, by (24), (13) is transformed into the following system of non-linear equations for $(u,\,v)$:

$$\begin{aligned}\left(\eta^{-1}\frac{d}{dt} - Q\right)\begin{pmatrix} u\theta \\ v\theta\end{pmatrix} \equiv &\left(\begin{pmatrix} -\alpha\sigma_1^\theta(u)u\theta \\ S(u,\,v)\end{pmatrix} + \eta^{-1}\begin{pmatrix} \varrho \\ \delta\end{pmatrix}\theta - uQ\begin{pmatrix} u\theta \\ v\theta\end{pmatrix}\right)\\ &- \left(u^2\begin{pmatrix} -\alpha\sigma_1^\theta(u) \\ 2\beta\sigma_1^\theta(u) + \alpha\sigma_1^\theta(v)\end{pmatrix} + \eta^{-1}\begin{pmatrix} 2\varrho u \\ \delta u + \varrho v\end{pmatrix}\right)\theta\\ &+ \eta^{-1}u\left(\varrho + \frac{d}{dt}\right)\begin{pmatrix} u \\ v\end{pmatrix}\theta,\end{aligned} \tag{33}$$

where ϱ and δ are defined by (27) respectively and $S(u,\,v)$ has the form

$$S(u,\,v) := \frac{1}{2}(-v,\,u)Q\begin{pmatrix} u\theta \\ v\theta\end{pmatrix} + \frac{1}{2}\alpha\sigma_1^\theta(u)v\theta + \frac{3}{2}\left(\alpha\sigma_1^\theta(v) + 2\beta\sigma_1^\theta(u)\right)u\theta. \tag{34}$$

4.4. Some essential properties associated with the map Q.

Similar to the results of [**2**] and [**20**], we construct a solution $(u,\,v)$ for (33) so that $(u,\,v)$ is expressed by a linear combination of eigenvector $A(\lambda)$'s of Q in the sense of $Q(A(\lambda)\theta) = \lambda A(\lambda)\theta$. As is shown below, the form of $A(\lambda)$'s depends on relations between coefficient p_j's in (14).

Let $x,\, y \in \Theta$. The equation $Q\begin{pmatrix} x\theta \\ y\theta\end{pmatrix} = \lambda\begin{pmatrix} x\theta \\ y\theta\end{pmatrix}$ is equivalent to

$$\left\{\begin{pmatrix} \alpha & 0 \\ 2\beta & \alpha\end{pmatrix} + M_\lambda\theta\right\}\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 2\beta & \alpha\end{pmatrix}\begin{pmatrix} \sigma_1^\theta(x) \\ \sigma_1^\theta(y)\end{pmatrix}\theta, \tag{35}$$

where

$$M_\lambda = \begin{pmatrix} \lambda + \sigma_1^\theta(\hat{f}_1) & 2p_2 \\ 2(p_1 - \sigma_1^\theta(\hat{f}_2)) & -(\lambda - \sigma_1^\theta(\hat{f}_1)) \end{pmatrix}.$$

We have

$$\det\left\{ \begin{pmatrix} \alpha & 0 \\ 2\beta & \alpha \end{pmatrix} + M_\lambda \theta \right\} = \alpha^2 + 2\left(\alpha\sigma_1^\theta(\hat{f}_1) - 2p_2\beta\right)\theta \\ - \left\{\lambda^2 - \sigma_1^\theta(\hat{f}_1)^2 + 4p_2(p_1 - \sigma_1^\theta(\hat{f}_2))\right\}\theta^2. \tag{36}$$

Therefore we need to consider separately the cases where $\alpha \neq 0$ or $\alpha = 0$.

4.4.1. Case I : $\alpha = p_3 + p_7 \neq 0$, $p_2 \neq 0$.

When $\alpha \neq 0$, the structure of system is almost the same as that of $(P_{\mathrm{II}})_m$ except for the existence of β. By calculating in a similar way to [**20**], we find that the eigenvector $A(\lambda)$ corresponding to an eigenvalue λ of Q has the form

$$A(\lambda) = \begin{pmatrix} a(\lambda) \\ \rho(\lambda)a(\lambda) \end{pmatrix} \quad \text{with} \quad a(\lambda) := \frac{\theta}{1 - \theta g(\lambda)}.$$

Here $\rho(\lambda)$ and $g(\lambda)$ are given by

$$\rho(\lambda) := -\frac{1}{2p_2}\left(\lambda + \frac{2p_2\beta}{\alpha} \pm \alpha G(\lambda)\right), \quad g(\lambda) := \frac{2p_2\beta - \alpha\sigma_1^\theta(\hat{f}_1)}{\alpha^2} \pm G(\lambda), \tag{37}$$

where $G(\lambda)$ is defined by

$$G(\lambda) := \frac{\sqrt{\alpha^2\lambda^2 + 4p_2\left(p_2\beta^2 - \alpha\beta\sigma_1^\theta(\hat{f}_1) - \alpha^2\left(\sigma_1^\theta(\hat{f}_2) - p_1\right)\right)}}{\alpha^2}$$

and both upper or both lower signs should be chosen in the double signs.

Now we explain how to determine λ. By the assumption $\sigma_{m+1}^\theta((1-\hat{u}_0)u) = 0$, the coefficient of θ^{m+1} in $(1-\hat{u}_0)A(\lambda)$ must be zero. Thus the following equation holds.

$$g(\lambda)^m - \sum_{k=1}^{m} \hat{u}_{k,0}\, g(\lambda)^{m-k} = 0, \tag{38}$$

where $\hat{u}_{k,0}$ is the coefficient of θ^k in $\hat{u}_0$ given by (19). Note that $g(\lambda)$ must satisfy

$$\alpha^2 g(\lambda)^2 + 2\left(\alpha\sigma_1^\theta(\hat{f}_1) - 2p_2\beta\right)g(\lambda) - \lambda^2 + \sigma_1^\theta(\hat{f}_1)^2 + 4p_2(\sigma_1^\theta(\hat{f}_2) - p_1) = 0. \tag{39}$$

Noticing (38) and (39), we define $\Lambda(\lambda, t)$ by the resultant of the following two polynomials of X:

$$X^m - \sum_{k=1}^{m} \hat{u}_{k,0} X^{m-k} = 0,$$

$$\alpha^2 X^2 + 2\left(\alpha\sigma_1^\theta(\hat{f}_1) - 2p_2\beta\right) X - \lambda^2 + \sigma_1^\theta(\hat{f}_1)^2 + 4p_2(\sigma_1^\theta(\hat{f}_2) - p_1) = 0.$$

We determine λ so that $\Lambda(\lambda, t)$ equals zero.

Finally, we remark the important relations which will be used several times in the following discussions.

$$\begin{aligned} \begin{pmatrix} a(\lambda) \\ 0 \end{pmatrix} &= \frac{p_2}{\lambda}\Big(\rho(-\lambda)A(\lambda) - \rho(\lambda)A(-\lambda)\Big), \\ \begin{pmatrix} 0 \\ a(\lambda) \end{pmatrix} &= -\frac{p_2}{\lambda}\Big(A(\lambda) - A(-\lambda)\Big) \end{aligned} \tag{40}$$

and

$$\rho(\lambda) - \rho(-\lambda) = -\frac{\lambda}{p_2}. \tag{41}$$

4.4.2. Case II : $\alpha = p_3 + p_7 = 0$, $\beta = p_6 + p_9 \neq 0$, $p_2 \neq 0$.

When $\alpha = 0$, we assume $\beta \neq 0$ so that the second term in the right-hand side of (36) does not vanish. By solving (35), we have

$$x = \frac{\theta}{1 - g(\lambda)\theta}\sigma_1^\theta(x), \quad y = \rho(\lambda)x.$$

Here $g(\lambda)$ and $\rho(\lambda)$ are defined by

$$g(\lambda) := \frac{\lambda^2 - (\sigma_1^\theta(\hat{f}_1))^2 + 4p_2\left(p_1 - \sigma_1^\theta(\hat{f}_2)\right)}{-4p_2\beta} \quad \text{and} \quad \rho(\lambda) := -\frac{\lambda + \sigma_1^\theta(\hat{f}_1)}{2p_2},$$

respectively. The eigenvalue λ of Q is a root of the following algebraic equation

$$\Lambda(\lambda, t) := g(\lambda)^m - \sum_{k=1}^{m} \hat{u}_{k,0}\, g(\lambda)^{m-k} = 0,$$

where $\hat{u}_{k,0}$ denotes the coefficient of θ^k in $\hat{u}_0$ of (19). Then the eigenvector $A(\lambda)$ corresponding to an eigenvalue λ has the form

$$A(\lambda) = \begin{pmatrix} a(\lambda) \\ \rho(\lambda)a(\lambda) \end{pmatrix} \text{ with } a(\lambda) := \frac{\theta}{1 - \theta g(\lambda)} = \sum_{j=0}^{\infty} g(\nu_k)^j \theta^{j+1}.$$

Remark that (40) and (41) also hold in the case II.

The rest of argument in Section 4.4 deals with case I and case II simultaneously. In the both cases, $\Lambda(\lambda, t)$ is an even function of λ. Let $\nu_{\pm 1}(t), \ldots, \nu_{\pm m}(t)$ be the roots of the algebraic equation $\Lambda(\lambda, t) = 0$ of λ with convention $\nu_k = -\nu_{-k}$ $(1 \leq k \leq m)$. Throughout this paper we always suppose

(S1) The roots $\nu_i(t)$'s $(1 \leq |i| \leq m)$ are mutually distinct for each t in Ω, i.e. t is neither a turning point of the first kind nor a turning point of the second kind.

(S2) The function $p_1\nu_1(t) + \cdots + p_m\nu_m(t)$ does not vanish identically on Ω for any $(p_1, \ldots, p_m) \in \mathbb{Z}^m \setminus \{0\}$.

The multiplicative relations in the following Lemma 4.1 are common to $(P_{\mathrm{J}})_m$ (J=I, II, IV, 34) (see [**2**], [**20**] and [**21**]). We claim that they also hold for our system (13) and Lemma 4.1 is a key of success in the construction of instanton-type solutions. For the proof, we refer the reader to Appendix A in [**2**].

Lemma 4.1. 1. *For any $k \neq j$ $(1 \leq k, j \leq m)$, we have*

$$a(\nu_k)a(\nu_j) = \frac{1}{g(\nu_k) - g(\nu_j)} \left(a(\nu_k) - a(\nu_j)\right). \tag{42}$$

Furthermore, for any integers $1 \leq i_1 < i_2 < \cdots < i_k \leq m$, we get

$$a(\nu_{i_1}) \cdots a(\nu_{i_k}) = \sum_{l=1}^{k} \frac{a(\nu_{i_l})}{(g(\nu_{i_l}) - g(\nu_{i_1})) \cdots (g(\nu_{i_l}) - g(\nu_{i_{l-1}}))(g(\nu_{i_l}) - g(\nu_{i_{l+1}})) \cdots (g(\nu_{i_l}) - g(\nu_{i_k}))}.$$

Note that these equations are strict (not $\equiv$).

2. *For any $1 \leq k \leq m$, we have*

$$a(\nu_k)^2 \equiv \sum_{j=1}^{m} h_{k,j} a(\nu_j), \tag{43}$$

where $h_{k,j}$ are defined by

$$h_{k,j} = \frac{\displaystyle\prod_{\substack{1 \leq l \leq m, \\ l \neq k, j}} (g(\nu_k) - g(\nu_l))}{\displaystyle\prod_{\substack{1 \leq l \leq m, \\ l \neq j}} (g(\nu_j) - g(\nu_l))} \quad (j \neq k), \quad h_{k,k} = \sum_{\substack{l=1, \\ l \neq k}}^{m} \frac{1}{g(\nu_k) - g(\nu_l)}. \tag{44}$$

3. *We have*

$$\frac{\partial a(\nu_k)}{\partial t} \equiv g(\nu_k)' \sum_{j=1}^{m} h_{k,j} a(\nu_j),$$

where $g(\nu_k)'$ denotes the derivative of $g(\nu_k(t))$ with respect to t.

4.5. Main theorem.

We denote by Ω an open subset in $\mathbb{C}_t$ satisfying (S1) and (S2) in the previous subsection and by $\mathcal{M}(\Omega)[[\theta]]$ the set of formal power series in θ with coefficients in multi-valued holomorphic functions with a finite number of branching points and poles on Ω. In what follows, we consider the case of $\ell = -1/2$ in (25). Let $\tau := (\tau_1, \ldots, \tau_m)$ be m-independent variables. Then we define the rings

$$\mathcal{A}_\ell(\Omega) := (\mathcal{M}(\Omega)[[\theta]])\left[\left[\eta^\ell e^{\tau_1},\,\ldots,\,\eta^\ell e^{\tau_m},\,\eta^\ell e^{-\tau_1},\,\ldots,\,\eta^\ell e^{-\tau_m}\right]\right].$$

We also define $\hat{\mathcal{A}}_\ell(\Omega)$ by the subset in $\mathcal{A}_\ell(\Omega)$ consisting of a formal power series of order less than or equal to ℓ with respect to η. We construct a solution in $\mathcal{A}_\ell(\Omega)$ for the system below. The operator P is defined by

$$P := \nu_1\frac{\partial}{\partial\tau_1} + \cdots + \nu_m\frac{\partial}{\partial\tau_m} - Q.$$

Then we obtain a partial differential equation associated with (33).

$$\begin{aligned} P\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} \equiv & \left(\begin{pmatrix} -\alpha\sigma_1^\theta(u)u\theta \\ S(u,\,v) \end{pmatrix} + \eta^{-1}\begin{pmatrix} \varrho \\ \delta \end{pmatrix}\theta + uP\begin{pmatrix} u\theta \\ v\theta \end{pmatrix}\right) \\ & - \left(u^2\begin{pmatrix} -\alpha\sigma_1^\theta(u) \\ 2\beta\sigma_1^\theta(u) + \alpha\sigma_1^\theta(v) \end{pmatrix} + \eta^{-1}\left(\begin{pmatrix} 2\varrho u \\ \delta u + \varrho v \end{pmatrix} + \frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix}\right)\right)\theta \\ & + \eta^{-1}u\left(\varrho + \frac{\partial}{\partial t}\right)\begin{pmatrix} u \\ v \end{pmatrix}\theta. \end{aligned} \tag{45}$$

Here $S(u,\,v)$, ϱ and δ have been given by (34) and (27).

By solving (45), we have the main result. Let us define the morphism ι by

$$\iota(\psi) = \psi\left(\eta\int^t \nu_1(s)ds,\,\ldots,\,\eta\int^t \nu_m(s)ds,\,t,\,\theta,\,\eta\right)$$

for $\psi(\tau_1,\,\ldots,\,\tau_m,\,t,\,\theta,\,\eta) \in \hat{\mathcal{A}}_\ell(\Omega)$. The main theorem of this paper is the following.

Theorem 4.2. *Let Ω be an open subset satisfying* ($S1$) *and* ($S2$). *Then we have instanton-type solutions for* (13) *with $2m$ free parameters $(\beta_{-m},\,\ldots,\,\beta_m) \in \mathbb{C}^{2m}[[\eta^{-1}]]$ in the case I and case II respectively*:

$$(U,\,V) = (\hat{u}_0,\,\hat{v}_0) + (1-\hat{u}_0)(\iota(u),\,\iota(v)),$$

where $(u,\,v)$ is a solution in $\hat{\mathcal{A}}_\ell^2(\Omega)$ for (45) *of the form*

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{1\le|k|\le m}\sum_{j=1}^{\infty}\left(\sum_{\ell\ge 0,\,p\in\mathbb{Z}^m,\,2\ell+|p|=j} f_{k,p,\ell}(t)e^{p\cdot\tau}\right)\eta^{-j/2}A(\nu_k).$$

Here ν_k and $A(\nu_k)$ are defined by the eigenvalue and the corresponding eigenvector of Q described in Section 4.4.

5. Proof for Theorem 4.2.

We apply the method of [**2**] and [**20**] by multiple-scale analysis for our system. As is described in [**19**], we derive non-secularity conditions $(\mathcal{E}_k)$ $(k = 1,\,2,\,\cdots)$ and by solving non-secularity conditions $(\mathcal{E}_k)$ we construct a solution with sufficiently many free parameters. Note that $(\mathcal{E}_1)$ is a system of non-linear equations and $(\mathcal{E}_k)$ $(k \ge 2)$ is a system of linear equations. If $(\mathcal{E}_1)$ is solved and has a solution with $2m$ free parameters in $\mathbb{C}^{2m}$, then we can construct a solution for (45) with $2m$ free parameters in $\mathbb{C}^{2m}[[\eta^{-1}]]$. Therefore the proof of Theorem 4.2 is completed by showing the solvability of the first member $(\mathcal{E}_1)$ of non-secularity conditions.

Assume that an element (u, v) in $\hat{\mathcal{A}}^2_\ell(\Omega)$ has the expansion

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{1\le|k|\le m} f_k(\tau, t; \eta)A(\nu_k) \quad \text{with} \quad f_k(\tau, t; \eta) := \sum_{j=1}^{\infty} f_{k,\, j\ell}(\tau, t)\eta^{j\ell}. \tag{46}$$

Here $A(\nu_k)$'s contain θ and f_k's are independent of θ.

Substituting (46) for (45) and looking at the coefficient of η^ℓ in both sides, we obtain

$$P\left(\sum_{1\le|k|\le m} f_{k,\,\ell}(\tau, t)A(\nu_k)\theta\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{47}$$

The following lemma is proved as Ker$P$ is equivalent to the subspace generated by the vectors $\eta^\ell e^{\tau_i}A(\nu_i)$ over $\mathcal{M}(\Omega)[[\eta^{-1}]]$.

Lemma 5.1. *We have a solution to* (47) *of the form*

$$f_{k,\,\ell} = \omega_k^{(1)} e^{\tau_k} \quad (1 \le |k| \le m),$$

where $\omega_k^{(1)}(t)$*'s* $(1 \le |k| \le m)$ *are arbitrary functions of* t.

The $\omega_k^{(1)}$'s are determined by the first member $(\mathcal{E}_1)$ of the non-secularity conditions and the forms of $(\mathcal{E}_1)$ corresponding to the cases I, II are given in the subsequent subsections.

5.1. Case I : $\boldsymbol{\alpha = p_3 + p_7 \neq 0,\ p_2 \neq 0}$.

Theorem 5.2. *The first member* $(\mathcal{E}_1)$ *of the non-secularity conditions is the following system of non-linear equations with* $2m$ *unknown functions* ω_k $(1 \le |k| \le m)$:

$$\frac{d\omega_k}{dt} = \frac{1}{\nu_k}\left(\sum_{j=1}^{m} \psi(k, j)\omega_j\omega_{-j} + J_k - \nu_k R_k\right)\omega_k, \tag{48}$$

$$\frac{d\omega_{-k}}{dt} = -\frac{1}{\nu_k}\left(\sum_{j=1}^{m} \psi(-k, j)\omega_j\omega_{-j} + J_{-k} + \nu_k R_{-k}\right)\omega_{-k} \tag{49}$$

for $1 \le k \le m$. *Here* $\psi(k, j)$*'s are rational functions of the variables* ν_i*'s,* J_k *and* R_k *are multi-valued functions of finite determination in* Ω. *They satisfy the conditions*:

$$\psi(k, j) = \psi(-k, j) \ \ (1 \le j \le m), \quad J_k = J_{-k}, \quad R_k = R_{-k}. \tag{$*$}$$

Remark that $\psi(k, j)$, J_k and R_k are given in Lemma 6.10. The proof will be done in Section 6 as it is lengthy. Since the equations (48) and (49) imply

$$\frac{d(\omega_k(t)\omega_{-k}(t))}{dt} = -2R_k\omega_k(t)\omega_{-k}(t),$$

$(\mathcal{E}_1)$ is solved globally and the leading term of ${}^t(u, v)$ is given as follows.

PROPOSITION 5.3. *The concrete form of the leading term of ${}^t(u, v)$ with respect to η is written as*

$$\eta^{-1/2}\sum_{|k|=1}^{m}\omega_k^{(1)}e^{\tau_k}A(\nu_k).$$

Here $\omega_k^{(1)}$, $\omega_{-k}^{(1)}$ $(1 \le k \le m)$ are multi-valued holomorphic functions on Ω in the form

$$\omega_k^{(1)} = \beta_k^{(1)}\exp\left(\int^t \frac{1}{\nu_k}\left(\sum_{j=1}^{m}\psi(k, j)\beta_j^{(1)}\beta_{-j}^{(1)}\exp\left(-2\int^t R_j dt\right) + J_k - \nu_k R_k\right)dt\right),$$

$$\omega_{-k}^{(1)} = \beta_{-k}^{(1)}\exp\left(\int^t -\frac{1}{\nu_k}\left(\sum_{j=1}^{m}\psi(k, j)\beta_j^{(1)}\beta_{-j}^{(1)}\exp\left(-2\int^t R_j dt\right) + J_k + \nu_k R_k\right)dt\right)$$

for $1 \le k \le m$ with $2m$ free parameters $(\beta_{-m}^{(1)}, \dots, \beta_m^{(1)}) \in \mathbb{C}^{2m}$.

In what follows, for simplicity, we use the notation below.

$$\rho_{k,\,j} := \rho(\nu_k) + \rho(\nu_j), \tag{50}$$

where $\rho(\nu_k)$'s have been defined by (37). We also have the following.

LEMMA 5.4. *The concrete form of the sub-leading term of ${}^t(u, v)$ with respect to η is written as*

$$\eta^{-1}\sum_{|k|=1}^{m} f_{k,\,-1}A(\nu_k),$$

where

$$\begin{aligned} f_{k,\,-1} :=& -p_2\sum_{\substack{1\le|j|\le m,\\ k+j\neq 0}}\left(\frac{2\nu_k+\nu_j}{\nu_k\nu_j(\nu_k+\nu_j)}(\alpha\rho_{k,\,j}+2\beta)+\frac{\alpha}{p_2\nu_j}\right)\omega_k^{(1)}\omega_j^{(1)}e^{\tau_k+\tau_j} \\ &+ p_2\sum_{\substack{1\le|j|\le m,\\ k+j\neq 0}}\frac{1}{\nu_k(\nu_k+\nu_j)}(\alpha\rho_{-k,\,-j}+2\beta)\omega_{-k}^{(1)}\omega_{-j}^{(1)}e^{-\tau_k-\tau_j} \\ &- \frac{p_2}{\nu_k^2}\left(\sum_{j=1}^{m}\frac{\nu_j^2}{p_2}h_{j,\,k}\omega_j^{(1)}\omega_{-j}^{(1)} - \left(3\alpha\rho_{k,\,-k}+6\beta+\frac{\alpha\nu_k}{p_2}\right)\omega_k^{(1)}\omega_{-k}^{(1)}\right) \\ &- \frac{p_2}{\nu_k^2}\left(\gamma_k\rho(\nu_{-k})-\delta_k\right). \end{aligned}$$

Here $h_{j,\,k}$'s are defined by (44) *with convention $h_{j,\,k} := h_{|j|,\,|k|}$ and γ_k, δ_k* $(1 \le k \le m)$ *are determined by*

$$\varrho \equiv \sum_{k=1}^{m} \gamma_k(t) a(\nu_k) \quad \text{and} \quad \delta \equiv \sum_{k=1}^{m} \delta_k(t) a(\nu_k) \tag{51}$$

with ϱ, δ of (27) *and $\gamma_{-k} := \gamma_k$, and $\delta_{-k} := \delta_k$.*

Remark that the explicit forms of γ_k and δ_k can be calculated by the same way as Appendix B in [**20**]. For the proof of Lemma 5.4, we refer the reader to Section 6.

5.2. Case II : $\alpha = p_3 + p_7 = 0$, $\beta = p_6 + p_9 \neq 0$, $p_2 \neq 0$.

THEOREM 5.5. *The first member $(\mathcal{E}_1)$ of the non-secularity conditions is the following system of non-linear equations with $2m$ unknown functions ω_k $(1 \le |k| \le m)$:*

$$\frac{d\omega_k}{dt} = \left(\frac{1}{\nu_k} \left(\sum_{j=1}^{m} \varphi(k,\,j) \omega_j \omega_{-j} + \Phi_k \right) - h_k \right) \omega_k,$$
$$\frac{d\omega_{-k}}{dt} = \left(-\frac{1}{\nu_k} \left(\sum_{j=1}^{m} \varphi(-k,\,j) \omega_j \omega_{-j} + \Phi_{-k} \right) - h_{-k} \right) \omega_{-k}$$

for $1 \le k \le m$. Here $\varphi(k,\,j)$'s are rational functions of the variables ν_i's, Φ_k and h_k are multi-valued functions of finite determination in Ω. They satisfy the conditions:

$$\varphi(k,\,j) = \varphi(-k,\,j) \ \ (1 \le j \le m), \quad \Phi_k = \Phi_{-k}, \quad h_k = h_{-k}. \tag{$*$}$$

The concrete forms of $\varphi(k,\,j)$'s, Φ_k and h_k can be calculated as in Case I. See Section 7. A key of the global solvability of $(\mathcal{E}_1)$ is the following relation:

$$\frac{d(\omega_k(t)\omega_{-k}(t))}{dt} = -2h_k \omega_k(t) \omega_{-k}(t). \tag{52}$$

By using (52), we have the explicit form of the leading term of ${}^t(u,\,v)$.

PROPOSITION 5.6. *The concrete form of the leading term of ${}^t(u,\,v)$ with respect to η is written as*

$$\eta^{-1/2} \sum_{|k|=1}^{m} \omega_k^{(1)} e^{\tau_k} A(\nu_k).$$

Here $\omega_k^{(1)}$, $\omega_{-k}^{(1)}$ $(1 \le k \le m)$ are multi-valued holomorphic functions on Ω in the form

$$\omega_k^{(1)} = \beta_k^{(1)} \exp\left(\int^t \left(\frac{1}{\nu_k}\sum_{j=1}^m \varphi(k,\,j)\beta_j^{(1)}\beta_{-j}^{(1)} \exp\left(-2\int^t h_j dt\right) + \frac{1}{\nu_k}\Phi_k - h_k\right) dt\right),$$

$$\omega_{-k}^{(1)} = \beta_{-k}^{(1)} \exp\left(\int^t - \left(\frac{1}{\nu_k}\sum_{j=1}^m \varphi(k,\,j)\beta_j^{(1)}\beta_{-j}^{(1)} \exp\left(-2\int^t h_j dt\right) + \frac{1}{\nu_k}\Phi_k + h_k\right) dt\right)$$

with $2m$ *free parameters* $(\beta_{-m}^{(1)},\,\dots,\,\beta_m^{(1)}) \in \mathbb{C}^{2m}$*, and* $\varphi(k,\,j)$*,* Φ_k*,* h_k *are given in Theorem* 5.5*.*

We also have the following.

Lemma 5.7. *The explicit form of the sub-leading term of* ${}^t(u,\,v)$ *with respect to* η *is written as*

$$\eta^{-1}\sum_{|k|=1}^m f_{k,\,-1}A(\nu_k),$$

where

$$\begin{aligned} f_{k,\,-1} :=& \sum_{\substack{1\le|j|\le m,\\ k+j\neq 0}} \frac{-2p_2\beta}{\nu_k\nu_j(\nu_k+\nu_j)}\left((2\nu_k+\nu_j)\omega_k^{(1)}\omega_j^{(1)}e^{\tau_k+\tau_j} - \nu_j\omega_{-k}^{(1)}\omega_{-j}^{(1)}e^{-\tau_k-\tau_j}\right) \\ &- \frac{1}{\nu_k^2}\left(\sum_{j=1}^m \nu_j^2 h_{j,\,k}\omega_j^{(1)}\omega_{-j}^{(1)} - 6p_2\beta\omega_k^{(1)}\omega_{-k}^{(1)}\right) - \frac{p_2}{\nu_k^2}\left(\gamma_k\rho(\nu_{-k}) - \delta_k\right). \end{aligned}$$

Here $h_{j,\,k}$*'s are defined by* (44) *with convention* $h_{j,\,k} := h_{|j|,\,|k|}$ *and* γ_k*,* δ_k $(1 \le k \le m)$ *are determined by the same forms as* (51) *with* ϱ*,* δ *of* (27) *and* $\gamma_{-k} := \gamma_k$*, and* $\delta_{-k} := \delta_k$*.*

For the method to obtain the explicit forms of γ_k and δ_k, we refer the reader to the proof for Appendix B in [**20**].

6. Proof of Theorem 5.2 in Case I.

Before we enter the proof of Theorem 5.2 in Case I, we prepare some lemmas. Thanks to (42) and (43), we can apply similar arguments as Lemma 3.4 in [**20**] to our case and we have Lemma 6.1.

Lemma 6.1. *We have*

$$\begin{pmatrix} -\alpha\sigma_1^\theta(u)u\,\theta \\ S(u,\,v) \end{pmatrix} \equiv \sum_{1\le|k|\le m} \frac{-p_2}{\nu_k}\Lambda_k(t)A(\nu_k)\theta,$$

where $\Lambda_k(t)$ *is expressed by*

$$\Lambda_k(t) := \sum_{\substack{1\le|j|\le m,\\ k+j\neq 0}} \frac{2\nu_k+\nu_j}{\nu_k+\nu_j}\left((\alpha\rho_{k,\,j}+2\beta)f_kf_j+(\rho_{-k,-j}+2\beta)f_{-k}f_{-j}\right)$$
$$+\frac{\alpha}{p_2}\sum_{\substack{1\le|j|\le m,\\ k+j\neq 0}}\nu_kf_kf_j-\frac{1}{p_2}\sum_{j=1}^{m}\nu_j^2h_{j,\,k}f_jf_{-j}+\left(3\alpha\rho_{k,-k}+6\beta+\frac{\alpha\nu_k}{p_2}\right)f_kf_{-k}.$$

Here $\rho_{k,\,j}$ *is defined by* (50).

A straightforward computation as Appendix B in [**20**] shows that ρ and δ of (27) can be written in (51) with multi-valued functions γ_k and δ_k of t. The following lemma follows from (51).

Lemma 6.2. *We obtain*

$$\begin{pmatrix}\varrho\\ \delta\end{pmatrix}\equiv\sum_{1\le|k|\le m}\frac{p_2}{\nu_k}(\gamma_k\rho(\nu_{-k})-\delta_k)A(\nu_k),$$

where γ_k, δ_k $(1\le k\le m)$ *are defined by* (51) *and* γ_{-k}, δ_{-k} *are given by*

$$\gamma_{-k}:=\gamma_k,\quad \delta_{-k}:=\delta_k\quad(1\le k\le m).$$

Proposition 6.3 is proved by Lemmas 6.1 and 6.2.

Proposition 6.3. *We have*

$$\begin{pmatrix}-\alpha\sigma_1^{\theta}(u)u\theta\\ S(u,\,v)\end{pmatrix}+\eta^{-1}\begin{pmatrix}\varrho\\ \delta\end{pmatrix}\theta$$
$$\equiv\sum_{1\le|k|\le m}\frac{-p_2}{\nu_k}\left(\Lambda_k(t)-\eta^{-1}(\gamma_k\rho(\nu_{-k})-\delta_k)\right)A(\nu_k)\theta. \tag{53}$$

Using Lemma 5.1 and Proposition 6.3, we can find the form of $f_{k,\,2\ell}$.

Lemma 6.4. *The* $f_{k,\,2\ell}$ *satisfies*

$$P\left(\sum_{1\le|k|\le m}f_{k,\,2\ell}A(\nu_k)\theta\right)\equiv\sum_{1\le|k|\le m}\frac{-p_2}{\nu_k}\Lambda_{k,\,2\ell}(t)A(\nu_k)\theta. \tag{54}$$

Here $\Lambda_{k,\,2\ell}$ *is defined by*

$$\Lambda_{k,\,2\ell}(t) := \sum_{\substack{1\le |j|\le m,\\ k+j\neq 0}} \left(\frac{2\nu_k+\nu_j}{\nu_k+\nu_j}(\alpha\rho_{k,\,j}+2\beta)+\frac{\alpha}{p_2}\nu_k\right)\omega_k^{(1)}\omega_j^{(1)}e^{\tau_k+\tau_j}$$
$$+\sum_{\substack{1\le |j|\le m,\\ k+j\neq 0}} \frac{2\nu_k+\nu_j}{\nu_k+\nu_j}(\alpha\rho_{-k,-j}+2\beta)\omega_{-k}^{(1)}\omega_{-j}^{(1)}e^{-\tau_k-\tau_j}$$
$$-\sum_{j=1}^{m}\frac{\nu_j^2}{p_2}h_{j,\,k}\omega_j^{(1)}\omega_{-j}^{(1)}+\left(3\alpha\rho_{k,-k}+6\beta+\frac{\alpha\nu_k}{p_2}\right)\omega_k^{(1)}\omega_{-k}^{(1)}-\gamma_k\rho(\nu_{-k})+\delta_k. \tag{55}$$

Therefore, by the same argument as Lemma 4.4 in [**20**], Lemma 5.4 is immediately proved.

In what follows, we shall prove Theorem 5.2. Firstly, we prepare a proposition whose proof is given by Appendix C in [**20**].

Proposition 6.5. *We have*

$$u\begin{pmatrix}\varrho\\ \delta\end{pmatrix} \equiv \sum_{1\le |k|\le m}\frac{p_2}{\nu_k}\Bigg(\sum_{j=1}^{m}(\gamma_j\rho(\nu_{-k})-\delta_j)(f_j+f_{-j})h_{j,\,k}$$
$$+\sum_{\substack{j=1,\\ j\neq \pm k}}^{m}\frac{(\gamma_j\rho(\nu_{-k})-\delta_j)(f_k+f_{-k})+(\gamma_k\rho(\nu_{-k})-\delta_k)(f_j+f_{-j})}{g(\nu_k)-g(\nu_j)}\Bigg)A(\nu_k) \tag{56}$$

and

$$\frac{\partial}{\partial t}\begin{pmatrix}u\\ v\end{pmatrix} \equiv \sum_{1\le |k|\le m}\left(\frac{p_2}{\nu_k}\widetilde{\Lambda}_k+\frac{\partial f_k}{\partial t}\right)A(\nu_k). \tag{57}$$

with

$$\widetilde{\Lambda}_k := \sum_{\substack{1\le |j|\le m,\\ j\neq \pm k}}(\rho(\nu_{-k})-\rho(\nu_j))g(\nu_j)'h_{j,\,k}f_j+\left(-\frac{\partial\rho(\nu_k)}{\partial t}+\frac{\nu_k}{p_2}g(\nu_k)'h_{k,\,k}\right)f_k$$
$$-\frac{\partial\rho(\nu_{-k})}{\partial t}f_{-k}. \tag{58}$$

The first member of the non-secularity conditions is derived by looking at the right-hand side of the equation for $f_{k,\,3\ell}$. Let us compare the coefficients of $\eta^{3\ell}$ in both sides of (45). Firstly, we have the following lemma.

Lemma 6.6. *We have*

$$(-\bar{v}_\ell,\ \bar{u}_\ell)Q\begin{pmatrix}\bar{u}_{2\ell}\ \theta\\ \bar{v}_{2\ell}\ \theta\end{pmatrix}+\frac{\bar{u}_\ell}{2}(-\bar{v}_\ell,\ \bar{u}_\ell)\,Q\begin{pmatrix}\bar{u}_\ell\ \theta\\ \bar{v}_\ell\ \theta\end{pmatrix}$$
$$+\frac{\alpha}{2}\bar{u}_\ell\Big(\sigma_1^\theta(\bar{u}_\ell)\bar{v}_\ell+\sigma_1^\theta(\bar{v}_\ell)\bar{u}_\ell\Big)\theta+\beta\bar{u}_\ell^2\sigma_1^\theta(\bar{u}_\ell)\theta$$

$$= \varrho\bar{v}_\ell - \delta\bar{u}_\ell + \sum_{\substack{1\le|k|\le m,\\ 1\le|j|\le m}} \sum_{\substack{1\le|i|\le m,\\ i\neq\pm k}} \frac{\alpha^2}{\nu_j}(\rho(\nu_k)-\rho(\nu_i))a(\nu_k)f_{i,\,\ell}f_{j,\,\ell}f_{k,\,\ell}\theta \quad (59)$$

with $\bar{u}_{j\ell} := \sigma^\eta_{j\ell}(u)$ *and* $\bar{v}_{j\ell} := \sigma^\eta_{j\ell}(v)$ $(j = 1,\, 2)$. *Here* $\sigma^\eta_{j\ell}(u)$ *denotes the coefficient of* $\eta^{j\ell}$ *in* u.

It follows from Lemma 6.6 that

$$\begin{aligned}
&P\left(\sum_{1\le|k|\le m} f_{k,\,3\ell}A(\nu_k)\theta\right)\\
&\quad\equiv \alpha\begin{pmatrix} -\sigma^\theta_1(\bar{u}_\ell)\bar{u}_{2\ell} - \sigma^\theta_1(\bar{u}_{2\ell})\bar{u}_\ell \\ \sigma^\theta_1(\bar{u}_{2\ell})\bar{v}_\ell + \sigma^\theta_1(\bar{v}_\ell)\bar{u}_{2\ell} + 2\sigma^\theta_1(\bar{v}_{2\ell})\bar{u}_\ell \end{pmatrix}\theta\\
&\qquad + 2\beta\begin{pmatrix} 0 \\ \sigma^\theta_1(\bar{u}_\ell)\bar{u}_{2\ell} + 2\sigma^\theta_1(\bar{u}_{2\ell})\bar{u}_\ell \end{pmatrix}\theta - \bar{u}_\ell\begin{pmatrix}\varrho\\ \delta\end{pmatrix}\theta - \frac{\partial}{\partial t}\begin{pmatrix}\bar{u}_\ell\\ \bar{v}_\ell\end{pmatrix}\theta\\
&\qquad + \sum_{1\le|k|\le m}\ \sum_{1\le|j|\le m}\ \sum_{\substack{1\le|i|\le m,\\ i\neq\pm k}} \frac{\alpha^2 p_2}{\nu_k\nu_j}\Big((\rho(\nu_i)-\rho(\nu_k))f_{i,\,\ell}f_{j,\,\ell}f_{k,\,\ell}\\
&\qquad\qquad - (\rho(\nu_{-i})-\rho(\nu_{-k}))f_{-i,\,\ell}f_{-j,\,\ell}f_{-k,\,\ell}\Big)A(\nu_k)\theta. \qquad (60)
\end{aligned}$$

Moreover, by some direct calculations, we obtain the concrete form of the right-hand side of (60). Set

$$\begin{aligned}
l(j,\,i) &:= \frac{1}{\nu_j+\nu_i}(\alpha\rho_{-j,\,-i}+2\beta), \quad \mu(j,\,i) := \nu_i^2 h_{i,\,j},\\
n(j) &:= 3\alpha\rho_{j,\,-j} + 6\beta + \frac{\alpha\nu_j}{p_2}, \quad r(j) := \gamma_j\rho(\nu_{-j}) - \delta_j.
\end{aligned}$$

Lemma 6.7 is proved in a way similar to that for Lemma E.1 in [**20**].

LEMMA 6.7. *For any* k $(1 \le |k| \le m)$, *there exist functions* $\varphi_1(k,\,j)$ *of the variables* ν_i*'s and multi-valued functions* $J_{k,\,1}$ *of finite determination in* Ω *satisfying*

$$\varphi_1(k,\,j) = \varphi_1(-k,\,j)\ \ (1\le j\le m), \quad J_{k,\,1} = J_{-k,\,1} \qquad (*)$$

such that the coefficient of $e^{\tau_k}A(\nu_k)$ *in the first and the fifth terms of the right-hand side of* (60) *is given by*

$$\frac{p_2\alpha}{\nu_k}\left(\sum_{j=1}^m \varphi_1(k,\,j)\omega^{(1)}_j\omega^{(1)}_{-j} + J_{k,\,1}\right)\omega^{(1)}_k. \qquad (61)$$

Moreover we have concrete forms of the coefficients of (61):

$$\begin{aligned}\varphi_1(k,\,j) := {} & \sum_{\substack{i=1,\\ i\neq|k|}}^{m} \frac{2p_2}{\nu_i^2}(\rho_{k,\,-k}+\rho_{i,\,-i})\mu(i,\,j) - \frac{6p_2}{\nu_j^2}(\rho_{k,\,-k}+\rho_{j,\,-j})(\alpha\rho_{j,\,-j}+2\beta) \\ & + \frac{p_2\alpha}{\nu_j^2-\nu_k^2}(\rho_{k,\,-k}+\rho_{j,\,-j})^2 - \frac{8p_2^2\beta}{\nu_j^2-\nu_k^2}\left(\frac{2\beta}{\alpha}+\rho_{k,\,-k}+\rho_{j,\,-j}\right) \\ & + \frac{6p_2}{\nu_k^2}\rho_{k,\,-k}\mu(k,\,j) + \frac{3\alpha}{p_2} \quad (j\neq|k|), \\ \varphi_1(k,\,|k|) := {} & \sum_{\substack{i=1,\\ i\neq|k|}}^{m} \frac{2p_2}{\nu_i^2}(\rho_{k,\,-k}+\rho_{i,\,-i})\mu(i,\,k) - \frac{16p_2\alpha}{\nu_k^2}(\rho_{k,\,-k})^2 + \frac{4p_2\alpha}{\nu_k^2}\rho(\nu_k)\rho(\nu_{-k}) \\ & + \frac{6p_2}{\nu_k^2}\rho_{k,\,-k}\mu(k,k) - \frac{28p_2\beta}{\nu_k^2}\rho_{k,\,-k} + \frac{4\alpha}{p_2}\end{aligned}$$

and

$$\begin{aligned}J_{k,\,1} := {} & \sum_{\substack{j=1,\\ j\neq|k|}}^{m} \frac{p_2}{\nu_j^2}\Big((\rho_{k,\,j}+\rho_{-k,\,j})r(j) + (\rho_{k,\,-j}+\rho_{-k,\,-j})r(-j)\Big) \\ & + \frac{3p_2\gamma_k}{\nu_k^2}(\rho_{k,\,-k})^2 - \frac{6p_2\delta_k}{\nu_k^2}\rho_{k,\,-k} - \frac{\gamma_k}{p_2}.\end{aligned}$$

Similarly, we can prove the following lemma.

Lemma 6.8. *For any k $(1\leq|k|\leq m)$, there exist functions $\varphi_2(k,\,j)$ of the variables ν_ℓ's and multi-valued functions $J_{k,\,2}$ of finite determination in Ω satisfying*

$$\varphi_2(k,\,j) = \varphi_2(-k,\,j) \ \ (1\leq j\leq m), \quad J_{k,\,2} = J_{-k,\,2} \tag{$*$}$$

such that the coefficient of $e^{\tau_k}A(\nu_k)$ in the second term of the right-hand side of (60) is given by

$$\frac{2p_2\beta}{\nu_k}\left(\sum_{j=1}^{m}\varphi_2(k,\,j)\omega_j^{(1)}\omega_{-j}^{(1)} + J_{k,\,2}\right)\omega_k^{(1)}.$$

The concrete forms of $\varphi_2(k,\,j)$ and $J_{k,\,2}$ are given by

$$\begin{aligned}\varphi_2(k,\,j) = {} & \frac{2p_2}{\nu_j^2-\nu_k^2}\left(\alpha(\rho_{k,\,-k}+\rho_{j,\,-j})+4\beta\right) + \frac{2p_2}{\nu_k^2}\mu(k,\,j) + \sum_{i=1}^{m}\frac{4p_2}{\nu_i^2}\mu(i,\,j) \\ & - \frac{12p_2}{\nu_j^2}(\alpha\rho_{j,\,-j}+2\beta) \quad (j\neq|k|), \\ \varphi_2(k,\,|k|) = {} & -\frac{15p_2}{\nu_k^2}(\alpha\rho_{k,\,-k}+2\beta) + \frac{2p_2}{\nu_k^2}\mu(k,\,k) + \sum_{j=1}^{m}\frac{4p_2}{\nu_j^2}\mu(j,\,k)\end{aligned}$$

and

$$J_{k,\,2}=\frac{p_2}{\nu_k^2}\left(\gamma_k\rho_{k,\,-k}-2\delta_k\right)+\sum_{j=1}^{m}\frac{2p_2}{\nu_j^2}\left(\gamma_j\rho_{j,\,-j}-2\delta_j\right),$$

respectively.

Proposition 6.5 implies Lemma 6.9.

LEMMA 6.9. *For any k $(1\le |k|\le m)$, there exist multi-valued functions $J_{k,\,2}$ and R_k of finite determination in Ω satisfying*

$$J_{k,\,3}=J_{-k,\,3},\quad R_k=R_{-k} \tag{$*$}$$

such that the coefficient of $e^{\tau_k}A(\nu_k)$ in the third and fourth terms of the right-hand side of (60) is given by

$$\left(\frac{p_2}{\nu_k}J_{k,\,3}-R_k\right)\omega_k-\frac{d\omega_k}{dt}.$$

The concrete forms of $J_{k,\,3}$ and R_k are given by

$$J_{k,\,3}:=\sum_{\substack{j=1,\\ j\neq|k|}}^{m}\frac{1}{g(\nu_k)-g(\nu_j)}\left(\delta_j-\frac{\gamma_j}{2}\rho_{k,\,-k}\right)+\left(\delta_k-\frac{\gamma_k}{2}\rho_{k,\,-k}\right)h_{k,\,k}+\frac{1}{2}\left(\rho_{k,\,-k}\right)',$$

$$R_k:=\frac{\nu_k'}{2\nu_k}+\sum_{\substack{j=1,\\ j\neq|k|}}^{m}\frac{\gamma_j}{2}\frac{1}{g(\nu_k)-g(\nu_j)}+\frac{1}{2}\gamma_k h_{k,\,k}+\left(g(\nu_k)\right)'h_{k,\,k}.$$

As a consequent of Lemmas 6.7, 6.8 and 6.9, the proof of Theorem 5.2 is now completed. Finally, we give the remark below.

LEMMA 6.10. *The coefficients appearing in Theorem 5.2 are explicitly given by*

$$\psi(k,\,j)=p_2\left(\alpha\varphi_1(k,\,j)+2\beta\varphi_2(k,\,j)\right),$$

$$J_k=p_2(\alpha J_{k,\,1}+2\beta J_{k,\,2}+J_{k,\,3}),$$

$$R_k=\frac{\nu_k'}{2\nu_k}+\sum_{\substack{j=1,\\ j\neq|k|}}^{m}\frac{\gamma_j}{2}\frac{1}{g(\nu_k)-g(\nu_j)}+\frac{1}{2}\gamma_k h_{k,\,k}+\left(g(\nu_k)\right)'h_{k,\,k}.$$

Here φ_1, φ_2 and $J_{k,\,i}$ $(i=1,\,2,\,3)$ are defined in Lemmas 6.7, 6.8 and 6.9.

7. Proof of Theorem 5.5 in Case II.

Note that we do not use the concrete form of $\rho(\nu_k)$ in (53)–(60) but use the relations of (40), (41) and Lemma 4.1. Therefore, we substitute $\alpha=0$ for the equations from (53) to (60). Putting $\alpha=0$ into (60), we have

$$
\begin{aligned}
&P\left(\sum_{1\le|k|\le m} f_{k,\,3\ell}A(\nu_k)\theta\right)\\
&\qquad \equiv 2\beta\begin{pmatrix}0\\ \sigma_1^\theta(\bar{u}_\ell)\bar{u}_{2\ell}+2\sigma_1^\theta(\bar{u}_{2\ell})\bar{u}_\ell\end{pmatrix}\theta-\bar{u}_\ell\begin{pmatrix}\varrho\\ \delta\end{pmatrix}\theta-\frac{\partial}{\partial t}\begin{pmatrix}\bar{u}_\ell\\ \bar{v}_\ell\end{pmatrix}\theta.
\end{aligned}
$$

To obtain the coefficients in the first member $(\mathcal{E}_1)$ of non-secularity conditions, we need the concrete form of $\rho(\nu_k)$. One of the differences between Case I and Case II is the form of $\rho(\nu_k)$. Taking this fact into account and using the computation in the proof of Lemmas 6.8 and 6.9, we can confirm Theorem 5.5. The details of the computations are omitted here.

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