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Equivalence of Littlewood–Paley square function and area function characterizations of weighted product Hardy spaces associated to operators

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Abstract. Let L_1 and L_2 be nonnegative self-adjoint operators acting on $L^2(X_1)$ and $L^2(X_2)$, respectively, where X_1 and X_2 are spaces of homogeneous type. Assume that L_1 and L_2 have Gaussian heat kernel bounds. This paper aims to study some equivalent characterizations of the weighted product Hardy spaces $H^p_{w,L_1,L_2}(X_1 \times X_2)$ associated to L_1 and L_2, for $p \in (0,\infty)$ and the weight w belongs to the product Muckenhoupt class $A_\infty(X_1 \times X_2)$. Our main result is that the spaces $H^p_{w,L_1,L_2}(X_1 \times X_2)$ introduced via area functions can be equivalently characterized by the Littlewood–Paley g-functions and $g^*_{\lambda_1,\lambda_2}$-functions, as well as the Peetre type maximal functions, without any further assumption beyond the Gaussian upper bounds on the heat kernels of L_1 and L_2. Our results are new even in the unweighted product setting.

1. Introduction.

The theory of Hardy spaces has been a successful story in modern harmonic analysis in the last fifty years. In the classical case of the Euclidean space $\mathbb{R}^n$, it is well known that among other equivalent characterizations the Hardy spaces $H^p(\mathbb{R}^n)$ can be characterized by area functions, by Littlewood–Paley g-functions and by atomic decomposition [**12**], [**21**]. Concerning Hardy spaces $H^p(X)$ on a space of homogeneous type X, a new approach to show the equivalence between characterizations of $H^p(X)$ by area functions and g-functions is to use the Plancherel–Polya type inequality, which requires the Hölder continuity and cancellation conditions [**6**]. About the more recent Hardy spaces $H^p_L(X)$ associated to an operator L on a space of homogeneous type X, one used to need extra assumptions to show that the characterizations by area functions and by Littlewood–Paley g-functions are equivalent, for example, Hölder continuity was assumed in [**8**] and Moser type estimate in [**10**]. Only recently, the equivalence of the characterizations of $H^p_L(X)$ by area functions and by Littlewood–Paley g-functions was obtained in [**16**] under no further assumption beyond the Gaussian heat kernel bounds. Actually, the work in [**16**] was done in the weighted setting.

The aim of the current paper is to prove the equivalence between the characterizations of the weighted product Hardy spaces $H^p_{w,L_1,L_2}(X_1 \times X_2)$ in terms of the area functions and Littlewood–Paley square functions, see Theorems 1.4 and 1.5, where we

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assume only that the operators L_1 and L_2 are nonnegative self-adjoint and have Gaussian upper bounds on their heat kernels. This extends the main result in [**16**] to the product setting. The strength of our results is that not only they are new for the setting of product spaces and cover larger classes of operators L_1 and L_2 but also recover a number of known results whose proofs rely on extra regularity of the semigroups. In particular, our Theorems 1.4 and 1.5

(i) give a direct proof for the equivalent characterizations via Littlewood–Paley square functions of the classical product Hardy space by Chang–Fefferman in [**5**],

(ii) provide a new proof of equivalent characterizations via Littlewood–Paley square functions of the product Hardy spaces on spaces of homogeneous type in [**15**] whose proofs require the Hölder continuity and cancellation conditions,

(iii) provide the missing characterizations of product Hardy spaces via Littlewood–Paley square functions in the setting developed in [**7**] and [**10**], and

(iv) recover the recent related known results in the setting of Bessel operators in [**9**] whose proofs relied on the Hölder regularity, and results for Bessel Schrödinger operators in [**1**] whose proofs used the Moser type inequality.

For more details and explanations of (iii) and (iv), we refer to Section 4.

We now recall some basic facts concerning spaces of homogeneous type. Let (X, ρ) be a metric space, and μ be a positive Radon measure on X. Write $V(x, r) := \mu(B(x, r))$, where $B(x, r)$ denotes the open ball centered at x with radius r. We say that (X, ρ, μ) is a space of homogeneous type if it satisfies the volume doubling property:

$$V(x, 2r) \leq V(x, r) \tag{1.1}$$

for all $x \in X$ and $r > 0$. An immediate consequence of (1.1) is that there exist constants C and n such that

$$V(x, \lambda r) \leq C\lambda^n V(x, r) \tag{1.2}$$

for all $x \in X$, $r > 0$ and $\lambda \geq 1$. The constant n plays the role of an upper bound of the dimension, though it need not even be an integer, and we want to take n as small as possible. There also exist constants C and D, $0 \leq D \leq n$, so that

$$V(y, r) \leq C\left(1 + \frac{\rho(x, y)}{r}\right)^D V(x, r) \tag{1.3}$$

uniformly for all $x, y \in X$ and $r > 0$. Indeed, property (1.3) with $D = n$ is a direct consequence of (1.2). In the case where X is the Euclidean space $\mathbb{R}^n$ or a Lie group of polynomial growth, D can be chosen to be 0.

Throughout this paper, we assume that, for $i = 1, 2$, (X_i, ρ_i, μ_i) is a space of homogenous type with $\mu(X_i) = \infty$. The constant n (resp. D) in (1.2) (resp. (1.3)) for (X_i, ρ_i, μ_i) is denoted by n_i (resp. D_i). Let L_i, $i = 1, 2$, be a linear operator on $L^2(X_i, d\mu_i)$ satisfying the following properties:

(H1) Each L_i is a nonnegative self-adjoint operator on $L^2(X_i, d\mu_i)$;

(H2) The kernel of the semigroup e^{-tL_i}, denoted by $p_t^{(i)}(x_i, y_i)$, is a measurable function on $X_i \times X_i$ and obeys a Gaussian upper bound, that is,

$$\left|p_t^{(i)}(x_i, y_i)\right| \le \frac{C_i}{V(x_i, \sqrt{t})} \exp\left(-\frac{\rho_i(x_i, y_i)^2}{c_i t}\right)$$

for all $t > 0$ and a.e. $(x_i, y_i) \in X_i \times X_i$, where C_i and c_i are positive constants, for $i = 1, 2$.

Definition 1.1. Let $\Phi_1, \Phi_2 \in \mathcal{S}(\mathbb{R})$.

a) Given a function $f \in L^2(X_1 \times X_2)$, we define the product type Littlewood–Paley g-function $g_{\Phi_1,\Phi_2,L_1,L_2}(f)$ associated to L_1 and L_2 by

$$g_{\Phi_1,\Phi_2,L_1,L_2}(f)(x_1, x_2) := \left(\int_0^\infty \int_0^\infty \left|\Phi_1(t_1\sqrt{L_1}) \otimes \Phi_2(t_2\sqrt{L_2}) f(x_1, x_2)\right|^2 \frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}.$$

b) The product type area function $S_{\Phi_1,\Phi_2,L_1,L_2}(f)$ associated to L_1 and L_2 is defined by

$$S_{\Phi_1,\Phi_2,L_1,L_2}(f)(x_1, x_2) := \left(\iint_{\Gamma_1(x_1)\times\Gamma_2(x_2)} \left|\Phi_1(t_1\sqrt{L_1}) \otimes \Phi_2(t_2\sqrt{L_2}) f(y_1, y_2)\right|^2 \frac{d\mu_1(y_1)dt_1}{V(x_1, t_1)t_1}\frac{d\mu_2(y_2)dt_2}{V(x_2, t_2)t_2}\right)^{1/2},$$

where $\Gamma_i(x_i) := \{(y_i, t_i) \in X_i \times (0, \infty) : \rho_i(x_i, y_i) < t_i\}$ for $i = 1, 2$.

c) For $\lambda_1, \lambda_2, t_1, t_2 > 0$, the product Peetre type maximal function associated to L_1 and L_2 is defined by

$$\left[\Phi_1(t_1\sqrt{L_1}) \otimes \Phi_2(t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2} f(x_1, x_2) := \sup_{(y_1,y_2)\in X_1\times X_2} \frac{\left|\Phi_1(t_1\sqrt{L_1}) \otimes \Phi_2(t_2\sqrt{L_2}) f(y_1, y_2)\right|}{(1 + t_1^{-1}\rho_1(x_1, y_1))^{\lambda_1}(1 + t_2^{-1}\rho_2(x_2, y_2))^{\lambda_2}},$$

for $(x_1, x_2) \in X_1 \times X_2$.

d) For $\lambda_1, \lambda_2 > 0$, the product type Littlewood–Paley $g^*_{\lambda_1,\lambda_2}$-function associated to L_1 and L_2 is defined by

$$g^*_{\Phi_1,\Phi_2,L_1,L_2,\lambda_1,\lambda_2}(f)(x_1, x_2) := \left(\int_0^\infty\int_0^\infty\int_{X_1}\int_{X_2} \frac{\left|\Phi_1(t_1\sqrt{L_1}) \otimes \Phi_2(t_2\sqrt{L_2}) f(y_1, y_2)\right|^2}{(1 + t_1^{-1}\rho_1(x_1, y_1))^{n_1\lambda_1}(1 + t_2^{-1}\rho_2(x_2, y_2))^{n_2\lambda_2}} \frac{d\mu_1(y_1)dt_1}{V(x_1, t_1)t_1}\frac{d\mu_2(y_2)dt_2}{V(x_2, t_2)t_2}\right)^{1/2},$$

for $(x_1, x_2) \in X_1 \times X_2$.

Following [**13**], [**14**], we introduce product Muckenhoupt weights on spaces of homogeneous type.

Definition 1.2. A nonnegative locally integrable function w on $X_1 \times X_2$ is said to belong to the product Muckenhoupt class $A_p(X_1 \times X_2)$ for a given $p \in (1, \infty)$, if there is a constant C such that for all balls $B_1 \subset X_1$ and $B_2 \subset X_2$,

$$\left(\frac{1}{\mu_1(B_1)\mu_2(B_2)}\iint_{B_1\times B_2} w(x_1,x_2)d\mu_1(x_1)d\mu_2(x_2)\right)$$
$$\times\left(\frac{1}{\mu_1(B_1)\mu_2(B_2)}\iint_{B_1\times B_2} w(x_1,x_2)^{-1/(p-1)}d\mu_1(x_1)d\mu_2(x_2)\right)^{p-1}\le C.$$

The class $A_1(X_1\times X_2)$ is defined to be the collection of all nonnegative locally integrable functions w on $X_1\times X_2$ such that

$$\left(\frac{1}{\mu_1(B_1)\mu_2(B_2)}\iint_{B_1\times B_2} w(x_1,x_2)d\mu_1(x_1)d\mu_2(x_2)\right)\|w^{-1}\|_{L^\infty(B_1\times B_2)}\le C$$

for all balls $B_1\subset X_1$ and $B_2\subset X_2$.

We let $A_\infty(X_1\times X_2):=\bigcup_{1\le p<\infty}A_p(X_1\times X_2)$ and, for any $w\in A_\infty(X_1\times X_2)$, define

$$q_w:=\inf\{q\in[1,\infty):w\in A_q(X_1\times X_2)\},$$

the critical index for w (see, for instance, [**14**]). For $1<p<\infty$, the weighted Lebesgue space $L^p_w(X_1\times X_2)$ is defined to be the collection of all measurable functions f on $X_1\times X_2$ for which

$$\|f\|_{L^p_w(X_1\times X_2)}:=\left(\iint_{X_1\times X_2}|f(x_1,x_2)|^p w(x_1,x_2)d\mu_1(x_1)d\mu_2(x_2)\right)^{1/p}<\infty.$$

We next introduce a class of functions on $\mathbb{R}$ which will play a significant role in our formulation.

DEFINITION 1.3. A function $\Phi\in\mathcal{S}(\mathbb{R})$ is said to belong to the class $\mathcal{A}(\mathbb{R})$ if it satisfies the Tauberian condition, namely,

$$|\Phi(\lambda)|>0\quad\text{on }\{\varepsilon/2<|\lambda|<2\varepsilon\}\tag{1.4}$$

for some $\varepsilon>0$.

Now we are ready to state our main results.

THEOREM 1.4. *Let $\Phi_1,\Phi_2,\widetilde{\Phi}_1,\widetilde{\Phi}_2\in\mathcal{A}(\mathbb{R})$ be even functions satisfying*

$$\Phi_1(0)=\Phi_2(0)=\widetilde{\Phi}_1(0)=\widetilde{\Phi}_2(0)=0.$$

Let $p\in(0,\infty)$ and $w\in A_\infty(X_1\times X_2)$. Then there exists a constant $C=C(p,w,\Phi_1,\Phi_2,\widetilde{\Phi}_1,\widetilde{\Phi}_2)$ such that for all $f\in L^2(X_1\times X_2)$,

$$C^{-1}\|g_{\widetilde{\Phi}_1,\widetilde{\Phi}_2,L_1,L_2}(f)\|_{L^p_w(X_1\times X_2)}\le\|g_{\Phi_1,\Phi_2,L_1,L_2}(f)\|_{L^p_w(X_1\times X_2)}$$
$$\le C\|g_{\widetilde{\Phi}_1,\widetilde{\Phi}_2,L_1,L_2}(f)\|_{L^p_w(X_1\times X_2)}.$$

THEOREM 1.5. *Let $\Phi_1,\Phi_2\in\mathcal{A}(\mathbb{R})$ be even functions. Let $p\in(0,\infty)$, $\lambda_i>2q_w/\min\{p,2\}$ and $\lambda_i'>(n_i+D_i)q_w/\min\{p,2\}$, $i=1,2$. Then for $f\in L^2(X_1\times X_2)$*

we have the following quasi-norm equivalence:

$$\begin{aligned}\|S_{\Phi_1,\Phi_2,L_1,L_2}(f)\|_{L^p_w(X_1\times X_2)} &\sim \|g_{\Phi_1,\Phi_2,L_1,L_2}(f)\|_{L^p_w(X_1\times X_2)}\\ &\sim \|g^*_{\Phi_1,\Phi_2,L_1,L_2,\lambda_1,\lambda_2}(f)\|_{L^p_w(X_1\times X_2)}\\ &\sim \left\|\left(\int_0^\infty\int_0^\infty \left|\left[\Phi_1(t_1\sqrt{L_1})\otimes\Phi_2(t_2\sqrt{L_2})\right]^*_{\lambda_1',\lambda_2'}f\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)}. \qquad (1.5)\end{aligned}$$

Having these results, one can introduce weighted product Hardy spaces associated to L_1 and L_2 as follows:

Definition 1.6. Let $p\in(0,\infty)$, $w\in A_\infty(X_1\times X_2)$, and $\Phi_1,\Phi_2\in\mathcal{A}(\mathbb{R})$ be even functions satisfying

$$\Phi_1(0)=\Phi_2(0)=0.$$

The weighted product Hardy space $H^p_{w,L_1,L_2}(X_1\times X_2)$ associated to L_1 and L_2 is defined to be the completion of the set

$$\left\{f\in L^2(X_1\times X_2): S_{\Phi_1,\Phi_2,L_1,L_2}(f)\in L^p_w(X_1\times X_2)\right\}$$

with respect to the (quasi-)norm

$$\|f\|_{H^p_{w,L_1,L_2}(X_1\times X_2)} := \|S_{\Phi_1,\Phi_2,L_1,L_2}(f)\|_{L^p_w(X_1\times X_2)}.$$

Remark 1.7. Combining Theorems 1.4 and 1.5 we see that the definition of $H^p_{w,L_1,L_2}(X_1\times X_2)$ is independent of the choice of the even functions Φ_1,Φ_2, as long as $\Phi_1,\Phi_2\in\mathcal{A}(\mathbb{R})$ and satisfy $\Phi_1(0)=\Phi_2(0)=0$. In particular, if one chooses $\Phi_1(\lambda)=\Phi_2(\lambda)=\lambda^2e^{-\lambda^2}$, then the (quasi-)norm of $H^p_{w,L_1,L_2}(X_1\times X_2)$ can be written as

$$\|f\|_{H^p_{w,L_1,L_2}(X_1\times X_2)} := \left\|\left(\iint_{\Gamma_1(x_1)\times\Gamma_2(x_2)}\left|(t_1^2L_1e^{-t_1^2L_1})\otimes(t_2^2L_2e^{-t_2^2L_2})f(y_1,y_2)\right|^2 \times\frac{d\mu_1(y_1)dt_1}{V(x_1,t_1)t_1}\frac{d\mu_2(y_2)dt_2}{V(x_2,t_2)t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)}.$$

Furthermore, from Theorem 1.5 we see that each quantity in (1.5) can be used as an equivalent (quasi-)norm of the space $H^p_{w,L_1,L_2}(X_1\times X_2)$.

As mentioned above, we make no further assumption on the heat kernel of L_1 or L_2 beyond the Gaussian upper bounds. Thus, the approach in [**8**] which uses a Plancherel–Polya type inequality and the approach in [**10**] which uses a discrete characterization can not be applied directly to our setting. To achieve our goal, we will follow the approach in [**2**], [**3**], [**18**], whose key ingredient is a sub-mean value property; see Lemma 3.4 below. This approach has recently been used in [**16**] to derive the equivalence of Littlewood–Paley g-function and area function characterizations of one-parameter Hardy spaces associated to operators. However, the Littlewood–Paley g-function and area function in

[**16**] are only defined via the heat semigroup, which are less general than those defined in the current paper.

We close this introduction by making some conventions. Throughout this paper, we denote by C and c (possibly with subscripts) constants that may vary from place to place and may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards. In some cases "sup" will mean "ess sup", which will be clear from the context.

2. Preliminaries.

In this section we collect some facts and technical results which will be needed in the subsequent section. We start by noting that, if (X, ρ, μ) is a space of homogeneous type, then for any $N > n$, there exists a constant $C = C(N)$ such that

$$\int_X \left(1 + \frac{\rho(x,y)}{t}\right)^{-N} d\mu(y) \le CV(x,t) \tag{2.1}$$

for all $x \in X$ and $t > 0$.

The following lemma is essentially [**4**, Lemma 2.3]. See also [**20**, Lemma 2.1].

LEMMA 2.1. *Assume that (X, ρ, μ) is a space of homogeneous type and L is a nonnegative self-adjoint operator on $L^2(X, d\mu)$ whose heat kernel obeys the Gaussian upper bound. Let $\Phi \in \mathcal{S}(\mathbb{R})$ be even functions. Then for every $N > 0$, there exists a constant $C = C(\Phi, N)$ such that the kernel $K_{\Phi(t\sqrt{L})}(x,y)$ of the operator $\Phi(t\sqrt{L})$ satisfies*

$$\left|K_{\Phi(t\sqrt{L})}(x,y)\right| \le \frac{C}{V(x,t)} \left(1 + \frac{\rho(x,y)}{t}\right)^{-N}.$$

LEMMA 2.2. *Assume that (X, ρ, μ) is a space of homogeneous type and L is a nonnegative self-adjoint operator on $L^2(X, d\mu)$ whose heat kernel obeys the Gaussian upper bound. Let $\Phi, \Psi \in \mathcal{S}(\mathbb{R})$ be even functions and let Ψ satisfy*

$$\Psi^{(\nu)}(0) = 0, \quad \nu = 0, 1, \ldots, m \tag{2.2}$$

for some positive odd integer m. Then for every $N > 0$, there exists a constant $C = C(\Phi, \Psi, N, m)$ such that for all $s \ge t > 0$,

$$\left|K_{\Phi(s\sqrt{L})\Psi(t\sqrt{L})}(x,y)\right| \le C \left(\frac{t}{s}\right)^{m+1} \frac{1}{V(x,s)} \left(1 + \frac{\rho(x,y)}{s}\right)^{-N}. \tag{2.3}$$

PROOF. First note that the property (2.2) implies that the function $\lambda \mapsto \lambda^{-(m+1)}\Psi(\lambda)$ is an even function, smooth at 0, and belongs to $\mathcal{S}(\mathbb{R})$. We set $\Phi_m(\lambda) := \lambda^{m+1}\Phi(\lambda)$ and $\Psi_m(\lambda) := \lambda^{-(m+1)}\Psi(\lambda)$ for $\lambda \in \mathbb{R}$. Then both Φ_m and Ψ_m are even functions and belong to $\mathcal{S}(\mathbb{R})$. Since

$$\Phi(s\sqrt{L})\Psi(t\sqrt{L}) = \left(\frac{t}{s}\right)^{m+1} \left[(s\sqrt{L})^{m+1}\Phi(s\sqrt{L})\right]\left[(t\sqrt{L})^{-(m+1)}\Psi(t\sqrt{L})\right]$$

$$= \left(\frac{t}{s}\right)^{m+1} \Phi_m(s\sqrt{L})\Psi_m(t\sqrt{L}),$$

it follows from Lemma 2.1 that

$$\begin{aligned}
&\left|K_{\Phi(s\sqrt{L})\Psi(t\sqrt{L})}(x,y)\right| \\
&\quad= \left(\frac{t}{s}\right)^{m+1} \left|K_{\Phi_m(s\sqrt{L})\Psi_m(t\sqrt{L})}(x,y)\right| \\
&\quad\le \left(\frac{t}{s}\right)^{m+1} \int_X \left|K_{\Phi_m(s\sqrt{L})}(x,z)K_{\Psi_m(t\sqrt{L})}(z,y)\right| d\mu(z) \\
&\quad\le C(\Phi,\Psi,N,m)\left(\frac{t}{s}\right)^{m+1} \int_X \frac{1}{V(x,s)}\left(1+\frac{\rho(x,z)}{s}\right)^{-N} \\
&\qquad\qquad \times \frac{1}{V(y,t)}\left(1+\frac{\rho(z,y)}{t}\right)^{-(N+n+1)} d\mu(z). \qquad (2.4)
\end{aligned}$$

For $s \ge t > 0$, we have

$$\left(1+\frac{\rho(x,z)}{s}\right)^{-N}\left(1+\frac{\rho(z,y)}{t}\right)^{-N} \le \left(1+\frac{\rho(x,y)}{s}\right)^{-N}.$$

This along with (2.1) yields

$$\begin{aligned}
&\int_X \left(1+\frac{\rho(x,z)}{s}\right)^{-N}\left(1+\frac{\rho(z,y)}{t}\right)^{-(N+n+1)} d\mu(z) \\
&\quad\le \left(1+\frac{\rho(x,y)}{s}\right)^{-N} \int_X \left(1+\frac{\rho(z,y)}{t}\right)^{-(n+1)} d\mu(z) \\
&\quad\le C\left(1+\frac{\rho(x,y)}{s}\right)^{-N} V(y,t). \qquad (2.5)
\end{aligned}$$

Combining (2.4) and (2.5) we obtain (2.3). □

Lemma 2.3. *Suppose $\Phi \in \mathcal{A}(\mathbb{R})$ is an even function. Then there exist even functions $\Psi, \Upsilon, \Theta \in \mathcal{S}(\mathbb{R})$ such that*

$$\begin{aligned}
&\operatorname{supp}\Upsilon \subset \{|\lambda| \le 2\varepsilon\}, \\
&\operatorname{supp}\Theta \subset \{\varepsilon/2 \le |\lambda| \le 2\varepsilon\}
\end{aligned}$$

and

$$\Psi(\lambda)\Upsilon(\lambda) + \sum_{k=1}^{\infty} \Phi(2^{-2k}\lambda)\Theta(2^{-2k}\lambda) = 1 \quad \textit{for all } \lambda \in \mathbb{R},$$

where ε is a constant from (1.4).

Proof. Define $\Psi(\lambda) := e^{-\lambda^2}$, $\lambda \in \mathbb{R}$. Obviously, $\Psi \in \mathcal{S}(\mathbb{R})$ and Ψ is even. Choose nonnegative even functions $\Omega, \Gamma \in \mathcal{S}(\mathbb{R})$ such that

$$\Omega(\lambda) \neq 0 \Longleftrightarrow |\lambda| < 2\varepsilon,$$
$$\Gamma(\lambda) \neq 0 \Longleftrightarrow \varepsilon/2 < |\lambda| < 2\varepsilon.$$

Then we set

$$\Xi(\lambda) := \Psi(\lambda)\Omega(\lambda) + \sum_{k=1}^{\infty} \Phi(2^{-k}\lambda)\Gamma(2^{-k}\lambda), \quad \lambda \in \mathbb{R}.$$

From the properties of Φ, Ψ, Ω and Γ it follows that Ξ is strictly positive on $\mathbb{R}$. In addition, from the properties of Ω and Γ we see that for any fixed $\lambda_0 \in \mathbb{R}\backslash\{0\}$, the number of those k's for which $\Phi(2^{-k}\lambda)\Gamma(2^{-k}\lambda)$ do not vanish identically in $(4\lambda_0/5, 6\lambda_0/5)$ is no more than 4, which implies that Ξ is smooth in $(4\lambda_0/5, 6\lambda_0/5)$ and hence $\Xi \in C^{\infty}(\mathbb{R}\backslash\{0\})$. It is obvious that Ξ is also smooth at the origin 0. Therefore $\Xi \in C^{\infty}(\mathbb{R})$. Now define the functions Υ and Θ respectively by

$$\Upsilon(\lambda) := \frac{\Omega(\lambda)}{\Xi(\lambda)} \quad \text{and} \quad \Theta(\lambda) := \frac{\Gamma(\lambda)}{\Xi(\lambda)}.$$

Then it is straightforward to verify that Ψ, Υ and Θ satisfy the desired properties. □

The following lemma is a homogeneous analogy of Lemma 2.3. It can be obtained by slightly modifying the argument of Lemma 2.3.

Lemma 2.4. *Suppose $\Phi \in \mathcal{A}(\mathbb{R})$ is an even function. Then there exists an even function $\Theta \in \mathcal{S}(\mathbb{R})$ such that*

$$\operatorname{supp} \Theta \subset \{\varepsilon/2 \leq |\lambda| \leq 2\varepsilon\}$$

and

$$\sum_{k=-\infty}^{\infty} \Phi(2^{-k}\lambda)\Theta(2^{-k}\lambda) = 1 \quad \text{for all } \lambda \in \mathbb{R}\backslash\{0\},$$

where ε is a constant from (1.4).

Lemma 2.5. *Assume that (X, ρ, μ) is a space of homogeneous type with $\mu(X) = \infty$ and L is a nonnegative self-adjoint operator on $L^2(X, d\mu)$ whose heat kernel obeys the Gaussian upper bound. Let $\{E(\lambda) : \lambda \geq 0\}$ be spectral resolution of L. Then the spectral measure of the set $\{0\}$ is zero, i.e., the point $\lambda = 0$ may be neglected in the spectral resolution.*

Proof. Assume by contradiction that $E(\{0\}) \neq 0$, then there exists $g \in L^2(X)$ such that $f := E(\{0\})g$ is not the zero element in $L^2(X, d\mu)$. Since $E(\{0\})$ is a an orthogonal projection,

$$E(\{0\})f = E(\{0\})E(\{0\})g = E(\{0\})g = f.$$

It follows that for all $t > 0$,

$$e^{-tL} f = \int_0^\infty e^{-t\lambda} dE(\lambda) f = \int_0^\infty e^{-t\lambda} dE(\lambda) E(\{0\}) f = \int_{\{0\}} e^{-t\lambda} dE(\lambda) f = E(\{0\}) f = f.$$

Hence, for a.e. $x \in X$ and all $t > 0$, we have

$$\begin{aligned} |f(x)| = \left|e^{-tL} f(x)\right| &\le \int_X |p_t(x,y)||f(y)| d\mu(y) \\ &\le \|f\|_{L^2(X,d\mu)} \left(\int_X |p_t(x,y)|^2 d\mu(y) \right)^{1/2} \\ &\le C\|f\|_{L^2(X,d\mu)} \left(\int_X \frac{1}{V(x,\sqrt{t})^2} \left(1 + \frac{\rho(x,y)}{\sqrt{t}}\right)^{-(n+1)} d\mu(y) \right)^{1/2} \\ &\le C\|f\|_{L^2(X,d\mu)} V(x,\sqrt{t})^{-1/2}. \end{aligned}$$

Since $\mu(X) = \infty$, letting $t \to \infty$ in the above inequalities yields that $f(x) = 0$. Hence $f = 0$ in $L^2(X, d\mu)$, which leads to a contradiction. Therefore we must have $E(\{0\}) = 0$. □

The following two lemmas are two-parameter counterparts of Lemma 2 and Lemma 3 in [**18**], respectively. These can be proved by slightly modifying the proofs of the corresponding one-parameter results. We omit the details here.

Lemma 2.6 ([**18**, Lemma 2]). *Let $0 < p, q < \infty$ and $\sigma_1, \sigma_2 > 0$. Let w be an arbitrary weight (i.e., nonnegative locally integrable function) on $X_1 \times X_2$. Let $\{g_{j_1,j_2}\}_{j_1,j_2=-\infty}^\infty$ be a sequence of nonnegative measurable functions on $X_1 \times X_2$ and put*

$$h_{j_1,j_2}(x_1,x_2) = \sum_{k_1=-\infty}^\infty \sum_{k_2=-\infty}^\infty 2^{-|k_1-j_1|\sigma_1} 2^{-|k_2-j_2|\sigma_2} g_{k_1,k_2}(x_1,x_2)$$

for $(x_1,x_2) \in X_1 \times X_2$ and $j_1, j_2 \in \mathbb{Z}$. Then, there exists a constant $C = C(q,\sigma_1,\sigma_2)$ such that

$$\left\| \{h_{j_1,j_2}\}_{j_1,j_2=-\infty}^\infty \right\|_{L^p_w(\ell^q)} \le C \left\| \{g_{j_1,j_2}\}_{j_1,j_2=-\infty}^\infty \right\|_{L^p_w(\ell^q)},$$

where

$$\begin{aligned} \left\| \{g_{j_1,j_2}\}_{j_1,j_2=-\infty}^\infty \right\|_{L^p_w(\ell^q)} &:= \left\| \left\| \{g_{j_1,j_2}\}_{j_1,j_2=-\infty}^\infty \right\|_{\ell_q} \right\|_{L^p_w(X_1\times X_2)} \\ &= \left\| \left(\sum_{j_1=-\infty}^\infty \sum_{j_2=-\infty}^\infty |g_{j_1,j_2}(x_1,x_2)|^q \right)^{1/q} \right\|_{L^p_w(X_1\times X_2)}. \end{aligned} \tag{2.6}$$

LEMMA 2.7 ([**18**, Lemma 3]). *Let* $0 < r \le 1$, *and let* $\{b_{j_1,j_2}\}_{j_1,j_2=-\infty}^{\infty}$ *and* $\{d_{j_1,j_2}\}_{j_1,j_2=-\infty}^{\infty}$ *be two sequences taking values in* $(0,\infty]$ *and* $(0,\infty)$ *respectively. Assume that there exists* $N_0 > 0$ *such that*

$$d_{j_1,j_2} = O(2^{j_1 N_0} 2^{j_2 N_0}), \quad j_1, j_2 \to \infty,$$

and that for every $N > 0$ *there exists a finite constant* $C = C_N$ *such that*

$$d_{j_1,j_2} \le C_N \sum_{k_1=j_1}^{\infty} \sum_{k_2=j_2}^{\infty} 2^{(j_1-k_1)N} 2^{(j_2-k_2)N} b_{k_1,k_2} d_{k_1,k_2}^{1-r}, \quad j_1, j_2 \in \mathbb{Z}.$$

Then for every $N > 0$,

$$d_{j_1,j_2}^r \le C_N \sum_{k_1=j_1}^{\infty} \sum_{k_2=j_2}^{\infty} 2^{(j_1-k_1)Nr} 2^{(j_2-k_2)Nr} b_{k_1,k_2}, \quad j_1, j_2 \in \mathbb{Z},$$

with the same constants C_N.

For a locally integrable function f on $X_1 \times X_2$, the strong maximal function is defined by

$$\mathcal{M}_s(f)(x_1,x_2) := \sup_{(x_1,x_2)\in B_1\times B_2} \frac{1}{\mu_1(B_1)\mu_2(B_2)} \iint_{B_1\times B_2} |f(y_1,y_2)| d\mu_1(y_1) d\mu_2(y_2),$$

where B_i runs over all balls in X_i, $i = 1, 2$. Using (1.3) and the volume doubling property, one can easily show that if $N_i > n_i + D_i$ for $i = 1, 2$, then

$$\iint_{X_1\times X_2} \frac{|f(y_1,y_2)|}{\prod_{i=1}^2 V(y_i,t_i)(1+t_i^{-1}\rho_i(x_i,y_i))^{N_i}} d\mu_1(y_1) d\mu_2(y_2) \le C\mathcal{M}_s(f)(x_1,x_2). \tag{2.7}$$

We will also need the following weighted vector-valued inequality for strong maximal functions on spaces of homogeneous type. See, for instance, [**14**] and [**19**].

LEMMA 2.8. *Suppose* $1 < p < \infty$, $1 < q \le \infty$ *and* $w \in A_p(X_1 \times X_2)$. *Then there exists a constant* C *such that*

$$\left\| \{\mathcal{M}_s(f_{j_1,j_2})\}_{j_1,j_2=-\infty}^{\infty} \right\|_{L_w^p(\ell^q)} \le C \left\| \{f_{j_1,j_2}\}_{j_1,j_2=-\infty}^{\infty} \right\|_{L_w^p(\ell^q)}$$

for all sequences $\{f_{j_1,j_2}\}_{j_1,j_2=-\infty}^{\infty}$ *on* $X_1 \times X_2$, *where the space* $L_w^p(\ell^q)$ *is defined by* (2.6).

3. Proofs of Theorems 1.4 and 1.5.

We divide the proofs of Theorems 1.4 and 1.5 into a sequence of lemmas.

The following lemma is standard; see, for instance, [**22**, Theorem 4 in Chapter 4].

LEMMA 3.1. *Let* $\Phi_1, \Phi_2 \in \mathcal{S}(\mathbb{R})$ *be even functions. Let* $p \in (0,\infty)$, $w \in A_\infty(X_1 \times X_2)$, *and* $\lambda_1, \lambda_2 > 2q_w/\min\{p,2\}$. *Then there exists a constant* C *such that for all* $f \in L^2(X_1 \times X_2)$,

$$\left\|g^*_{\Phi_1,\Phi_2,L_1,L_2,\lambda_1,\lambda_2}(f)\right\|_{L^p_w(X_1\times X_2)} \le C\left\|S_{\Phi_1,\Phi_2,L_1,L_2}(f)\right\|_{L^p_w(X_1\times X_2)}.$$

Lemma 3.2. *Let $\Phi_1, \Phi_2 \in \mathcal{S}(\mathbb{R})$ be even functions. Let $p \in (0,\infty)$, $\lambda_1, \lambda_2 > 0$, and w be an arbitrary weight (i.e., nonnegative locally integrable function) on $X_1 \times X_2$. Then there exists a constant C such that for all $f \in L^2(X_1 \times X_2)$,*

$$\begin{aligned}&\|S_{\Phi_1,\Phi_2,L_1,L_2}(f)\|_{L^p_w(X_1\times X_2)}\\ &\quad\le C\left\|\left(\int_0^\infty\int_0^\infty\left|\left[\Phi_1(t_1\sqrt{L_1})\otimes\Phi_2(t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2}f\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)}.\end{aligned}$$

Proof. Observe that for all $\lambda_1, \lambda_2, t_1, t_2 > 0$ and all $(x_1, x_2) \in X_1 \times X_2$,

$$\begin{aligned}&\frac{1}{V(x_1,t_1)V(x_2,t_2)}\iint_{B(x_1,t_1)\times B(x_2,t_2)}\left|\Phi_1(t_1\sqrt{L_1})\otimes\Phi_2(t_2\sqrt{L_2})f(y_1,y_2)\right|^2 d\mu_1(y_1)d\mu_2(y_2)\\ &\qquad\le \sup_{(y_1,y_2)\in B(x_1,t_1)\times B(x_2,t_2)}\left|\Phi_1(t_1\sqrt{L_1})\otimes\Phi_2(t_2\sqrt{L_2})f(y_1,y_2)\right|^2\\ &\qquad\le 2^{2\lambda_1}2^{2\lambda_2}\left|\left[\Phi_1(t_1\sqrt{L_1})\otimes\Phi_2(t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2}f(x_1,x_2)\right|^2.\end{aligned}$$

Taking the norm $\int_0^\infty\int_0^\infty|\cdot|(dt_1/t_1)(dt_2/t_2)$ on both sides gives the pointwise estimate

$$\begin{aligned}&\left[S_{\Phi_1,\Phi_2,L_1,L_2}(f)(x_1,x_2)\right]^2\\ &\quad\le 2^{2\lambda_1}2^{2\lambda_2}\int_0^\infty\int_0^\infty\left|\left[\Phi_1(t_1\sqrt{L_1})\otimes\Phi_2(t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2}f(x_1,x_2)\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2},\end{aligned}$$

which readily yields the desired estimate. □

Lemma 3.3. *Suppose $\Phi_1, \Phi_2, \widetilde{\Phi}_1, \widetilde{\Phi}_2 \in \mathcal{A}(\mathbb{R})$ are even functions satisfying*

$$\Phi_1(0) = \Phi_2(0) = \widetilde{\Phi}_1(0) = \widetilde{\Phi}_2(0) = 0.$$

Let $p \in (0,\infty)$, $\lambda_1, \lambda_2 > 0$, and w be an arbitrary weight (i.e., nonnegative locally integrable function) on $X_1 \times X_2$. Then there exists a constant C such that for all $f \in L^2(X_1 \times X_2)$,

$$\begin{aligned}&\left\|\left(\int_0^\infty\int_0^\infty\left|\left[\widetilde{\Phi}_1(t_1\sqrt{L_1})\otimes\widetilde{\Phi}_2(t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2}f\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)}\\ &\quad\sim\left\|\left(\int_0^\infty\int_0^\infty\left|\left[\Phi_1(t_1\sqrt{L_1})\otimes\Phi_2(t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2}f\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)}.\end{aligned}$$

Proof. For $i = 1, 2$, since $\Phi_i \in \mathcal{A}(\mathbb{R})$ and Φ_i is even, by Lemma 2.4 there exists an even function $\Theta_i \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp}\Theta_i \subset \{\varepsilon_i/2 \le |\lambda| \le 2\varepsilon_i\}$ and

$$\sum_{k=-\infty}^{\infty}\Phi_i(2^{-k}\lambda)\Theta_i(2^{-k}\lambda) = 1 \quad \text{for } \lambda \in \mathbb{R}\backslash\{0\},$$

where ε_i is the constant in the Tauberian condition (1.4) corresponding to Φ_i. Hence it follows from Lemma 2.5 and the spectral theorem that for all $f \in L^2(X_1 \times X_2)$ and $t_1, t_2 \in [1,2]$,

$$f = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \big(\Phi_1(2^{-k_1}t_1\sqrt{L_1})\Theta_1(2^{-k_1}t_1\sqrt{L_1})\big)\otimes\big(\Phi_2(2^{-k_2}t_2\sqrt{L_2})\Theta_2(2^{-k_2}t_2\sqrt{L_2})\big) f$$

with convergence in the sense of $L^2(X_1 \times X_2)$ norm. Consequently, for all $j_1, j_2 \in \mathbb{Z}$, all $t_1, t_2 \in [1,2]$ and a.e. $(y_1, y_2) \in X_1 \times X_2$,

$$\begin{aligned}
&\widetilde{\Phi}_1(2^{-j_1}t_1\sqrt{L_1}) \otimes \widetilde{\Phi}_2(2^{-j_2}t_2\sqrt{L_2}) f(y_1,y_2) \\
&\quad = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \big(\widetilde{\Phi}_1(2^{-j_1}t_1\sqrt{L_1})\Phi_1(2^{-k_1}t_1\sqrt{L_1})\Theta_1(2^{-k_1}t_1\sqrt{L_1})\big) \\
&\qquad\qquad \otimes \big(\widetilde{\Phi}_2(2^{-j_2}t_2\sqrt{L_2})\Phi_2(2^{-k_2}t_2\sqrt{L_2})\Theta_2(2^{-k_2}t_2\sqrt{L_2})\big) f(y_1,y_2) \\
&\quad = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \iint_{X_1\times X_2} K_{\widetilde{\Phi}_1(2^{-j_1}t_1\sqrt{L_1})\Theta_1(2^{-k_1}t_1\sqrt{L_1})}(y_1,z_1) \\
&\qquad\qquad \times K_{\widetilde{\Phi}_2(2^{-j_2}t_2\sqrt{L_2})\Theta_2(2^{-k_2}t_2\sqrt{L_2})}(y_2,z_2) \\
&\qquad\qquad \times \big(\Phi_1(2^{-k_1}t_1\sqrt{L_1}) \otimes \Phi_2(2^{-k_2}t_2\sqrt{L_2})\big) f(z_1,z_2) d\mu_1(z_1) d\mu_2(z_2). \qquad (3.1)
\end{aligned}$$

Since $\widetilde{\Phi}_i$ is even on $\mathbb{R}$, we have $\widetilde{\Phi}_i'(0) = 0$, for $i = 1,2$. Thus $\widetilde{\Phi}_i(0) = \widetilde{\Phi}_i'(0) = 0$ for $i = 1,2$. On the other hand, since Θ_i vanishes near the origin, we have $\Theta_i^{(\nu)}(0) = 0$ for every non-negative integer ν. Hence it follows from Lemma 2.2 that for any positive integer m and any $N > 0$,

$$\begin{aligned}
&\Big|K_{\widetilde{\Phi}_i(2^{-j_i}t_i\sqrt{L_i})\Theta_i(2^{-k_i}t_i\sqrt{L_i})}(y_i,z_i)\Big| \\
&\quad \le \begin{cases} C(\widetilde{\Phi}_i,\Theta_i,N)2^{-2|j_i-k_i|}V(y_i,2^{-k_i}t_i)^{-1}(1+2^{k_i}t_i^{-1}\rho_i(y_i,z_i))^{-N}, & j_i \ge k_i, \\ C(\widetilde{\Phi}_i,\Theta_i,N,m)2^{-m|j_i-k_i|}V(y_i,2^{-j_i}t_i)^{-1}(1+2^{j_i}t_i^{-1}\rho_i(y_i,z_i))^{-N}, & j_i < k_i. \end{cases} \qquad (3.2)
\end{aligned}$$

Choose $N \ge \max\{\lambda_1 + n_1 + 1, \lambda_2 + n_2 + 1\}$, then from (3.1), (3.2) and the inequality

$$\begin{aligned}
&\Big|\big(\Phi_1(2^{-k_1}t_1\sqrt{L_1}) \otimes \Phi_2(2^{-k_2}t_2\sqrt{L_2})\big) f(z_1,z_2)\Big| \\
&\quad \le \big[\Phi_1(2^{-k_1}t_1\sqrt{L_1}) \otimes \Phi_2(2^{-k_2}t_2\sqrt{L_2})\big]^*_{\lambda_1,\lambda_2} f(x_1,x_2) \\
&\qquad \times (1+2^{k_1}t_1^{-1}\rho_1(x_1,z_1))^{\lambda_1}(1+2^{k_2}t_2^{-1}\rho_2(x_2,z_2))^{\lambda_2},
\end{aligned}$$

we infer that

$$\begin{aligned}
&\big[\widetilde{\Phi}_1(2^{-j_1}t_1\sqrt{L_1}) \otimes \widetilde{\Phi}_2(2^{-j_2}t_2\sqrt{L_2})\big]^*_{\lambda_1,\lambda_2} f(x_1,x_2) \\
&\quad \le \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \gamma_{j_1,k_1,j_2,k_2} \big[\Phi_1(2^{-k_1}t_1\sqrt{L_1}) \otimes \Phi_2(2^{-k_2}t_2\sqrt{L_2})\big]^*_{\lambda_1,\lambda_2} f(x_1,x_2)
\end{aligned}$$

$$\times \sup_{(y_1,y_2)\in X_1\times X_2} \iint_{X_1\times X_2} \frac{(1+2^{k_i}t_i^{-1}\rho_i(x_i,z_i))^{\lambda_i}}{\prod_{i=1}^2 (1+2^{j_i}t_i^{-1}\rho_i(x_i,y_i))^{\lambda_i}(1+2^{j_i\wedge k_i}t_i^{-1}\rho_i(y_i,z_i))^{\lambda_i+n_i+1}} \times \frac{d\mu_1(z_1)d\mu_2(z_2)}{V(y_1,2^{-(j_1\wedge k_1)}t_1)V(y_2,2^{-(j_2\wedge k_2)}t_2)}$$

where $j_i \wedge k_i := \min\{j_i, k_i\}$ and

$$\gamma_{j_1,k_1,j_2,k_2} := \begin{cases} 2^{-2|j_1-k_1|}2^{-2|j_2-k_2|} & \text{if } j_1 \ge k_1 \text{ and } j_2 \ge k_2, \\ 2^{-2|j_1-k_1|}2^{-m|j_2-k_2|} & \text{if } j_1 \ge k_1 \text{ and } j_2 < k_2, \\ 2^{-m|j_1-k_1|}2^{-2|j_2-k_2|} & \text{if } j_1 < k_1 \text{ and } j_2 \ge k_2, \\ 2^{-m|j_1-k_1|}2^{-m|j_2-k_2|} & \text{if } j_1 \ge k_1 \text{ and } j_2 \ge k_2. \end{cases}$$

Using (2.1) and the fundamental inequality

$$(1+2^{k_i}t_i^{-1}\rho_i(x_i,z_i))^{\lambda_i} \le \begin{cases} (1+2^{j_i}t_i^{-1}\rho_i(x_i,y_i))^{\lambda_i}(1+2^{k_i}t_i^{-1}\rho_i(y_i,z_i))^{\lambda_i}, & j_i \ge k_i, \\ 2^{(k_i-j_i)\lambda_i}(1+2^{j_i}t_i^{-1}\rho_i(x_i,y_i))^{\lambda_i}(1+2^{j_i}t_i^{-1}\rho_i(y_i,z_i))^{\lambda_i}, & j_i < k_i, \end{cases}$$

it follows that

$$\big[\widetilde{\Phi}_1(2^{-j_1}t_1\sqrt{L_1}) \otimes \widetilde{\Phi}_2(2^{-j_2}t_2\sqrt{L_2})\big]^*_{\lambda_1,\lambda_2} f(x_1,x_2) \le \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} \gamma'_{j_1,k_1,j_2,k_2}\big[\Phi_1(2^{-k_1}t_1\sqrt{L_1}) \otimes \Phi_2(2^{-k_2}t_2\sqrt{L_2})\big]^*_{\lambda_1,\lambda_2} f(x_1,x_2), \quad (3.3)$$

where

$$\gamma'_{j_1,k_1,j_2,k_2} := \begin{cases} 2^{-2|j_1-k_1|}2^{-2|j_2-k_2|} & \text{if } j_1 \ge k_1 \text{ and } j_2 \ge k_2, \\ 2^{-2|j_1-k_1|}2^{-(m-\lambda_2)|j_2-k_2|} & \text{if } j_1 \ge k_1 \text{ and } j_2 < k_2, \\ 2^{-(m-\lambda_1)|j_1-k_1|}2^{-2|j_2-k_2|} & \text{if } j_1 < k_1 \text{ and } j_2 \ge k_2, \\ 2^{-(m-\lambda_1)|j_1-k_1|}2^{-(m-\lambda_2)|j_2-k_2|} & \text{if } j_1 \ge k_1 \text{ and } j_2 \ge k_2. \end{cases}$$

Now let us choose $m > \max\{\lambda_1, \lambda_2\}$ and set $\sigma := \min\{m-\lambda_1, m-\lambda_2, 2\}$. Then (3.3) implies that

$$\big[\widetilde{\Phi}_1(2^{-j_1}t_1\sqrt{L_1}) \otimes \widetilde{\Phi}_2(2^{-j_2}t_2\sqrt{L_2})\big]^*_{\lambda_1,\lambda_2} f(x_1,x_2) \le \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} 2^{-|j_1-k_1|\sigma}2^{-|j_2-k_2|\sigma}\big[\widetilde{\Phi}_1(2^{-k_1}t_1\sqrt{L_1}) \otimes \widetilde{\Phi}_2(2^{-k_2}t_2\sqrt{L_2})\big]^*_{\lambda_1,\lambda_2} f(x_1,x_2).$$

Taking on both sides the norm $(\int_1^2\int_1^2 |\cdot|^2 (dt_1/t_1)(dt_2/t_2))^{1/2}$ and using Minkowski's inequality, we get

$$\left(\int_1^2\int_1^2 \Big\{\big[\widetilde{\Phi}_1(2^{-j_1}t_1\sqrt{L_1}) \otimes \widetilde{\Phi}_2(2^{-j_2}t_2\sqrt{L_2})\big]^*_{\lambda_1,\lambda_2} f(x_1,x_2)\Big\}^2 \frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}$$

$$
\begin{aligned}
&\le C \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} 2^{-|j_1-k_1|\sigma} 2^{-|j_2-k_2|\sigma} \\
&\quad \times \left(\int_1^2 \int_1^2 \left\{ \left[\Phi_1(2^{-k_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-j_2} t_2 \sqrt{L_2}) \right]^*_{\lambda_1,\lambda_2} f(x_1,x_2) \right\}^2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^{1/2}.
\end{aligned}
$$

Finally, applying Lemma 2.6 in $L^p(\ell^2)$ yields

$$
\begin{aligned}
&\left\| \left(\int_0^\infty \int_0^\infty \left| \left[\widetilde{\Phi}_1(t_1\sqrt{L_1}) \otimes \widetilde{\Phi}_2(t_2\sqrt{L_2}) \right]^*_{\lambda_1,\lambda_2} f \right|^2 \frac{dt_1}{t_1}\frac{dt_2}{t_2} \right)^{1/2} \right\|_{L^p_w(X_1\times X_2)} \\
&\quad \le C \left\| \left(\int_0^\infty \int_0^\infty \left| \left[\Phi_1(t_1\sqrt{L_1}) \otimes \Phi_2(t_2\sqrt{L_2}) \right]^*_{\lambda_1,\lambda_2} f \right|^2 \frac{dt_1}{t_1}\frac{dt_2}{t_2} \right)^{1/2} \right\|_{L^p_w(X_1\times X_2)}. \qquad (3.4)
\end{aligned}
$$

By symmetry, the converse inequality of (3.4) also holds. The proof of the lemma is complete. □

LEMMA 3.4. *Let $\Phi_1, \Phi_2 \in \mathcal{A}(\mathbb{R})$ be even functions. Then for any $r > 0$, $\sigma > 0$, $\lambda_1 > D_1/2$ and $\lambda_2 > D_2/2$, there exists a constant C such that for all $f \in L^2(X_1 \times X_2)$, all $(x_1, x_2) \in X_1 \times X_2$ and all $t_1, t_2 \in [1,2]$,*

$$
\begin{aligned}
&\left\{ \left[\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-j_2} t_2 \sqrt{L_2}) \right]^*_{\lambda_1,\lambda_2} f(x_1,x_2) \right\}^r \\
&\quad \le C \sum_{k_1=j_1}^{\infty} \sum_{k_2=j_2}^{\infty} 2^{(j_1-k_1)\sigma} 2^{(j_2-k_2)\sigma} \iint_{X_1\times X_2} \\
&\quad \times \frac{\left| \Phi_1(2^{-k_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-k_2} t_2 \sqrt{L_2}) f(z_1,z_2) \right|^r d\mu_1(z_1) d\mu_2(z_2)}{V(z_1, 2^{-k_1}t_1)(1+2^{k_1}t_1^{-1}\rho(x_1,z_1))^{\lambda_1 r} V(z_2, 2^{-k_2}t_2)(1+2^{k_2}t_2^{-1}\rho_2(x_2,z_2))^{\lambda_2 r}}. \qquad (3.5)
\end{aligned}
$$

PROOF. By Lemma 2.3, for $i = 1, 2$ there exist even functions $\Psi_i, \Upsilon_i, \Theta_i \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp} \Upsilon_i \subset \{|\lambda| \le 2\varepsilon_i\}$, $\operatorname{supp} \Theta_i \subset \{\varepsilon_i/2 \le |\lambda| \le 2\varepsilon_i\}$, and

$$
\Psi_i(\lambda)\Upsilon_i(\lambda) + \sum_{k_i=1}^{\infty} \Phi_i(2^{-k_i}\lambda)\Theta_i(2^{-k_i}\lambda) = 1 \quad \text{for all } \lambda \in \mathbb{R}, \qquad (3.6)
$$

where ε_i is the constant in the Tauberian condition (1.4) corresponding to Φ_i. Replacing λ with $2^{-j_i}t_i\lambda$ in (3.6), we see that for all $j_i \in \mathbb{Z}$ and $t_i \in [1,2]$,

$$
\Psi_i(2^{-j_i}t_i\lambda)\Upsilon_i(2^{-j_i}t_i\lambda) + \sum_{k_i=1}^{\infty} \Phi_i(2^{-(k_i+j_i)}t_i\lambda)\Theta_i(2^{-(k_i+j_i)}t_i\lambda) = 1.
$$

It then follows from the spectral theorem that for all $f \in L^2(X_1 \times X_2)$, all $j_1, j_2 \in \mathbb{Z}$ and all $t_1, t_2 \in [1,2]$,

$$
f = \left(\Psi_1(2^{-j_1}t_1\sqrt{L_1})\Upsilon_1(2^{-j_1}t_1\sqrt{L_1}) \right) \otimes \left(\Psi_2(2^{-j_2}t_2\sqrt{L_2})\Upsilon_2(2^{-j_2}t_2\sqrt{L_2}) \right) f
$$

$$+\sum_{k_1=1}^{\infty}\big(\Phi_1(2^{-(k_1+j_1)}t_1\sqrt{L_1})\Theta_1(2^{-(k_1+j_1)}t_1\sqrt{L_1})\big)$$
$$\otimes\big(\Psi_2(2^{-j_2}t_2\sqrt{L_2})\Upsilon_2(2^{-j_2}t_2\sqrt{L_2})\big)f$$
$$+\sum_{k_2=1}^{\infty}\big(\Psi_1(2^{-j_1}t_1\sqrt{L_1})\Upsilon_1(2^{-j_1}t_1\sqrt{L_1})\big)$$
$$\otimes\big(\Phi_2(2^{-(k_2+j_2)}t_2\sqrt{L_2})\Theta_2(2^{-(k_2+j_2)}t_2\sqrt{L_2})\big)f$$
$$+\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\big(\Phi_1(2^{-(k_1+j_1)}t_1\sqrt{L_1})\Theta_1(2^{-(k_1+j_1)}t_1\sqrt{L_1})\big)$$
$$\otimes\big(\Phi_2(2^{-(k_2+j_2)}t_2\sqrt{L_2})\Theta_2(2^{-(k_2+j_2)}t_2\sqrt{L_2})\big)f$$

with convergence in the sense of $L^2(X_1\times X_2)$ norm. Hence, for all $j_1, j_2\in\mathbb{Z}$ and a.e. $(y_1,y_2)\in X_1\times X_2$, we have

$$\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})f(y_1,y_2)$$
$$=\big(\Phi_1(2^{-j_1}t_1\sqrt{L_1})\Psi_1(2^{-j_1}t_1\sqrt{L_1})\Upsilon_1(2^{-j_1}t_1\sqrt{L_1})\big)$$
$$\otimes\big(\Phi_2(2^{-j_2}t_2\sqrt{L_2})\Psi_2(2^{-j_2}t_2\sqrt{L_2})\Upsilon_2(2^{-j_2}t_2\sqrt{L_2})\big)f(y_1,y_2)$$
$$+\sum_{k_1=1}^{\infty}\big(\Phi_1(2^{-j_1}t_1\sqrt{L_1})\Phi_1(2^{-(k_1+j_1)}t_1\sqrt{L_1})\Theta_1(2^{-(k_1+j_1)}t_1\sqrt{L_1})\big)$$
$$\otimes\big(\Phi_2(2^{-j_2}t_2\sqrt{L_2})\Psi_2(2^{-j_2}t_2\sqrt{L_2})\Upsilon_2(2^{-j_2}t_2\sqrt{L_2})\big)f(y_1,y_2)$$
$$+\sum_{k_2=1}^{\infty}\big(\Phi_1(2^{-j_1}t_1\sqrt{L_1})\Psi_1(2^{-j_1}t_1\sqrt{L_1})\Upsilon_1(2^{-j_1}t_1\sqrt{L_1})\big)$$
$$\otimes\big(\Phi_2(2^{-j_2}t_2\sqrt{L_2})\Phi_2(2^{-(k_2+j_2)}t_2\sqrt{L_2})\Theta_2(2^{-(k_2+j_2)}t_2\sqrt{L_2})\big)f$$
$$+\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\big(\Phi_1(2^{-j_1}t_1\sqrt{L_1})\Phi_1(2^{-(k_1+j_1)}t_1\sqrt{L_1})\Theta_1(2^{-(k_1+j_1)}t_1\sqrt{L_1})\big)$$
$$\otimes\big(\Phi_2(2^{-j_2}t_2\sqrt{L_2})\Phi_2(2^{-(k_2+j_2)}t_2\sqrt{L_2})\Theta_2(2^{-(k_2+j_2)}t_2\sqrt{L_2})\big)f(y_1,y_2)$$
$$=\iint_{X_1\times X_2}K_{\Psi_1(2^{-j_1}t_1\sqrt{L_1})\Upsilon_1(2^{-j_1}t_1\sqrt{L_1})}(y_1,z_1)$$
$$\times K_{\Psi_2(2^{-j_2}t_2\sqrt{L_2})\Upsilon_2(2^{-j_2}t_2\sqrt{L_2})}(y_2,z_2)$$
$$\times\big(\Phi_1(2^{-(0+j_1)}t_1\sqrt{L_1})\otimes\Phi_2(2^{-(0+j_2)}t_2\sqrt{L_1})\big)f(z_1,z_2)d\mu_1(z_1)d\mu_2(z_2)$$
$$+\sum_{k_1=1}^{\infty}\iint_{X_1\times X_2}K_{\Phi_1(2^{-j_1}t_1\sqrt{L_1})\Theta_1(2^{-(k_1+j_1)}t_1\sqrt{L_1})}(y_1,z_1)$$
$$\times K_{\Psi_2(2^{-j_2}t_2\sqrt{L_2})\Upsilon_2(2^{-j_2}t_2\sqrt{L_2})}(y_2,z_2)$$
$$\times\big(\Phi_1(2^{-(k_1+j_1)}t_1\sqrt{L_1})\otimes\Phi_2(2^{-(0+j_2)}t_2\sqrt{L_1})\big)f(z_1,z_2)d\mu_1(z_1)d\mu_2(z_2)$$
$$+\sum_{k_2=1}^{\infty}\iint_{X_1\times X_2}K_{\Psi_1(2^{-j_1}t_1\sqrt{L_1})\Upsilon_1(2^{-j_1}t_1\sqrt{L_1})}(y_1,z_1)$$

$$\times K_{\Phi_2(2^{-j_2}t_2\sqrt{L_2})\Theta_2(2^{-(k_2+j_2)}t_2\sqrt{L_2})}(y_2,z_2)$$
$$\times \big(\Phi_1(2^{-(0+j_1)}t_1\sqrt{L_1})\otimes\Phi_2(2^{-(k_2+j_2)}t_2\sqrt{L_1})\big)f(z_1,z_2)d\mu_1(z_1)d\mu_2(z_2)$$
$$+\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\iint_{X_1\times X_2}K_{\Phi_1(2^{-j_1}t_1\sqrt{L_1})\Theta_1(2^{-(k_1+j_1)}t_1\sqrt{L_1})}(y_1,z_1)$$
$$\times K_{\Phi_2(2^{-j_2}t_2\sqrt{L_2})\Theta_2(2^{-(k_2+j_2)}t_2\sqrt{L_2})}(y_2,z_2)$$
$$\times \big(\Phi_1(2^{-(k_1+j_1)}t_1\sqrt{L_1})\otimes\Phi_2(2^{-(k_2+j_2)}t_2\sqrt{L_2})\big)f(z_1,z_2)d\mu_1(z_1)d\mu_2(z_2). \quad (3.7)$$

For $i=1,2$, let $N_i\ge\lambda_i$ and m_i be any integer such that $m_i-\lambda_i-n_i/r>0$. Since Θ_i vanishes near the origin, it follows from Lemma 2.2 that there exists a constant $C=C(\Phi_i,\Theta_i,m_i,N_i)$ such that for all $j_i\in\mathbb{Z}$, all $k_i\in\{1,2,\dots\}$, and all $t_i\in[1,2]$,

$$\big|K_{\Phi_i(2^{-j_i}t_i\sqrt{L_i})\Theta_i(2^{-(k_i+j_i)}t_i\sqrt{L_i})}(y_i,z_i)\big| \le C2^{-k_im_i}V(z_i,2^{-j_i}t_i)^{-1}(1+2^{j_i}t_i^{-1}\rho_i(y_i,z_i))^{-N_i}. \quad (3.8)$$

Analogously, for $i=1,2$, we have

$$\big|K_{\Psi_i(2^{-j_i}t_i\sqrt{L_i})\Upsilon_i(2^{-j_i}t_i\sqrt{L_i})}(y_i,z_i)\big|\le CV(z_i,2^{-j_i}t_i)^{-1}(1+2^{j_i}t_i^{-1}\rho_i(y_i,z_i))^{-N_i}. \quad (3.9)$$

Putting (3.8) and (3.9) into (3.7), we obtain

$$\Big|\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})f(y_1,y_2)\Big|$$
$$\le C\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}2^{-k_1m_1}2^{-k_2m_2}$$
$$\times\iint_{X_1\times X_2}\frac{\big|\Phi_1(2^{-(k_1+j_1)}t_1\sqrt{L_1})\otimes\Phi_2(2^{-(k_2+j_2)}t_2\sqrt{L_2})f(z_1,z_2)\big|}{\prod_{i=1}^2V(z_i,2^{-j_i}t_i)(1+2^{j_i}t_i^{-1}\rho_i(y_i,z_i))^{N_i}}d\mu_1(z_1)d\mu_2(z_2)$$
$$=C\sum_{k_1=j_1}^{\infty}\sum_{k_2=j_2}^{\infty}2^{(j_1-k_1)m_1}2^{(j_2-k_2)m_2}$$
$$\times\iint_{X_1\times X_2}\frac{\big|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f(z_1,z_2)\big|}{\prod_{i=1}^2V(z_i,2^{-j_i}t_i)(1+2^{j_i}t_i^{-1}\rho_i(y_i,z_i))^{N_i}}d\mu_1(z_1)d\mu_2(z_2). \quad (3.10)$$

To prove the desired inequality, we first consider the case $0<r\le1$. Dividing both sides of (3.10) by $(1+2^{j_1}t_1^{-1}\rho_1(x_1,y_1))^{\lambda_1}(1+2^{j_2}t_2^{-1}\rho_2(x_2,y_2))^{\lambda_2}$, taking the supremum over $(y_1,y_2)\in X_1\times X_2$ in the left-hand side, and using the inequalities $V(z,2^{-j_i}t_i)\ge V(z_i,2^{-k_i}t_i)$ $(\forall k_i\ge j_i)$ and $(1+2^{j_i}t_i^{-1}\rho_i(x_i,y_i))(1+2^{j_i}t_i^{-1}\rho_i(y_i,z_i))\ge(1+2^{j_i}t_i^{-1}\rho_i(x_i,z_i))$ $(\forall t_i\in[1,2])$ in the right-hand side, we get that, for all $t_i\in[1,2]$ and $x_i\in X_i$,

$$\big[\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})\big]^*_{\lambda_1,\lambda_2}f(x_1,x_2)$$
$$\le C\sum_{k_1=j_1}^{\infty}\sum_{k_2=j_2}^{\infty}2^{(j_1-k_1)m_1}2^{(j_2-k_2)m_2}$$

$$\times \iint_{X_1\times X_2} \frac{\left|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f(z_1,z_2)\right|}{\prod_{i=1}^2 V(z_i,2^{-k_i}t_i)(1+2^{j_i}t_i^{-1}\rho_i(x_i,z_i))^{\lambda_i}}d\mu_1(z_1)d\mu_2(z_2). \quad (3.11)$$

To proceed further, we note that

$$\begin{aligned}
&\left|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f(z_1,z_2)\right| \\
&\quad\le \left|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f(z_1,z_2)\right|^r \\
&\qquad\times\left\{\left[\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2}f(x_1,x_2)\right\}^{1-r} \\
&\qquad\times(1+2^{k_1}t_1^{-1}\rho_1(x_1,z_1))^{\lambda_1(1-r)}(1+2^{k_2}t_2^{-1}\rho_2(x_2,z_2))^{\lambda_2(1-r)}. \qquad (3.12)
\end{aligned}$$

From (3.11), (3.12), and the inequality

$$(1+2^{k_i}t_i^{-1}\rho_i(x_i,z_i))^{\lambda_i}\le 2^{(k_i-j_i)\lambda_i}(1+2^{j_i}t_i^{-1}\rho_i(x_i,z_i))^{\lambda_i}\quad (\forall k_i\ge j_i,\ \forall t_i\in[1,2]),$$

it follows that

$$\begin{aligned}
&\left[\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2}f(x_1,x_2) \\
&\quad\le C\sum_{k_1=j_1}^{\infty}\sum_{k_2=j_2}^{\infty}2^{(j_1-k_1)(m_1-\lambda_1)}2^{(j_2-k_2)(m_2-\lambda_2)} \\
&\qquad\times\iint_{X_1\times X_2}\frac{\left|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f(z_1,z_2)\right|^r}{\prod_{i=1}^2 V(z_i,2^{-k_i}t_i)(1+2^{k_i}t_i^{-1}\rho_i(x_i,z_i))^{\lambda_i r}}d\mu_1(z_1)d\mu_2(z_2) \\
&\qquad\times\left\{\left[\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2}f(x_1,x_2)\right\}^{1-r}. \qquad (3.13)
\end{aligned}$$

We claim that for any $f\in L^2(X_1\times X_2)$, $\lambda_i>D_i/2$, $x_i\in X_i$, $t_i\in[1,2]$, and $j_i\in\mathbb{Z}$,

$$\left[\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2}f(x_1,x_2)<\infty, \qquad (3.14)$$

and there exists $N_0>0$ such that

$$\left[\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2}f(x_1,x_2)=O(2^{j_1N_0}2^{j_2N_0}) \qquad (3.15)$$

as $j_1,j_2\to+\infty$. Indeed, for $i=1,2$, by Lemma 2.1 we have

$$\left|K_{\Phi_i(2^{-j_i}t_i\sqrt{L_i})}(y_i,z_i)\right|\le CV(y_i,2^{-j_i}t_i)^{-1}(1+2^{j_i}t_i^{-1}\rho(y_i,z_i))^{-(n_i+1)/2}.$$

Hence, by the Cauchy–Schwartz inequality and (2.1), we have

$$\begin{aligned}
&\left|\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})f(y_1,y_2)\right| \\
&\quad\le C\iint_{X_1\times X_2}\left|K_{\Phi_1(2^{-j_1}t_1\sqrt{L_1})}(y_1,z_1)\right|\left|K_{\Phi_2(2^{-j_2}t_2\sqrt{L_2})}(y_2,z_2)\right||f(z_1,z_2)|d\mu_1(z_1)d\mu_2(z_2) \\
&\quad\le C\|f\|_{L^2(X_1\times X_2)}V(y_1,2^{-j_1}t_1)^{-1/2}V(y_2,2^{-j_2}t_1)^{-1/2}.
\end{aligned}$$

This along with (1.3) yields that for $\lambda_i \geq D_i/2$,

$$\begin{aligned}&\big[\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})\big]^*_{\lambda_1,\lambda_2}f(x_1,x_2)\\&\quad\leq C\sup_{(y_1,y_2)\in X_1\times X_2}\frac{\|f\|_{L^2(X_1\times X_2)}}{\prod_{i=1}^2 V(y_i,2^{-j_i}t_i)^{-1/2}(1+2^{j_i}t_i^{-1}\rho_i(x_i,y_i))^{\lambda_i}}\\&\quad\leq C\|f\|_{L^2(X_1\times X_2)}V(x_1,2^{-j_1}t_1)^{-1/2}V(x_2,2^{-j_2}t_2)^{-1/2}.\end{aligned}$$

Hence (3.14) is true. Moreover, if $j_1, j_2 \geq 1$, by (1.2) we have

$$\begin{aligned}&\big[\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})\big]^*_{\lambda_1,\lambda_2}f(x_1,x_2)\\&\quad\leq C\|f\|_{L^2(X_1\times X_2)}V(x_1,2^{-j_1}t_1)^{-1/2}V(x_2,2^{-j_2}t_2)^{-1/2}\\&\quad\leq C2^{j_1n_1/2}2^{j_2n_2/2}\|f\|_{L^2(X_1\times X_2)}V(x_1,1)^{-1/2}V(x_2,1)^{-1/2},\end{aligned}$$

which verifies (3.15) with $N_0 = \max\{n_1/2, n_2/2\}$.

Since m_1, m_2 in (3.13) can be chosen to be arbitrarily large, it follows from (3.13), (3.14), (3.15) and Lemma 2.7 that for any $\sigma > 0$,

$$\begin{aligned}&\Big\{\big[\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})\big]^*_{\lambda_1,\lambda_2}f(x_1,x_2)\Big\}^r\\&\quad\leq C\sum_{k_1=j_1}^{\infty}\sum_{k_2=j_2}^{\infty}2^{(j_1-k_1)\sigma}2^{(j_2-k_2)\sigma}\iint_{X_1\times X_2}\\&\qquad\times\frac{\big|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f(z_1,z_2)\big|^r d\mu_1(z_1)d\mu_2(z_2)}{V(z_1,2^{-k_1}t_1)(1+2^{k_1}t_1^{-1}\rho_1(x_1,z_1))^{\lambda_1 r}V(z_2,2^{-k_2}t_2)(1+2^{k_2}t_2^{-1}\rho_2(x_2,z_2))^{\lambda_2 r}}.\end{aligned}$$

This proves (3.5) for $0 < r \leq 1$.

Next we show (3.5) for $r > 1$. Indeed, from (3.10) with $m_i \geq \sigma + \lambda_i r + \varepsilon$ and $N_i \geq \lambda_i + (D_i + n_i + 1)/r'$, where ε is any fixed positive number and r' is a number such that $1/r + 1/r' = 1$, it follows that

$$\begin{aligned}&\Big|\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})f(y_1,y_2)\Big|\\&\quad\leq C\sum_{k_1=j_1}^{\infty}\sum_{k_2=j_2}^{\infty}2^{(j_1-k_1)(\sigma+\lambda_1 r+\varepsilon)}2^{(j_2-k_2)(\sigma+\lambda_2 r+\varepsilon)}\\&\qquad\times\iint_{X_1\times X_2}\frac{\big|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f(z_1,z_2)\big|}{\prod_{i=1}^2 V(z_i,2^{-j_i}t_i)(1+2^{j_i}t_i^{-1}\rho_i(y_i,z_i))^{\lambda_i+(D_i+n_i+1)/r'}}d\mu_1(z_1)d\mu_2(z_2)\\&\quad\leq C\sum_{k_1=j_1}^{\infty}\sum_{k_2=j_2}^{\infty}2^{(j_1-k_1)(\sigma+\lambda_1 r+\varepsilon)}2^{(j_2-k_2)(\sigma+\lambda_2 r+\varepsilon)}\\&\qquad\times\bigg(\iint_{X_1\times X_2}\frac{\big|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f(z_1,z_2)\big|^r}{\prod_{i=1}^2 V(z_i,2^{-j_i}t_i)(1+2^{j_i}t_i^{-1}\rho_i(y_i,z_i))^{\lambda_i r}}d\mu_1(z_1)d\mu_2(z_2)\bigg)^{1/r}\\&\quad\leq C\bigg(\sum_{k_1=j_1}^{\infty}\sum_{k_2=j_2}^{\infty}2^{(j_1-k_1)(\sigma_1+\lambda_1 r)}2^{(j_2-k_2)(\sigma_2+\lambda_2 r)}\end{aligned}$$

$$\times \iint_{X_1\times X_2} \frac{\left|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f(z_1,z_2)\right|^r}{\prod_{i=1}^2 V(z_i,2^{-j_i}t_i)(1+2^{j_i}t_i^{-1}\rho_i(y_i,z_i))^{\lambda_i r}}d\mu_1(z_1)d\mu_2(z_2)\Bigg)^{1/r},$$

where we applied Hölder's inequality for the integrals and the sums, and used (1.3) and (2.1). Raising both sides to the power r, dividing both sides by $(1+2^{j_1}t_1^{-1}\rho_1(x_1,y_1))^{\lambda_1 r}(1+2^{j_2}t_2^{-1}\rho_2(x_2,y_2))^{\lambda_2 r}$, in the left-hand side taking the supremum over $(y_1,y_2)\in X_1\times X_2$, and in the right-hand side using the inequalities

$$\begin{aligned}&(1+2^{j_i}t_i^{-1}\rho_i(x_i,y_i))^{\lambda_i r}(1+2^{j_i}t_i^{-1}\rho_i(y_i,z_i))^{\lambda_i r}\\ &\quad\ge (1+2^{j_i}t_i^{-1}\rho_i(x_i,z_i))^{\lambda_i r}\\ &\quad\ge 2^{(j_i-k_i)\lambda_i r}(1+2^{k_i}t_i^{-1}\rho_i(x_i,z_i))^{\lambda_i r}\quad (\forall k_i\ge j_i)\end{aligned}$$

and $V(z_i,2^{-j_i}t_i)\ge V(z_i,2^{-k_i}t_i)$ $(\forall k_i\ge j_i)$, we obtain (3.5) for $r>1$. □

LEMMA 3.5. *Let $\Phi_1,\Phi_2\in\mathcal{A}(\mathbb{R})$ be even functions. Let $p\in(0,\infty)$ and $\lambda_i>(n_i+D_i)q_w/\min\{p,2\}$, $i=1,2$. Then there exists a constant C such that for all $f\in L^2(X_1\times X_2)$,*

$$\begin{aligned}&\left\|\left(\int_0^\infty\int_0^\infty\left|\left[\Phi_1(t_1\sqrt{L_1})\otimes\Phi_2(t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2}f\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)}\\ &\qquad\le C\left\|\left(\int_0^\infty\int_0^\infty\left|\Phi_1(t_1\sqrt{L_1})\otimes\Phi_2(t_2\sqrt{L_2})f\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)}.\end{aligned}$$

PROOF. Since $\lambda_i>(n_i+D_i)q_w/\min\{p,2\}$, there exists a number r such that $0<r<\min\{p,2\}/q_w$ and $\lambda_i r>n_i+D_i$. From Lemma 3.4 we see that for any $\sigma>0$ there exists a constant C such that for all $f\in L^2(X_1\times X_2)$, $j_i\in\mathbb{Z}$, $x_i\in X_i$ and $t_i\in[1,2]$,

$$\begin{aligned}&\left\{\left[\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2}f(x_1,x_2)\right\}^r\\ &\le C\sum_{k_1=j_1}^\infty\sum_{k_2=j_2}^\infty 2^{(j_1-k_1)\sigma}2^{(j_2-k_2)\sigma}\\ &\times\iint_{X_1\times X_2}\frac{\left|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f(z_1,z_2)\right|^r d\mu_1(z_1)d\mu_2(z_2)}{V(z_1,2^{-k_1}t_1)(1+2^{k_1}t_1^{-1}\rho_1(x_1,z_1))^{\lambda_1 r}V(z_2,2^{-k_2}t_2)(1+2^{k_2}t_2^{-1}\rho_2(x_2,z_2))^{\lambda_2 r}}.\end{aligned}$$

Taking the norm $(\int_1^2\int_1^2|\cdot|^{2/r}(dt_1/t_1)(dt_2/t_2))^{r/2}$ on both sides, applying Minkowski's inequality, and then using (2.7), we get

$$\begin{aligned}&\left(\int_1^2\int_1^2\left|\left[\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2}f(x_1,x_2)\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{r/2}\\ &\le C\sum_{k_1=j_1}^\infty\sum_{k_2=j_2}^\infty 2^{(j_1-k_1)\sigma}2^{(j_2-k_2)\sigma}\end{aligned}$$

$$\times \iint_{X_1\times X_2} \frac{\left(\int_1^2\int_1^2 \left|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f(z_1,z_2)\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{r/2} d\mu_1(z_1)d\mu_2(z_2)}{V(z_1,2^{-k_1}t_1)(1+2^{k_1}t_1^{-1}\rho_1(x_1,z_1))^{\lambda_1 r}V(z_2,2^{-k_2}t_2)(1+2^{k_2}t_2^{-1}\rho_2(x_2,z_2))^{\lambda_2 r}}$$

$$\le C\sum_{k_1=j_1}^{\infty}\sum_{k_2=j_2}^{\infty} 2^{(j_1-k_1)\sigma}2^{(j_2-k_2)\sigma}$$

$$\times \mathcal{M}_s\left[\left(\int_1^2\int_1^2 \left|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{r/2}\right](x_1,x_2)$$

$$\le C\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} 2^{-|k_1-j_1|\sigma}2^{-|k_2-j_2|\sigma}$$

$$\times \mathcal{M}_s\left[\left(\int_1^2\int_1^2 \left|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{r/2}\right](x_1,x_2).$$

It then follows from Lemma 2.6 and Lemma 2.8 that

$$\left\|\left(\int_0^\infty\int_0^\infty \left|\left[\Phi_1(t_1\sqrt{L_1})\otimes\Phi_2(t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2}f\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)}$$

$$= \left\|\left\{\left(\int_1^2\int_1^2 \left|\left[\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2}f\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{r/2}\right\}_{j_1,j_2\in\mathbb{Z}}\right\|^{1/r}_{L^{p/r}_w(\ell^{2/r})}$$

$$\le C\left\|\left\{\mathcal{M}_s\left[\left(\int_1^2\int_1^2 \left|\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})f\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{r/2}\right]\right\}_{j_1,j_2\in\mathbb{Z}}\right\|^{1/r}_{L^{p/r}_w(\ell^{2/r})}$$

$$\le C\left\|\left\{\left(\int_1^2\int_1^2 \left|\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})f\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{r/2}\right\}_{j_1,j_2\in\mathbb{Z}}\right\|^{1/r}_{L^{p/r}_w(\ell^{2/r})}$$

$$= C\left\|\left(\int_0^\infty\int_0^\infty \left|\Phi_1(t_1\sqrt{L_1})\otimes\Phi_2(t_2\sqrt{L_2})f\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)},$$

where we used the fact that $p/r > q_w$ (which implies $w \in A_{p/r}(X_1\times X_2)$) and $2/r > 1$. □

LEMMA 3.6. *Let $\Phi_1,\Phi_2 \in \mathcal{A}(\mathbb{R})$ be even functions. Let $p \in (0,\infty)$ and $\lambda_i > 0$, $i = 1,2$. Let w be an arbitrary weight (i.e., nonnegative locally integrable function) on $X_1 \times X_2$. Then there exists a constant C such that for all $f \in L^2(X_1 \times X_2)$,*

$$\left\|\left(\int_0^\infty\int_0^\infty \left|\left[\Phi_1(t_1\sqrt{L_1})\otimes\Phi_2(t_2\sqrt{L_2})\right]^*_{\lambda_1+D_1/2,\lambda_2+D_2/2}f\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)}$$

$$\le C\left\|g^*_{\Phi_1,\Phi_2,L_1,L_2,(2/n_1)\lambda_1,(2/n_2)\lambda_2}(f)\right\|_{L^p_w(X_1\times X_2)}.$$

PROOF. Let $\sigma > 0$. By Lemma 3.4 with $r = 2$, we see that there exists a constant C such that for all $f \in L^2(X_1 \times X_2)$, $j_i \in \mathbb{Z}$ and $t_i \in [1,2]$,

$$
\begin{aligned}
&\Big\{\big[\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})\big]^*_{\lambda_1+D_1/2,\lambda_2+D_2/2}f(x_1,x_2)\Big\}^2\\
&\quad\le C\sum_{k_1=j_1}^{\infty}\sum_{k_2=j_2}^{\infty}2^{(j_1-k_1)\sigma}2^{(j_2-k_2)\sigma}\\
&\quad\times\iint_{X_1\times X_2}\frac{\big|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f(z_1,z_2)\big|^2}{\prod_{i=1}^2 V(z_i,2^{-k_i}t_i)(1+2^{k_i}t_i^{-1}\rho_i(x_i,z_i))^{2\lambda_i+D_i}}d\mu_1(z_1)d\mu_2(z_2)\\
&\quad\le C\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}2^{-|k_1-j_1|\sigma}2^{-|k_2-j_2|\sigma}\\
&\quad\times\iint_{X_1\times X_2}\frac{\big|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f(z_1,z_2)\big|^2}{\prod_{i=1}^2(1+2^{k_i}t_i^{-1}\rho(x_i,z_i))^{2\lambda_i}}\\
&\qquad\qquad\times\frac{d\mu_1(z_1)d\mu_2(z_2)}{V(x_1,2^{-k_1}t_1)V(x_2,2^{-k_2}t_2)},
\end{aligned}\tag{3.16}
$$

where for the last line we used (1.3). Taking the norm $\int_1^2\int_1^2|\cdot|(dt_1/t_1)(dt_2/t_2)$ on both sides of (3.16) gives

$$
\begin{aligned}
&\int_1^2\int_1^2\Big\{\big[\Phi_1(2^{-2j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})\big]^*_{\lambda_1+D_1/2,\lambda_2+D_2/2}f(x_1,x_2)\Big\}^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\\
&\quad\le C\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}2^{-|k_1-j_1|\sigma}2^{-|k_2-j_2|\sigma}\\
&\quad\times\int_1^2\int_1^2\int_{X_1}\int_{X_2}\frac{\big|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f(z_1,z_2)\big|^2}{(1+2^{k_1}t_1^{-1}\rho(x_1,z_1))^{2\lambda_1}(1+2^{k_2}t_2^{-1}\rho_2(x_2,z_2))^{2\lambda_2}}\\
&\qquad\qquad\times\frac{d\mu_1(z_1)dt_1d\mu_2(z_2)dt_2}{V(x_1,2^{-k_1}t_1)t_1V(x_2,2^{-k_2}t_2)t_2}.
\end{aligned}
$$

Applying Lemma 2.6 in $L^{p/2}_w(\ell^1)$ we obtain

$$
\begin{aligned}
&\bigg\|\bigg(\int_0^\infty\int_0^\infty\Big|\big[\Phi_1(t_1\sqrt{L_1})\otimes\Phi_2(t_2\sqrt{L_2})\big]^*_{\lambda_1+D_1/2,\lambda_2+D_2/2}f\Big|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\bigg)^{1/2}\bigg\|_{L^p_w(X_1\times X_2)}\\
&=\bigg\|\bigg\{\int_1^2\int_1^2\Big\{\big[\Phi_1(2^{-j_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-j_2}t_2\sqrt{L_2})\big]^*_{\lambda_1+D_1/2,\lambda_2+D_2/2}f(x_1,x_2)\Big\}^2\\
&\qquad\qquad\times\frac{dt_1}{t_1}\frac{dt_2}{t_2}\bigg\}_{j_1,j_2\in\mathbb{Z}}\bigg\|^{1/2}_{L^{p/2}_w(\ell^1)}\\
&\le C\bigg\|\bigg\{\int_1^2\int_1^2\int_X\int_X\frac{\big|\Phi_1(2^{-k_1}t_1\sqrt{L_1})\otimes\Phi_2(2^{-k_2}t_2\sqrt{L_2})f(z_1,z_2)\big|^2}{(1+2^{k_1}t_1^{-1}\rho(x_1,z_1))^{2\lambda_1}(1+2^{k_2}t_2^{-1}\rho_2(x_2,z_2))^{2\lambda_2}}\\
&\qquad\qquad\times\frac{d\mu_1(z_1)dt_1}{V(x_1,2^{-k_1}t_1)t_1}\frac{d\mu_2(z_2)dt_2}{V(x_2,2^{-k_2}t_2)t_2}\bigg\}_{k_1,k_2\in\mathbb{Z}}\bigg\|^{1/2}_{L^{p/2}_w(\ell^1)}\\
&=C\Big\|g^*_{\Phi_1,\Phi_2,L_1,L_2,(2/n_1)\lambda_1,(2/n_2)\lambda_2}(f)\Big\|_{L^p_w(X_1\times X_2)},
\end{aligned}
$$

as desired. □

Having the above lemmas, we are ready to give the proofs of Theorems 1.4 and 1.5.

PROOF OF THEOREM 1.4. Let $\Phi_1, \Phi_2, \widetilde{\Phi}_1, \widetilde{\Phi}_2 \in \mathcal{A}(\mathbb{R})$ be even functions satisfying

$$\Phi_1(0) = \Phi_2(0) = \widetilde{\Phi}_1(0) = \widetilde{\Phi}_2(0) = 0.$$

Let $p \in (0, \infty)$ and $\lambda_i > (n_i + D_i)q_w/\min\{p, 2\}$, $i = 1, 2$. Note that for a.e. $(x_1, x_2) \in X_1 \times X_2$,

$$\widetilde{\Phi}_1(t_1\sqrt{L_1}) \otimes \widetilde{\Phi}_2(t_2\sqrt{L_2})f(x_1, x_2) \leq \left[\widetilde{\Phi}_1(t_1\sqrt{L_1}) \otimes \widetilde{\Phi}_2(t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2} f(x_1, x_2). \quad (3.17)$$

Using (3.17), Lemma 3.3 and Lemma 3.5, we infer

$$\begin{aligned}
&\|g_{\widetilde{\Phi}_1,\widetilde{\Phi}_2,L_1,L_2}(f)\|_{L^p_w(X_1\times X_2)} \\
&\quad = \left\|\left(\int_0^\infty \int_0^\infty \left|\widetilde{\Phi}_1(t_1\sqrt{L_1}) \otimes \widetilde{\Phi}_2(t_2\sqrt{L_2})f\right|^2 \frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)} \\
&\quad \leq \left\|\left(\int_0^\infty \int_0^\infty \left|\left[\widetilde{\Phi}_1(t_1\sqrt{L_1}) \otimes \widetilde{\Phi}_2(t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2} f\right|^2 \frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)} \\
&\quad \leq C\left\|\left(\int_0^\infty \int_0^\infty \left|\left[\Phi_1(t_1\sqrt{L_1}) \otimes \Phi_2(t_2\sqrt{L_2})\right]^*_{\lambda_1,\lambda_2} f\right|^2 \frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)} \\
&\quad \leq C\left\|\left(\int_0^\infty \int_0^\infty \left|\Phi_1(t_1\sqrt{L_1}) \otimes \Phi_2(t_2\sqrt{L_2})f\right|^2 \frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)} \\
&\quad = C\|g_{\Phi_1,\Phi_2,L_1,L_2}(f)\|_{L^p_w(X_1\times X_2)}.
\end{aligned}$$

By symmetry, there also holds that

$$\|g_{\Phi_1,\Phi_2,L_1,L_2}(f)\|_{L^p_w(X_1\times X_2)} \leq C\|g_{\Psi_1,\Psi_2,L_1,L_2}(f)\|_{L^p_w(X_1\times X_2)}.$$

Hence the assertion of Theorem 1.4 is true. □

PROOF OF THEOREM 1.5. Let $\Phi_1, \Phi_2 \in \mathcal{A}(\mathbb{R})$ be even functions. Let $p \in (0, \infty)$, $\lambda_i > 2q_w/\min\{p, 2\}$ and $\lambda_i' > (n_i + D_i)q_w/\min\{p, 2\}$, $i = 1, 2$. Then, for all $f \in L^2(X_1 \times X_2)$, by (3.17), Lemma 3.6, Lemma 3.1, Lemma 3.2 and Lemma 3.5, we have

$$\begin{aligned}
&\|g_{\Phi_1,\Phi_2,L_1,L_2}(f)\|_{L^p_w(X_1\times X_2)} \\
&\quad \leq \left\|\left(\int_0^\infty \int_0^\infty \left|\left[\Phi_1(t_1\sqrt{L_1}) \otimes \Phi_2(t_2\sqrt{L_2})\right]^*_{(n_1/2)\lambda_1+D_1/2,(n_2/2)\lambda_2+D_2/2} f\right|^2\right.\right. \\
&\qquad\qquad \left.\left.\times \frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)} \\
&\quad \leq C\left\|g^*_{\Phi_1,\Phi_2,L_1,L_2,\lambda_1,\lambda_2}(f)\right\|_{L^p_w(X_1\times X_2)}
\end{aligned}$$

$$
\begin{aligned}
&\le C\big\|S_{\Phi_1,\Phi_2,L_1,L_2}(f)\big\|_{L^p_w(X_1\times X_2)}\\
&\le C\left\|\left(\int_0^\infty\int_0^\infty\left|\left[\Phi_1(t_1\sqrt{L_1})\otimes\Phi_2(t_2\sqrt{L_2})\right]^*_{\lambda_1',\lambda_2'}f\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}\right\|_{L^p_w(X_1\times X_2)}\\
&\le C\|g_{\Phi_1,\Phi_2,L_1,L_2}(f)\|_{L^p_w(X_1\times X_2)},
\end{aligned}
$$

which yields (1.5). The proof of Theorem 1.5 is complete. □

4. Applications of Theorems 1.4 and 1.5.

1. In [**7**] and [**11**], the theory of product Hardy space $H^1_{L_1,L_2}(\mathbb{R}^n\times\mathbb{R}^m)$ via the Littlewood–Paley area functions was established, where L_1 and L_2 are two nonnegative self-adjoint operators that satisfy only the Gaussian heat kernel bound. To be more specific, $H^1_{L_1,L_2}(\mathbb{R}^n\times\mathbb{R}^m)$ is defined as the closure of

$$\{f\in L^2(\mathbb{R}^n\times\mathbb{R}^m): S_{L_1,L_2}(f)\in L^1(\mathbb{R}^n\times\mathbb{R}^m)\}$$

under the norm $\|f\|_{H^1_{L_1,L_2}(\mathbb{R}^n\times\mathbb{R}^m)} := \|S_{L_1,L_2}(f)\|_{H^1_{L_1,L_2}(\mathbb{R}^n\times\mathbb{R}^m)}$, where

$$
\begin{aligned}
&S_{L_1,L_2}(f)(x_1,x_2)\\
&:=\left(\iint_{\Gamma_1(x_1)\times\Gamma_2(x_2)}\left|(t_1^2L_1e^{-t_1^2L_1})\otimes(t_2^2L_2e^{-t_2^2L_2})f(y_1,y_2)\right|^2\frac{dy_1dt_1}{t_1^{n+1}}\frac{dy_2dt_2}{t_2^{m+1}}\right)^{1/2}.
\end{aligned}
$$

Then, by applying our main result Theorem 1.5 (also Remark 1.7), we obtain the characterization of $H^1_{L_1,L_2}(\mathbb{R}^n\times\mathbb{R}^m)$ via the Littlewood–Paley square function as follows, which is missing in [**7**] and [**11**], i.e., $H^1_{L_1,L_2}(\mathbb{R}^n\times\mathbb{R}^m)$ is equivalent to the closure of

$$\big\{f\in L^2(\mathbb{R}^n\times\mathbb{R}^m): g_{L_1,L_2}(f)\in L^1(\mathbb{R}^n\times\mathbb{R}^m)\big\}$$

under the norm $\|g_{L_1,L_2}(f)\|_{H^1_{L_1,L_2}(\mathbb{R}^n\times\mathbb{R}^m)}$, where

$$g_{L_1,L_2}(f)(x_1,x_2)=\left(\int_0^\infty\int_0^\infty\left|(t_1^2L_1e^{-t_1^2L_1})\otimes(t_2^2L_2e^{-t_2^2L_2})f(x_1,x_2)\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{1/2}.$$

2. In 1965, Muckenhoupt and Stein in [**17**] introduced a notion of conjugacy associated with the Bessel operator Δ_λ on $\mathbb{R}_+ := (0,\infty)$ defined by

$$\Delta_\lambda f(x) := -\frac{d^2}{dx^2}f(x)-\frac{2\lambda}{x}\frac{d}{dx}f(x),\quad x>0,$$

and the Bessel Schrödinger operator S_λ on $\mathbb{R}_+$

$$S_\lambda f(x) := -\frac{d^2}{dx^2}f(x)+\frac{\lambda^2-\lambda}{x^2}f(x),\quad x>0.$$

In [**9**], Duong *et al.* established the product Hardy space $H^p_{\Delta_\lambda}(\mathbb{R}_+\times\mathbb{R}_+)$ associated with Δ_λ via the Littlewood–Paley area function and square functions. Note that the measure on $\mathbb{R}_+$ related to Δ_λ is $d\mu_\lambda(x) = x^{2\lambda}dx$. We point out that the kernel of $t^2\Delta_\lambda e^{-t^2\Delta_\lambda}$ satisfies the Gaussian upper bounds with respect to the measure $d\mu_\lambda$, the Hölder regularity and the cancellation property. Hence, by using the approach in [**15**] via the Plancherel–Polya type inequality, they obtained the equivalence of the characterizations of $H^p_{\Delta_\lambda}(\mathbb{R}_+ \times \mathbb{R}_+)$ via Littlewood–Paley area function and square functions. By applying our main result Theorem 1.5 (also Remark 1.7), we obtain a direct proof of the equivalence without using the Hölder regularity and the cancellation property.

In [**1**], Betancor *et al.* established the product Hardy space $H^p_{S_\lambda}(\mathbb{R}_+ \times \mathbb{R}_+)$ associated with Δ_λ via the Littlewood–Paley area function and square functions. To prove the equivalence, they need to use the Poisson semigroup $\{e^{-t\sqrt{S_\lambda}}\}$, the subordination formula and the Moser type inequality as a bridge. By applying our main result Theorem 1.5 (also Remark 1.7), we obtain a direct proof of this equivalence without using the Moser type inequality.

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