

# Dynamics of isolated left orders

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**Abstract.** A left order of a countable group  $G$  is called isolated if it is an isolated point in the compact space  $LO(G)$  of all the left orders of  $G$ . We study properties of a dynamical realization of an isolated left order. Especially we show that it acts on  $\mathbb{R}$  cocompactly. As an application, we give a dynamical proof of the Tararin theorem which characterizes those countable groups which admit only finitely many left orders. We also show that the braid group  $B_3$  admits countably many isolated left orders which are not the automorphic images of the others.

## 1. Introduction.

Throughout this paper, all the groups considered are countable. Given a group  $G$ , a total order  $<_\lambda$  on  $G$  is called a *left order* if for any  $f, g, h \in G$ ,  $f <_\lambda g$  implies  $hf <_\lambda hg$ . An element  $g \in G$  is called  $\lambda$ -positive if  $g >_\lambda e$ . The set of all the  $\lambda$ -positive elements is called the *positive cone of  $\lambda$*  and is denoted by  $P_\lambda$ . It is a subsemigroup and  $P_\lambda \sqcup P_\lambda^{-1} = G \setminus \{e\}$ .

Given a left order  $<_\lambda$ , we define  $\lambda : G \setminus \{e\} \rightarrow \{\pm 1\}$  by  $\lambda(g) = 1$  if and only if  $g \in P_\lambda$ . Then we have

$$\lambda(f) = 1, \lambda(g) = 1 \Rightarrow \lambda(fg) = 1, \text{ and } \lambda(f^{-1}) = -\lambda(f). \quad (1.1)$$

Conversely given a map  $\lambda : G \setminus \{e\} \rightarrow \{\pm 1\}$  which satisfies (1.1), we get a left order  $<_\lambda$  by setting  $f <_\lambda g$  if  $\lambda(f^{-1}g) = 1$ . The map  $\lambda$  is also referred to as a left order. Thus the set  $LO(G)$  of the left orders on  $G$  is viewed as a closed subset of the space  $\{\pm 1\}^{G \setminus \{e\}}$  with the pointwise convergence topology. This yields a totally disconnected compact metrizable topology on  $LO(G)$  (metrizable since  $G$  is countable). It is either finite or uncountably many [9]. We call  $\lambda \in LO(G)$  *isolated* if it is an isolated point in the space  $LO(G)$ .

Given  $\lambda \in LO(G)$ , there is defined a dynamical realization

$$\rho_\lambda : G \rightarrow \text{Homeo}_+(\mathbb{R})$$

based at  $x_0 \in \mathbb{R}$  such that  $f <_\lambda g$  if and only if  $fx_0 < gx_0$ . We discuss its fundamental properties in Section 2. Especially we show that the dynamical realization is tight at the base point. See Definition 2.1.

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In this paper, we are mainly interested in isolated orders, since in this case, the dynamical realizations display a certain kind of rigidity, and vice versa. In [10, Theorems 1.2 and 3.11], the relation between the isolation of left orders and the rigidity of the dynamical realization is described, as well as for circular orders.

An action  $\rho : G \rightarrow \text{Homeo}_+(\mathbb{R})$  is said to be *cocompact* if there is a compact interval  $I$  such that any orbit  $\rho(G)x$  intersects  $I$ . Our first result, proved in Section 3, is the following.

**THEOREM 1.** *If  $\lambda \in LO(G)$  is isolated, then its dynamical realization  $\rho_\lambda$  is cocompact.*

In fact if the group  $G$  is finitely generated, the dynamical realization is cocompact for any left order, isolated or not. (See Lemma 3.1 below.) Therefore Theorem 1 is mainly concerned with non finitely generated groups. By Theorem 1, the dynamical realization of an isolated order admits a minimal set  $\mathcal{M}$ , which is shown to be unique unless  $G \cong \mathbb{Z}$ . In Section 4, we show that if  $\mathcal{M} = \mathbb{R}$ , then the group is rational (Theorem 4.1).

Given  $\lambda \in LO(G)$ , a subgroup  $H$  of  $G$  is called  $\lambda$ -convex, if whenever  $h_1, h_2 \in H$ ,  $g \in G$  and  $h_1 <_\lambda g <_\lambda h_2$ , we have  $g \in H$ . The set of convex subgroups is totally ordered by the inclusion. The following theorem is shown in Section 5.

**THEOREM 2.** *If  $\lambda \in LO(G)$  is isolated, then there are only finitely many  $\lambda$ -convex subgroups.*

This is known to specialists (see for example [4, Exercise 3.3.15]). However, our strategy of the proof is different from that mentioned in [4].

Theorem 2 enables us to define the maximal sequence of convex subgroups of an isolated left order. As an application of our method, we give a dynamical proof of the Tararin theorem which characterizes the groups with finitely many left orders in Section 6. In Section 7, the maximal Tararin subgroup of an isolated left order is defined, and is shown to be equal to the Conradian soul [12].

Last Sections 8 and 9 are more or less independent of the previous sections. Dubrovina–Dubrovin [2] constructed an isolated order  $\lambda_n$  on the braid group  $B_n$ ,  $n \geq 3$ . In Section 9, we show:

**THEOREM 3.** *There are countably many isolated orders in  $LO(B_3)$  which are not the automorphic images<sup>1</sup> of the others.*

The method is a modification of the proof of [10, Theorem 1.4]. The following theorem is the starting point of the proof of Theorem 3. Let

$$G = \langle a, b \mid a^2 = b^3 \rangle, \quad \overline{G} = \langle \alpha, \beta \mid \alpha^2 = \beta^3 = e \rangle,$$

and  $q : G \rightarrow \overline{G}$  the surjective homomorphism defined by  $q(a) = \alpha$  and  $q(b) = \beta$ . Notice that  $G$  is isomorphic to  $B_3$ , and  $\overline{G}$  to  $PSL(2, \mathbb{Z})$ . We denote by  $CO(\overline{G})$  the space of the

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<sup>1</sup>Given  $\phi \in \text{Aut}(G)$  and  $\lambda \in LO(G)$ , the left order  $\phi^*\lambda \in LO(G)$  defined by  $g <_{\phi^*\lambda} g'$  if and only if  $\phi(g) <_\lambda \phi(g')$  is called an automorphic image of  $\lambda$ . For example, the reciprocal of the natural order  $\lambda$  of  $\mathbb{Z}$  is an automorphic image of  $\lambda$ .

left invariant circular orders of  $\overline{G}$  (see Section 8). In Section 9, we show:

**THEOREM 4.** *The homomorphism  $q$  induces a homeomorphism  $q_* : LO(G) \rightarrow CO(\overline{G})$ .*

Isolated left orders are often induced from isolated circular orders of the group quotiented by the center. See [10, Section 5], for example. The above theorem is also a typical example. However there is an example of a group with isolated left orders which admits no center, constructed in [6, 3.2].

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## 2. Dynamical realization.

In this section, we define a dynamical realization of a left order  $\lambda \in LO(G)$  and study its fundamental properties. Fix an enumeration of  $G$ :  $G = \{g_i \mid i \in \mathbb{N}\}$  such that  $g_1 = e$ . We define an order preserving embedding  $\iota : G \rightarrow \mathbb{R}$  inductively as follows. Define  $\iota(g_1) = x_0$ , where  $x_0$  is some point in  $\mathbb{R}$ . Assume we have defined  $\iota$  on the subset  $\{g_1, \dots, g_n\}$ ,  $n \geq 1$ , and let us define  $\iota(g_{n+1})$ . Order the subset  $\{g_1, \dots, g_n\}$  as

$$g_{i_1} <_\lambda g_{i_2} <_\lambda \dots <_\lambda g_{i_n}.$$

- If  $g_{n+1} <_\lambda g_{i_1}$ , define  $\iota(g_{n+1}) = \iota(g_{i_1}) - 1$ ,
- if  $g_{i_n} <_\lambda g_{n+1}$ ,  $\iota(g_{n+1}) = \iota(g_{i_n}) + 1$ ,
- and if  $g_{i_k} <_\lambda g_{n+1} <_\lambda g_{i_{k+1}}$ ,  $\iota(g_{n+1}) = (1/2)(\iota(g_{i_k}) + \iota(g_{i_{k+1}}))$ .

Then we have  $\inf \iota(G) = -\infty$  and  $\sup \iota(G) = \infty$ . The left translation of  $G$  yields an order preserving action of  $G$  on  $\iota(G)$ , which extends to a continuous action on the closure  $Cl(\iota(G))$ . (See Proposition 2.2 and the proof of Corollary 2.3.) Extend it further to a continuous action on  $\mathbb{R}$  by setting that the action on gaps of  $Cl(\iota(G))$  be linear.

This action is called the *dynamical realization of  $\lambda$  based at  $x_0$* , and is denoted by  $\rho_\lambda$ . The dynamical realization depends on the choice of the enumeration of  $G$ . Soon later, we shall show that any two dynamical realizations are mutually topologically conjugate.

**DEFINITION 2.1.** An action  $\rho : G \rightarrow \text{Homeo}_+(\mathbb{R})$  is called *tight at  $x_0 \in \mathbb{R}$*  if

- (1)  $\rho$  is free at  $x_0$  i.e. the stabilizer at  $x_0$  is trivial,
- (2)  $\inf \rho(G)x_0 = -\infty$ ,  $\sup \rho(G)x_0 = \infty$ , and
- (3) whenever  $Cl(\rho(G)x_0) \cap [a, b] = \{a, b\}$  for any  $a < b$ , we have  $\{a, b\} \subset \rho(G)x_0$ .

**PROPOSITION 2.2.** *The dynamical realization  $\rho_\lambda$  based at  $x_0$  is tight at  $x_0$ .*

**PROOF.** All that needs proof is (3). Let  $a < b$  be as in (3). The proof is by contradiction. Assume, to fix the idea, that  $a \notin \rho(G)x_0 = \iota(G)$ . (Notice that  $\iota(g) = \rho_\lambda(g)x_0$ .) Choose  $\epsilon$  small enough compared with  $b - a$ , and choose  $\iota(g_1) \in (a - \epsilon, a)$  and  $\iota(g_2) \in [b, b + \epsilon)$ . Recall that the dynamical realization is defined via an enumeration of  $G$ . One may assume that there is no point in  $(\iota(g_1), \iota(g_2)) \cap \iota(G)$  which is enumerated before  $g_1$  or  $g_2$ , since otherwise one may pass to that point. Since  $a \in Cl(\iota(G)) \setminus \iota(G)$ , there is a point  $\iota(g_3)$  in  $(\iota(g_1), \iota(g_2)) \cap \iota(G)$  which is enumerated for the first time after

$g_1$  and  $g_2$ . Then  $\iota(g_3)$  is the midpoint of  $\iota(g_1)$  and  $\iota(g_2)$  and must be fallen in  $(a, b)$  since  $\epsilon$  is small. A contradiction.  $\square$

**COROLLARY 2.3.** *The dynamical realizations defined via two different enumerations of  $G$  are mutually conjugate by an orientation and base point preserving homeomorphism of  $\mathbb{R}$ .*

**PROOF.** Let  $\iota$  and  $\iota'$  be two embeddings of  $G$  obtained by different enumerations of  $G$ . There is an orientation preserving bijection  $h : \iota(G) \rightarrow \iota'(G)$  defined by  $h(\iota(g)) = \iota'(g)$  ( $g \in G$ ). By the tightness,  $h$  extends, first of all, to a homeomorphism  $h : Cl(\iota(G)) \rightarrow Cl(\iota'(G))$ , and then to a homeomorphism of  $\mathbb{R}$  linearly on gaps. The extended  $h$  yields the required conjugacy.  $\square$

The proof of the previous corollary also yields the following result, which will be used in Section 9.

**COROLLARY 2.4.** *Let  $\mathcal{H}$  be the set of the orientation and base point  $x_0$  preserving topological conjugacy classes of the homomorphisms  $G \rightarrow \text{Homeo}_+(\mathbb{R})$  which are tight at  $x_0$ . Then the dynamical realization at  $x_0$  induces a bijection of  $LO(G)$  onto  $\mathcal{H}$ .*

A left order  $<_\lambda$  is called *discrete* if there is a minimal  $\lambda$ -positive element, and *indiscrete* otherwise.

**COROLLARY 2.5.** *If  $\lambda \in LO(G)$  is indiscrete, then the orbit  $\rho_\lambda(G)x_0$  of the base point  $x_0$  is dense in  $\mathbb{R}$ .*

**PROOF.** Assume  $Cl(\rho_\lambda(G)x_0) \neq \mathbb{R}$  and let  $(a, b)$  be a gap of  $Cl(\rho_\lambda(G)x_0)$ . Then by the previous lemma, we have  $a, b \in \rho_\lambda(G)x_0$ . That is,  $a = \iota(g_1)$  and  $b = \iota(g_2)$  for some  $g_1, g_2 \in G$ . Then  $g_1^{-1}g_2$  is the minimal positive element, and  $\lambda$  is discrete.  $\square$

### 3. Proof of Theorem 1.

We begin with two lemmas. The first one can be found in [11, Proposition 2.1.12].

**LEMMA 3.1.** *Let  $G$  be a finitely generated group which acts on  $\mathbb{R}$  without global fixed points. Then the action is cocompact.*

**PROOF.** We identify  $\mathbb{R} \approx (0, 1)$ . Let  $G_0$  be a finite generating set of  $G$ . Define

$$a = \sup_{s \in G_0} \sup_{x \in (0, 1)} |sx - x|.$$

Choose a compact interval  $J \subset (0, 1)$  such that  $|J| > a$ . Given any point  $x \in (0, 1)$ , we have  $\inf Gx = 0$  and  $\sup Gx = 1$  since there is no global fixed point. Considering the Schreier graph of  $Gx$ , one can show that  $Gx \cap J \neq \emptyset$ .  $\square$

**LEMMA 3.2.** *Let  $G$  be a group acting on  $\mathbb{R}$  and let  $y_0 \in \mathbb{R}$ . Denote by  $G_{y_0}$  the stabilizer of  $G$  at  $y_0$ . Assume  $G_{y_0} \neq G$ . Given  $\lambda_0 \in LO(G_{y_0})$ , there are at least two orders in  $LO(G)$  which restrict to  $\lambda_0$  on  $G_{y_0}$ .*

PROOF. Let  $\mu$  be the  $G$ -invariant order on  $G/G_{y_0}$  given by the natural order of the orbit  $Gy_0 \approx G/G_{y_0}$  in  $\mathbb{R}$ . Then  $\lambda_0$  and  $\mu$  determines a left order on  $G$  lexicographically (Lemma 5.1). If we consider the reciprocal order  $-\mu$ , we get another one.  $\square$

Assume  $\lambda \in LO(G)$  is an isolated left order on  $G$ . Since we are considering the pointwise convergence topology, this is equivalent to the following condition  $(\star)$ .

$(\star)$  There is a finite subset  $S \subset P_\lambda$  such that  $\lambda$  is the only element in  $LO(G)$  which contains  $S$  in its positive cone.

Such a subset  $S$  is called a *characteristic positive set* of  $\lambda$ .

PROOF OF THEOREM 1. By the dynamical realization of the isolated left order  $\lambda$ , the group  $G$  acts on  $\mathbb{R}$ . Let  $H$  be the subgroup of  $G$  generated by a characteristic positive set  $S$  of  $\lambda$ . If there is no global fixed point by the action of  $H$ , then  $H$  acts on  $\mathbb{R}$  cocompactly by Lemma 3.1, and hence also  $G$ , finishing the proof. In the remaining case, choose a global fixed point  $y_0$  of  $H$  and consider  $G_{y_0}$ . We have  $G_{y_0} \neq G$  since the dynamical realization has no global fixed point, by its tightness. By the previous lemma, the restriction of  $\lambda$  to  $G_{y_0}$  extends to two left orders of  $G$ . But we have  $S \subset H \subset G_{y_0}$  and hence  $S$  is contained in the positive cone of both orders. A contradiction.  $\square$

REMARK 3.3. The condition that  $\lambda$  be isolated is actually necessary for Theorem 1. To show this, let  $G$  be the infinite direct sum of  $\mathbb{Z}$ , i.e.

$$G = \{(a_n)_{n \in \mathbb{N}} \mid a_n \in \mathbb{Z}, a_n = 0 \text{ but for finitely many } n\}.$$

Define a left order on  $G$  by setting  $0 < (a_n)$  if  $0 < a_N$ , where  $N$  is the largest number such that  $a_N \neq 0$ . Then its dynamical realization is not cocompact. To show this, define for  $m \in \mathbb{N}$ ,

$$G_m = \{(a_n) \mid a_n = 0, \forall n > m\}.$$

Then  $G_m$ 's form an exhausting increasing sequence of convex subgroups. Consider the dynamical realization  $\rho_\lambda$  based at  $x_0$ . The points

$$\xi_n = \inf \rho_\lambda(G_n)x_0 \quad \text{and} \quad \eta_n = \sup \rho_\lambda(G_n)x_0$$

are fixed points of  $\rho_\lambda(G_n)$ . They satisfy  $\xi_n \searrow -\infty$  and  $\eta_n \nearrow \infty$  by condition (2) of Definition 2.1, since  $G_n$  is exhausting. This implies that  $\rho_\lambda$  is not cocompact.

Theorem 1 implies that there is a minimal set  $\mathcal{M}$  for the dynamical realization of an isolated left order.<sup>2</sup> There is a trichotomy for  $\mathcal{M}$  ([3, Proposition 6.1]).

- (I)  $\mathcal{M} = \mathbb{R}$ .
- (II)  $\mathcal{M}$  is infinite and discrete in  $\mathbb{R}$ .

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<sup>2</sup>The results of the remaining part of this section (trichotomy, Lemma 3.4 and Corollary 3.5) and Lemma 5.4 (1) (2) hold true whenever the left order  $\lambda$  admits a cocompact dynamical realization, especially when  $G$  is finitely generated. But we shall state it only for an isolated left order  $\lambda$ .

(III)  $\mathcal{M}$  is locally Cantor. In this case, if  $X$  is a nonempty closed subset of  $\mathbb{R}$  invariant by the dynamical realization of  $G$ , then  $\mathcal{M} \subset X$ . Especially,  $\mathcal{M}$  is the unique minimal set.

LEMMA 3.4. *Let  $\lambda \in LO(G)$  be isolated, with  $\mathcal{M}$  an associated minimal set. Assume (III) above, or (II) and  $G \not\cong \mathbb{Z}$ . Then the base point  $x_0$  is contained in a gap  $I_1$  of  $\mathcal{M}$ , the stabilizer  $G_{I_1}$  is nontrivial, and there is no gap of  $\mathcal{M}$  other than the orbit of  $I_1$ .*

PROOF. We give a proof only for case (III). Case (II) can be treated much easier. Notice that if  $\rho(G)x_0$  is discrete, then  $G \cong \mathbb{Z}$ . Assume that the base point  $x_0$  is contained in  $\mathcal{M}$  and let  $(a, b)$  be a gap of  $\mathcal{M}$ . Since the dynamical realization  $\rho_\lambda$  is tight, we have  $a, b \in \rho_\lambda(G)x_0$ . But there is no orientation preserving homeomorphism leaving  $\mathcal{M}$  invariant and mapping  $a$  to  $b$ . The contradiction shows that  $x_0$  is contained in a gap  $I_1$  of  $\mathcal{M}$ .

If  $G_{I_1}$  is trivial, then  $\rho_\lambda(G)x_0 \cap I_1 = \{x_0\}$ . Again by the tightness, the boundary points of  $I_1$  must belong to  $\rho_\lambda(G)x_0$ . A contradiction. The last statement follows similarly from the tightness.  $\square$

COROLLARY 3.5. *If  $G \not\cong \mathbb{Z}$ , then the minimal set  $\mathcal{M}$  of the dynamical realization  $\rho_\lambda$  of an isolated left order  $\lambda$  is unique.*

PROOF. All that needs proof is the case where  $\mathcal{M}$  is discrete, since a locally Cantor minimal set is always unique. We still use the notation of the previous lemma. If there is another minimal set  $\mathcal{M}'$ , then  $\mathcal{M}' \cap I_1$  must be one point, say  $y_0$ , which is fixed by  $\rho_\lambda(G_{I_1})$ . But then  $\rho_\lambda(G)x_0 \cap I_1 = \rho_\lambda(G_{I_1})x_0$  must be contained in an open subinterval of  $I_1$  delimited by  $y_0$ , contrary to the tightness.  $\square$

#### 4. The case $\mathcal{M} = \mathbb{R}$ .

This section is devoted to the proof of the following theorem.

THEOREM 4.1. *Let  $\lambda \in LO(G)$  be isolated and assume that the dynamical realization  $\rho_\lambda$  is minimal. Then the group  $G$  is isomorphic to an additive subgroup  $A$  of  $\mathbb{Q}$  such that  $A \not\cong \mathbb{Z}$ , and  $\lambda$  is either the natural left order given by  $A \subset \mathbb{Q} \subset \mathbb{R}$  or its reciprocal.*

This theorem might be known among specialists, but the author cannot locate it in the literature.

Let  $\lambda$  be an element of  $LO(G)$  which satisfies the hypothesis of Theorem 4.1. We shall abbreviate the notations  $\rho_\lambda(g)x$  by  $gx$ , and  $\rho_\lambda(G) \subset \text{Homeo}_+(\mathbb{R})$  by  $G$ . Let  $Z$  be the centralizer of  $G$  in  $\text{Homeo}_+(\mathbb{R})$ .

LEMMA 4.2. *The centralizer  $Z$  is an abelian group which acts freely on  $\mathbb{R}$ .*

PROOF. For  $\zeta \in Z \setminus \{\text{id}\}$ ,  $\text{Fix}(\zeta)$  is a closed set which is invariant by  $G$ . Since the  $G$ -action is minimal, we have  $\text{Fix}(\zeta) = \emptyset$ . By Hölder's theorem (e.g. [12]), any group acting freely on  $\mathbb{R}$  is abelian.  $\square$

Let  $x_0$  be the base point of the dynamical realization. Choose  $x_n \in Gx_0$ ,  $n \in \mathbb{N}$ , such that  $x_n \rightarrow x_0$ ,  $x_n \neq x_0$ . Notice that  $G$  acts freely at  $x_n$ . Let  $\lambda_n \in LO(G)$  be the order determined by  $x_n$ :  $g >_{\lambda_n} e$  if and only if  $gx_n > x_n$ . Then  $\lambda_n \rightarrow \lambda$  in  $LO(G)$ . Since  $\lambda$  is isolated,  $\lambda_n = \lambda$  for any large  $n$ . We assume  $\lambda_n = \lambda$  for all  $n$ . Define an order preserving bijection  $\zeta_n : Gx_0 \rightarrow Gx_n$  by  $\zeta_n(gx_0) = gx_n$ . Since  $Gx_0 = Gx_n$  is dense in  $\mathbb{R}$ , the map  $\zeta_n$  extends to an orientation preserving homeomorphism of  $\mathbb{R}$ , denoted by the same letter  $\zeta_n$ . Clearly  $\zeta_n \neq id$ .

LEMMA 4.3. *We have  $\zeta_n \in Z$ .*

PROOF. Given any  $g \in G$ , it suffices to show that  $\zeta_n g = g \zeta_n$  on the dense subset  $Gx_0$ . For any  $hx_0 \in Gx_0$ , we have

$$\zeta_n g(hx_0) = \zeta_n((gh)x_0) = ghx_n = g(hx_n) = g \zeta_n(hx_0),$$

as is required. □

LEMMA 4.4. *The action of  $Z$  is minimal, and is conjugate to translations.*

PROOF. By Lemmas 4.2 and 4.3, there is an element in  $Z$  which acts freely on  $\mathbb{R}$ . This implies that the action of  $Z$  is cocompact. Let  $\mathcal{N}$  be a minimal set of  $Z$ . If it is locally Cantor, then  $\mathcal{N}$  is the unique minimal set, and must be invariant by  $G$ . But  $G$ -action is minimal by the assumption. A contradiction. Next assume  $\mathcal{N}$  is discrete. Then since the  $Z$ -action is free, we must have  $Z \cong \mathbb{Z}$ , contradicting Lemma 4.3. Therefore  $Z$  must act minimally on  $\mathbb{R}$ .

Choose any  $\zeta_0 \in Z \setminus \{id\}$ . Since the action of the group  $\langle \zeta_0 \rangle$  is free and  $Z$  is abelian, the group  $Z/\langle \zeta_0 \rangle$  acts on  $\mathbb{R}/\langle \zeta_0 \rangle \approx S^1$ . Since  $Z/\langle \zeta_0 \rangle$  is amenable, there is an  $Z/\langle \zeta_0 \rangle$ -invariant probability measure. It lifts to a locally finite  $Z$ -invariant measure  $\mu$  on  $\mathbb{R}$ . Since the action of  $Z$  is minimal,  $\mu$  is atomless and fully supported. Thus there is a homeomorphism  $h$  such that  $h_*\mu$  is the Lebesgue. Conjugating the  $Z$ -action by  $h$ , we obtain an action by translations. □

PROOF OF THEOREM 4.1. By changing the coordinate, we assume that the action of  $Z$  is by translations. Since the  $Z$ -action is minimal, any element of  $G$ , commuting with  $Z$ , acts also by translations. Then we have an injective homomorphism  $\phi : G \rightarrow \mathbb{R}$  defined by the translation length. We shall show that  $\phi$  embeds  $G$  into  $\mathbb{Q}$ . Assume not. Then  $G$  is a nontrivial direct sum:  $G = G_1 \oplus G_2$ . Given any  $a \in \mathbb{R}$ , we obtain a homomorphism  $\phi_a : G \rightarrow \mathbb{R}$  by setting  $\phi_a = \phi$  on  $G_1$  and  $\phi_a = a\phi$  on  $G_2$ . There is  $a$  arbitrarily near 1 such that  $\phi_a$  is injective. But  $\phi_a$  yields a left order different from  $\lambda$  and arbitrarily near  $\lambda$ . This contradicts the assumption that  $\lambda$  is isolated, finishing the proof that  $G$  is isomorphic to an additive subgroup  $A$  of  $\mathbb{Q}$ . The last statement of the theorem follows at once. □

### 5. Convex subgroups.

We shall prove Theorem 2 in this section. First we begin with fundamental properties of convex subgroups. For the definition of convex subgroups, see Introduction. We begin

with a well known easy fact.

LEMMA 5.1. *Let  $H$  be a subgroup of  $G$ . For any  $\lambda_0 \in LO(H)$  and any  $G$ -invariant total order  $\lambda_1$  on  $G/H$ , there is a unique order  $\lambda \in LO(G)$  such that  $H$  is  $\lambda$ -convex, that  $\lambda|_H = \lambda_0$ , and that for  $g \notin H$ ,  $g >_\lambda e$  if and only if  $gH >_{\lambda_1} H$ .  $\square$*

Such an order  $\lambda$  is said to be *determined lexicographically* by  $\lambda_0$  and  $\lambda_1$ .

LEMMA 5.2. *Let  $\lambda \in LO(G)$  and  $H$  a  $\lambda$ -convex subgroup of  $G$ . Then there is a  $G$ -invariant total order  $\lambda_1$  on  $G/H$  such that  $\lambda$  is determined lexicographically by  $\lambda|_H$  and  $\lambda_1$ .*

PROOF. Define a total order  $\lambda_1$  on  $G/H$  by setting  $g_1H <_{\lambda_1} g_2H$  if  $e <_\lambda g_1^{-1}g_2$  and  $g_1^{-1}g_2 \notin H$ . The convexity of  $H$  shows that this is a well defined  $G$ -invariant order.  $\square$

If  $G$  is isomorphic to  $\mathbb{Z}$  or if the minimal set of the dynamical realization of  $\lambda$  is  $\mathbb{R}$ , then there is no proper  $\lambda$ -convex subgroups, and Theorem 2 holds true. Henceforth in this section we work under the following assumption.

- ASSUMPTION 5.3. (1)  $\lambda \in LO(G)$  is isolated with a characteristic positive set  $S$ .  
 (2)  $G$  is not isomorphic to  $\mathbb{Z}$ .  
 (3) The minimal set  $\mathcal{M}$  of the dynamical realization is not  $\mathbb{R}$ .

Denote by  $I_1 = (y_0, z_0)$  the gap of  $\mathcal{M}$  which contains the base point  $x_0$  (Lemma 3.4), and by  $G_1$  the stabilizer of  $I_1$ .

- LEMMA 5.4. (1)  $G_1$  is proper and nontrivial.  
 (2)  $G_1$  is the maximal proper  $\lambda$ -convex subgroup of  $G$ .  
 (3) The restricted order  $\lambda|_{G_1}$  is isolated with characteristic positive set  $S \cap G_1$ .  
 (4)  $S \cap (G \setminus G_1) \neq \emptyset$ .

PROOF. The subgroup  $G_1$  is clearly proper. It is nontrivial by Lemma 3.4. Also  $G_1$  is convex. Let  $H$  be an arbitrary proper  $\lambda$ -convex subgroup of  $G$ . We shall show that  $H \subset G_1$ . Consider first the case where  $\mathcal{M}$  is discrete. By looking at the action of  $G$  on  $\mathcal{M}$ , one can define a surjective homomorphism  $\phi : G \rightarrow \mathbb{Z}$  such that  $\text{Ker}(\phi) = G_1$ . If  $\phi(H)$  is nontrivial, then clearly we have  $H = G$  since  $H$  is convex. If  $\phi(H)$  is trivial, then  $H \subset G_1$ , as is required.

So in the rest, we assume that  $\mathcal{M}$  is locally Cantor. Let  $\mathcal{H}$  be the convex hull of  $Hx_0$  in  $\mathbb{R}$ . Then  $\mathcal{H}$  is a bounded open interval of  $\mathbb{R}$ . The boundedness follows from the convexity and the properness of  $H$ . The convexity of  $H$  implies that for any  $g \in G$ , we have either  $g\mathcal{H} = \mathcal{H}$  or  $g\mathcal{H} \cap \mathcal{H} = \emptyset$ . Thus the closed set

$$X = \mathbb{R} \setminus \bigcup_{g \in G} g\mathcal{H}$$

is  $G$ -invariant and nonempty. Therefore we have  $\mathcal{M} \subset X$ , which implies  $\mathcal{H} \subset I_1$ , showing that  $H \subset G_1$ .

Let us show that  $S \cap G_1$  is a characteristic positive set of  $\lambda|_{G_1}$ . If not, there is a left order  $\lambda'_0$  ( $\lambda'_0 \neq \lambda|_{G_1}$ ) of  $G_1$  such that  $S \cap G_1$  is contained in the positive cone of  $\lambda'_0$ . Let  $\lambda_1$  be the  $G$ -invariant total order on  $G/G_1$  obtained by Lemma 5.2. Let  $\lambda' \in LO(G)$  be the order determined lexicographically by  $\lambda'_0$  and  $\lambda_1$ . Then  $\lambda'$  contains  $S$  in its positive cone and  $\lambda' \neq \lambda$ , contradicting that  $S$  is a characteristic positive set of  $\lambda$ .

Finally let us show that  $S \cap (G \setminus G_1)$  is nonempty. If it is empty, then  $\lambda|_{G_1}$  and  $-\lambda_1$  lexicographically determine  $\lambda' \in LO(G)$ , where  $-\lambda_1$  is the reciprocal of the order  $\lambda_1$  constructed in Lemma 5.2. But  $S$  is contained in the positive cone of  $\lambda'$ . A contradiction.  $\square$

PROOF OF THEOREM 2. By Lemma 5.4, we obtain the maximal proper convex subgroup  $G_1$ . If  $G_1$  is not isomorphic to  $\mathbb{Z}$  and the minimal set of the dynamical realization of  $\lambda|_{G_1}$  is not the whole  $\mathbb{R}$ , then we can repeat the process and obtain the second maximal proper convex subgroup  $G_2$ . This process ends at finite steps since each time the number of elements of positive characteristic set decreases.  $\square$

DEFINITION 5.5. The sequence

$$G = G_0 > G_1 > \dots > G_n > \{e\}$$

of all the  $\lambda$ -convex subgroups is called the *maximal convex sequence* of the isolated order  $\lambda$ . The number  $n$  is called the *height* of  $\lambda$ .

Thus an isolated left order with minimal dynamical realization has height 0. Let  $\mathcal{M}_0$  be the minimal set of  $G$  and  $I_1$  the gap of  $\mathcal{M}_0$  containing the base point  $x_0$ . Then the maximal proper  $\lambda$ -convex subgroup  $G_1$  is the stabilizer of  $I_1$ . By Lemma 5.4 (3),  $\lambda|_{G_1}$  is isolated, and there is a minimal set  $\mathcal{M}_1$  of the  $G_1$ -action on  $I_1$ . Next consider the gap  $I_2$  of  $\mathcal{M}_1$  in  $I_1$  containing  $x_0$ . Continuing this way, we get a decreasing sequence of open intervals

$$\mathbb{R} \supset I_1 \supset \dots \supset I_n.$$

Each subgroup  $G_i$  is the stabilizer of  $I_i$ , and each  $\mathcal{M}_i$  is a minimal set of  $G_i$  in  $I_i$ . The pair  $(I_i, \mathcal{M}_i)$  is called the  *$i$ -th internal pair associated with the maximal convex sequence*. There are only two possibilities for the last group  $G_n$ :

- (A)  $\mathcal{M}_n = I_n$ ,
- (B)  $G_n = \mathbb{Z}$ .

In (A), the order  $\lambda$  is indiscrete and in (B), it is discrete.

As a corollary of Theorem 4.1, we get the following proposition, which will be used in the next section.

PROPOSITION 5.6. *If an isolated order  $\lambda$  has height 0, i.e. if there is no proper  $\lambda$ -convex subgroup, then the group  $G$  is rational and the order  $\lambda$  is the natural order of  $G \subset \mathbb{Q} \subset \mathbb{R}$  or its reciprocal.*

**6. Tararin groups.**

DEFINITION 6.1. A group  $G$  is called a *Tararin group* if  $|LO(G)| < \infty$ .

Of course any left order of a Tararin group is isolated. In this section, we shall give a dynamical proof of the following theorem by Tararin [14]. See also [4, Theorem 2.2.13] or [8].

THEOREM 6.2. (I) Assume  $|LO(G)| < \infty$ . Then the following holds.

(1) There is a unique rational series<sup>3</sup>

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n \triangleright G_{n+1} = \{e\}. \tag{6.1}$$

(The uniqueness implies that each subgroup  $G_i$  is characteristic, i.e. invariant by any automorphism of  $G$ . Especially it is a normal subgroup of  $G$ .)

(2) There are elements  $s_i \in G_i \setminus G_{i+1}$  for each  $i \in \{0, 1, \dots, n\}$  such that for any map  $\epsilon : \{0, 1, \dots, n\} \rightarrow \{\pm 1\}$ , there is exactly one order  $\lambda_\epsilon$  such that  $s_i^{\epsilon(i)}$  is positive. Thus

$$LO(G) = \{\lambda_\epsilon \mid \epsilon \in \{\pm 1\}^{\{0,1,\dots,n\}}\}.$$

(3) The sequence (6.1) is the maximal convex sequence for any  $\lambda_\epsilon$ .

(4) The quotient group  $G_i/G_{i+2}$ ,  $i \in \{0, \dots, n-1\}$ , is not bi-orderable.

(II) Conversely, if a group  $G$  admits a rational series (6.1) such that  $G_{i+2}$  is a normal subgroup of  $G_i$  and  $G_i/G_{i+2}$  is not bi-orderable ( $0 \leq i \leq n-1$ ), then  $|LO(G)| = 2^{n+1}$ .

PROOF. First of all, let us show (II). It suffices to prove that  $G_1$  is  $\lambda$ -convex for any  $\lambda \in LO(G)$ . In fact, this implies that any  $\lambda$  is constructed in a lexicographical way, and thus  $|LO(G)| = |LO(G_1)| \cdot |LO(G/G_1)|$ . On the other hand, we have  $|LO(G/G_1)| = 2$ . An induction on  $n$  shows that  $|LO(G)| = 2^{n+1}$ . We use the following easy fact.

If  $A$  is a rational group and  $\phi : A \rightarrow \{\pm 1\}$  is a nontrivial homomorphism, then for any nontrivial element  $g \in A$ , there are  $g_0 \in A$  and  $n \geq 1$  such that  $g = g_0^n$  and  $\phi(g_0) = -1$ .

Fix  $\lambda \in LO(G)$ . We shall show that  $G_1$  is  $\lambda$ -convex by an induction on  $n$ . Consider an exact sequence

$$1 \rightarrow G_1/G_2 \rightarrow G/G_2 \rightarrow G/G_1 \rightarrow 1.$$

By the induction hypothesis,  $G_2$  is  $\lambda|_{G_1}$ -convex and there is a left order  $<$  on  $G_1/G_2$  induced from  $\lambda|_{G_1}$ . One can define a homomorphism  $\phi' : G \rightarrow \{\pm 1\}$  according as the conjugation by an element of  $G$  preserves the order  $<$  on  $G_1/G_2$  or not. (Notice that there are only two orders on  $G_1/G_2$ .) Since  $G_1/G_2$  is abelian,  $\phi'$  induces a homomorphism  $\phi : G/G_1 \rightarrow \{\pm 1\}$ . Should  $\phi$ , equivalently  $\phi'$ , be trivial, the order on  $G/G_2$  constructed

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<sup>3</sup>Rational series means that for any  $i$ ,  $G_i/G_{i+1}$  is a rational group, i.e. an abelian group embeddable into  $\mathbb{Q}$ .

lexicographically from  $<$  and an order of  $G/G_1$  would be a bi-order. This shows that  $\phi$  is nontrivial.

To complete the proof, let us show that for any element  $g \in G \setminus G_1$ ,  $g >_\lambda e$ , we have  $g^{-1} <_\lambda G_1 <_\lambda g$ . There exist  $g_0 \in G$  and  $n \geq 1$  such that  $g \equiv g_0^n \pmod{G_1}$  and  $\phi'(g_0) = -1$ . Then for any  $h \in G_1 \setminus G_2$ ,  $h >_\lambda e$  if and only if  $g_0^{-1}hg_0 <_\lambda e$ .

Assume for a while that  $g_0 >_\lambda e$ . Then if  $h >_\lambda e$ ,

$$e <_\lambda h <_\lambda hg_0 <_\lambda g_0.$$

Applying  $h$  successively, we obtain

$$e <_\lambda h <_\lambda h^2 <_\lambda \cdots <_\lambda h^2g_0 <_\lambda hg_0 <_\lambda g_0.$$

If we put  $h_1 = g_0^{-1}hg_0$ , then

$$e <_\lambda h <_\lambda h^2 <_\lambda \cdots <_\lambda g_0h_1^2 <_\lambda g_0h_1 <_\lambda g_0. \tag{6.2}$$

By an analogous argument, we have

$$g_0^{-1} <_\lambda g_0^{-1}h_1^{-1} <_\lambda g_0^{-1}h_1^{-2} <_\lambda \cdots <_\lambda h^{-2} <_\lambda h^{-1} <_\lambda e. \tag{6.3}$$

The elements  $h, h_1 \in G_1 \setminus G_2$  are  $\lambda|_{G_1}$ -cofinal<sup>4</sup> by the assumption that  $G_2$  is  $\lambda|_{G_1}$ -convex and  $G_1/G_2$  is rational. Therefore by (6.2) and (6.3), we obtain  $g_0^{-1}G_1 <_\lambda G_1 <_\lambda g_0G_1$ . For our initial  $g$ , since  $g^{\pm 1}G_1 = g_0^{\pm n}G_1$ , we have  $g^{-1}G_1 <_\lambda G_1 <_\lambda gG_1$ , as is required. On the other hand, if  $g_0 <_\lambda e$ , then the same argument shows that  $gG_1 <_\lambda G_1 <_\lambda g^{-1}G_1$ , contradicting the hypothesis  $g >_\lambda e$ . This finishes the proof of (II).

Now we shall proceed to the proof of (I). For a Tararin group  $G$ , let  $n(G)$  be the minimal height of all the elements of  $LO(G)$ . We shall show (I) by the induction on  $n(G)$ . This is already shown for  $n(G) = 0$  by Proposition 5.6. Let  $G$  be a Tararin group,  $\lambda \in LO(G)$  with height  $n = n(G)$ , and  $G_1$  the maximal proper  $\lambda$ -convex subgroup. Then the lexicographic construction shows that  $G_1$  is also Tararin, and  $n(G_1) \leq n - 1$ . Therefore by the induction hypothesis, the maximal convex sequence of  $\lambda|_{G_1}$

$$G_1 \triangleright G_2 \triangleright \cdots \triangleright G_{n+1} = \{e\}$$

is a unique rational series of  $G_1$  and  $G_i/G_{i+2}$  is not bi-orderable ( $1 \leq i \leq n - 1$ ).

First of all, let us show that  $G_1$  is a normal subgroup of  $G$ , and  $G/G_1$  is a rational group. But this is clear if the minimal set  $\mathcal{M}$  of the dynamical realization  $\rho_\lambda$  is discrete. So assume  $\mathcal{M}$  is a locally Cantor set. Let  $x_0$  be the base point of  $\rho_\lambda$ , and choose  $g_k \in G$  so that  $\rho_\lambda(g_k)x_0 \rightarrow \exists y_0 \in \mathcal{M}$  as  $k \rightarrow \infty$ . One may assume that  $\rho_\lambda(g_k)x_0$  belongs to a distinct gap of  $\mathcal{M}$  for each  $k$ . The left orders of  $G$  induced by the  $\rho_\lambda(G)$ -orbit of  $\rho_\lambda(g_k)x_0$  are finite in number. So one may assume, by passing to a subsequence, that the left orders are the same. By the same argument as in Theorem 4.1, one can construct order preserving homeomorphisms  $h_{k,k'}$  of  $Cl(\rho_\lambda(G))x_0$  which commute with any  $\rho_\lambda(g)$

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<sup>4</sup>For  $\lambda \in LO(G)$ , an element  $h \in G$  is said to be  $\lambda$ -cofinal if for any  $g \in G$ , there are  $n, m \in \mathbb{Z}$  such that  $h^n <_\lambda g <_\lambda h^m$ .

such that  $h_{k,k'}(\rho_\lambda(g_k)x_0) = \rho_\lambda(g_{k'})x_0$ . The map  $h_{k,k'}$  leaves the unique minimal set  $\mathcal{M}$  of  $\rho_\lambda(G)$  invariant.

Consider the quotient space  $\mathcal{R}$  of  $\mathbb{R}$  obtained by collapsing each gap of  $\mathcal{M}$  to a point. Then  $h_{k,k'}$  induces an orientation preserving homeomorphism of  $\mathcal{R}$  commuting with the induced action of  $\rho_\lambda(G)$ . Let  $Z$  be the centralizer of the action on  $\mathcal{R}$  induced from  $\rho_\lambda(G)$  in the space of the orientation preserving homeomorphisms of  $\mathcal{R}$ . Then since the induced action of  $G$  is minimal,  $Z$  acts freely on  $\mathcal{R}$ . In fact, if an element of  $Z$  has nonempty fixed point set, then the fixed point set must be  $G$ -invariant and coincides with  $\mathcal{R}$ . Thus the action of  $Z$  is topologically conjugate to translations.

By the choices of  $k, k'$ , there are arbitrarily small translations. That is, the action of  $Z$  must be minimal. This shows that the induced  $G$ -action on  $\mathcal{R}$  itself is also by translations. Therefore  $G_1$  is the kernel of the induced  $G$ -action, and is a normal subgroup of  $G$ . Finally, the left order of  $G/G_1$  induced by  $\lambda$  must be isolated, and hence by Theorem 4.1,  $G/G_1$  is rational.

Since  $G_1$  is a normal subgroup of  $G$ , and  $G_i$  ( $i \geq 2$ ) is a characteristic subgroup of  $G_1$  by the induction hypothesis,  $G_i$ , especially  $G_2$ , is a normal subgroup of  $G$ . Next let us show that  $H = G/G_2$  is not bi-orderable. Denote  $A = G_1/G_2$  and  $B = G/G_1$ . There is an exact sequence

$$1 \rightarrow A \rightarrow H \xrightarrow{q} B \rightarrow 1. \tag{6.4}$$

Notice that  $H$  is Tarsian, since otherwise lexicographic construction would yield infinitely many left orders on  $G$ . The conjugation yields a homomorphism from  $H$  to  $\text{Aut}(A)$ , which projects to a homomorphism  $\phi : B \rightarrow \text{Aut}(A)$  since  $A$  is abelian. Any automorphism of  $A \subset \mathbb{Q}$  is the multiplication by a nonzero rational number. Thus we get  $\phi : B \rightarrow \mathbb{Q}^\times$ . If  $\phi$  takes a negative value, then  $H$  does not admit a bi-order, and we are done. If  $\phi$  is trivial, then projecting  $H = A \times B \subset \mathbb{Q}^2 \subset \mathbb{R}^2$  to  $\mathbb{R}$  along one dimensional linear subspaces of irrational slope yields embeddings of  $H$  into  $\mathbb{R}$ , from which we obtain infinitely many left orders on  $H$ . A contradiction.

Assume  $\phi$  is positive valued and nontrivial. Let  $\{B_i\}$  be an exhausting increasing sequence of subgroups of  $B$  which are isomorphic to  $\mathbb{Z}$ , and let  $H_i = q^{-1}(B_i)$ . Then the exact sequence

$$1 \rightarrow A \rightarrow H_i \rightarrow B_i \rightarrow 1$$

is split. There is a representation  $f_i : H_i \rightarrow \text{Aff}_+(\mathbb{R})$  to the group of the orientation preserving affine transformations of the real line such that  $A$  is mapped to translations (by  $A \subset \mathbb{Q}$  itself) and that the split image of  $B_i$  is mapped to the homotheties of ratio  $\phi(B_i)$  at some point of  $\mathbb{R}$ . Two such representations are mutually conjugate by translations (regardless of the choice of the splittings). Therefore we can arrange so that  $f_{i+1}$  is an extension of  $f_i$ . As the direct limit, we get a faithful representation  $f : H \rightarrow \text{Aff}_+(\mathbb{R})$ . By considering the orbit of various points of  $\mathbb{R}$  at which  $f(H)$  acts freely, we get various left orders of  $H$ , leading to a contradiction. This finishes the proof that  $H$  is not bi-orderable.

Finally let us show that a rational series of  $G$  is unique. By the induction hypothesis,

the groups  $G_i/G_{i+2}$ ,  $1 \leq i \leq n - 1$ , are also not bi-orderable. So the sequence

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n \triangleright G_{n+1} = \{e\} \tag{6.5}$$

satisfies the hypothesis of (II). We already know that the cardinality of  $LO(G)$  is  $2^{n+1}$ .

Choose  $s_i \in G_i \setminus G_{i+1}$  and let  $S = \{s_0, \dots, s_n\}$ . For any  $\epsilon : S \rightarrow \{\pm 1\}$ , define

$$S^\epsilon = \{s_i^{\epsilon(s_i)} \mid i = 0, \dots, n\}.$$

For any  $\epsilon$ , we can construct a left order  $\lambda^\epsilon$  whose positive cone contains  $S^\epsilon$ , lexicographically using sequence (6.5). Such left orders exhaust  $LO(G)$ , since  $|LO(G)| = 2^{n+1}$ . This shows that a rational series of  $G$  is unique. In fact, any such series gives birth to a left order lexicographically. The series is the maximal convex sequence of that order, but all the  $2^{n+1}$  orders have (6.5) as the maximal convex sequence.  $\square$

REMARK 6.3. Let  $(I_i, \mathcal{M}_i)$  be the  $i$ -th internal pair associated with the maximal convex sequence (6.1) of a Tararin group  $G$ . The next subgroup  $G_{i+1}$  leaves the gap  $I_{i+1}$  of  $\mathcal{M}_i$  in  $I_i$  invariant. But because  $G_{i+1}$  is a normal subgroup of  $G_i$ , it leaves all the iterates of  $I_{i+1}$  under  $G_i$  invariant. By Lemma 3.4, these are the only gaps of  $\mathcal{M}_i$ . Therefore  $G_{i+1}$  acts trivially on  $\mathcal{M}_i$ . That is, there is an induced action of  $G_i/G_{i+1}$  on  $\mathcal{M}_i$ . If  $\mathcal{M}_i$  is discrete, then  $G_i/G_{i+1} \cong \mathbb{Z}$ , and the action on  $\mathcal{M}_i$  is by translation. Assume  $\mathcal{M}_i$  is locally Cantor. Let  $\mathcal{R}_i$  be the quotient space obtained by  $I_i$  by collapsing each gap of  $\mathcal{M}_i$  to a point. It is homeomorphic to  $\mathbb{R}$ . The quotient group  $G_i/G_{i+1}$  acts on  $\mathcal{R}_i$  minimally and freely. The whole action of  $G$  on  $\mathbb{R}$  is a “pileup” of translations. Any left order is discrete if and only if the last group  $G_n$  is isomorphic to  $\mathbb{Z}$ .

### 7. Maximal convex sequence.

We shall raise one more example (other than the Tararin groups) of isolated orders whose height is as big as possible. Let  $B_n$  be the braid group of  $n$  strings, with the standard generators  $\sigma_1, \dots, \sigma_{n-1}$ . Define

$$z_1 = \sigma_1 \cdots \sigma_{n-1}, \quad z_2 = \sigma_2 \cdots \sigma_{n-1}, \quad \dots, \quad z_{n-2} = \sigma_{n-2} \sigma_{n-1}, \quad z_{n-1} = \sigma_{n-1},$$

and  $y_i = z_i^{(-1)^{i-1}}$ . Let  $P_n$  be the subsemigroup of  $B_n$  generated by  $y_i$ 's. Based upon a result of Dehornoy [1], Dubrovina and Dubrovin [2] have shown a remarkable fact that  $P_n \sqcup P_n^{-1} = B_n \setminus \{e\}$ . The left order  $\lambda_n$  whose positive cone is  $P_n$  is called the *Dubrovina–Dubrovin order*. Since  $S = \{y_1, \dots, y_{n-1}\}$  generates  $P_n$ , the order  $\lambda_n$  is isolated with characteristic positive set  $S$ . Moreover  $\lambda_n$  can be defined lexicographically as a twist of the Dehornoy order [1], and the subgroups

$$B_{n-k}^* = \langle y_{k+1}, \dots, y_{n-1} \rangle = \langle \sigma_{k+1}, \dots, \sigma_{n-1} \rangle$$

are  $\lambda_n$ -convex. Since  $|S| = n - 1$ , they are the only convex subgroups by Lemma 5.4, and the maximal convex sequence is given by

$$B_n > B_{n-1}^* > \cdots > B_2^* > \{e\}. \tag{7.1}$$

The height of  $\lambda_n$  is  $n-2$ . The order  $\lambda_n$  is discrete since  $B_2^* \cong \mathbb{Z}$ . The  $i$ -th minimal set  $\mathcal{M}_i$  of the  $i$ -th internal pair  $(I_i, \mathcal{M}_i)$  is locally Cantor, since each term  $B_{n-k}^*$  in (7.1) is not a normal subgroup of the previous term  $B_{n-(k-1)}^*$ , because  $\sigma_{k+1} \in B_{n-k}^*$ ,  $\sigma_k \in B_{n-(k-1)}^*$ , and  $\sigma_k \sigma_{k+1} \sigma_k^{-1} \notin B_{n-k}^*$ .

We shall construct countably many isolated orders of  $B_3$  in Section 9.

For an isolated order  $\lambda \in LO(G)$ , we can define the *maximal Tararin subgroup*  $G_i$  in its maximal convex sequence

$$G > G_1 > \cdots > G_n > \{e\}. \tag{7.2}$$

For  $\lambda_n$ , the maximal Tararin subgroup is  $B_2^* \cong \mathbb{Z}$ , and its height is 0. We shall raise questions about the isolated orders of non Tararin groups.

QUESTION 7.1. Is there a non Tararin group with an isolated order whose maximal Tararin subgroup has height  $\geq 1$ ?

QUESTION 7.2. Is there a non Tararin group with an isolated and indiscrete order?

There is a sufficient condition for a group to be Tararin in terms of an isolated order on it.

PROPOSITION 7.3. *If the maximal convex sequence of an isolated order  $\lambda \in LO(G)$  is subnormal,<sup>5</sup> then  $G$  is a Tararin group.*

PROOF. The proof is an induction on the height of  $\lambda$ . For height 0, this is true by Proposition 5.6. Assume the height is  $\geq 1$  and consider the maximal convex sequence of  $\lambda$ :

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_n \triangleright G_{n+1} = \{e\}. \tag{7.3}$$

By the induction hypothesis,  $G_1$  is a Tararin group and the subsequence of (7.3) that begins with  $G_1$  is the unique rational series in Theorem 6.2. Since each  $G_i$   $2 \leq i \leq n$ , is a characteristic subgroup of  $G_1$  and since  $G_1$  is a normal subgroup of  $G$ ,  $G_i$  is a normal subgroup of  $G$ . By virtue of lemmas 5.1 and 5.2, the order induced from  $\lambda$  on  $G/G_1$  is isolated, and of height 0. Therefore  $G/G_1$  is a rational group, by virtue of Proposition 5.6. That is, the sequence (7.3) is a rational series.

Finally let us show that  $H = G/G_2$  is not bi-orderable. Let  $A = G_1/G_2$ ,  $B = G/G_1$  and consider the exact sequence

$$1 \rightarrow A \rightarrow H \rightarrow B \rightarrow 1.$$

As in the proof of Theorem 6.2, the conjugation defines a homomorphism  $\phi : B \rightarrow \text{Aut}(A) \subset \mathbb{Q}^\times$ . If  $\phi$  attains a negative value, then  $H$  is not bi-orderable, and we are done.

The order  $\lambda$  induces a left order  $\lambda_0$  of  $H$ , which is the lexicographical order given by the orders of  $B$  and  $A$ . To fix the idea, assume that these two orders are the natural

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<sup>5</sup>Each term is a normal subgroup of the previous term.

one given by the inclusions  $B \subset \mathbb{Q}$  and  $A \subset \mathbb{Q}$ . Notice also that  $\lambda_0$  is isolated, since  $\lambda$  is isolated.

If  $\phi$  is trivial, then  $H = A \times B$ . Consider the embeddings

$$A \times B \subset \mathbb{Q}^2 \subset \mathbb{R}^2.$$

Let  $\pi_n : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the projection along a one dimensional subspace of irrational slope  $k_n$ . The projection  $\pi_n$  maps  $A \times B$  injectively to  $\mathbb{R}$ , and this gives a left order  $\lambda_n$  of  $A \times B$ . Then  $\lambda_n \rightarrow \lambda_0$  as  $k_n \downarrow 0$ : the  $y$ -coordinate becomes more and more important as  $k_n \downarrow 0$ , and  $\lambda_0$  is the lexicographical order for which  $B$ -factor ( $y$ -coordinate) is of the primary importance. Thus  $\lambda_0$  is not isolated.

If  $\phi$  is nontrivial and positive valued, there is an embedding  $\phi$  of  $H$  into  $\text{Aff}_+(\mathbb{R})$  (Proof of Theorem 6.2). Points  $x_n \in \mathbb{R}$  at which  $\phi(H)$  acts freely yield left orders  $\lambda_n$  on  $H$ . As is observed by Rivas [13], we have  $\lambda_n \rightarrow \lambda_0$  as  $x_n \rightarrow \infty$  (the slope of affine transformations becomes more and more important).  $\square$

**COROLLARY 7.4.** *Let  $\lambda \in LO(G)$  be isolated of height 1. If the minimal set of the dynamical realization is discrete, then  $G$  is a Tararin group.*

**PROOF.** If the minimal set is discrete, then we get a surjective homomorphism  $\phi : G \rightarrow \mathbb{Z}$  and its kernel is a convex subgroup. By the previous proposition,  $G$  is a Tararin group.  $\square$

**EXAMPLE 7.5.** The above corollary does not hold if we remove the condition that  $\lambda$  is height 1. Let us construct an example of isolated order  $\lambda \in LO(G)$  of height 2 with discrete minimal set, where  $G$  is non Tararin. We start with the braid group  $B_3$ . The subsemigroup  $P$  generated by  $y_1 = \sigma_1\sigma_2$  and  $y_2 = \sigma_2^{-1}$  is the positive cone of the Dubrovina–Dubrovin order  $\lambda_3$ . The group  $B_3$  is described as

$$B_3 = \langle y_1, y_2 \mid y_2 y_1^2 y_2 = y_1 \rangle.$$

There is an automorphism  $\phi$  of  $B_3$  which satisfies  $\phi(y_1) = y_1^{-1}$  and  $\phi(y_2) = y_2^{-1}$ . Therefore if we define a group  $G$  by

$$G = \langle x, y_1, y_2 \mid y_2 y_1^2 y_2 = y_1, x y_1 x^{-1} = y_1^{-1}, x y_2 x^{-1} = y_2^{-1} \rangle,$$

then  $B_3$  is a subgroup of  $G$  [5]. Let  $\hat{P}$  be the subsemigroup of  $G$  generated by  $x$  and  $P$ . Then we have  $B_3 = P \sqcup P^{-1} \sqcup \{e\}$ ,  $xP = P^{-1}x$ , and  $G = \hat{P} \sqcup \hat{P}^{-1} \sqcup \{e\}$ . To show the last statement, denote by  $\langle x \rangle_{\pm}$  the subsemigroup generated by  $x^{\pm 1}$ . Then  $\langle x \rangle_+ P^{-1} = P \langle x \rangle_+ \subset \hat{P}$  and  $\langle x \rangle_- P = P^{-1} \langle x \rangle_- \subset \hat{P}^{-1}$ . Since  $B_3$  is a normal subgroup of  $G$ , we have

$$\begin{aligned} G &= \langle x \rangle B_3 = (\langle x \rangle_+ \sqcup \langle x \rangle_- \sqcup \{e\})(P \sqcup P^{-1} \sqcup \{e\}) \\ &= \langle x \rangle_+ P \sqcup \langle x \rangle_- P \sqcup P \sqcup \langle x \rangle_+ P^{-1} \sqcup \langle x \rangle_- P^{-1} \sqcup P^{-1} \sqcup \langle x \rangle_+ \sqcup \langle x \rangle_- \sqcup \{e\}, \end{aligned}$$

and each term except  $\{e\}$  is contained either in  $\hat{P}$  or in  $\hat{P}^{-1}$ .

The left order  $\lambda$  on  $G$  determined by  $\hat{P}$  has  $B_3$  as a  $\lambda$ -convex normal subgroup. In

fact,

$$B_3^{-1}x = (P \sqcup P^{-1} \sqcup \{e\})x = Px \sqcup P^{-1}x \sqcup \{x\} = Px \sqcup xP \sqcup \{x\} \subset \hat{P}$$

and likewise  $B_3^{-1}x^{-1} \subset \hat{P}^{-1}$ , which means  $x^{-1} <_\lambda B_3 <_\lambda x$ . Since  $G/B_3 \cong \mathbb{Z}$ , the minimal set associated to  $\lambda$  is discrete. The dynamics of  $\lambda$  is as depicted in Figure 1.

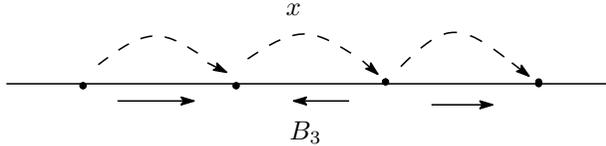


Figure 1. The dotted points form the minimal set  $\mathcal{M}$ . The element  $x$  moves these points one to the right. The intervals bounded by the points are invariant by  $B_3$ . The actions of  $B_3$  are opposite in neighbouring intervals, showing the stability of the action.

Navas [12] has defined the Conradian soul  $C_\lambda$  for any  $\lambda \in LO(G)$ . Let us recall it briefly. A left order  $\lambda \in LO(G)$  of a group  $G$  is called *Conradian* if we have  $g^{-1}hg^2 >_\lambda e$  whenever  $g >_\lambda e$  and  $h >_\lambda e$ . Thus a bi-invariant order is Conradian. Given an action of  $G$  on  $\mathbb{R}$ , a point  $x \in \mathbb{R}$ , is called *resilient* if there are an element  $h$  of the stabilizer of  $x$  and a point  $y \in Gx \setminus \{x\}$  such that  $h^n y \rightarrow x$  as  $n \rightarrow \infty$ . It is shown [12] that  $\lambda \in LO(G)$  is Conradian if and only if the dynamical realization of  $\lambda$  admits no resilient point.

For a general left order  $\lambda \in LO(G)$ , a subgroup  $H < G$  is called  $\lambda$ -Conradian if the restriction of  $\lambda$  to  $H$  is Conradian. The *Conradian soul*  $C_\lambda$  of  $\lambda$  is defined to be the maximal convex Conradian subgroup. In other words, it is the union of all the convex Conradian subgroups. The following proposition is a consequence of [12], Proposition 4.1, which states that if a group  $G$  is non Tararin, a Conradian order of  $G$  can never be isolated. Here we will give a proof based upon Proposition 7.3.

PROPOSITION 7.6. *If  $\lambda$  is isolated, the maximal Tararin subgroup of  $\lambda$  coincides with the Conradian soul of  $\lambda$ .*

PROOF. In the maximal convex sequence (7.2) of  $\lambda$ , let  $G_i$  be the maximal Tararin subgroup. It follows from Remark 6.3, that the dynamical realization of  $\lambda|_{G_i}$  is a pileup of translations, and cannot have a resilient point. Thus  $G_i$  is  $\lambda$ -Conradian. So it suffices to show that  $G_{i-1}$  is not  $\lambda$ -Conradian, that is, the dynamical realization of  $\lambda|_{G_{i-1}}$  admits a resilient point. It is no loss of generality to assume that  $i = 1$ . That is, we assume that  $G$  is not a Tararin group, while its maximal convex subgroup  $G_1$  is. By Proposition 7.3,  $G_1$  is not a normal subgroup of  $G$ . Then the minimal set  $\mathcal{M}$  is not discrete, and the action of  $G_1$  on  $\mathcal{M}$  is nontrivial. Choose  $g \in G_1$  which acts nontrivially on  $\mathcal{M}$ . Since  $G_1$  leaves invariant the gap  $I_1$  of  $\mathcal{M}$  containing the base point  $x_0$ , we have  $\text{Fix}(g) \cap \mathcal{M} \neq \emptyset$ . Then there are distinct points  $x, y \in \mathcal{M}$  such that  $g(x) = x$  and either  $g^n(y) \rightarrow x$  or  $g^{-n}(y) \rightarrow x$  as  $n \rightarrow \infty$ . Since the action of  $G$  on  $\mathcal{M}$  is minimal, the point  $y$  is accumulated by the orbit of  $x$ . This shows that the point  $x$  is resilient.  $\square$

### 8. Circular orders.

In this section, we provide preliminary facts about circular orders.

DEFINITION 8.1. For a countable group  $\overline{G}$ , a map  $c : \overline{G}^3 \rightarrow \{0, 1, -1\}$  is called a *left invariant circular order of  $\overline{G}$*  if it satisfies the following conditions.

- (1)  $c(g_1, g_2, g_3) = 0$  if and only if  $g_i = g_j$  for some  $i \neq j$ .
- (2) For any  $g_1, g_2, g_3, g_4 \in \overline{G}$ , we have

$$c(g_2, g_3, g_4) - c(g_1, g_3, g_4) + c(g_1, g_2, g_4) - c(g_1, g_2, g_3) = 0.$$

- (3) For any  $g_1, g_2, g_3, g_4 \in \overline{G}$ , we have

$$c(g_4g_1, g_4g_2, g_4g_3) = c(g_1, g_2, g_3).$$

DEFINITION 8.2. Given a finite set  $F$  of  $\overline{G}$ , a *configuration of  $F$  in  $S^1$*  is an equivalence class of injections  $\iota : F \rightarrow S^1$ , where two injections  $\iota$  and  $\iota'$  are said to be equivalent if there is an orientation preserving homeomorphism  $h$  of  $S^1$  such that  $\iota' = h\iota$ .

Given a left invariant circular order  $c$  of  $\overline{G}$ , the configuration of the set  $\{g_1, g_2, g_3\}$  of three points is determined by the rule that  $g_1, g_2$  and  $g_3$  are positioned anticlockwise if  $c(g_1, g_2, g_3) = 1$ , and clockwise if  $c(g_1, g_2, g_3) = -1$ . By condition (2) of Definition 8.1, this is well defined. But (2) says more. One can show the following proposition by an easy induction on the cardinality of  $F$ .

PROPOSITION 8.3. *Given a left invariant circular order of  $\overline{G}$ , the configuration of any finite set  $F$  in  $S^1$  is determined.* □

Denote by  $CO(\overline{G})$  the set of all the left invariant circular orders. It is equipped with a totally disconnected compact metrizable topology, just as  $LO(\overline{G})$ . An *isolated left invariant circular order* is defined using this topology. If  $c \in CO(\overline{G})$  is isolated, then there is a finite set  $\overline{S}$  of  $\overline{G}$ , called a *determining set*, such that any left invariant circular order which gives the same configuration of  $\overline{S}$  as  $c$  is  $c$ .

Given  $c \in CO(\overline{G})$ , we define a dynamical realization  $\rho_c : \overline{G} \rightarrow \text{Homeo}_+(S^1)$  based at  $y_0 \in S^1$  as follows. Fix an enumeration of  $\overline{G}$ :  $\overline{G} = \{g_i \mid i \in \mathbb{N}\}$  such that  $g_1 = e$ . Define an embedding  $\iota : \overline{G} \rightarrow S^1$  inductively as follows. First, set  $\iota(g_1) = y_0$  and  $\iota(g_2) = y_0 + 1/2$ . If  $\iota$  is defined on  $\{g_1, \dots, g_n\}$ , then there is a connected component of  $S^1 \setminus \{\iota(g_1), \dots, \iota(g_n)\}$  where the point  $g_{n+1}$  should be embedded, by virtue of Proposition 8.3. Define  $\iota(g_{n+1})$  to be the midpoint of that interval. Using the injection  $\iota$ , we can define the action of  $\overline{G}$  on  $S^1$  just as in the case of left orders. The action is called the *dynamical realization of  $c$  based at  $y_0$*  and denoted by  $\rho_c$ . We shall raise fundamental properties of  $\rho_c$ . The proof is completely parallel to the case of left orders.

LEMMA 8.4. *The dynamical realization  $\rho_c$  is tight at the base point  $y_0$ , i.e. it is free at  $y_0$  and if  $I$  is a connected component of  $S^1 \setminus Cl(\rho_c(\overline{G})y_0)$ , then  $\partial I \subset \rho_c(\overline{G})y_0$ .*

LEMMA 8.5. *Two dynamical realizations obtained via different enumerations of  $\overline{G}$  are mutually conjugate by an orientation and base point preserving homeomorphism of*

$S^1$ .

Let  $\mathcal{M}$  be a minimal set of the dynamical realization  $\rho_c$  of an isolated circular order  $c$ . It is shown by Mann and Rivas [10] that (unlike left orders)  $\mathcal{M}$  is always a proper subset of  $S^1$ . Summarizing with other properties, we get:

LEMMA 8.6. *If  $\overline{G}$  is not finite cyclic, the minimal set  $\mathcal{M}$  of the dynamical realization  $\rho_c$  of any isolated circular order  $c \in CO(\overline{G})$  is unique. It is either a finite set or a Cantor set.*

PROOF. If  $\mathcal{M}$  is a Cantor set, then a standard argument shows that it is the unique minimal set. So suppose  $\mathcal{M}$  is a finite set of cardinality, say  $n$ . Then by looking at the action of  $\overline{G}$  on  $\mathcal{M}$ , one can define a surjective homomorphism  $\phi : \overline{G} \rightarrow \mathbb{Z}/n\mathbb{Z}$ . Since  $\overline{G}$  is not finite cyclic, the base point  $y_0$  of the dynamical realization  $\rho_c$  must be contained in a gap, say  $I_1 = (a, b)$ <sup>6</sup> of  $\mathcal{M}$ . The stabilizer  $\overline{G}_1$  of  $I_1$  is the kernel of  $\phi$ , and  $\rho_c(\overline{G})y_0 \cap I_1 = \rho_c(\overline{G}_1)y_0$ . Now if we set  $x = \inf(\rho_c(\overline{G}_1)y_0)$ , then  $x = a$ : for, otherwise,  $x \in \rho_c(\overline{G})y_0$  by the tightness, contradicting that  $x$  is the infimum. Likewise we have  $\sup(\rho_c(\overline{G}_1)y_0) = b$ . This shows that all the orbits other than  $\mathcal{M}$  is infinite and contains  $\mathcal{M}$  in its closure, that is,  $\mathcal{M}$  is the unique minimal set.  $\square$

Taking into account the Cantor minimal set case, the similar argument shows the following.

LEMMA 8.7. *If  $\overline{G}$  is not finite cyclic and  $c$  is isolated, then the base point  $y_0$  of the dynamical realization is contained in a gap  $I$  of the minimal set  $\mathcal{M}$ , the stabilizer  $\overline{G}_I$  of  $I$  is nontrivial, and there is no gap of  $\mathcal{M}$  other than the orbit of  $I$ .*

Here is an analogue of Corollary 2.4 for circular orders.

LEMMA 8.8. *Let  $\overline{\mathcal{H}}$  be the set of the orientation and base point  $y_0$  preserving topological conjugacy classes of the homomorphisms  $\overline{G} \rightarrow \text{Homeo}_+(S^1)$  which are tight at  $y_0$ . Then the dynamical realization at  $y_0$  induces a bijection of  $CO(\overline{G})$  onto  $\overline{\mathcal{H}}$ .*

DEFINITION 8.9. Let  $c$  be a circular order of  $\overline{G}$ , isolated or not, and  $H$  a nontrivial subgroup of  $\overline{G}$ .  $H$  is said to be  $c$ -convex if  $\rho_c(H)$  acts with global fixed points, and  $\rho_c(\overline{G})y_0 \cap I_H = \rho_c(H)y_0$ , where  $I_H$  denotes the connected component of the complement of the global fixed point set of  $\rho_c(H)$  containing  $y_0$ . The configuration of  $\rho_c(H)y_0$  in  $I_H$  defines a left order  $\lambda$  on  $H$ , which we call the left order on  $H$  induced from  $c$ . The trivial subgroup is said to be  $c$ -convex.

As shown in [10, Lemma 3.15], there is a unique maximal  $c$ -convex subgroup for any  $c \in CO(\overline{G})$ , which we call the *linear part* of  $c$ . By virtue of Lemmas 8.6 and 8.7, we get the following lemma.

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<sup>6</sup>Given two points  $a, b \in S^1$ , we define  $(a, b) = \{t \in S^1 \mid a \prec t \prec b\}$ , where  $\prec$  is the anticlockwise circular order of  $S^1$ .

LEMMA 8.10. *Assume  $\overline{G}$  is not finite cyclic. Let  $\mathcal{M}$  be the minimal set of the dynamical realization of an isolated circular order  $c \in CO(\overline{G})$ , and  $I$  the gap of  $\mathcal{M}$  which contains the base point  $y_0$ . Then the linear part of  $c$  coincides with the stabilizer of  $I$ .*

### 9. Isolated left orders on $B_3$ .

In this section, using a method of [10], we construct countably many isolated left orders on the braid group  $B_3$ , which are not the automorphic images of the others. The group  $B_3$  has the following presentations.

$$\begin{aligned} B_3 &= \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle \\ &= \langle y_1, y_2 \mid y_2y_1^2y_2 = y_1 \rangle \\ &= \langle a, b \mid a^2 = b^3 \rangle, \end{aligned}$$

where the generators are related by

$$y_1 = \sigma_1\sigma_2, \quad y_2 = \sigma_2^{-1}, \quad a = y_2y_1^2, \quad b = y_1.$$

The Dubrovina–Dubrovin order  $\lambda_3$  is the unique left order on  $B_3$  which satisfies  $y_1 >_{\lambda_3} e$  and  $y_2 >_{\lambda_3} e$ , equivalently  $e <_{\lambda_3} a <_{\lambda_3} b$ . To show the equivalence, assume  $y_1 >_{\lambda_3} e$  and  $y_2 >_{\lambda_3} e$ . Then

$$a = y_2y_1^2 >_{\lambda_3} e, \quad \text{and} \quad a^{-1}b = y_1^{-2}y_2^{-1}y_1 = y_1^{-2}y_2^{-1}(y_2y_1^2y_2) = y_2 >_{\lambda_3} e.$$

The converse is shown similarly.

Henceforth in this section we denote by  $G$  the braid group  $B_3$  and by  $\overline{G}$  its quotient by the center. Namely, we put

$$G = \langle a, b, t \mid a^2 = b^3 = t \rangle, \quad \overline{G} = \langle \alpha, \beta \mid \alpha^2 = \beta^3 = e \rangle,$$

and  $q : G \rightarrow \overline{G}$  to be the surjective homomorphism satisfying  $q(a) = \alpha$  and  $q(b) = \beta$ .

The first half of this section is devoted to show that there is a homeomorphism between  $LO(G)$  and  $CO(\overline{G})$ . Thus the construction of isolated orders in  $LO(G)$  reduces to the construction of isolated orders in  $CO(\overline{G})$ , which is easier, thanks to well developed theory of Fuchsian groups. The last half is devoted to this construction.

First of all, notice that  $t$  is  $\lambda$ -cofinal for any  $\lambda \in LO(G)$ . In fact, assume, to fix the idea, that  $t >_{\lambda} e$ . Then  $a >_{\lambda} e$  and  $b >_{\lambda} e$ . Since  $t$  is in the center of  $G$ , any element  $g \in G$  can be written as  $g = t^N a^{i_1} b^{j_1} \dots a^{i_r} b^{j_r}$  for an integer  $N$  and negative integers  $i_\nu$  and  $j_\nu$ , showing that  $g <_{\lambda} t^N$ . Likewise, there is  $M \in \mathbb{Z}$  such that  $t^M <_{\lambda} g$ .

Let  $\tau$  be the translation of  $\mathbb{R}$  by 1, and  $p : \mathbb{R} \rightarrow S^1 = \mathbb{R}/\langle \tau \rangle$  the canonical projection. Denote by  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$  the group of all the homeomorphisms of  $\mathbb{R}$  which commute with  $\tau$ . It is the universal covering group of  $\text{Homeo}_+(S^1)$ . Denote by  $\pi : \text{Homeo}_{\mathbb{Z}}(\mathbb{R}) \rightarrow \text{Homeo}_+(S^1)$  the covering map. Let

$$LO^+(G) = \{ \lambda \in LO(G) \mid t >_{\lambda} e \} \quad \text{and} \quad LO^-(G) = \{ \lambda \in LO(G) \mid t <_{\lambda} e \}.$$

Likewise, let

$$CO^\pm(\overline{G}) = \{c \in CO(\overline{G}) \mid c(e, \beta, \beta^2) = \pm 1\}.$$

In order to show that  $LO^\pm(G)$  is homeomorphic to  $CO^\pm(\overline{G})$ , it is easier and more natural to consider the sets of the conjugacy classes of certain homeomorphisms: one is  $\mathcal{H}^\pm$  with a bijection to  $LO^\pm(G)$  and the other is  $\overline{\mathcal{H}}^\pm$  with a bijection to  $CO^\pm(\overline{G})$ . Next we shall construct a natural bijection from  $\mathcal{H}^\pm$  to  $\overline{\mathcal{H}}^\pm$ . Finally, we shall show that the induced bijection from  $LO^\pm(G)$  to  $CO^\pm(\overline{G})$  is a homeomorphism (Theorem 9.1).

Let  $H^\pm$  be the set of the homomorphisms  $\rho : G \rightarrow \text{Homeo}_{\mathbb{Z}}(\mathbb{R})$  which are tight at a prescribed base point  $x_0$  and satisfy  $\rho(t) = \tau^{\pm 1}$ , and let  $\mathcal{H}^\pm$  be the set of the orientation and base point preserving topological conjugacy classes of the elements of  $H^\pm$ . Then we have  $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$  for  $\mathcal{H}$  in Corollary 2.4. In fact, for any class  $[\rho]$  of  $\mathcal{H}$ ,  $\rho(t)$  is fixed point free since  $t$  is cofinal for any left orders. Thus  $[\rho]$  has a representative  $\rho$  such that  $\rho(t) = \tau^{\pm 1}$ .

A dynamical realization at  $x_0$  of any element of  $LO^\pm(G)$  represents a unique element of  $\mathcal{H}^\pm$  (Corollary 2.4). That is, we get a map  $\phi : LO^\pm(G) \rightarrow \mathcal{H}^\pm$ . Clearly  $\phi$  is injective. To show the surjectivity, let  $\rho \in H^\pm$ , let  $\lambda \in LO^\pm(G)$  be the left order of  $G$  defined by the natural order of the orbit  $\rho(G)x_0$ , and let  $\rho_\lambda \in H^\pm$  a representative of the conjugacy class of the dynamical realization of  $\lambda$ . There is an order and the base point preserving equivariant bijection from  $\rho_\lambda(G)x_0$  to  $\rho(G)x_0$ , which can be extended to a homeomorphism between the closures, thanks to the tightness. Finally this homeomorphism can be extended to an equivariant homeomorphism of  $\mathbb{R}$ . We have completed the proof that  $\phi$  is a bijection.

Denote by  $\overline{H}^\pm$  the set of the homomorphisms  $\overline{\rho} : \overline{G} \rightarrow \text{Homeo}_+(S^1)$  which are tight at  $y_0 = p(x_0)$  and satisfy  $\text{rot}(\overline{\rho}(\beta)) = \pm 1/3$ , and let  $\overline{\mathcal{H}}^\pm$  the set of the orientation and the base point preserving topological conjugacy classes of the elements of  $\overline{H}^\pm$ . Then  $\overline{\mathcal{H}}^\pm$  is identified with  $CO^\pm(\overline{G})$  (Lemma 8.8).

Define a map  $q_* : H^\pm \rightarrow \overline{H}^\pm$  by  $(q_*\rho)(\overline{g}) = \pi(\rho(g))$ , where  $\rho \in H^\pm$ ,  $\overline{g} \in \overline{G}$  and  $g \in G$  is any element such that  $q(g) = \overline{g}$ . There is a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \text{Homeo}_{\mathbb{Z}}(\mathbb{R}) \\ \downarrow q & & \downarrow \pi \\ \overline{G} & \xrightarrow{q_*\rho} & \text{Homeo}_+(S^1). \end{array}$$

Define a map  $\pi^* : \overline{H}^\pm \rightarrow H^\pm$  for  $\overline{\rho} \in \overline{H}^\pm$  by

- $(\pi^*\overline{\rho})(a)$  is the lift of  $\overline{\rho}(\alpha)$  to  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$  whose square is  $\tau^{\pm 1}$ , and
- $(\pi^*\overline{\rho})(b)$  is the lift of  $\overline{\rho}(\beta)$  to  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$  whose cube is  $\tau^{\pm 1}$ .

Also we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi^*\overline{\rho}} & \text{Homeo}_{\mathbb{Z}}(\mathbb{R}) \\ \downarrow q & & \downarrow \pi \\ \overline{G} & \xrightarrow{\overline{\rho}} & \text{Homeo}_+(S^1). \end{array}$$

It is clear that  $q_*$  and  $\pi^*$  map the conjugacy classes to the conjugacy classes. That is, we have maps (denoted by the same letters)  $q_* : \mathcal{H}^\pm \rightarrow \overline{\mathcal{H}}^\pm$  and  $\pi^* : \overline{\mathcal{H}}^\pm \rightarrow \mathcal{H}^\pm$ . We have

$\pi^*q_* = q_*\pi^* = id$ . Thus we obtain a bijection  $q_* : LO(G) \rightarrow CO(\overline{G})$  and its inverse  $\pi^*$ . We shall show the following theorem (Theorem 4 in the introduction).

**THEOREM 9.1.** *The map  $q_* : LO(G) \rightarrow CO(\overline{G})$  is a homeomorphism.*

**PROOF.** Let us show that  $q_*$  is continuous. For any  $\lambda \in LO(G)$ , let  $c = q_*(\lambda) \in CO(\overline{G})$ . Choose arbitrary elements  $\overline{g}_1, \dots, \overline{g}_n$  of  $\overline{G}$  and consider their configuration in  $S^1$  with respect to  $c$ . This is the same as the configuration of  $\overline{\rho}(\overline{g}_1)y_0, \dots, \overline{\rho}(\overline{g}_n)y_0$  in  $S^1$ , where  $\overline{\rho} \in \overline{\mathcal{H}}$  is a dynamical realization of  $c$ . Let  $\rho = \pi^*(\overline{\rho}) \in \mathcal{H}$ , a dynamical realization of  $\lambda$ . Choose  $g_i \in G$  such that  $q(g_i) = \overline{g}_i$  and  $e \leq_\lambda g_i <_\lambda t$  ( $1 \leq i \leq n$ ). The configuration of  $\rho(g_1)x_0, \dots, \rho(g_n)x_0$  in  $\mathbb{R}$  coincides with the configuration of  $g_1, \dots, g_n$  with respect to  $\lambda$ . Choose any  $\lambda' \in LO(G)$  whose configuration of  $e, g_1, \dots, g_n, t$  is the same as  $\lambda$ . Then the configuration of  $\overline{g}_1, \dots, \overline{g}_n$  of  $q_*(\lambda')$  is the same as  $c$ , showing the continuity of  $q_*$ . Thus the compact metrizable sets  $LO(G)$  and  $CO(\overline{G})$  are homeomorphic by  $q_*$ .  $\square$

By virtue of the previous theorem, Theorem 3 in the introduction reduces to the following theorem. This is because any automorphism of  $G$ , preserving the center, induces an automorphism of  $\overline{G}$ .

**THEOREM 9.2.** *There are isolated circular orders  $c^{(k)} \in CO(\overline{G})$ , ( $k > 0, k \equiv \pm 1 \pmod 6$ ) which are not the automorphic images of the others.*

The rest of this section is devoted to the proof of this theorem. Our argument is based upon Fuchsian representations. The Lie group  $PSL(2, \mathbb{R})$  is the group of the orientation preserving isometries of the Poincaré upper half plane  $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ , acting by linear fractional transformations. We consider  $\mathbb{H}$  to be an open half disk in the Riemann sphere  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . Then  $PSL(2, \mathbb{R})$  acts on  $\hat{\mathbb{C}}$  by linear fractional transformations and leaves invariant the oriented boundary  $\partial\mathbb{H}$ , which we identify with  $S^1$ . This gives an inclusion  $PSL(2, \mathbb{R}) \subset \text{Homeo}_+(S^1)$ . Denote by  $\overline{\mathbb{H}}$  the closure of  $\mathbb{H}$  in  $\hat{\mathbb{C}}$ .

Let  $\Gamma$  be a nonamenable countable group. A representation  $\rho : \Gamma \rightarrow PSL(2, \mathbb{R})$  is called *Fuchsian* if it is faithful and the image  $\rho(\Gamma)$  is discrete in  $PSL(2, \mathbb{R})$ . For a Fuchsian representation  $\rho$ , its *limit set*  $L_\rho \subset \partial\mathbb{H} \approx S^1$  is, by definition, the set of the accumulation points of an orbit  $\rho(\Gamma)z_0$  ( $z_0 \in \mathbb{H}$ ). It does not depend on the choice of  $z_0$ . It is also characterized as the unique minimal set of the representation  $\Gamma \xrightarrow{\rho} PSL(2, \mathbb{R}) \subset \text{Homeo}_+(S^1)$ . It is either the whole  $S^1$  or a Cantor set (by the nonamenability of  $\Gamma$ ). In the former case,  $\rho$  is called *of the first kind*, and in the latter *of the second kind*.

Let us return to our group  $\overline{G}$ . There is an isomorphism  $\iota : \overline{G} \cong PSL(2, \mathbb{Z})$  which satisfies

$$\iota(\alpha) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \iota(\beta) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let us define a homomorphism  $\rho_M : \overline{G} \rightarrow PSL(2, \mathbb{R})$ , called the *modular representation*, as the composite

$$\overline{G} \xrightarrow{\iota} PSL(2, \mathbb{Z}) \subset PSL(2, \mathbb{R}).$$

For the dynamics of the modular representation  $\rho_M$ , see Figure 2. The open disk bounded by the circle is the image of  $\mathbb{H}$  by the stereographic projection from  $-i$ . The element

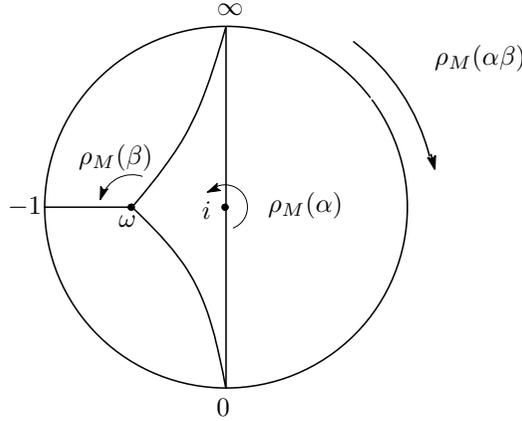


Figure 2.

$\rho_M(\alpha)$  is the  $1/2$ -rotation around  $i$ , and  $\rho_M(\beta)$  the  $1/3$ -rotation around  $\omega = (-1 + \sqrt{-3})/2$ . The element  $\rho_M(\alpha\beta)$  is a parabolic transformation which fixes the point  $0$  and moves points on  $S^1 \setminus \{0\}$  clockwise, as is depicted in Figure 2. The Fuchsian group  $\rho_M(\overline{G})$  is of the first kind.

Let us define another Fuchsian representation  $\rho : \overline{G} \rightarrow PSL(2, \mathbb{R})$ , a deformation of  $\rho_M$ . Choose a point  $\omega'$  on the geodesic which passes through  $i$  and  $\omega$ , but slightly farther than  $\omega$  from  $i$ :  $d(\omega', i) > d(\omega, i)$ . See Figure 3. We set  $\rho(\alpha)$  to be the same as  $\rho_M(\alpha)$ , the  $1/2$ -rotation around  $i$ , and  $\rho(\beta)$  the  $1/3$ -rotation around  $\omega'$ . We put the base point  $y_0 = 0$ .

Consider the 4-gon  $P$  depicted in Figure 3, a closed subset of  $\overline{\mathbb{H}}$ . It is routine to show, using hyperbolic metric of  $\mathbb{H}$ , that the translates of  $P \cap \mathbb{H}$  tessellate  $\mathbb{H}$ , that is,  $\bigcup_{\gamma \in \overline{G}} \rho(\gamma)(P \cap \mathbb{H}) = \mathbb{H}$  and  $\rho(\gamma)(\text{Int}P) \cap \text{Int}P = \emptyset$  if  $\gamma \neq e$ . This shows that  $\rho$  is a Fuchsian representation. Moreover it is of the second kind since the tessellation implies that  $L_\rho \cap P = \emptyset$ . But we need a bit more: we shall show that the translates of  $P$  tessellate the whole  $\overline{\mathbb{H}} \setminus L_\rho$ , that is,  $\bigcup_{\gamma \in \overline{G}} \rho(\gamma)P = \overline{\mathbb{H}} \setminus L_\rho$ .

Endow the half disk  $\overline{\mathbb{H}}$  with the restriction of the spherical metric of the Riemann sphere  $\mathbb{C} \cup \{\infty\}$ . Then for any  $g \in PSL(2, \mathbb{R})$  and any  $z \in \overline{\mathbb{H}}$ , the absolute value of the derivative  $|g'(z)|$  is well defined. Given  $g \in PSL(2, \mathbb{R}) \setminus PSO(2)$ , define  $I(g) = \{z \in \overline{\mathbb{H}} \mid |g'(z)| \geq 1\}$ . It is a subset of  $\overline{\mathbb{H}}$  delimited by a circle perpendicular to  $\partial\mathbb{H}$ , whose radius is denoted by  $\text{rad}(I(g))$ . It satisfies the following properties: (1)  $gI(g) = \overline{\mathbb{H}} \setminus I(g^{-1})$ . (2)  $g$  is hyperbolic if and only if  $I(g) \cap I(g^{-1}) = \emptyset$ . (3)  $g_n \rightarrow \infty$  in  $PSL(2, \mathbb{R})$  if and only if  $\text{rad}(I(g_n)) \rightarrow 0$ .

Returning to our representation  $\rho : \overline{G} \rightarrow PSL(2, \mathbb{R})$ , there is a purely hyperbolic subgroup of  $\rho(\overline{G})$  of index 6, as is shown later. On the other hand, the limit set does not

<sup>7</sup>To show “only if part”, let  $\ell$  be the axis of a hyperbolic element  $g$ . Let  $m$  be the midpoint of  $\ell$  and let  $p, q \in \ell$  be points such that  $d(p, m) = d(q, m)$  and  $g(p) = q$ . Then  $I(g)$  (resp.  $I(g^{-1})$ ) is bounded by a circle crossing  $\ell$  perpendicularly at  $p$  (resp.  $q$ ). Thus  $I(g) \cap I(g^{-1}) = \emptyset$ .

change if we pass to a finite index subgroup. Thus we get:

*For a point  $x \in \partial\mathbb{H}$ ,  $x \in L_\rho$  if and only if  $\rho(\gamma_n)z_0 \rightarrow x$  for some hyperbolic elements  $\rho(\gamma_n)$ , where  $z_0$  is any prescribed point in  $\mathbb{H}$ .*

Now if  $\rho(\gamma)$  is hyperbolic, then  $I(\rho(\gamma))$ , as well as  $I(\rho(\gamma)^{-1})$ , contains a fixed point of  $\rho(\gamma)$ . In particular,  $I(\rho(\gamma)) \cap L_\rho \neq \emptyset$  and  $I(\rho(\gamma)^{-1}) \cap L_\rho \neq \emptyset$ . Moreover if  $\rho(\gamma)$  is sufficiently far away from the identity, then  $I(\rho(\gamma)) \cap P = \emptyset$  because  $\text{rad}(I(\rho(\gamma)))$  is small and  $P \cap L_\rho = \emptyset$ . Therefore the translate  $\rho(\gamma)P$  is contained in  $I(\rho(\gamma)^{-1})$  and has small diameter. This is true not only for elements of a subgroup of index 6, but also for any element of  $\rho(\overline{G})$ : if  $\gamma_n \rightarrow \infty$  in the word norm, then  $\text{diam}(\rho(\gamma_n)P) \rightarrow 0$ . This yields yet another characterization of the limit set.

*For  $x \in \partial\mathbb{H}$ ,  $x \in L_\rho$  if and only if any neighbourhood of  $x$  in  $\overline{\mathbb{H}}$  intersects infinitely many translates of  $P$ .*

Finally, this, together with the fact that the translates of  $P \cap \mathbb{H}$  tessellate  $\mathbb{H}$ , implies that the translates of  $P$  tessellate  $\overline{\mathbb{H}} \setminus L_\rho$ . This is what we wanted to show.

Henceforth we denote the composite

$$\overline{G} \xrightarrow{\rho} PSL(2, \mathbb{R}) \subset \text{Homeo}_+(S^1)$$

also by  $\rho$ , and the unique minimal set  $L_\rho$  of the homomorphism  $\rho$  by  $\mathcal{M}$ . The translates of the interval  $P|_{S^1}$  tessellate  $S^1 \setminus \mathcal{M}$ , where  $P|_{S^1}$  is the component of  $P \cap S^1$  which is homeomorphic to a closed interval. Let  $y_0$  be a point in  $S^1$  which corresponds to  $0 \in \partial\mathbb{H}$  and is depicted by  $e$  in Figure 3. Since  $y_0 \in P$ , it lies in a gap of  $\mathcal{M}$ , say  $I_1 = (\sigma_-, \sigma_+)$ . The element  $\alpha\beta \in \overline{G}$  sends  $y_0$  to a point slightly right to itself ( $e$  to  $\alpha\beta$  in Figure 3). As can be seen by Figure 3, the iterates of  $\rho(\alpha)P|_{S^1}$  by  $\rho(\alpha\beta)^n$ ,  $n \in \mathbb{Z}$ , tessellate an open interval of  $S^1 \setminus \mathcal{M}$ . Therefore  $\rho(\alpha\beta)^n y_0$  lie in  $I_1$ . We also have  $\lim_{n \rightarrow \pm\infty} \rho(\alpha\beta)^n y_0 = \sigma_\pm$ , since the limits, being fixed points of  $\rho(\alpha\beta)$ , must be contained in  $\mathcal{M} = L_\rho$ . Thus  $\alpha\beta$  generates the stabilizer of  $I_1$ . Since  $\rho$  is a slight perturbation of  $\rho_M$ , we can assume that  $\sigma_\pm$  are very near to  $e$  (in Figure 3).

Since the translates of  $P|_{S^1}$  tessellate  $S^1 \setminus \mathcal{M}$ , there are no gaps of  $\mathcal{M}$  other than the translates of  $I_1$ . In particular, any gap of  $Cl(\rho(\overline{G})y_0)$  is a gap of  $\rho(\overline{G})y_0$ . This, together with the freeness of the action  $\rho$  at  $y_0$ , shows that  $\rho$  acts tightly at  $y_0$ . Thus  $\rho$  is topologically conjugate to a dynamical realization of a circular order  $c \in CO(\overline{G})$ . The linear part of  $c$  is the subgroup  $\langle \alpha\beta \rangle$  by Lemma 8.10.

Let us show first of all that  $c$  is isolated. In [10, Proposition 3.3], the authors showed that the dynamical realization is continuous. More precisely, they showed the following.

**PROPOSITION 9.3.** *Given any neighbourhood  $U$  of  $\rho$  in  $\text{Hom}(\overline{G}, \text{Homeo}_+(S^1))$ , there is a neighbourhood  $V$  of  $c$  in  $CO(\overline{G})$  such that any element in  $V$  has a conjugate of its dynamical realization contained in  $U$ .*

In Figure 3, six translates of  $P$  are depicted. Let  $Q$  be their union:

$$Q = P \cup \beta P \cup \beta^2 P \cup \alpha P \cup \alpha\beta P \cup \alpha\beta^2 P.$$

The convex set  $Q \cap \mathbb{H}$  has four sides. Let  $\gamma_1 = \beta^2\alpha\beta\alpha$  and  $\gamma_2 = \alpha\beta\alpha\beta^2$ . Then  $\rho(\gamma_i)$  maps

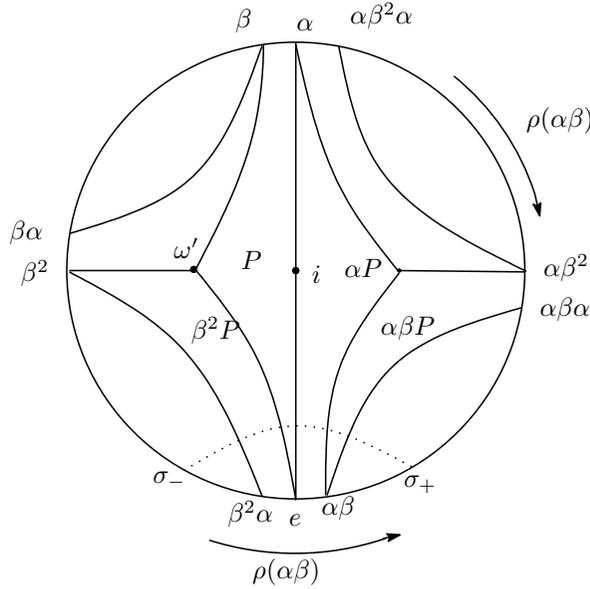


Figure 3.  $e$  indicates the base point  $y_0 = 0$  and  $g$  indicates the point  $\rho(g)y_0$ .  $P$  is a fundamental domain of  $\rho(\overline{G})$ .

a side of  $Q \cap \mathbb{H}$  onto its opposite side, as indicated in Figure 4. By the Klein criterion (also known as the ping-pong lemma),  $\gamma_1$  and  $\gamma_2$  are free generators of a free subgroup  $H$  of  $\overline{G}$ . Moreover, for any nontrivial element  $\gamma$  of  $H$ ,  $\rho(\gamma)$  is a hyperbolic transformation. The subgroup  $H$  coincides with the commutator subgroup  $[\overline{G}, \overline{G}]$ . To show this, notice that  $\gamma_i$  are commutators and thus  $H \subset [\overline{G}, \overline{G}]$ . On the other hand,  $H$  is an index 6 subgroup of  $\overline{G}$ , since the fundamental domain  $Q$  of  $\rho(H)$  consists of 6 iterates of  $P$ , and  $[\overline{G}, \overline{G}]$  is also of index 6, since the abelianization of  $\overline{G}$  is  $\mathbb{Z}/6\mathbb{Z}$ .

Below we indicate the point  $\rho(g)y_0$  simply by  $g$ , as we already did in the figures. Let us define four intervals of  $S^1$ :

$$K_1^- = [\alpha\beta^2, \alpha\beta^2\alpha], \quad K_1^+ = [\beta^2, \beta^2\alpha], \quad K_2^- = [\beta, \beta\alpha], \quad K_2^+ = [\alpha\beta, \alpha\beta\alpha].$$

Then  $\rho(\gamma_1)(S^1 \setminus K_1^-) = \text{Int}(K_1^+)$  and  $\rho(\gamma_2)(S^1 \setminus K_2^-) = \text{Int}(K_2^+)$ . Define open intervals  $J_1^-, J_1^+, J_2^-$  and  $J_2^+$ , slightly bigger than  $K_1^-, K_1^+, K_2^-$  and  $K_2^+$ . See Figure 4. ( $J_i^\pm$  and  $K_i^\pm$  are actually intervals of  $S^1$ .) Recall the interval  $I_1$ , the gap of  $\mathcal{M}$  containing  $y_0$ , and let  $I_2 = \rho(\alpha\beta^2)I_1$ ,  $I_3 = \rho(\alpha)I_1$  and  $I_4 = \rho(\beta\alpha)I_1$  (Figure 4). They are gaps of the minimal set  $\mathcal{M}$  and the orbit  $\rho(\overline{G})y_0$  is discrete in each of them. Since the stabilizer of  $I_1$  is generated by  $\alpha\beta$ , the three points in  $I_1$ ,  $\beta^2\alpha$ ,  $e$  and  $\alpha\beta$  are consecutive points of  $\rho(\overline{G})y_0$  contained in  $I_1$ . Their images by  $\rho(\alpha)$ ,  $\alpha\beta^2\alpha$ ,  $\alpha$  and  $\beta$  are consecutive points of  $\rho(\overline{G})y_0$  contained in  $I_3$ . Likewise  $\alpha\beta\alpha$  and  $\alpha\beta^2$  are consecutive in  $I_2$ , and  $\beta\alpha$  and  $\beta^2$  are consecutive in  $I_4$ . One chooses  $J_2^+$  so that the point  $\alpha\beta$  (resp.  $\alpha\beta\alpha$ ) is the leftmost (resp. rightmost) point of  $\rho(\overline{G})y_0 \cap J_2^+$ . More generally, the intervals  $J_i^\pm$  are so chosen that points of  $\partial K_i^\pm$  are extremal in  $\rho(\overline{G})y_0 \cap J_i^\pm$ . These points are called the *guardians* of the interval with respect to  $\rho$ . Notice that there are just two points of  $\rho(\overline{G})y_0$  outside

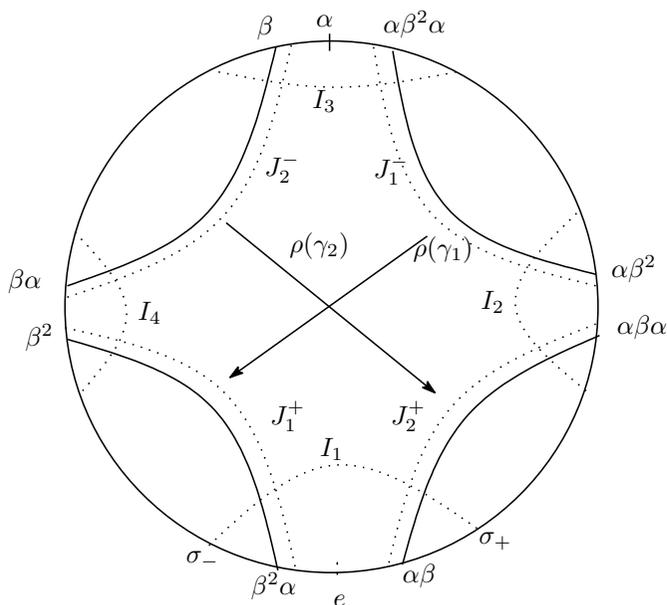


Figure 4.  $\rho(\gamma_1) = \rho(\beta^2\alpha\beta\alpha)$  maps  $\alpha\beta^2\alpha$  to  $\beta^2$  and  $\alpha\beta^2$  to  $\beta^2\alpha$ .  
 $\rho(\gamma_2) = \rho(\alpha\beta\alpha\beta^2)$  maps  $\beta$  to  $\alpha\beta\alpha$  and  $\beta\alpha$  to  $\alpha\beta$ .

of  $\bigcup_{i,\pm} J_i^\pm$ , namely  $y_0$  and  $\rho(\alpha)y_0$  (denoted by  $e$  and  $\alpha$  in Figure 4).

Define a neighbourhood  $U$  of  $\rho$  in  $\text{Hom}(\overline{G}, \text{Homeo}_+(S^1))$  such that each element  $\rho' \in U$  satisfies the following conditions.

- (1)  $\rho'(\gamma_i)$  maps the closed set  $S^1 \setminus J_i^-$  into the open set  $J_i^+$  ( $i = 1, 2$ ).
- (2) The configuration of  $\rho'(g)y_0$  in  $S^1$  for ten elements

$$g = e, \alpha\beta, \alpha\beta\alpha, \alpha\beta^2, \alpha\beta^2\alpha, \alpha, \beta, \beta\alpha, \beta^2, \beta^2\alpha \tag{9.1}$$

is the same as for  $\rho$  (Figure 4), as well as their configuration with respect to the four intervals  $J_i^\pm$ .

Take a neighbourhood  $V$  of  $c$  as in Proposition 9.3, and for any  $c' \in V$ , let  $\rho'$  be a conjugate of a dynamical realization of  $c'$  which is contained in  $U$ . Then by the ping-pong argument, the circular order of the orbit  $\rho'(\overline{G})y_0$  is uniquely determined. Let us show this a bit in detail. Call ten points  $\rho'(g)y_0$  ( $g$  as in (9.1)) *of the first generation*. The images of points of the first generation by  $\gamma_i^{\pm 1}$  which are not themselves of first generation are called *of the second generation*. Then the configuration of the points of first and second generations are uniquely determined. In fact, the guardians  $\rho'(\alpha\beta^2)y_0$  and  $\rho'(\alpha\beta^2\alpha)y_0$  of the interval  $J_1^-$  are mapped by  $\rho'(\gamma_1)$  to the guardians  $\rho'(\beta^2\alpha)y_0$  and  $\rho'(\beta^2)y_0$  of  $J_1^+$ , and all the other eight points are mapped into the interval in  $J_1^+$  bounded by the latter guardians. The same is true for  $\gamma_1^{-1}$  and  $\gamma_2^{\pm 1}$ . Since  $\gamma_i^{\pm 1}$  are orientation preserving, the configuration of the points of the first and second generation is uniquely determined. Next we define points of third generation in a similar way. These points are contained in  $\bigcup_{i,\pm} J_i^\pm$ . For example, those contained in  $J_1^+$  are the images of the points of second

generation in  $S^1 \setminus J_1^-$  by  $\rho'(\gamma_1)$ . The configuration of these points, together with the points of first and second generations, is uniquely determined. On the other hand, any point in  $\rho'(\overline{G})y_0$  is a point of some generation. This is because the ten elements of (9.1) exhaust the set of right cosets  $[\overline{G}, \overline{G}] \setminus \overline{G}$ , and the elements  $\gamma_1$  and  $\gamma_2$  generate  $[\overline{G}, \overline{G}]$ . Therefore continuing this way, we see that the natural circular order of the whole orbit  $\rho'(\overline{G})y_0$  is uniquely determined, that is, the same as  $\rho(\overline{G})y_0$ . This shows that  $c' = c$ , i.e.  $c$  is isolated. Define  $c^{(1)}$  in Theorem 9.2 to be this  $c$ .

For  $k > 1$ , denote by  $p_k : S^1 \rightarrow S^1$  the  $k$ -fold covering map. A representation  $\rho^{(k)} : \overline{G} \rightarrow \text{Homeo}_+(S^1)$  is called a  $k$ -fold lift of  $\rho$  if  $p_k \rho^{(k)}(g) = \rho(g)p_k$  holds for any  $g \in \overline{G}$ . There is a  $k$ -fold lift  $\rho^{(k)}$  of our representation  $\rho$  if and only if  $k \equiv \pm 1 \pmod 6$ , and it is unique if it exists. Computation shows that if  $k = 6\ell \pm 1$ , then

$$\text{rot}(\rho^{(k)}(\alpha\beta)) = \mp \ell/k. \tag{9.2}$$

Notice that  $(k, \ell) = 1$ . We fix such  $k$ .

Let  $y_0^\mu$ ,  $1 \leq \mu \leq k$ , be the lifts of the point  $y_0$  by  $p_k$ . The natural circular order of the orbit  $\rho^{(k)}(\overline{G})y_0^\mu$  in  $S^1$  is the same for any  $\mu$ . Denote it by  $c^{(k)} \in CO(\overline{G})$ . It is easy to show that  $\rho^{(k)}$  is tight at  $y_0^\mu$ . Thus  $\rho^{(k)}$  is topologically conjugate to a dynamical realization of  $c^{(k)}$  at  $y_0^\mu$  by an orientation and base point preserving homeomorphism. Let us show that  $c^{(k)}$  is isolated. Let  $J_{i,\mu}^\pm$  ( $\mu = 1, \dots, k$ ) be the connected components of  $p_k^{-1}(J_i^\pm)$ .

Define a neighbourhood  $U^{(k)}$  of  $\rho^{(k)}$  such that each element  $\rho' \in U^{(k)}$  satisfies the following conditions.

- (1)  $\rho'(\gamma_i)$  maps each component of  $S^1 \setminus \bigcup_\mu J_{i,\mu}^-$  into  $J_{i,\nu}^+$  ( $i = 1, 2$ ), where  $\nu$  is determined so that  $\rho^{(k)}(\gamma_i)$  maps the same component into  $J_{i,\nu}^+$ .
- (2) The configuration of  $10k$  points  $\rho'(g)y_0^\mu$  in  $S^1$  ( $g$  as in (9.1) and  $1 \leq \mu \leq k$ ) is the same as  $\rho^{(k)}$ . Their configuration relative to  $J_{i,\mu}^\pm$  is also the same.

Then the same ping-pong argument as before shows that the natural circular order of  $\rho'(\overline{G})(p_k^{-1}(y_0))$  for  $\rho' \in U^{(k)}$  is uniquely determined. In particular, the natural circular order of  $\rho'(\overline{G})y_0^\mu$  is the same as for  $\rho^{(k)}$ , showing that  $c^{(k)}$  is isolated.

Finally let us show that  $c^{(k)}$ 's are not the automorphic images of the others, by considering their linear parts. In  $\overline{G} = PSL(2, \mathbb{Z})$ , any element of infinite order is a multiple of a unique primitive element. This can be shown by considering the modular representation  $\rho_M$ : the fixed point set of any element of infinite order is either one point of  $\partial\mathbb{H}$  or a two point set of  $\partial\mathbb{H}$ , and the isotropy group of the fixed point set is infinite cyclic.

As we have seen above, the linear part of  $c^{(1)}$  is generated by a primitive element  $(\alpha\beta)^{\pm 1}$ . The equality (9.2) shows that the linear part of  $c^{(k)}$  is generated by  $(\alpha\beta)^{\pm k}$ . For different choices of  $k$  and  $k'$ , there is no automorphism of  $\overline{G}$  which maps  $(\alpha\beta)^{\pm k}$  to  $(\alpha\beta)^{\pm k'}$ . In fact, any automorphism of  $\overline{G}$  maps a primitive element to a primitive element, and thus it should map  $(\alpha\beta)^{\pm k}$  to a  $k$ -multiple of a primitive element. This finishes the proof of Theorem 9.2. □

REMARK 9.4. The left order  $\lambda = \pi^*c^{(1)} \in LO(G)$  is the Dubrovina–Dubrovin order, since it satisfies  $e <_\lambda a <_\lambda b$ . It can be shown that  $\lambda' = \pi^*c^{(5)}$  is the unique left

order which satisfies

$$(ab)^5 t^{-4} <_{\lambda'} e <_{\lambda'} a <_{\lambda'} (ab)^5 t^{-4} a.$$

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